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The Hartle-Hawking-Israel state on spacetimes with stationary bifurcate Killing horizons

Christian GÉRARD

ABSTRACT. We consider a free massive quantized Klein-Gordon field in a spacetime (M, \mathbf{g}) containing a stationary bifurcate Killing horizon, i.e. a bifurcate Killing horizon whose Killing vector field is globally time-like in the right wedge \mathcal{M}^+ associated to the horizon.

We prove the existence of the *Hartle-Hawking-Israel* vacuum state, which is a pure state on the whole spacetime whose restriction to \mathcal{M}^+ is a thermal state ω_{T_H} for the time-like Killing field at Hawking temperature $T_H = \kappa(2\pi)^{-1}$, where κ is the surface gravity of the horizon.

We show that the HHI state is a Hadamard state and is the unique Hadamard state which is equal to the double T_H^{-1} -KMS state in the double wedge $\mathcal{M}^- \cup \mathcal{M}^+$. We construct the HHI state by Wick rotation in Killing time coordinates, using the notion of the Calderón projector for elliptic boundary value problems.

1. INTRODUCTION

In this paper we consider a free quantized Klein-Gordon field in a spacetime (M, \mathbf{g}) containing a stationary bifurcate Killing horizon. The problem we are interested in originates from the works by Hartle and Hawking [HH] and Israel [I].

Hartle and Hawking considered a free Klein-Gordon field in the Schwarzschild spacetime, and used formal path integral arguments to construct a distinguished Feynman propagator, which essentially amounts to constructing a distinguished state. This Feynman propagator is obtained by Wick rotation arguments, from a corresponding inverse for an elliptic operator. They showed that the state obtained from this formal procedure is a thermal state in the exterior region (or right wedge) at Hawking temperature $T_H = \kappa(2\pi)^{-1}$, where κ is the surface gravity of the horizon. Israel discussed the extension of this state to the left wedge of the Kruskal extension of the Schwarzschild spacetime. We refer the reader to [W, Chap. 5] for a more detailed discussion.

The first rigorous construction of this so-called Hartle-Hawking-Israel (HHI) state in the double wedge region of the Kruskal spacetime is due to Kay [K3]. This construction was valid for any temperature, the resulting state being an example of a *double KMS state*.

Later on Kay and Wald [KW] and Kay [K4] made a systematic study of states for linear scalar quantum fields in a globally hyperbolic spacetime with a bifurcate Killing horizon. They gave a rigorous definition of the *Hadamard condition* and emphasized its importance in this context.

They constructed subalgebras \mathcal{A} , resp. $\mathcal{A}^{L/R}$ of the free field algebra which are in some sense attached to the horizon resp. to its left/right portions, and showed

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that all states which are both invariant under the Killing isometries and Hadamard near the horizon have the same restriction to these subalgebras. This restriction must moreover be a KMS state (with respect to the Killing vector field generating the horizon) at the Hawking temperature on the right algebra \mathcal{A}^R .

They also showed non-existence results in suitable globally hyperbolic regions of the maximally extended Schwarzschild-de Sitter and Kerr spacetimes, (which both contain bifurcate Killing horizons), by proving that there does not exist *any* invariant state (not necessarily KMS) which is of Hadamard form near the horizon. (In the case of the Kerr spacetime the result needs a physically reasonable, but still unproved hypothesis of existence of some superradiant solutions of the wave equation.)

The first global construction of the HHI state is due to Sanders [S1], who considered spacetimes with a *static* bifurcate Killing horizon, i.e. such that the Killing vector field V is *static* in the exterior region. Sanders proved in [S1] the existence of the HHI state and showed that it is a pure Hadamard state. The proof in [S1] relied on the *Wick rotation* in the Killing time coordinates, which was also the basis for the heuristic arguments in [HH] and which we will also use in this paper.

In [G] we gave another proof of the Hadamard property of the HHI state in the situation considered in [S1], by combining the Wick rotation with a tool which is familiar in elliptic boundary value problems, namely the *Calderón projector*, see 1.2.2, which was also used in [GW2] to construct *analytic Hadamard states* on general analytic spacetimes. The use of Calderón projectors allows to construct the HHI state directly on a Cauchy surface Σ and avoids needing to consider its behavior near the Killing horizon.

In the present paper we consider the more general *stationary case*, and give a construction of the HHI state for spacetimes with a stationary bifurcate Killing horizon. At the end of Subsect. 1.1 we will comment on the relationships between the situations considered in [KW] or [S1] and the one considered in this paper.

1.1. Results. We now present in greater detail the main result of this paper.

1.1.1. *Bifurcate Killing horizons.* Let (M, \mathbf{g}) be a globally hyperbolic spacetime with a complete Killing vector field V . We use the 'mostly +' convention for the metric signature.

(M, \mathbf{g}) admits a *bifurcate Killing horizon* see [B, KW], if the *bifurcation surface* $\mathcal{B} = \{x \in M : V(x) = 0\}$ is an orientable submanifold of codimension 2, and if there exists a smooth space-like Cauchy surface Σ containing \mathcal{B} . We will always assume that \mathcal{B} is *compact and connected*.

M splits then into four globally hyperbolic regions, the *right/left wedges* \mathcal{M}^+ , \mathcal{M}^- and the *future/past cones* \mathcal{F} , \mathcal{P} , each invariant under the flow of V .

The *bifurcate Killing horizon* is then $\mathcal{H} = \partial(\mathcal{F} \cup \mathcal{P})$. An important object related with the bifurcate Killing horizon is its *surface gravity* κ , which is a scalar, constant over all of \mathcal{H} .

One also assumes the existence of a *wedge reflection* $R : M \rightarrow M$ which is an isometry of $(\mathcal{M}^- \cup U \cup \mathcal{M}^+, \mathbf{g})$, where U is a neighborhood of \mathcal{B} in M , such that $R \circ R = Id$, $R = Id$ on \mathcal{B} , R reverses the time orientation and $R^*V = V$. In concrete situations, the left wedge \mathcal{M}^- is actually constructed by reflection of the right wedge \mathcal{M}^+ , so the existence of a wedge reflection does not seem to be such a strong hypothesis.

The bifurcate Killing horizon \mathcal{H} is *stationary* resp. *static* if there exists a Cauchy surface satisfying the above requirements such that, in addition, V is *time-like* on $\Sigma \setminus \mathcal{B}$, resp. orthogonal to $\Sigma \setminus \mathcal{B}$. For technical reasons, we require V to be *uniformly time-like* near infinity on Σ , see Subsect. 2.4. This condition is imposed

only far away from the bifurcation surface \mathcal{B} and will hold for example if (M, \mathbf{g}) is asymptotically flat near spatial infinity.

We consider on (M, \mathbf{g}) a free quantum Klein-Gordon field associated to the Klein-Gordon equation

$$-\square_{\mathbf{g}}\phi(x) + m(x)\phi(x) = 0,$$

where $m \in C^\infty(M, \mathbb{R})$ is invariant under V and R . We assume that $m(x) \geq m_0^2 > 0$ i.e. the Klein-Gordon field is massive.

1.1.2. *The double β -KMS state.* Since $(\mathcal{M}^+, \mathbf{g}, V)$ is a stationary spacetime, there exists (see [S2]) for any $\beta > 0$ a *thermal state* ω_β at temperature β^{-1} with respect to the group of Killing isometries of $(\mathcal{M}^+, \mathbf{g})$ generated by V .

The wedge reflection $R : \mathcal{M}^+ \xrightarrow{\sim} \mathcal{M}^-$ allows to extend ω_β to the *double β -KMS state* $\omega_{\mathcal{D}}$ on $\mathcal{M}^+ \cup \mathcal{M}^-$. This extension exists for any $\beta > 0$ and is a pure state in $\mathcal{M}^+ \cup \mathcal{M}^-$.

1.1.3. *Main result.* We prove in this paper the following theorem.

Theorem 1.1. *Let (M, \mathbf{g}, V) be a globally hyperbolic spacetime with a stationary bifurcate Killing horizon and a wedge reflection. Let $P = -\square_{\mathbf{g}} + V$ be a Klein-Gordon operator invariant under the Killing vector field V and the wedge reflection R . Assume moreover that the Cauchy surface Σ containing \mathcal{B} can be chosen so that conditions (H1)—(H4) in Subsects. 2.1, 2.3, 2.4 are satisfied.*

Then there exists a state ω_{HHI} for P in (M, \mathbf{g}) called the Hartle-Hawking-Israel state such that:

- (1) ω_{HHI} is a pure Hadamard state in M ,
- (2) the restriction of ω_{HHI} to $\mathcal{M}^+ \cup \mathcal{M}^-$ is the double β -KMS state $\omega_{\mathcal{D}}$ at Hawking temperature $T_{\text{H}} = \kappa(2\pi)^{-1}$ where κ is the surface gravity of the horizon,
- (3) ω_{HHI} is the unique extension of $\omega_{\mathcal{D}}$ such that its spacetime covariances Λ^\pm map $C_c^\infty(M)$ into $C^\infty(M)$ continuously. In particular it is the unique Hadamard extension of $\omega_{\mathcal{D}}$.

Thm. 1.1 will be proved in Sects. 9 and 10.

1.1.4. *Some comments.* Let us now comment on the hypotheses (H1)—(H4) that we impose in this paper. Conditions (H1i) and H2i) are the standard conditions needed to setup the problem, namely existence of a bifurcate Killing horizon with a wedge reflection, and invariance of the Klein-Gordon operator under the Killing field and wedge reflection.

Condition (H1ii) requires that the bifurcation surface \mathcal{B} is compact and connected.

The fact that \mathcal{B} is compact allows to introduce Gaussian normal coordinates to \mathcal{B} in the Cauchy surface Σ and to construct the smooth extension $(M_{\text{ext}}^{\text{eucl}}, \mathbf{g}_{\text{ext}}^{\text{eucl}})$ of the Wick rotated manifold $(\mathbb{S}_\beta \times \Sigma^+, \mathbf{g}^{\text{eucl}})$, (for β equal to the inverse Hawking temperature), where \mathbf{g}^{eucl} is the complex metric obtained from \mathbf{g} by Wick rotation in the Killing time coordinate.

This assumption excludes of course the Minkowski spacetime, when the Killing vector field X is a boost generator, for which the above results are very well-known.

The fact that \mathcal{B} is connected implies that the surface gravity is constant on \mathcal{B} . This excludes for example the extended Schwarzschild-de Sitter spacetime, see [KW], where the bifurcation surface has two connected components. It is quite likely that the results of this paper can be extended if \mathcal{B} is compact with several connected components, with the same surface gravity.

Finally let us recall that for the extended Kerr spacetime, the Killing vector field becomes space-like at space-like infinity. The horizon is not stationary and hence the Kerr spacetime is also outside the scope of this paper. Note that it has been

shown in [KW, Sect. 6.2] that if the Killing field is not everywhere time-like then there exists no associated KMS state in the exterior region, hence in particular no HHI state.

Condition (H2ii) requires that the Klein-Gordon field is massive. Note that in [S1], the weaker condition $m(x) > 0$ was assumed.

Conditions (H3 H4) concern the behavior of the metric \mathbf{g} and of the Killing vector field V near infinity on Σ . (H4), i.e. completeness of (Σ, \mathbf{h}) , where \mathbf{h} is the Riemannian metric induced by \mathbf{g} on Σ is needed to prove the purity of the HHI state.

1.2. Main ideas of the construction. We now outline the construction of the HHI state ω_{HHI} . We look for ω_{HHI} as an extension to M of the double β -KMS state ω_{D} on $\mathcal{M}^- \cup \mathcal{M}^+$, where $\beta^{-1} = \kappa(2\pi)^{-1}$ is the Hawking temperature. The first step consists in understanding in sufficient detail the β -KMS state in \mathcal{M}^+ .

Writing the metric \mathbf{g} in \mathcal{M}^+ using the Killing time coordinate associated to V and Σ , \mathcal{M}^+ is identified with $\mathbb{R} \times \Sigma^+$, where $\Sigma^+ = \mathcal{M}^+ \cap \Sigma$ and the metric \mathbf{g} becomes

$$(1.1) \quad \mathbf{g} = -N^2(y)dt^2 + \mathbf{h}_{ij}(y)(dy^i + \mathbf{w}^i(y)dt)(dy^j + \mathbf{w}^j(y)dt),$$

where N is the lapse function, \mathbf{w} the shift vector field, \mathbf{h} the induced metric on Σ . The Killing field V is simply $\frac{\partial}{\partial t}$. The fact that V is time-like in \mathcal{M}^+ is equivalent to the inequality $N^2(y) > \mathbf{w}^i(y) \cdot \mathbf{h}_{ij}(y) \mathbf{w}^j(y)$ for $y \in \Sigma^+$.

The Klein-Gordon operator P associated to \mathbf{g} can be written as:

$$(1.2) \quad P = (\partial_t + w^*)N^{-2}(\partial_t - w) + h_0,$$

where $w = \mathbf{w}^i \cdot \partial_{y^i}$, (considered as a first order differential operator), and $h_0 = \nabla^* \mathbf{h}^{-1} \nabla + m$ is an elliptic operator on Σ .

1.2.1. The Wick rotation. The *Wick rotation* consists in replacing t by is and produces the *complex metric*

$$(1.3) \quad \mathbf{g}^{\text{eucl}} = N^2(y)ds^2 + \mathbf{h}_{ij}(y)(dy^i + i\mathbf{w}^i(y)ds)(dy^j + i\mathbf{w}^j(y)ds).$$

In the static case considered in [S1, G] \mathbf{w} vanishes and \mathbf{g}^{eucl} is Riemannian. The fact that \mathbf{g}^{eucl} is now a complex metric causes several new difficulties. Performing the same transformation on P yields the *Wick rotated operator*

$$K = -(\partial_s + iw^*)N^{-2}(\partial_s + iw) + h_0.$$

There are several different linear operators that can be associated to the formal expression K . The first one consists in working on $L^2(\mathbb{R} \times \Sigma^+)$, using the sesquilinear form

$$Q_\infty(u, u) = \|N^{-1}\partial_s u\|^2 + (u|hu) - i(N^{-1}\partial_s u|N^{-1}wu) - i(N^{-1}wu|N^{-1}\partial_s u),$$

where $h = h_0 - w^*N^{-2}w$, with $\text{Dom}Q_\infty = C_c^\infty(\mathbb{R} \times \Sigma^+)$. Another possibility is to work on $L^2(\mathbb{S}_\beta \times \Sigma^+)$ where $\mathbb{S}_\beta = [-\frac{\beta}{2}, \frac{\beta}{2}]$ is the circle of length β . The sesquilinear form Q_β has the same expression as Q_∞ but the domain is now $\text{Dom}Q_\beta = C_c^\infty(\mathbb{S}_\beta \times \Sigma^+)$, which corresponds to imposing β -periodic boundary conditions on K .

Since we have assumed that V is *uniformly time-like* near infinity, see Subject. 2.4, one can show that the sesquilinear forms Q_∞, Q_β are closeable and *sectorial* and hence generate *injective* linear operators K_∞, K_β . Their inverses $K_\infty^{-1}, K_\beta^{-1}$ are then well defined between abstract Sobolev spaces, using the Lax-Milgram theorem.

1.2.2. *Calderón projectors.* Let $\Omega \subset \mathbb{R}^n$ an open set with smooth *compact* boundary and $P = P(x, \partial_x)$ a second order elliptic operator on \mathbb{R}^n . Let us set $\Omega^+ = \Omega, \Omega^- = \mathbb{R}^n \setminus \Omega^{\text{cl}}$. If $u \in \overline{\mathcal{D}'(\Omega^\pm)}$ is an extendible distribution in Ω^\pm such that $Pu = 0$ in Ω^\pm then its traces $\gamma^\pm u = \begin{pmatrix} u|_{\partial\Omega} \\ \pm \nu \cdot du|_{\partial\Omega} \end{pmatrix}$, where ν is a vector field transverse to $\partial\Omega$, are well defined in $\mathcal{D}'(\partial\Omega; \mathbb{C}^2)$.

The spaces $Z^\pm = \{\gamma^\pm u : u \in \overline{\mathcal{D}'(\Omega^\pm)}, Pu = 0 \text{ in } \Omega^\pm\}$ are two complementary spaces in $\mathcal{D}'(\partial\Omega; \mathbb{C}^2)$ and the associated projections c^\pm are called *Calderón projectors* for P and $\partial\Omega$. Assuming P to be invertible, they can be easily expressed in terms of P^{-1} , see eg. Def. 9.4 below.

The expressions giving c^\pm still make sense if $\partial\Omega$ is not compact, for example as operators $c^\pm : \mathcal{E}'(\partial\Omega; \mathbb{C}^2) \rightarrow \mathcal{D}'(\partial\Omega; \mathbb{C}^2)$ but they are not projections on $\mathcal{E}'(\partial\Omega; \mathbb{C}^2)$ since they do not preserve this space. Nevertheless we will still call c^\pm Calderón projectors.

We will use Calderón projectors denoted by $c_\beta^\pm, \beta \in]0, +\infty]$ for the open sets $\Omega_\infty =]0, +\infty[\times \Sigma^+, \Omega_\beta =]0, \frac{\beta}{2}[\times \Sigma^+$ and elliptic operators K_∞, K_β, ν being the exterior unit normal for \mathbf{g}^{eucl} to $\partial\Omega_\beta, \beta \in]0, +\infty]$. Note that ν is a *complex* vector field, but its imaginary part is tangent to $\partial\Omega_\beta$. The expression of c_β^\pm in terms of the inverse K_β^{-1} is given in Subsect. 8.7.

1.2.3. *Vacuum and double β -KMS states.* If $\beta = \infty$, the boundary $\partial\Omega_\infty$ equals Σ^+ , and one can try to construct a state in \mathcal{M}^+ by defining its covariances on Σ^+ as

$$\lambda_\infty^\pm = \pm q \circ c_\infty^\pm,$$

where $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the charge defining the symplectic structure on the space $C_c^\infty(\Sigma^+; \mathbb{C}^2)$ of Cauchy data on Σ^+ . It turns out that λ_∞^\pm are actually the covariances of the *vacuum state* ω_{vac} in \mathcal{M}^+ .

Of course the study of the vacuum state ω_{vac} , corresponding to $\beta = \infty$, is not necessary for the construction of the HHI state, but gives a nice introduction to the more complicated case $\beta < \infty$.

If $\beta < \infty$, the boundary $\partial\Omega_\beta$ has two components, both isomorphic to Σ^+ . The state ω_{D} obtained similarly from the Calderón projectors c_β^\pm is now the *double β -KMS state* ω_{D} in $\mathcal{M}^- \cup \mathcal{M}^+$, modulo the identification of Σ^+ with Σ^- by the wedge reflection.

The proof of these facts takes up a large part of the paper. First of all we reduce ourselves to the situation $N(y) = 1$ by considering $\tilde{P} = NPN$ and $\tilde{K}_\beta = NK_\beta N$, the last identity taking a rather transparent form if we use the framework of sesquilinear forms, see Subsect. 8.6. The covariances of $\omega_{\text{vac}}, \omega_{\text{D}}$ for the Klein-Gordon operator P can similarly be deduced from those of the analogous states $\tilde{\omega}_{\text{vac}}, \tilde{\omega}_{\text{D}}$ for \tilde{P} .

The operator \tilde{P} can be written as $(\partial_t + \tilde{w}^*)(\partial_t - \tilde{w}) + \tilde{h}_0$, and the computations of $\tilde{\omega}_{\text{vac}}, \tilde{\omega}_{\text{D}}$ can be done by reducing the Klein-Gordon equation $\tilde{P}\tilde{\phi} = 0$ to a first order system $\partial_t f - iHf = 0$, see Sects. 6, 7. This system is an example of a *stable symplectic dynamics*, which is studied in Sects. 4, 5.

1.2.4. *The surface gravity and the extended Euclidean metric.* All the constructions up to now are valid for any value of the inverse temperature β . If $\beta = (2\pi)\kappa^{-1}$, i.e. if β^{-1} equals the *Hawking temperature* $\kappa(2\pi)^{-1}$, where κ is the *surface gravity* of the horizon, one can show that $(\mathbb{S}_\beta \times \Sigma^+, \mathbf{g}^{\text{eucl}})$ has a unique extension $(M_{\text{ext}}^{\text{eucl}}, \mathbf{g}_{\text{ext}}^{\text{eucl}})$, which corresponds exactly to passing from polar to cartesian coordinates in the plane.

1.2.5. *The Hartle-Hawking-Israel state.* The open set $]0, \frac{\beta}{2}[\times \Sigma^+$ extends as an open set Ω_{ext} with boundary isomorphic to the *full Cauchy surface* Σ . The Wick rotated operator K_β extends as an elliptic operator K_{ext} acting on $M_{\text{ext}}^{\text{eucl}}$, and one can consider the Calderón projectors c_{ext}^\pm associated to K_{ext} and Ω_{ext} .

One defines the covariances on Σ

$$\lambda_{\text{HHI}}^\pm = \pm q \circ c_{\text{ext}}^\pm,$$

and one can rather easily show that λ_{HHI}^\pm are the covariances of a quasi-free state ω_{HHI} defined on the whole of M . One uses that the restriction of λ_{HHI}^\pm to $C_c^\infty(\Sigma \setminus \mathcal{B})$ are precisely the covariances of the double β -KMS state ω_{D} , and some continuity properties of Calderón projectors and density results in Sobolev spaces, see Subsect. 9.2.

One can also prove that the HHI state ω_{HHI} is a Hadamard state, by an argument already used in [G] in the static case, relying on the fact that the covariances of any Hadamard state on Σ are matrices of pseudodifferential operators.

The proof of the purity of ω_{HHI} is more complicated and relies on arguments from [GW2].

1.3. Notations. We now collect some notation.

We set $\langle \lambda \rangle = (1 + \lambda^2)^{\frac{1}{2}}$ for $\lambda \in \mathbb{R}$.

We write $A \Subset B$ if A is relatively compact in B .

If X, Y are sets and $f : X \rightarrow Y$ we write $f : X \xrightarrow{\sim} Y$ if f is bijective. If X, Y are equipped with topologies, we write $f : X \rightarrow Y$ if the map is continuous, and $f : X \xrightarrow{\sim} Y$ if it is a homeomorphism.

1.3.1. *Duals and antiduals.* Let \mathcal{X} be a real vector space. Its dual will be denoted by $\mathcal{X}^\#$. Let \mathcal{Y} be a complex vector space. We denote by $\mathcal{Y}_{\mathbb{R}}$ its *real form*, i.e. \mathcal{Y} as a vector space over \mathbb{R} . We denote by $\mathcal{Y}^\#$ its dual, i.e. the space of \mathbb{C} -linear forms on \mathcal{Y} and by \mathcal{Y}^* its anti-dual, i.e. the space of \mathbb{C} -antilinear forms on \mathcal{Y} .

We denote by $\overline{\mathcal{Y}}$ the *conjugate vector space* to \mathcal{Y} , i.e. $\overline{\mathcal{Y}} = \mathcal{Y}_{\mathbb{R}}$ as a \mathbb{R} -vector space, equipped with the complex structure $-i$, if $i \in L(\mathcal{Y}_{\mathbb{R}})$ is the complex structure of \mathcal{Y} . The (anti-linear) identity map $Id : \mathcal{Y} \rightarrow \overline{\mathcal{Y}}$ will be denoted by $y \mapsto \overline{y}$, i.e. \overline{y} equals y but considered as an element of $\overline{\mathcal{Y}}$.

If \mathcal{Y} is a Hilbert space, then $\overline{\mathcal{Y}}$ inherits also a Hilbert space structure by

$$(\overline{y_1} | \overline{y_2})_{\overline{\mathcal{Y}}} := \overline{(y_1 | y_2)_{\mathcal{Y}}}.$$

By definition we have $\mathcal{Y}^* = \overline{\mathcal{Y}^\#}$. Note that we have a \mathbb{C} -linear identification $\overline{\mathcal{Y}^\#} \sim \mathcal{Y}^*$ defined as follows: if $y \in \mathcal{Y}$ and $w \in \mathcal{Y}^\#$ then

$$\overline{w \cdot y} := \overline{w} \cdot y$$

This identifies $\overline{w} \in \overline{\mathcal{Y}^\#}$ with an element of \mathcal{Y}^* . Similarly we have a \mathbb{C} -linear identification $\overline{\mathcal{Y}^*} \sim \mathcal{Y}^\#$.

1.3.2. *Linear operators.* If \mathcal{X}_i , $i = 1, 2$ are real or complex vector spaces and $a \in L(\mathcal{X}_1, \mathcal{X}_2)$ we denote by $a^\# \in L(\mathcal{X}_2^\#, \mathcal{X}_1^\#)$ its transpose. If \mathcal{Y}_i , $i = 1, 2$ are complex vector spaces we denote by $a^* \in L(\mathcal{Y}_2^*, \mathcal{Y}_1^*)$ its adjoint, and by $\overline{a} \in L(\overline{\mathcal{Y}}_1, \overline{\mathcal{Y}}_2)$ its *conjugate*, defined by $\overline{a} \overline{y_1} = \overline{a y_1}$. With the above identifications we have $a^* = \overline{a^\#} = \overline{a^\#}$.

1.3.3. *Bilinear and sesquilinear forms.* If \mathcal{X} is a real or complex vector space, a bilinear form on \mathcal{X} is given by $a \in L(\mathcal{X}, \mathcal{X}^\#)$, its action on a couple (x_1, x_2) is denoted by $x_1 \cdot a x_2$. We denote by $L_{\text{s/a}}(\mathcal{X}, \mathcal{X}^\#)$ the symmetric/antisymmetric forms on \mathcal{X} . a is *non-degenerate* if $\text{Ker } a = \{0\}$. An antisymmetric, non-degenerate form σ is called a *symplectic form* on \mathcal{X} .

Similarly if \mathcal{Y} is a complex vector space, a sesquilinear form on \mathcal{Y} is given by $a \in L(\mathcal{Y}, \mathcal{Y}^*)$, its action on a couple (y_1, y_2) is denoted by $\overline{y_1} \cdot a y_2$, the last notation being a reminder that $\mathcal{Y}^* = \overline{\mathcal{Y}^\#}$. We denote by $L_{h/a}(\mathcal{Y}, \mathcal{Y}^*)$ the Hermitian/anti-Hermitian forms on \mathcal{Y} . Non-degenerate forms are defined as in the real case. An anti-Hermitian, non-degenerate form σ is called a (complex) *symplectic form* on \mathcal{Y} .

If $a \in L(\mathcal{Y}, \mathcal{Y}^*)$ then $\overline{a} \in L(\overline{\mathcal{Y}}, \overline{\mathcal{Y}^*})$ and with the above identifications we have $y_1 \cdot \overline{a y_2} = \overline{\overline{y_1} \cdot a y_2}$ for $y_1, y_2 \in \mathcal{Y}$.

1.3.4. *Linear operators on Hilbert spaces.* The domain of a closed, densely defined operator a on a Hilbert space \mathcal{H} will be denoted by Doma , equipped with the graph norm, its spectrum by $\sigma(a)$ and its resolvent set by $\text{res}(a)$. We will similarly denote by $\text{Dom}Q$ the domain of a sesquilinear form Q . If Q is closeable we denote by Q^{cl} its closure.

1.3.5. *Scale of Hilbert spaces associated to a selfadjoint operator.* If a is selfadjoint on \mathcal{H} , we write $a > 0$ if $a \geq 0$ and $\text{Ker } a = \{0\}$. If $a > 0$ and $s \in \mathbb{R}$ we denote by $a^s \mathcal{H}$ the completion of Doma^{-s} for the norm $\|a^{-s}u\|_{\mathcal{H}}$. Equipped with the scalar product $(u|v)_s = (a^{-s}u|a^{-s}v)_{\mathcal{H}}$, it is a Hilbert space. The spaces $a^s \mathcal{H}$ and $a^{-s} \mathcal{H}$ form a dual pair for the duality pairing $\langle u|v \rangle = (a^{-s}u|a^s v)_{\mathcal{H}}$.

We define similarly the spaces $\langle a \rangle^s \mathcal{H}$ for any selfadjoint operator a on \mathcal{H} . We have $\langle a \rangle^{-s} \mathcal{H} = \text{Dom}|a|^s$ for $s > 0$. We have $\langle a \rangle^{-s} \mathcal{H} \subset \mathcal{H} \subset \langle a \rangle^s \mathcal{H}$ for $s \geq 0$ and $\langle a \rangle^s \mathcal{H} = |a|^s \mathcal{H}$ if $0 \notin \sigma(a)$.

The notation $\langle a \rangle^s \mathcal{H}$ or $a^s \mathcal{H}$ is very convenient but somewhat ambiguous because usually $a\mathcal{H}$ denotes the image of \mathcal{H} under the linear operator a . Let us explain how to reconcile these two meanings.

Let \mathcal{H}_c be the space of $u \in \mathcal{H}$ such that $u = \mathbb{1}_I(a)u$, for some compact $I \subset \mathbb{R} \setminus \{0\}$. We equip \mathcal{H}_c with its natural topology by saying that $u_n \rightarrow u$ in \mathcal{H}_c if there exists $I \subset \mathbb{R} \setminus \{0\}$ compact such that $u_n = \mathbb{1}_I(a)u_n$ for all n and $u_n \rightarrow u$ in \mathcal{H} . We denote by \mathcal{H}_{loc} the topological anti-dual of \mathcal{H}_c . Then $|a|^s$ and $\langle a \rangle^s$ preserve \mathcal{H}_c and \mathcal{H}_{loc} , and $\langle a \rangle^s \mathcal{H}$, resp. $|a|^s \mathcal{H}$ are the images in \mathcal{H}_{loc} of \mathcal{H} under $\langle a \rangle^s$, resp. $|a|^s$. It follows that these spaces are subspaces (equipped with finer topologies) of \mathcal{H}_{loc} , in particular they are pairwise compatible and one can consider their intersections inside \mathcal{H}_{loc} . It is easy to verify, using the spectral decomposition of a that for example

$$(1.4) \quad \mathcal{H} \cap a^s \mathcal{H} = \text{Doma}^{-s}, \quad s \in \mathbb{R}$$

if we equip $\mathcal{H} \cap a^s \mathcal{H}$ with the norm $\|\cdot\| + \|\cdot\|_s$.

1.3.6. *Operator inequalities.* If a_1, a_2 are selfadjoint on \mathcal{H} with $a_1, a_2 > 0$ we write $a_1 \lesssim a_2$ if $\text{Doma}_1^{\frac{1}{2}} \supset \text{Doma}_2^{\frac{1}{2}}$ and $a_1 \leq c a_2$ on $\text{Doma}_2^{\frac{1}{2}}$ for some $c > 0$. We write $a_1 \sim a_2$ if $a_1 \lesssim a_2$ and $a_2 \lesssim a_1$.

If $a_1 \sim a_2$ the Kato-Heinz theorem implies that $a_2^{-1} \sim a_1^{-1}$ and that $a_1^s \mathcal{H} = a_2^s \mathcal{H}$ as Banach spaces for $s \in [-\frac{1}{2}, \frac{1}{2}]$.

Similarly if q_1, q_2 are two positive quadratic forms with $q_i(u, u) = 0 \Rightarrow u = 0$, we write $q_1 \lesssim q_2$ if $\text{Dom}q_1 \supset \text{Dom}q_2$ and $q_1 \leq c q_2$ on $\text{Dom}q_2$ and we write $q_1 \sim q_2$ if $q_1 \lesssim q_2$ and $q_2 \lesssim q_1$.

1.3.7. *Differential operators on manifolds.* If X is a smooth manifold and a, b are differential operators on X the composition $a \circ b$ is denoted by ab . If a is a differential operator on X and $u \in C^\infty(X)$, then au denotes the composition of a with the operator of multiplication by u , while $(au) \in C^\infty(X)$ denotes the image of u under a .

1.3.8. *Spaces of distributions.* Let X be a smooth manifold. Fixing a smooth density we identify distributions and distributional densities on X . If $\Omega \subset X$ is an open set with smooth boundary and $F(X) \subset \mathcal{D}'(X)$ is a vector space, we denote by $\overline{F}(\Omega) \subset \mathcal{D}'(\Omega)$ the space of *restrictions* of elements of $F(X)$ to Ω .

We denote by $\delta_a \in \mathcal{D}'(\mathbb{R})$ the Dirac distribution at $a \in \mathbb{R}$.

2. SPACETIMES WITH A STATIONARY BIFURCATE KILLING HORIZON

In this section we recall the definition of spacetimes with bifurcate Killing horizons, see [B, KW]. We express various natural objects, like the lapse function, shift vector field and induced Riemannian metric in Gaussian coordinates near the bifurcation surface.

We then consider the *Wick rotated metric* \mathbf{g}^{eucl} , obtained by the Wick rotation $t \rightarrow is$ in the Killing time t , and show that if s belongs to the circle $\mathbb{S}_{(2\pi)\kappa^{-1}}$ of length $(2\pi)\kappa^{-1}$, for κ the surface gravity of the horizon, \mathbf{g}^{eucl} has a smooth extension up to the bifurcation surface \mathcal{B} . This fundamental fact, already known for static horizons, see [S1, Sect. 2.2] lies at the basis of the construction of the HHI state in later sections.

2.1. Bifurcate Killing horizons.

Definition 2.1. *A spacetime with a bifurcate Killing horizon is a triple (M, \mathbf{g}, V) such that :*

- (1) (M, \mathbf{g}) is a globally hyperbolic spacetime,
- (2) V is a smooth, complete Killing vector field on (M, \mathbf{g}) ,
- (3) $\mathcal{B} := \{x \in M : V(x) = 0\}$ is an orientable submanifold of codimension 2, called the bifurcation surface,
- (4) there exists a smooth, space-like Cauchy hypersurface Σ with $\mathcal{B} \subset \Sigma$.

If n is the future directed unit normal vector field to Σ , one defines the *lapse function* $N \in C^\infty(\Sigma)$ and *shift vector field* \mathbf{w} , which is a smooth vector field tangent to Σ , by

$$V = Nn + \mathbf{w} \text{ on } \Sigma,$$

ie

$$N := -V \cdot \mathbf{g}n, \quad \mathbf{w} := V - Nn \text{ on } \Sigma.$$

Let us denote by y the elements of Σ . The Cauchy surface Σ is then decomposed as

$$\Sigma = \Sigma^- \cup \mathcal{B} \cup \Sigma^+, \quad \Sigma^\pm := \{y \in \Sigma : \pm N(y) > 0\},$$

ie V is future/past directed over Σ^\pm .

The spacetime M splits as

$$M = \mathcal{M}^+ \cup \mathcal{M}^- \cup \overline{\mathcal{F}} \cup \overline{\mathcal{P}},$$

where the *future cone* $\mathcal{F} := I^+(\mathcal{B})$, the *past cone* $\mathcal{P} := I^-(\mathcal{B})$, the *right/left wedges* $\mathcal{M}^\pm := D(\Sigma^\pm)$, are all globally hyperbolic when equipped with \mathbf{g} .

The future cone \mathcal{F} may be a black hole. The *bifurcate Killing horizon* is then

$$\mathcal{H} := \partial\mathcal{F} \cup \partial\mathcal{P}.$$

The Killing vector field V is tangent to \mathcal{H} . In Figure 1 below the vector field V is represented by arrows.

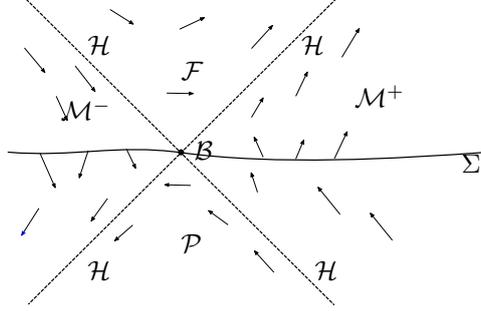


FIGURE 1.

The following definition is due to Sanders [S1].

Definition 2.2. A triple (M, \mathbf{g}, V) as in Def. 2.1 is called a spacetime with a stationary, resp. static bifurcate Killing horizon if one can find a Cauchy hypersurface Σ as in Def. 2.1 such that in addition V is time-like on $\Sigma \setminus \mathcal{B}$, resp. g -orthogonal to $\Sigma \setminus \mathcal{B}$.

2.2. Wedge reflection. Additionally one assumes the existence of a *wedge reflection*, see [S1, Def. 2.6].

Definition 2.3. A wedge reflection R for a spacetime (M, \mathbf{g}, V) with a stationary Killing horizon is a diffeomorphism $R : \mathcal{M}^- \cup U \cup \mathcal{M}^+ \xrightarrow{\sim} \mathcal{M}^- \cup U \cup \mathcal{M}^+$, where U is a neighborhood of \mathcal{B} in M such that:

- (1) R is an isometry of $(\mathcal{M}^- \cup U \cup \mathcal{M}^+, \mathbf{g})$ which reverses the time orientation,
- (2) $R \circ R = Id$, $R = Id$ on \mathcal{B} ,
- (3) $R^*V = V$.

2.2.1. Weak wedge reflection. It is known, see [S1, Prop. 2.7] that if R is a wedge reflection, one can find a Cauchy surface Σ as in Def. 2.1 such that $R : \Sigma \xrightarrow{\sim} \Sigma$. The map $r := R|_{\Sigma}$ is called a *weak wedge reflection*. If the Riemannian metric \mathbf{h} is the restriction of \mathbf{g} to Σ , one has:

- (1) r is an isometry of (Σ, \mathbf{h}) with $r \circ r = Id$,
- (2) $r = Id$ on \mathcal{B} ,
- (3) $r^*N = -N$, $r^*\mathbf{w} = \mathbf{w}$.

By (3) above we have $r : \Sigma^{\pm} \xrightarrow{\sim} \Sigma^{\mp}$. Henceforth, Σ will denote a Cauchy surface with all these properties.

Let us summarize the assumptions we will make on the spacetime (M, \mathbf{g}, V) :

- (H1i) (M, g, V) is a spacetime with a stationary bifurcate Killing horizon, admitting a wedge reflection R ,
- (H1ii) the bifurcation surface \mathcal{B} is *compact and connected*.

2.3. Klein-Gordon operators. We fix a real function $m \in C^{\infty}(M)$. We will assume the following:

- (H2i) $V^a \nabla_a m(x) = 0$, $m \circ R(x) = m(x)$, $x \in \mathcal{M}^+ \cup \mathcal{M}^- \cup U$,
- (H2ii) $m(x) \geq m_0^2 > 0$, $x \in M$,

ie as in [S1] we assume that m is stationary w.r.t. the Killing vector field V and invariant under the wedge reflection, but we consider only *massive* Klein-Gordon fields, where in [S1] the weaker condition $m(x) > 0$ as allowed.

The Klein-Gordon operator is

$$(2.1) \quad P = -\square_{\mathbf{g}} + m.$$

2.4. Conditions near infinity on Σ . It will be necessary, in order to control various energy spaces in Sect. 8, to impose conditions on the Killing vector field V near infinity on Σ .

$\exists U$ neighborhood of \mathcal{B} in Σ such that:

$$(H3i) \quad V + \delta \mathbf{w} \text{ is time-like on } \Sigma \setminus U \text{ for some } \delta > 0,$$

$$(H3ii) \quad N^{-2} \mathbf{w}^i \cdot (\nabla_i^{\mathbf{h}} N), \quad N^{-1} \nabla_i^{\mathbf{h}} \mathbf{w}^i \text{ are bounded on } \Sigma \setminus U,$$

where we recall that \mathbf{h} is the restriction of \mathbf{g} to Σ .

(H3i) has a clear geometrical meaning since it means that V is *uniformly time-like* near infinity on Σ . (H3ii) depends only on the asymptotic behavior of \mathbf{g} on Σ near infinity, so it is easy to check in practice.

Finally in order to prove the purity of the HHI state constructed in Sect. 9, we will need the following condition

$$(H4) \quad (\Sigma, \mathbf{h}) \text{ is complete.}$$

2.5. The surface gravity. The surface gravity is defined by:

$$\kappa^2 = -\frac{1}{2} (\nabla^{(\mathbf{g})b} V^a \nabla_b^{(\mathbf{g})} V_a)|_{\mathcal{B}}, \quad \kappa > 0.$$

It is a fundamental fact, see [KW, Sect. 2], that κ is constant on \mathcal{B} and actually on the whole horizon \mathcal{H} .

For $\omega \in \mathcal{B}$ let $n_\omega \in T_\omega \Sigma$ be the unit normal to \mathcal{B} for \mathbf{h} pointing towards Σ^+ . We introduce Gaussian normal coordinates to \mathcal{B} in (Σ, \mathbf{h}) by:

$$\chi: \quad] - \delta, \delta[\times \mathcal{B} \rightarrow \Sigma \\ (u, \omega) \mapsto \exp_{\omega}^{\mathbf{h}}(un_\omega)$$

which is a smooth diffeomorphism from $] - \delta, \delta[\times \mathcal{B}$ to a relatively compact neighborhood U of \mathcal{B} in Σ . In the next proposition we express \mathbf{h} , N , w and the wedge reflection r in the local coordinates (u, ω) on U . We recall that the elements of Σ are denoted by y .

Proposition 2.4. *On U one has:*

$$(2.2) \quad r(u, \omega) = (-u, \omega),$$

and

$$(2.3) \quad \mathbf{h}_{ij}(y) dy^i dy^j = du^2 + \mathbf{k}_{\alpha\beta}(u, \omega) d\omega^\alpha d\omega^\beta, \\ \mathbf{w}^i(y) \partial_{y^i} = \mathbf{w}^0(u, \omega) \partial_u + \mathbf{w}^\alpha(u, \omega) \partial_{\omega^\alpha}, \\ N(y) = N(u, \omega), \\ m(y) = m(u, \omega),$$

where $\mathbf{k}_{\alpha\beta}(u, \omega) d\omega^\alpha d\omega^\beta$ is a smooth, u -dependent Riemannian metric on \mathcal{B} with:

$$(2.4) \quad N(u, \omega) = u(\kappa + u^2 d(u^2, \omega)), \\ \mathbf{w}^0(u, \omega) = u^3 b(u^2, \omega), \quad \mathbf{w}^\alpha(u, \omega) = u^2 c^\alpha(u^2, \omega), \\ \mathbf{k}_{\alpha\beta}(u, \omega) = d_{\alpha\beta}(u^2, \omega), \\ m(u, \omega) = n(u^2, \omega)$$

for smooth maps $b, d, n, c^\alpha, d_{\alpha\beta}$ defined on $[-\epsilon, \epsilon] \times \mathcal{B}$ for some $\epsilon > 0$ with

$$n(0, \omega) \geq c > 0 \text{ for some } c > 0.$$

The proof of Prop. 2.4 is given in Appendix A.1.

2.6. The metric in \mathcal{M}^+ . Let us denote by Φ_t the flow of the Killing vector field V . We identify $\mathbb{R} \times \Sigma^+$ with \mathcal{M}^+ by

$$\chi : \mathbb{R} \times \Sigma^+ \ni (t, y) \mapsto \Phi_t(y) \in \mathcal{M}^+.$$

We have $\chi^*V = \frac{\partial}{\partial t}$ and

$$\begin{aligned} \chi^*\mathbf{g} &= -N^2(y)dt^2 + (dy^i + \mathbf{w}^i(y)dt)\mathbf{h}_{ij}(y)(dy^j + \mathbf{w}^j(y)dt) \\ &= -v^2(y)dt^2 + \mathbf{w}_i(y)dy^i dt + \mathbf{w}_j(y)dt dy^j + \mathbf{h}_{ij}(y)dy^i dy^j, \end{aligned}$$

for $v^2(y) = (N^2(y) - \mathbf{w}^i(y)\mathbf{h}_{ij}(y)\mathbf{w}^j(y))$. Note that the fact that V is time-like in \mathcal{M}^+ is equivalent to

$$(2.5) \quad N^2(y) > \mathbf{w}^i(y)\mathbf{h}_{ij}(y)\mathbf{w}^j(y), y \in \Sigma^+.$$

The unit normal vector field to the foliation $\Sigma_t = \{t\} \times \Sigma$ is

$$(2.6) \quad n = N^{-1}\left(\frac{\partial}{\partial t} - w\right),$$

Denoting $\chi^*\mathbf{g}$ on $\mathbb{R} \times \Sigma^+$ simply by \mathbf{g} , we have $|\mathbf{g}| = N^2|\mathbf{h}|$ and

$$(2.7) \quad \mathbf{g}^{-1} = -N^{-2}\partial_t^2 + N^{-2}(\mathbf{w}^i\partial_{y^i}\partial_t + \mathbf{w}^j\partial_t\partial_{y^j}) + (\mathbf{h}^{ij} - N^{-2}\mathbf{w}^i\mathbf{w}^j)\partial_{y^i}\partial_{y^j}.$$

Since the potential m is invariant under the Killing vector field, we have $m = m(y)$.

2.7. The Wick rotated metric.

2.7.1. Complex metrics. If X is a smooth manifold, we denote by $\mathbb{T}_q^p(X)$ the space of smooth, real (p, q) tensors on X and by $\mathbb{CT}_q^p(X)$ its complexification. An element $\mathbf{k} = \mathbf{k}_{ab}(x)dx^a dx^b$ of $\mathbb{CT}_2^0(X)$ which is symmetric and non-degenerate will be called a *complex metric* on X .

2.7.2. The Wick rotated metric. We denote by $\mathbb{S}_\beta = [-\frac{\beta}{2}, \frac{\beta}{2}[$ with endpoints identified the circle of length β and

$$M^{\text{eucl}} := \mathbb{S}_\beta \times \Sigma^+,$$

with variables (s, y) . Replacing t by s we obtain the complex metric on M^{eucl} :

$$(2.8) \quad \begin{aligned} \mathbf{g}^{\text{eucl}} &= N^2(y)ds^2 + (dy^j + i\mathbf{w}^j(y)ds)\mathbf{h}_{jk}(y)(dy^k + i\mathbf{w}^k(y)ds) \\ &= v^2(y)ds^2 + i\mathbf{w}_j(y)dy^j ds + i\mathbf{w}_j(y)ds dy^j + \mathbf{h}_{jk}(y)dy^j dy^k. \end{aligned}$$

We embed $\Sigma \setminus \mathcal{B}$ into $M^{\text{eucl}} = \mathbb{S}_\beta \times \Sigma^+$ by the map

$$\hat{\psi} : y \mapsto \begin{cases} (0, y) & \text{for } y \in \Sigma^+, \\ (\frac{\beta}{2}, r(y)) & \text{for } y \in \Sigma^-, \end{cases}$$

where $r : \Sigma \rightarrow \Sigma$ is the weak wedge reflection.

2.8. The smooth extension.

Proposition 2.5. *Assume that $\beta = (2\pi)\kappa^{-1}$. Then there exists a smooth manifold $M_{\text{ext}}^{\text{eucl}}$ equipped with a smooth complex metric $\mathbf{g}_{\text{ext}}^{\text{eucl}}$ and*

- (1) *a smooth embedding $\psi : \Sigma \rightarrow M_{\text{ext}}^{\text{eucl}}$,*
 - (2) *a smooth isometric embedding $\chi : (M^{\text{eucl}}, \mathbf{g}^{\text{eucl}}) \rightarrow (M_{\text{ext}}^{\text{eucl}} \setminus \mathcal{B}_{\text{ext}}, \mathbf{g}_{\text{ext}}^{\text{eucl}})$, where $\mathcal{B}_{\text{ext}} = \psi(\mathcal{B})$,*
 - (3) *an open set Ω_{ext} such that $\partial\Omega_{\text{ext}} = \psi(\Sigma)$ and $\chi :]0, \frac{\beta}{2}[\times \Sigma^+ \xrightarrow{\sim} \Omega_{\text{ext}} \setminus \mathcal{B}_{\text{ext}}$,*
 - (4) *a smooth function $m_{\text{ext}} : M_{\text{ext}}^{\text{eucl}} \rightarrow \mathbb{R}$ with $m_{\text{ext}} \geq m_0^2 > 0$,*
- such that:*

$$\psi|_{\Sigma \setminus \mathcal{B}} = \chi \circ \hat{\psi}, \quad \chi^* m_{\text{ext}} = m|_{M^{\text{eucl}}}.$$

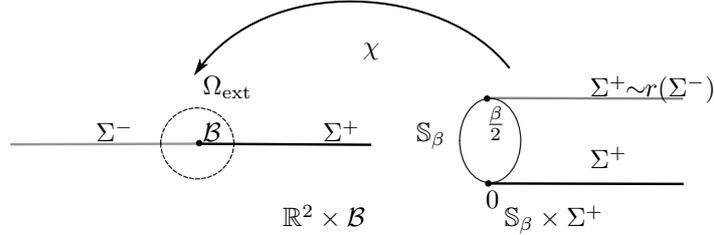


FIGURE 2. The embedding χ

The proof of Prop. 2.5 is given in Appendix A.2.

3. FREE KLEIN-GORDON FIELDS

In this section we briefly recall some well-known background material on free quantum Klein-Gordon fields on globally hyperbolic spacetimes. We follow the presentation in [GW1, Sect. 2] based on *charged fields*. There is no loss of generality to restrict oneself to charged fields and gauge invariant states, see eg the discussion in [GW1, Sect. 2].

3.1. CCR*-algebras.

3.1.1. Polynomial CCR *-algebra. Let \mathcal{Y} a complex vector space and $q \in L_{\text{h}}(\mathcal{Y}, \mathcal{Y}^*)$ a Hermitian form on \mathcal{Y} , possibly degenerate. Setting $\mathcal{X} = \mathcal{Y}_{\mathbb{R}}, \sigma = \text{Im}q$, (\mathcal{X}, σ) is a pre-symplectic space.

The *CCR *-algebra* $\text{CCR}(\mathcal{Y}, q)$ is the complex *-algebra generated by symbols $\mathbb{1}, \psi(y), \psi^*(y), y \in \mathcal{Y}$ and the relations

$$\begin{aligned} \psi(y_1 + \lambda y_2) &= \psi(y_1) + \bar{\lambda}\psi(y_2), \\ \psi^*(y_1 + \lambda y_2) &= \psi^*(y_1) + \lambda\psi^*(y_2), \quad y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C}, \\ [\psi(y_1), \psi(y_2)] &= [\psi^*(y_1), \psi^*(y_2)] = 0, \\ [\psi(y_1), \psi^*(y_2)] &= \bar{y}_1 \cdot qy_2 \mathbb{1}, \quad y_1, y_2 \in \mathcal{Y}, \\ \psi(y)^* &= \psi^*(y), \quad y \in \mathcal{Y}. \end{aligned}$$

3.1.2. *Weyl *-algebra.* The *Weyl *-algebra* $\text{Weyl}(\mathcal{Y}, q)$ is the *-algebra generated by the elements $\mathbb{1}, W(x)$, $x \in \mathcal{Y}_{\mathbb{R}}$, with relations

$$\begin{aligned} W(0) &= \mathbb{1}, \quad W(x)^* = W(-x), \\ W(x_1)W(x_2) &= e^{-\frac{1}{2}x_1 \cdot \sigma x_2} W(x_1 + x_2), \end{aligned} \quad x, x_1, x_2 \in \mathcal{Y}_{\mathbb{R}},$$

where we recall that $\sigma = \text{Im}q$.

3.1.3. *Weyl C^* -algebra.* Setting

$$\|A\| = \sup_{\omega \in \mathcal{F}} \omega(A^*A)^{\frac{1}{2}}, \quad A \in \text{Weyl}(\mathcal{Y}, q),$$

where \mathcal{F} is the set of states on $\text{Weyl}(\mathcal{Y}, q)$ one can show that $\|\cdot\|$ is a C^* -norm, see eg [MSTV] and we denote by $\text{Weyl}^{C^*}(\mathcal{Y}, q)$ the completion of $\text{Weyl}(\mathcal{Y}, q)$ for this norm.

3.1.4. *Quasi-free states.*

Definition 3.1. *A state ω on $\text{CCR}(\mathcal{Y}, q)$ is (gauge invariant) quasi-free if*

$$\omega\left(\prod_{i=1}^p \psi(y_i) \prod_{j=1}^q \psi^*(y'_j)\right) = \begin{cases} 0 & \text{if } p \neq q, \\ \sum_{\sigma \in S_p} \prod_{i=1}^p \omega(\psi(y_i) \psi^*(y'_{\sigma(i)})) & \text{if } p = q. \end{cases}$$

The (complex) covariances $\lambda^{\pm} \in L_{\text{h}}(\mathcal{Y}, \mathcal{Y}^*)$ of ω are defined as:

$$\begin{aligned} \omega(\psi(y_1) \psi^*(y_2)) &=: \bar{y}_1 \cdot \lambda^+ y_2, \\ \omega(\psi^*(y_2) \psi(y_1)) &=: \bar{y}_1 \cdot \lambda^- y_2, \end{aligned} \quad y_1, y_2 \in \mathcal{Y}.$$

The following result is well-known, see eg [DG, Sect. 17.1].

Proposition 3.2. *Two Hermitian forms $\lambda^{\pm} \in L_{\text{h}}(\mathcal{Y}, \mathcal{Y}^*)$ are the covariances of a quasi-free state ω on $\text{CCR}(\mathcal{Y}, q)$ iff*

$$(3.1) \quad \lambda^{\pm} \geq 0, \quad \lambda^+ - \lambda^- = q.$$

If we set

$$(3.2) \quad \eta = \frac{1}{2} \text{Re}(\lambda^+ + \lambda^-),$$

then we have

$$\eta \geq 0, \quad |x_1 \cdot x_2| \leq 2(x_1 \cdot \eta x_1)^{\frac{1}{2}} (x_2 \cdot x_2)^{\frac{1}{2}}, \quad x_1, x_2 \in \mathcal{Y}_{\mathbb{R}}.$$

It follows that ω induces a quasi-free state on $\text{Weyl}(\mathcal{Y}, q)$, or equivalently on $\text{Weyl}^{C^*}(\mathcal{Y}, q)$ defined by

$$\omega(W(x)) = e^{-\frac{1}{2}x\eta x}, \quad x \in \mathcal{Y}_{\mathbb{R}}.$$

3.1.5. *Pure quasi-free states.*

Definition 3.3. *A quasi-free state ω on $\text{CCR}(\mathcal{Y}, q)$ is said pure if the induced state on $\text{Weyl}^{C^*}(\mathcal{Y}, q)$ is pure.*

In the sequel we consider a quasi-free state ω on $\text{CCR}(\mathcal{Y}, q)$ with covariances λ^{\pm} . We will assume that

$$\text{Ker}(\lambda^+ + \lambda^-) = \{0\}.$$

By Prop. 3.2 this is true if q is non degenerate.

Let \mathcal{Y}^{cl} be the completion of \mathcal{Y} for the norm $\|y\|_{\omega} = (\bar{y} \cdot \lambda^+ y + \bar{y} \cdot \lambda^- y)^{\frac{1}{2}}$. Since $\lambda^{\pm} \geq 0$ and $q = \lambda^+ - \lambda^-$, q, λ^{\pm} uniquely extend as bounded Hermitian forms $q^{\text{cl}}, \lambda^{\pm \text{cl}}$ on \mathcal{Y}^{cl} . Note that q^{cl} may be degenerate, even if q is not.

Proposition 3.4. *The state ω on $\text{CCR}(\mathcal{Y}, q)$ is pure iff there exist projections $c^\pm \in L(\mathcal{Y}^{\text{cl}})$ such that*

$$(3.3) \quad c^+ + c^- = \mathbb{1}, \lambda^{\pm \text{cl}} = \pm q^{\text{cl}} \circ c^\pm.$$

Remark 3.5. *If ω is pure on $\text{CCR}(\mathcal{Y}, q)$ then q^{cl} is non degenerate. In fact since $\lambda^{\pm \text{cl}}$ and q^{cl} are Hermitian we obtain $\lambda^{\pm \text{cl}} = \pm c^{\pm*} q^{\text{cl}}$ hence $\text{Ker} q^{\text{cl}} \subset \text{Ker}(\lambda^{+\text{cl}} + \lambda^{-\text{cl}}) = \{0\}$.*

Proof of Prop. 3.4. Assume first that \mathcal{Y} is complete for $\|\cdot\|_\omega$ ie $\mathcal{Y}^{\text{cl}} = \mathcal{Y}$. This means that $\mathcal{Y}_\mathbb{R}$ is complete for the euclidean scalar product η in (3.2). In that situation we know from [DG, Thm. 17.13] that ω is pure iff $(2\eta, \sigma)$ is Kähler, ie if there exists an anti-involution j on $\mathcal{Y}_\mathbb{R}$ with $j^\# \sigma j = \sigma$ and $2\eta = \sigma j$. This is equivalent to the existence of projections c^\pm satisfying

$$(3.4) \quad c^+ + c^- = \mathbb{1}, c^{+*} q c^- = 0, \lambda^\pm = \pm q \circ c^\pm,$$

as requested, see [GW1, Prop. 2.7].

Let us now consider the general case. We recall from [BR, Thm. 2.3.19] that a state ω on a C^* -algebra \mathfrak{A} is pure iff its GNS representation (\mathcal{H}, π) is irreducible, i.e. iff \mathcal{H} does not contain non-trivial closed subspaces invariant under $\pi(\mathfrak{A})$.

We set $\mathfrak{A} = \text{Weyl}^{C^*}(\mathcal{Y}, q)$, $\mathfrak{A}^{\text{cl}} = \text{Weyl}^{C^*}(\mathcal{Y}^{\text{cl}}, q^{\text{cl}})$ ω^{cl} the quasi-free state on \mathfrak{A}^{cl} with covariances $\lambda^{\pm \text{cl}}$ and let $(\mathcal{H}^{(\text{cl})}, \pi^{(\text{cl})}, \Omega^{(\text{cl})})$ be the GNS triple for $(\mathfrak{A}^{(\text{cl})}, \omega^{(\text{cl})})$. Using that \mathcal{Y} is dense in \mathcal{Y}^{cl} for $\|\cdot\|_\omega$, we first obtain that $\mathcal{H} = \mathcal{H}^{\text{cl}}$, $\Omega = \Omega^{\text{cl}}$ and $\pi^{\text{cl}}|_{\mathfrak{A}} = \pi$.

We also easily obtain that $\pi(\mathfrak{A})$ is strongly dense in $\pi^{\text{cl}}(\mathfrak{A}^{\text{cl}})$. In fact, if $A = \sum_1^N \alpha_i \pi^{\text{cl}}(W(y_i)) \in \pi^{\text{cl}}(\mathfrak{A}^{\text{cl}})$ and $y_{i,n} \in \mathcal{Y}$ with $y_{i,n} \rightarrow y_i$ for $\|\cdot\|_\omega$, we obtain that $A_n = \sum_1^N \alpha_i \pi(W(y_{i,n}))$ is bounded in norm by $\sum_1^N |\alpha_i|$ and converges strongly to A on the dense subspace $\pi(\mathfrak{A})\Omega$, hence on \mathcal{H}^{cl} .

From this fact we see that a closed subspace $\mathcal{K} \subset \mathcal{H} = \mathcal{H}^{\text{cl}}$ is invariant under $\pi(\mathfrak{A})$ iff it is invariant under $\pi^{\text{cl}}(\mathfrak{A}^{\text{cl}})$, hence ω is pure iff its extension ω^{cl} to \mathfrak{A}^{cl} is pure. We are hence reduced to the already treated case when $\mathcal{Y} = \mathcal{Y}^{\text{cl}}$. \square

Let us now state another criterion for purity.

Prop. 3.6 below is due to Kay and Wald [KW] in the real case (see [GW2, Prop. 2.1] for the complex case).

Proposition 3.6. *The state ω with covariances Λ^\pm is pure iff:*

$$(3.5) \quad \overline{y}_1 \cdot (\lambda^+ + \lambda^-) y_1 = \sup_{y_2 \in \mathcal{Y}, y_2 \neq 0} \frac{|\overline{y}_1 \cdot q y_2|^2}{\overline{y}_2 \cdot (\lambda^+ + \lambda^-) y_2}, \quad \forall y_1 \in \mathcal{Y}.$$

Proof. From [KW] we know that ω is a pure state iff

$$x_1 \cdot \eta x_1 = \frac{1}{4} \sup_{x_2 \in \mathcal{Y}_\mathbb{R}, x_2 \neq 0} \frac{|x_1 \cdot \text{Im} q x_2|^2}{x_2 \cdot \eta x_2}.$$

Using that $\eta = \frac{1}{2} \text{Re}(\lambda^+ + \lambda^-)$ and that q is sesquilinear, this is equivalent to (3.5). \square

3.2. Free Klein-Gordon fields. Let $P = -\square_{\mathbf{g}} + m$, $m \in C^\infty(M, \mathbb{R})$ be a Klein-Gordon operator on a globally hyperbolic spacetime (M, \mathbf{g}) . Let $G_{\text{ret/adv}}$ be the retarded/advanced inverses of P and $G := G_{\text{ret}} - G_{\text{adv}}$. We apply the above framework to

$$\mathcal{Y} = \frac{C_c^\infty(M)}{PC_c^\infty(M)}, \quad \overline{[u]} \cdot q[v] = i(u|Gv)_M,$$

where $(u|v)_M = \int_M \bar{u}v dVol_{\mathbf{g}}$. Denoting by $\text{Sol}_{\text{sc}}(P)$ the space of smooth space-compact solutions of $P\phi = 0$, it is well known that

$$[G] : \left(\frac{C_c^\infty(M)}{PC_c^\infty(M)}, i(\cdot|G\cdot)_M \right) \ni [u] \mapsto Gu \in (\text{Sol}_{\text{sc}}(P), q)$$

is unitary for

$$(3.6) \quad \bar{\phi}_1 \cdot q\phi_2 := i \int_{\Sigma} (\nabla_{\mu} \bar{\phi}_1 \phi_2 - \bar{\phi}_1 \nabla_{\mu} \phi_2) n^{\mu} d\sigma_{\Sigma},$$

where Σ is any spacelike Cauchy hypersurface, n^{μ} is the future directed unit normal vector field to Σ and $d\sigma_{\Sigma}$ the induced surface density. Setting

$$\varrho : C_{\text{sc}}^\infty(M) \ni \phi \mapsto \left(i^{-1} n^{\mu} \partial_{\mu} \phi|_{\Sigma} \right) = f \in C_c^\infty(\Sigma; \mathbb{C}^2)$$

the map

$$\left(\frac{C_c^\infty(M)}{PC_c^\infty(M)}, i(\cdot|G\cdot)_M \right) \ni [u] \mapsto \varrho Gu \in (C_c^\infty(\Sigma; \mathbb{C}^2), q)$$

is unitary for

$$(3.7) \quad \bar{f} \cdot qf := \int_{\Sigma} (\bar{f}_1 f_0 + \bar{f}_0 f_1) d\sigma_{\Sigma}, \quad f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}.$$

In the sequel the $*$ -algebra $\text{CCR}(\mathcal{Y}, q)$ where (\mathcal{Y}, q) is any of the above equivalent non degenerate Hermitian spaces will be denoted by $\text{CCR}(P)$.

3.3. Quasi-free states. One restricts attention to quasi-free states on $\text{CCR}(P)$ whose covariances are given by distributions on $M \times M$, i.e. such that there exists $\Lambda^{\pm} \in \mathcal{D}'(M \times M)$ with

$$(3.8) \quad \begin{aligned} \omega(\psi([u_1])\psi^*([u_2])) &= (u_1|\Lambda^+u_2)_M, \\ \omega(\psi^*([u_2])\psi([u_1])) &= (u_1|\Lambda^-u_2)_M, \end{aligned} \quad u_1, u_2 \in C_c^\infty(M).$$

In the sequel the distributions $\Lambda^{\pm} \in \mathcal{D}'(M \times M)$ will be called the *spacetime covariances* of the state ω .

In (3.8) we identify distributions on M with distributional densities using the density $dVol_{\mathbf{g}}$ and use the notation $(u|\varphi)_M$, $u \in C_c^\infty(M)$, $\varphi \in \mathcal{D}'(M)$ for the duality bracket. We have then

$$(3.9) \quad \begin{aligned} P(x, \partial_x)\Lambda^{\pm}(x, x') &= P(x', \partial_{x'})\Lambda^{\pm}(x, x') = 0, \\ \Lambda^+(x, x') - \Lambda^-(x, x') &= iG(x, x'). \end{aligned}$$

3.4. Cauchy surface covariances. Using $(C_c^\infty(\Sigma; \mathbb{C}^2), q)$ instead of $(\frac{C_c^\infty(M)}{PC_c^\infty(M)}, i(\cdot|G\cdot)_M)$ one can associate to a quasi-free state its *Cauchy surface covariances* λ^{\pm} defined by:

$$(3.10) \quad \lambda^{\pm} := (\varrho_{\Sigma}^*)^* \Lambda^{\pm} (\varrho_{\Sigma}^* q).$$

Using the canonical scalar product $(f|f)_{\Sigma} := \int_{\Sigma} (\bar{f}_1 f_1 + \bar{f}_0 f_0) d\sigma_{\Sigma}$ we identify λ^{\pm} with operators, still denoted by $\lambda^{\pm} : C_c^\infty(\Sigma; \mathbb{C}^2) \rightarrow \mathcal{D}'(\Sigma; \mathbb{C}^2)$. Conversely we obtain Λ^{\pm} from λ^{\pm} by:

$$\Lambda^{\pm} = (\varrho G)^* \lambda^{\pm} (\varrho G).$$

3.5. Hadamard states. A quasi-free state is called a *Hadamard state*, (see [R] for the neutral case and [GW1] for the complex case) if

$$(3.11) \quad \text{WF}(\Lambda^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm,$$

where $\text{WF}(\Lambda)'$ denotes the 'primed' wavefront set of Λ , i.e. $S' := \{((x, \xi), (x', -\xi')) : ((x, \xi), (x', \xi')) \in S\}$ for $S \subset T^*M \times T^*M$, and \mathcal{N}^\pm are the two connected components (positive/negative energy shell) of the characteristic manifold:

$$(3.12) \quad \mathcal{N} := \{(x, \xi) \in T^*M \setminus o : \xi_\mu g^{\mu\nu}(x) \xi_\nu = 0\}.$$

We recall that $T^*X \setminus o$ denotes the cotangent bundle of X with the zero section removed.

Large classes of Hadamard states were constructed in terms of their Cauchy surface covariances in [GW1, GOW] using pseudodifferential calculus on Σ , see below for a short summary.

3.6. Pseudodifferential operators. We briefly recall the notion of (classical) pseudodifferential operators on a manifold, referring to [Sh, Sect. 4.3] for details.

For $m \in \mathbb{R}$ we denote by $\Psi^m(\mathbb{R}^d)$ the space of classical pseudodifferential operators on \mathbb{R}^d , associated with poly-homogeneous symbols of order m , see eg [Sh, Sect. 3.7].

Let X be a smooth, d -dimensional manifold. Let $U \subset X$ be a precompact chart open set and $\psi : U \rightarrow \tilde{U}$ a chart diffeomorphism, where $\tilde{U} \subset \mathbb{R}^d$ is precompact, open. We denote by $\psi^* : C_c^\infty(\tilde{U}) \rightarrow C_c^\infty(U)$ the map $\psi^*u(x) := u \circ \psi(x)$.

Definition 3.7. A linear continuous map $A : C_c^\infty(X) \rightarrow C^\infty(X)$ belongs to $\Psi^m(X)$ if the following condition holds:

(C) Let $U \subset X$ be precompact open, $\psi : U \rightarrow \tilde{U}$ a chart diffeomorphism, $\chi_1, \chi_2 \in C_c^\infty(U)$ and $\tilde{\chi}_i = \chi_i \circ \psi^{-1}$. Then there exists $\tilde{A} \in \Psi^m(\mathbb{R}^d)$ such that

$$(3.13) \quad (\psi^*)^{-1} \chi_1 A \chi_2 \psi^* = \tilde{\chi}_1 \tilde{A} \tilde{\chi}_2.$$

Elements of $\Psi^m(X)$ are called (classical) pseudodifferential operators of order m on X .

The subspace of $\Psi^m(X)$ of pseudodifferential operators with properly supported kernels is denoted by $\Psi_c^m(X)$.

Note that if $\Psi_c^\infty(X) := \bigcup_{m \in \mathbb{R}} \Psi_c^m(X)$, then $\Psi_c^\infty(X)$ is an algebra, but $\Psi^\infty(X)$ is not, since without the proper support condition, pseudodifferential operators cannot in general be composed.

To $A \in \Psi^m(X)$ one can associate its *principal symbol* $\sigma_{\text{pr}}(A) \in C^\infty(T^*X \setminus o)$, which is homogeneous of degree m in the fiber variable ξ in $T^*X \setminus o$. A is called *elliptic* in $\Psi^m(X)$ at $(x_0, \xi_0) \in T^*X \setminus o$ if $\sigma_{\text{pr}}(A)(x_0, \xi_0) \neq 0$.

If $A \in \Psi^m(X)$ there exists (many) $A_c \in \Psi_c^m(X)$ such that $A - A_c$ has a smooth kernel.

3.7. The Cauchy surface covariances of Hadamard states. We now state a result which follows directly from a construction of Hadamard states in [GW1, Subsect. 8.2].

Theorem 3.8. Let ω be any Hadamard state for the free Klein-Gordon field on (M, \mathbf{g}) and Σ a smooth space-like Cauchy surface. Then its Cauchy surface covariances λ^\pm are 2×2 matrices with entries in $\Psi^\infty(\Sigma)$.

We refer the reader to [G, Thm. 3.2] for the proof.

4. GREEN OPERATORS AND CALDERÓN PROJECTORS

In this section we collect some formulas expressing the Green operators, i.e. inverses for abstract operators of the form $\partial_s + b$, where s belongs either to \mathbb{R} or to the circle \mathbb{S}_β . We also compute various Calderón projectors. The formulas in this section will be used later in Sect. 6 to express Calderón projectors for second order elliptic operators obtained from abstract Klein-Gordon operators by Wick rotation.

4.1. Green operators and Calderón projectors. Let b be a selfadjoint operator on a Hilbert space \mathfrak{h} with $\text{Ker } b = \{0\}$. We recall that $\mathbb{S}_\beta = [-\frac{\beta}{2}, \frac{\beta}{2}[$ is the circle of length β . For $0 < \beta \leq \infty$ we set

$$(4.1) \quad \mathfrak{h}_\beta = L^2(\mathbb{S}_\beta) \otimes \mathfrak{h}, \text{ for } \beta < \infty, \quad \mathfrak{h}_\infty = L^2(\mathbb{R}) \otimes \mathfrak{h}.$$

The operator ∂_s is anti-selfadjoint on \mathfrak{h}_β with its natural domain. Denoting still by b the extension of b to \mathfrak{h}_β we see that $B_\beta = \partial_s + b$ with domain $\text{Dom } \partial_s \cap \text{Dom } b$ is normal.

To better understand the operator B_β and its inverse B_β^{-1} one can of course use the spectral decomposition of b , or the Fourier transform in s (on \mathbb{R} or \mathbb{S}_β), or both together.

4.1.1. Green operators. If $0 \in \sigma(b)$ then $0 \in \sigma(B_\beta)$ but we can still make sense out of B_β^{-1} as

$$B_\beta^{-1} : \mathfrak{h}_\beta \rightarrow (-\partial_s^2 + b^2)^{-\frac{1}{2}} \mathfrak{h}_\beta,$$

or

$$B_\beta^{-1} : (-\partial_s^2 + b^2)^{\frac{1}{2}} \mathfrak{h}_\beta \rightarrow \mathfrak{h}_\beta.$$

Note that

$$\begin{aligned} u = B_\beta^{-1}v, \quad v \in \mathfrak{h}_\beta &\Leftrightarrow u \in |b|^{-1}\mathfrak{h}_\beta, \partial_s u \in \mathfrak{h}_\beta, (\partial_s + b)u = v \in \mathfrak{h}_\beta \\ &\Leftrightarrow u \in |b|^{-1}\mathfrak{h}_\beta, (\partial_s + b)u = v \in \mathfrak{h}_\beta. \end{aligned}$$

Let us set:

$$(4.2) \quad G_\infty(s) := e^{-sb} (\mathbb{1}_{\mathbb{R}^+}(s)\mathbb{1}_{\mathbb{R}^+}(b) - \mathbb{1}_{\mathbb{R}^-}(s)\mathbb{1}_{\mathbb{R}^-}(b)).$$

Note that

$$(4.3) \quad \int_{\mathbb{R}} |G_\infty(s)| ds = |b|^{-1},$$

where $|G_\infty(s)| \in B(\mathfrak{h})$ is defined by functional calculus. A straightforward computation shows then that:

$$(4.4) \quad B_\infty^{-1}f(s) = \int_{\mathbb{R}} G_\infty(s-s')f(s')ds', \quad f \in L^2(\mathbb{R}; \mathfrak{h}) = \mathfrak{h}_\infty.$$

The identity (4.3) and the well-known fact that $L^1(\mathbb{R}) \star L^2(\mathbb{R}) \subset L^2(\mathbb{R})$, where \star denotes the convolution, show that

$$B_\infty^{-1} : \mathfrak{h}_\infty = L^2(\mathbb{R}; \mathfrak{h}) \rightarrow L^2(\mathbb{R}; |b|^{-1}\mathfrak{h}) = |b|^{-1}\mathfrak{h}_\infty.$$

Similarly for $\beta < \infty$ let us define $G_\beta(s)$ as follows: we set

$$G_\beta(s) := e^{-sb} (\mathbb{1}_{\mathbb{R}^+}(s)(1 - e^{-\beta b})^{-1} - \mathbb{1}_{\mathbb{R}^-}(s)(1 - e^{\beta b})^{-1}), \quad s \in [-\frac{\beta}{2}, \frac{\beta}{2}],$$

(note that $G_\beta(\frac{\beta}{2}) = G_\beta(-\frac{\beta}{2})$) and extend it to $s \in \mathbb{R}$ by β -periodicity. In particular we have:

$$(4.5) \quad G_\beta(s) = e^{-sb} (\mathbb{1}_{\mathbb{R}^+}(s)(1 - e^{-\beta b})^{-1} - \mathbb{1}_{\mathbb{R}^-}(s)(1 - e^{\beta b})^{-1}), \quad s \in [-\beta, \beta].$$

we have

$$(4.6) \quad \int_{\mathbb{R}} |G_\beta(s)| ds = |b|^{-1},$$

and

$$(4.7) \quad B_\beta^{-1}f(s) = \int_{\mathbb{S}_\beta} G_\beta(s-s')f(s')ds', \quad f \in L^2(\mathbb{S}_\beta; \mathfrak{h}) = \mathfrak{h}_\beta.$$

Again from (4.6) we obtain that

$$B_\beta^{-1} : \mathfrak{h}_\beta = L^2(\mathbb{S}_\beta; \mathfrak{h}) \rightarrow L^2(\mathbb{S}_\beta; |b|^{-1}\mathfrak{h}) = |b|^{-1}\mathfrak{h}_\beta.$$

The r.h.s. of (4.4), resp. (4.7) extends to $f \in \mathcal{E}'(\mathbb{R}; \mathfrak{h})$, resp. $f \in \mathcal{D}'(\mathbb{S}_\beta; \mathfrak{h})$ and give unique extensions of B_∞^{-1} , resp. B_β^{-1} with

$$(4.8) \quad \begin{aligned} B_\infty^{-1} &: \mathcal{E}'(\mathbb{R}; \mathfrak{h}) \rightarrow \mathcal{D}'(\mathbb{R}; |b|^{-1}\mathfrak{h}) \text{ continuously,} \\ B_\beta^{-1} &: \mathcal{D}'(\mathbb{S}_\beta; \mathfrak{h}) \rightarrow \mathcal{D}'(\mathbb{S}_\beta; |b|^{-1}\mathfrak{h}) \text{ continuously.} \end{aligned}$$

4.1.2. Calderón projectors for B_∞ . Let us first motivate the definition of Calderón projectors in Prop. 4.1 below. We set $I_\infty^\pm = \pm]0, +\infty[$. For $F \in \overline{C^0}(I_\infty^\pm; \mathfrak{h})$ we set

$$\Gamma_\infty^\pm F = F(0^\pm) = \lim_{s \rightarrow 0^\pm} F(s).$$

Assume that $(\partial_s + b)F = 0$ in $\mathcal{D}'(I_\infty^\pm; \mathfrak{h})$. Denoting by $i_\infty^\pm F$ the extension of F by 0 in $\mathbb{R} \setminus I_\infty^\pm$ we have

$$(\partial_s + b)i_\infty^\pm F = \pm\delta_0(s) \otimes \Gamma_\infty^\pm F \text{ in } \mathcal{D}'(\mathbb{R}; \mathfrak{h}).$$

This implies formally that $i_\infty^\pm F = \pm B_\infty^{-1}(\delta_0(s) \otimes f)$ for $f = \Gamma_\infty^\pm F$ and hence:

$$f = \pm \Gamma_\infty^\pm \circ B_\infty^{-1}(\delta_0(s) \otimes f)$$

if $f = \Gamma_\infty^\pm F$ for F solving $(\partial_s + b)F = 0$ in I_∞^\pm .

Note that $B_\infty^{-1}(\delta_0(s) \otimes f)$ for $f \in \mathfrak{h}$ is well defined by the discussion at the end of 4.1.1.

Proposition 4.1. (1) $B_\infty^{-1}(\delta_0(s) \otimes f)$ belongs to $\overline{C^0}(I_\infty^\pm; \mathfrak{h})$ for $f \in \mathfrak{h}$. It follows that

$$(4.9) \quad C_\infty^\pm f = \pm \Gamma_\infty^\pm \circ B_\infty^{-1}(\delta_0(s) \otimes f), \quad f \in \mathfrak{h}$$

are well defined.

(2) One has

$$(4.10) \quad C_\infty^\pm = \mathbb{1}_{\mathbb{R}^\pm}(b),$$

It follows that $C_\infty^\pm \in B(\mathfrak{h})$ are bounded complementary projections on \mathfrak{h} , called Calderón projectors.

Proof. Let $\chi_n(s) = n\chi(ns)$ for $\chi \in C_c^\infty(\mathbb{R})$ with $\int \chi(s)ds = 1$. Clearly $\chi_n(s) \otimes f \rightarrow \delta_0(s) \otimes f$ in $\mathcal{E}'(\mathbb{R}; \mathfrak{h})$ and hence

$$B_\infty^{-1}(\delta_0(s) \otimes f) = \lim_{n \rightarrow \infty} B_\infty^{-1}(\chi_n(s) \otimes f) \text{ in } \mathcal{D}'(\mathbb{R}; |b|^{-1}\mathfrak{h}).$$

For $s > 0$, we have

$$B_\infty^{-1}(\chi_n \otimes f)(s) = \int e^{-(s-s')b} \mathbb{1}_{\mathbb{R}^+}(s-s') \mathbb{1}_{\mathbb{R}^+}(b) \chi_n(s') \otimes f ds'$$

for n large enough. Letting $n \rightarrow \infty$ we obtain that

$$B_\infty^{-1}(\delta_0 \otimes f)(s) = e^{-sb} \mathbb{1}_{\mathbb{R}^+}(b) f \text{ in } s > 0,$$

hence $B_\infty^{-1}(\delta_0 \otimes f) \in \overline{C^0}(I_\infty^+; \mathfrak{h})$ and $\Gamma_\infty^+ \circ B_\infty^{-1}(\delta_0(s) \otimes f) = \mathbb{1}_{\mathbb{R}^+}(b)f$. We use the same argument for C_∞^- . \square

4.1.3. *Calderón projectors for B_β .* For $\beta < \infty$ we set $I_\beta^\pm = \pm]0, \frac{\beta}{2}[$. The boundary ∂I_β^\pm has two connected components $\{s = 0\}$ and $\{s = \beta/2\}$, which complicates a little the computation of the Calderón projectors. Moreover the kernel G_β of B_β^{-1} has a infrared singularity at $b = 0$, coming from the factors $(1 - e^{\mp\beta b})^{-1}$ in (4.5).

As before we start with the heuristic motivation for the definition of Calderón projectors in Prop. 4.2 below. For $F \in \overline{C^0}(I_\beta^\pm; \mathfrak{h})$ we set:

$$(4.11) \quad \begin{aligned} \Gamma_\beta^+ F &:= F(0^+) \oplus F(\frac{\beta}{2}^-) =: \Gamma_\beta^{(0)^+} F \oplus \Gamma_\beta^{(\frac{\beta}{2})^+} F, \\ \Gamma_\beta^- F &:= F(0^-) \oplus F(-\frac{\beta}{2}^+) =: \Gamma_\beta^{(0)^-} F \oplus \Gamma_\beta^{(\frac{\beta}{2})^-} F. \end{aligned}$$

Assume that $(\partial_s + b)F = 0$ in $\mathcal{D}'(I_\beta^\pm; \mathfrak{h})$. Then denoting by $i_\beta^\pm F$ the extension of F by 0 in $\mathbb{S}_\beta \setminus I_\beta^\pm$, we have

$$(\partial_s + b)i_\beta^\pm F = \pm(\delta_0(s) \otimes \Gamma_\beta^{(0)^\pm} F - \delta_{\frac{\beta}{2}}(s) \otimes \Gamma_\beta^{(\frac{\beta}{2})^\pm} F) \text{ in } \mathcal{D}'(\mathbb{S}_\beta; \mathfrak{h}),$$

which formally implies that

$$i_\beta^\pm F = B_\beta^{-1}(\delta_0(s) \otimes f^{(0)} - \delta_{\frac{\beta}{2}}(s) \otimes f^{(\frac{\beta}{2})})$$

for $f = f^{(0)} \oplus f^{(\frac{\beta}{2})} = \Gamma_\beta^\pm F$. Again the r.h.s. above is well defined by the discussion at the end of 4.1.1.

Proposition 4.2. (1) $B_\beta^{-1}(\delta_0(s) \otimes f^{(0)} - \delta_{\frac{\beta}{2}}(s) \otimes f^{(\frac{\beta}{2})})$ belongs to $\overline{C^0}(I_\beta^\pm; (1 + |b|^{-1})\mathfrak{h})$ for $f = f^{(0)} \oplus f^{(\frac{\beta}{2})} \in \mathfrak{h} \oplus \mathfrak{h}$. It follows that

$$(4.12) \quad C_\beta^\pm f := \pm \Gamma_\beta^\pm \circ B_\beta^{-1}(\delta_0(s) \otimes f^{(0)} - \delta_{\frac{\beta}{2}}(s) \otimes f^{(\frac{\beta}{2})}), \quad f = f^{(0)} \oplus f^{(\frac{\beta}{2})} \in \mathfrak{h} \oplus \mathfrak{h}$$

are well defined and belong to $B(\mathfrak{h} \oplus \mathfrak{h}, (1 + |b|^{-1})\mathfrak{h} \oplus (1 + |b|^{-1})\mathfrak{h})$.

(2) One has:

$$(4.13) \quad \begin{aligned} C_\beta^+ &= \begin{pmatrix} (1 - e^{-\beta b})^{-1} & (1 - e^{\beta b})^{-1} e^{\frac{\beta}{2} b} \\ (1 - e^{-\beta b})^{-1} e^{-\frac{\beta}{2} b} & (1 - e^{\beta b})^{-1} \end{pmatrix}, \\ C_\beta^- &= \begin{pmatrix} (1 - e^{\beta b})^{-1} & -e^{\frac{\beta}{2} b} (1 - e^{\beta b})^{-1} \\ -e^{-\frac{\beta}{2} b} (1 - e^{-\beta b})^{-1} & (1 - e^{-\beta b})^{-1} \end{pmatrix}. \end{aligned}$$

(3) On $\mathbb{1}_I(b)\mathfrak{h} \oplus \mathbb{1}_I(b)\mathfrak{h}$ for any interval $I \Subset \mathbb{R}^*$ one has:

$$C_\beta^\pm C_\beta^\pm = C_\beta^\pm, \quad C_\beta^+ + C_\beta^- = \mathbb{1},$$

It follows that C_β^\pm are complementary projections on $\mathbb{1}_I(b)\mathfrak{h} \oplus \mathbb{1}_I(b)\mathfrak{h}$ called Calderón projectors.

Proof. The proof of (1) is analogous to the proof of (1) in Prop. 4.1. The fact that \mathfrak{h} is replaced by $(1 + |b|^{-1})\mathfrak{h}$ comes from the extra 'infrared singularity' of $(1 - e^{\mp\beta b})^{-1}$, since $(1 - e^{-\beta\lambda})^{-1}$ behaves like λ^{-1} near $\lambda = 0$.

(2) is a routine computation using (4.5). We check (3) using the identity $(1 - a)^{-1} + (1 - a^{-1})^{-1} = 1$ for $a = e^{-\beta b}$. \square

5. VACUA AND KMS STATES FOR STABLE SYMPLECTIC DYNAMICS

In this section we recall well-known formulas for the covariances of the vacuum and KMS states associated to a symplectic flow on a symplectic space. The symplectic flow has to be *stable*, i.e. generated by a positive classical energy. In concrete situations the symplectic flow is generated by a time-like Killing vector field. We also recall the definition of the *double* KMS state, due to Kay [K1, K2], which is related to the Araki-Woods representation of a KMS state.

The new result of this section is that the covariances of the vacuum and double KMS states can be expressed by the *Calderón projectors* introduced in Sect. 4. Note that only the double KMS states will be important for the construction of the HHI state later on. Nevertheless the case of vacuum state is simpler and we include it for pedagogical reasons.

5.1. Weakly stable symplectic dynamics. We describe now a framework for symplectic dynamics, which can be found in [DG, Sect. 18.2.1], called there a *weakly stable symplectic dynamics*.

Let (\mathcal{Y}, q) be a non-degenerate Hermitian space and $E \in L_{\text{h}}(\mathcal{Y}, \mathcal{Y}^*)$ with $E > 0$, the function $\mathcal{Y} \ni y \mapsto \bar{y} \cdot E y$ being the *classical energy*. The *energy space* \mathcal{Y}_{en} is the completion of \mathcal{Y} for the scalar product $(y_1 | y_2)_{\text{en}} = \bar{y}_1 \cdot E y_2$ and is a complex Hilbert space.

Let $r_t = e^{itb}$ be a strongly continuous unitary group on \mathcal{Y}_{en} with selfadjoint generator b . We assume that $r_t : \mathcal{Y} \rightarrow \mathcal{Y}$, $\mathcal{Y} \subset \text{Dom} b$. From Nelson's invariant domain theorem it follows that b is essentially selfadjoint on \mathcal{Y} . We assume also that

$$(5.1) \quad \text{Ker} b = \{0\}.$$

If one applies this abstract framework to Klein-Gordon equations on stationary spacetimes, the stronger condition $0 \notin \sigma(b)$ correspond to the massive case.

5.1.1. Dynamical Hilbert space. It is convenient, in connection with the quantization of the symplectic flow $\{r_t\}_{t \in \mathbb{R}}$, to introduce the *dynamical Hilbert space*

$$\mathcal{Y}_{\text{dyn}} := |b|^{\frac{1}{2}} \mathcal{Y}_{\text{en}},$$

see [DG, Subsect. 18.2.1], equipped with the scalar product

$$(y_1 | y_2)_{\text{dyn}} = (|b|^{-\frac{1}{2}} y_1 | |b|^{-\frac{1}{2}} y_2)_{\text{en}}.$$

The group $\{r_t\}_{t \in \mathbb{R}}$ extends obviously as a unitary group on \mathcal{Y}_{dyn} . If we denote the generator of r_t on $\mathcal{Y}_{\text{en/dyn}}$ by $b_{\text{en/dyn}}$ then $b_{\text{en}} = b$ and $b_{\text{dyn}} = |b|^{\frac{1}{2}} b_{\text{en}} |b|^{-\frac{1}{2}}$ since $|b|^{-\frac{1}{2}} : \mathcal{Y}_{\text{dyn}} \rightarrow \mathcal{Y}_{\text{en}}$ is unitary. Therefore we will often denote both generators by the same letter b , when there is no risk of confusion.

We will assume that

$$(5.2) \quad \mathcal{Y} \subset \mathcal{Y}_{\text{dyn}}, \text{ i.e. } \mathcal{Y} \subset \text{Dom} |b|^{-\frac{1}{2}},$$

where we consider b as acting on \mathcal{Y}_{en} , and that:

$$(5.3) \quad \bar{y}_1 \cdot q y_2 = (y_1 | b^{-1} y_2)_{\text{en}} = (y_1 | \text{sgn}(b) y_2)_{\text{dyn}}, \quad y_1, y_2 \in \mathcal{Y}.$$

Note that if $b : \mathcal{Y} \rightarrow \mathcal{Y}$ this implies that

$$\bar{y}_1 \cdot E y_2 = \bar{y}_1 \cdot q b y_2, \quad y_1, y_2 \in \mathcal{Y},$$

which means that $\{r_s\}_{s \in \mathbb{R}}$ can be seen as the symplectic evolution group associated to the classical energy $\bar{y} \cdot E y$ and the symplectic form $\sigma = i^{-1} q$.

For Klein-Gordon equations on stationary spacetimes, conditions (5.2), (5.3) will be checked in Lemma 8.3.

We can equip \mathcal{Y}_{dyn} with the bounded Hermitian form

$$(5.4) \quad \bar{y}_1 \cdot q_{\text{dyn}} y_2 := (y_1 | \text{sgn}(b) y_2)_{\text{dyn}},$$

which is non-degenerate by (5.1), and we have $q = q_{\text{dyn}}$ on \mathcal{Y} . Note that one cannot extend q as a bounded Hermitian form on \mathcal{Y}_{en} , unless $0 \notin \sigma(b)$.

5.2. Vacuum state. We now recall the definition of the vacuum state ω_{vac} associated to the dynamics $\{r_t\}_{t \in \mathbb{R}}$.

Definition 5.1. *The vacuum state ω_{vac} is the quasi-free state on $\text{CCR}(\mathcal{Y}_{\text{dyn}}, q_{\text{dyn}})$ defined by the covariances:*

$$(5.5) \quad \bar{y}_1 \cdot \lambda_{\text{vac}}^{\pm} y_2 = (y_1 | \mathbb{1}_{\mathbb{R}^{\pm}}(b) y_2)_{\text{dyn}}.$$

Remark 5.2. *Since $\mathcal{Y} \subset \mathcal{Y}_{\text{dyn}}$ and $q_{\text{dyn}} = q$ on \mathcal{Y} , we see that ω_{vac} is also a quasi-free state on $\text{CCR}(\mathcal{Y}, q)$. By Prop. 3.4 ω_{vac} is a pure state on $\text{CCR}(\mathcal{Y}_{\text{dyn}}, q_{\text{dyn}})$. However this does not imply that it is a pure state on $\text{CCR}(\mathcal{Y}, q)$.*

From (5.4) we obtain that if c_{vac}^{\pm} are defined as in Prop. 3.4 by

$$\lambda_{\text{vac}}^{\pm} =: \pm q_{\text{dyn}} \circ c_{\text{vac}}^{\pm},$$

one has:

$$(5.6) \quad c_{\text{vac}}^{\pm} = \mathbb{1}_{\mathbb{R}^{\pm}}(b).$$

It follows from Prop. 4.1 that:

$$(5.7) \quad c_{\text{vac}}^{\pm} = C_{\infty}^{\pm},$$

where the Calderón projectors C_{∞}^{\pm} are defined in Prop. 4.1, for $\mathfrak{h} = \mathcal{Y}_{\text{dyn}}$.

5.3. KMS state. Let us now define the β -KMS state ω_{β} associated to the dynamics $\{r_t\}_{t \in \mathbb{R}}$. We introduce the *thermal Hilbert space*:

$$(5.8) \quad \mathcal{Y}_{\text{th}} := \mathcal{Y}_{\text{dyn}} \cap |b|^{\frac{1}{2}} \mathcal{Y}_{\text{dyn}} = \text{Dom}|b|^{-\frac{1}{2}},$$

see (1.4).

Definition 5.3. *The β -KMS state ω_{β} is the quasi-free state on $\text{CCR}(\mathcal{Y}_{\text{th}}, q_{\text{dyn}})$ defined by the covariances:*

$$(5.9) \quad \begin{aligned} \bar{y}_1 \cdot \lambda_{\beta}^{+} y_2 &= \bar{y}_1 \cdot q_{\text{dyn}} (1 - e^{-\beta b})^{-1} y_2, \\ \bar{y}_1 \cdot \lambda_{\beta}^{-} y_2 &= \bar{y}_1 \cdot q_{\text{dyn}} (e^{\beta b} - 1)^{-1} y_2. \end{aligned}$$

Again if $0 \in \sigma(b)$, then using that $(1 - e^{\lambda})^{-1}$ behaves like λ^{-1} near $\lambda = 0$, we see that λ_{β}^{\pm} are defined only on \mathcal{Y}_{th} .

Remark 5.4. *In order for λ_{β}^{\pm} to define a state on $\text{CCR}(\mathcal{Y}, q)$ we need the stronger infrared condition*

$$(5.10) \quad \mathcal{Y} \subset \mathcal{Y}_{\text{th}}.$$

Again for Klein-Gordon equations on stationary spacetimes (5.10) will be checked in Lemma 8.3.

5.4. Double β -KMS states. The *double β -KMS state* see [K1, K2] can easily be related to the *Araki-Woods* representation of ω_{β} , see eg [DG, Subsect. 17.1.5], that we first briefly recall.

5.4.1. Araki-Woods representation. Let us denote by \mathcal{Z} the space \mathcal{Y}_{dyn} as a real vector space, equipped with the complex structure

$$j := i \circ \text{sgn}(b)$$

and the scalar product:

$$(z_1 | z_2)_{\mathcal{Z}} := (y_{1+} | y_{2+})_{\text{dyn}} + (y_{2-} | y_{1-})_{\text{dyn}},$$

for $y_{\pm} := \mathbb{1}_{\mathbb{R}^{\pm}}(b)y$ and $z = y$ (considered as an element of \mathcal{Z}). \mathcal{Z} is a Hilbert space equal to $\mathcal{Y}_{\text{dyn}+} \oplus \overline{\mathcal{Y}}_{\text{dyn}-}$ for $\mathcal{Y}_{\text{dyn}\pm} = \mathbb{1}_{\mathbb{R}^{\pm}}(b)\mathcal{Y}_{\text{dyn}}$. Note that since $[b, j] = 0$, b induces a selfadjoint operator on \mathcal{Z} , still denoted by b . We set

$$(5.11) \quad \rho := (e^{\beta|b|} - 1)^{-1},$$

which is a selfadjoint operator on \mathcal{Z} . We have $\text{Dom}\rho^{\frac{1}{2}} = \mathcal{Z} \cap |b|^{\frac{1}{2}}\mathcal{Z}$, which also equals \mathcal{Y}_{th} as real vector spaces.

We also introduce the Hilbert space $\mathcal{Z} \oplus \overline{\mathcal{Z}}$ and the bosonic Fock space $\Gamma_{\mathfrak{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$, see eg [DG, Subsect. 3.3.1]. For $(z_1, \overline{z}_2) \in \mathcal{Z} \oplus \overline{\mathcal{Z}}$ we denote by $a^{(*)}(z_1, \overline{z}_2)$ the Fock creation/annihilation operators acting on $\Gamma_{\mathfrak{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$.

The *left/right Araki-Woods* creation/annihilation operators are defined for $z \in \text{Dom}\rho^{\frac{1}{2}}$ by:

$$(5.12) \quad \begin{aligned} a_1^*(z) &= a^*((1+\rho)^{\frac{1}{2}}z, 0) + a(0, \overline{\rho^{\frac{1}{2}}z}), \\ a_1(z) &= a((1+\rho)^{\frac{1}{2}}z, 0) + a^*(0, \overline{\rho^{\frac{1}{2}}z}), \\ a_r^*(\overline{z}) &= a(\rho^{\frac{1}{2}}z, 0) + a^*(0, (1+\overline{\rho})^{\frac{1}{2}}\overline{z}), \\ a_r(\overline{z}) &= a^*(\rho^{\frac{1}{2}}z, 0) + a(0, (1+\overline{\rho})^{\frac{1}{2}}\overline{z}). \end{aligned}$$

One has:

$$[a_1(z_1), a_1^*(z_2)] = (z_1|z_2)_{\mathcal{Z}}\mathbb{1}, \quad [a_r(\overline{z}_1), a_r^*(\overline{z}_2)] = (\overline{z}_1|\overline{z}_2)_{\overline{\mathcal{Z}}}\mathbb{1}, \quad z_1, z_2 \in \text{Dom}\rho^{\frac{1}{2}},$$

all other commutators being equal to 0. Setting $z_{\pm} = y_{\pm}$ for $y \in \mathcal{Y}_{\text{dyn}}$ we set

$$(5.13) \quad \begin{aligned} \psi_1^*(y) &:= a_1^*(z_+) + a_1(z_-), \quad \psi_1(y) := a_1(z_+) + a_1^*(z_-) \\ \psi_r^*(y) &:= a_r^*(\overline{z}_-) + a_r(\overline{z}_+), \quad \psi_r(y) := a_r(\overline{z}_-) + a_r^*(\overline{z}_+), \end{aligned} \quad y \in \mathcal{Y}_{\text{th}}.$$

An easy computation show that

$$(5.14) \quad [\psi_1(y_1), \psi_1^*(y_2)] = \overline{y}_1 \cdot q_{\text{dyn}} y_2, \quad [\psi_r(y_1), \psi_r^*(y_2)] = -\overline{y}_1 \cdot q_{\text{dyn}} y_2,$$

all other commutators being equal to 0. Moreover $\mathcal{Y}_{\text{th}} \ni y \mapsto \psi_{1/r}^*(y)$ is \mathbb{C} -linear.

This means that $\mathcal{Y}_{\text{th}} \ni y \mapsto \psi_{1/r}^{(*)}(y)$ induces two commuting representations of $\text{CCR}(\mathcal{Y}_{\text{th}}, \pm q_{\text{dyn}})$.

From (5.13) we obtain that:

$$\begin{aligned} (\Omega|\psi_1(y_1)\psi_1^*(y_2)\Omega)_{\Gamma_{\mathfrak{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})} &= \overline{y}_1 \cdot \lambda_{\beta}^+ y_2, \\ (\Omega|\psi_1^*(y_2)\psi_1(y_1)\Omega)_{\Gamma_{\mathfrak{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})} &= \overline{y}_1 \cdot \lambda_{\beta}^- y_2, \\ (\Omega|\psi_1(y_2)\psi_1(y_1)\Omega)_{\Gamma_{\mathfrak{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})} &= (\Omega|\psi_1^*(y_2)\psi_1^*(y_1)\Omega)_{\Gamma_{\mathfrak{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})} = 0, \end{aligned}$$

where Ω is the vacuum vector in $\Gamma_{\mathfrak{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}})$. If $\pi_{\text{AW},1}$ is the representation of $\text{CCR}(\mathcal{Y}_{\text{dyn}} \cap |b|^{\frac{1}{2}}\mathcal{Y}_{\text{dyn}}, q_{\text{dyn}})$ defined by $\pi_{\text{AW},1}(\psi^{(*)}(y)) = \psi_{1/r}^{(*)}(y)$, then $(\pi_{\text{AW},1}, \Gamma_{\mathfrak{s}}(\mathcal{Z} \oplus \overline{\mathcal{Z}}), \Omega)$ is the GNS representation associated to the β -KMS state ω_{β} .

5.4.2. *The double β -KMS state.* To define the double β -KMS state associated to ω_{β} we set

$$(\mathcal{X}, Q) := (\mathcal{Y}_{\text{th}} \oplus \mathcal{Y}_{\text{th}}, q_{\text{dyn}} \oplus -q_{\text{dyn}}).$$

Recalling that $\sigma = i^{-1}q_{\text{dyn}}$, this corresponds to adding to the real symplectic space $(\mathcal{Y}_{\text{th},\mathbb{R}}, \text{Re}\sigma)$ its anti-symplectic copy $(\mathcal{Y}_{\text{th},\mathbb{R}}, -\text{Re}\sigma)$. From (5.14) we see that $\mathcal{X} \ni x \mapsto \Psi_{\text{AW}}^{(*)}(x)$ for

$$(5.15) \quad \Psi_{\text{AW}}^{(*)}(x) := \psi_1^{(*)}(y) + \psi_r^{(*)}(y'), \quad x = (y, y') \in \mathcal{X}$$

induces a representation of $\text{CCR}(\mathcal{X}, Q)$.

Definition 5.5. The double β -KMS state ω_d is the quasi-free state on $\text{CCR}(\mathcal{X}, Q)$ defined by

$$\omega_d(\Psi^{(*)}(x_1)\Psi^{(*)}(x_2)) := (\Omega|\Psi_{\text{AW}}^{(*)}(x_1)\Psi_{\text{AW}}^{(*)}(x_2)\Omega)_{\Gamma_{\mathfrak{s}}(\mathcal{Z}\oplus\bar{\mathcal{Z}})}, \quad x_1, x_2 \in \mathcal{X}.$$

Proposition 5.6. ω_d is a pure, gauge invariant quasi-free state on $\text{CCR}(\mathcal{X}, Q)$. If λ_d^\pm are the covariances of ω_d we have

$$\bar{x}_1 \cdot \lambda_d^\pm x_2 = \pm \bar{x}_1 \cdot Q C_\beta^\pm x_2, \quad x_1, x_2 \in \mathcal{X}.$$

where C_β^\pm are the Calderón projectors for B_β defined in Prop. 4.2.

Remark 5.7. Let us denote $(\mathcal{Y}_{\text{th}}, q_{\text{dyn}})$ by (\mathcal{Y}_1, q_1) and let (\mathcal{Y}_2, q_2) be another Hermitian space with $I : (\mathcal{Y}_2, q_2) \rightarrow (\mathcal{Y}_1, -q_1)$ unitary. Then ω_d induces a quasi-free state on $\text{CCR}(\mathcal{Y}_1 \oplus \mathcal{Y}_2, q_1 \oplus q_2)$. Its covariances are

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & I^* \end{pmatrix} \lambda_d^\pm \begin{pmatrix} \mathbb{1} & 0 \\ 0 & I \end{pmatrix} = \pm \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & I^{-1} \end{pmatrix} C_\beta^\pm \begin{pmatrix} \mathbb{1} & 0 \\ 0 & I \end{pmatrix}.$$

Remark 5.8. If $\mathcal{Y} \subset \mathcal{Y}_{\text{th}}$ then ω_d is also a (non necessarily pure) state on $\text{CCR}(\mathcal{Y} \oplus \mathcal{Y}, q \oplus -q)$.

Proof of Prop. 5.6. We obtain from (5.13), (5.12):

$$\begin{aligned} & \Psi_{\text{AW}}^*(x)\Omega = a_1^*(z_+)\Omega + a_1(z_-)\Omega + a_r^*(\bar{z}'_+)\Omega + a_r(\bar{z}'_-)\Omega \\ (5.16) \quad & = ((\rho+1)^{\frac{1}{2}}z_+ + \rho^{\frac{1}{2}}z'_+, \bar{\rho}^{\frac{1}{2}}\bar{z}'_- + (\bar{\rho}+1)^{\frac{1}{2}}\bar{z}'_-), \\ & \Psi_{\text{AW}}(x)\Omega = a_1(z_+)\Omega + a_1^*(z_-)\Omega + a_r(\bar{z}'_-)\Omega + a_r^*(\bar{z}'_+)\Omega \\ & = ((\rho+1)^{\frac{1}{2}}z_- + \rho^{\frac{1}{2}}z'_-, \bar{\rho}^{\frac{1}{2}}\bar{z}'_+ + (\bar{\rho}+1)^{\frac{1}{2}}\bar{z}'_+), \end{aligned}$$

as elements of $\mathcal{Z} \oplus \bar{\mathcal{Z}}$. From (5.16) we immediately obtain that

$$\omega_d(\Psi_{\text{AW}}(x_1)\Psi_{\text{AW}}(x_2)) = \omega_d(\Psi_{\text{AW}}^*(x_1)\Psi_{\text{AW}}^*(x_2)) = 0,$$

ie ω_d is gauge invariant for the complex structure $i \oplus i$ of \mathcal{X} . We have next

$$\begin{aligned} \omega_d(\Psi_{\text{AW}}(x_1)\Psi_{\text{AW}}^*(x_2)) &= (\Psi_{\text{AW}}^*(x_1)\Omega|\Psi_{\text{AW}}^*(x_2)\Omega)_{\mathcal{Z}\oplus\bar{\mathcal{Z}}} \\ &= \left((\rho+1)^{\frac{1}{2}}y_{1+} + \rho^{\frac{1}{2}}y'_{1+} | (\rho+1)^{\frac{1}{2}}y_{2+} + \rho^{\frac{1}{2}}y'_{2+} \right)_{\mathcal{Y}} \\ &\quad + \left(\rho^{\frac{1}{2}}y_{1-} + (\rho+1)^{\frac{1}{2}}y'_{1-} | \rho^{\frac{1}{2}}y_{2-} + (\rho+1)^{\frac{1}{2}}y'_{2-} \right)_{\mathcal{Y}} \end{aligned}$$

If $\lambda_d^+ =: Q \circ C_d^+$, where $Q = q \oplus -q$, we obtain from (5.4) that:

$$C_d^+ = \begin{pmatrix} (\rho+1)\mathbb{1}_+ - \rho\mathbb{1}_- & -\rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}\mathbb{1}_+ + \rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}\mathbb{1}_- \\ \rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}\mathbb{1}_+ - \rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}\mathbb{1}_- & -\rho\mathbb{1}_+ + (\rho+1)\mathbb{1}_- \end{pmatrix},$$

for $\mathbb{1}_\pm = \mathbb{1}_{\mathbb{R}^\pm}(b)$. We compute:

$$\begin{aligned} & (1+\rho)\mathbb{1}_+ - \rho\mathbb{1}_- \\ &= (1 - e^{-\beta b})^{-1}\mathbb{1}_+ - e^{\beta b}(1 - e^{\beta b})^{-1}\mathbb{1}_- = (1 - e^{-\beta b})^{-1}; \\ & -\rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\mathbb{1}_+ + \rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\mathbb{1}_- \\ &= -e^{-\beta b/2}(1 - e^{-\beta b})^{-1} + e^{\beta b/2}(1 - e^{\beta b})^{-1} = e^{\beta b/2}(1 - e^{\beta b})^{-1}; \\ & \rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\mathbb{1}_+ - \rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\mathbb{1}_- \\ &= -e^{\beta b/2}(1 - e^{\beta b})^{-1} = e^{-\beta b/2}(1 - e^{-\beta b})^{-1}; \\ & -\rho\mathbb{1}_+ + (1+\rho)\mathbb{1}_- \\ &= -e^{-\beta b}(1 - e^{-\beta b})^{-1}\mathbb{1}_+ + (1 - e^{\beta b})^{-1}\mathbb{1}_- = (1 - e^{\beta b})^{-1}. \end{aligned}$$

Therefore $C_d^+ = C_\beta^+$. Since $C_d^+ + C_d^- = \mathbb{1}$, we have also $C_d^- = C_\beta^-$.

To see that ω_d is pure, we have to check that the representation of the Weyl algebra $\text{CCR}^{\text{Weyl}}(\mathcal{X}, Q)$ associated to $\Psi_{\text{AW}}^{(*)}(x)$, $x \in \mathcal{X}$ is irreducible. This follows from the definition (5.15) of $\Psi_{\text{AW}}^{(*)}$ and statements (5), (7) in [DG, Thm. 17.24]. \square

6. ABSTRACT KLEIN-GORDON EQUATIONS

In this section we collect some results about abstract Klein-Gordon equations of the form

$$(6.1) \quad (\partial_t + \tilde{w}^*)(\partial_t - \tilde{w})\tilde{\phi} + \tilde{h}_0\tilde{\phi} = 0,$$

where $\tilde{\phi} : \mathbb{R} \rightarrow \tilde{\mathcal{H}}$, $\tilde{\mathcal{H}}$ is some Hilbert space and \tilde{h}_0, \tilde{w} are linear operators on $\tilde{\mathcal{H}}$. Such Klein-Gordon equations arise from stationary metrics on a spacetime $M = \mathbb{R} \times S$, with Killing vector field equal to $\frac{\partial}{\partial t}$, when \tilde{w} represent the shift vector field and the lapse function is equal to 1. The case of general stationary Klein-Gordon operators will be considered later in Sect. 8.

We will also consider the Wick rotated operator \tilde{K}_β obtained by setting $t = is$, where s belongs either to \mathbb{R} or to \mathbb{S}_β . Using sesquilinear form techniques we give a rigorous meaning to its inverse \tilde{K}_β^{-1} and relate it to the Green operators in Sect. 4.

6.1. Hypotheses. We will assume the following hypotheses:

- $$(6.2)$$
- i) \tilde{h}_0 is selfadjoint on $\tilde{\mathcal{H}}$ and $\tilde{h}_0 > 0$,
 - ii) $\text{Dom}\tilde{w} = \text{Dom}\tilde{h}_0^{\frac{1}{2}}$, $\text{Dom}\tilde{h}_0^{\frac{1}{2}} \subset \text{Dom}\tilde{w}^*$ and $\tilde{w}\tilde{h}_0^{-\frac{1}{2}}, \tilde{w}^*\tilde{h}_0^{-\frac{1}{2}} \in B(\tilde{\mathcal{H}})$,
 - iii) if $\tilde{h} := \tilde{h}_0 - \tilde{w}^*\tilde{w}$, considered as a quadratic form on $\text{Dom}\tilde{h}_0^{\frac{1}{2}}$, then $\tilde{h} \sim \tilde{h}_0$.

We can rewrite (6.1) as

$$(6.3) \quad \partial_t^2\tilde{\phi} - 2i\tilde{k}\partial_t\tilde{\phi} + \tilde{h}\tilde{\phi} = 0,$$

where $\tilde{k} = (2i)^{-1}(\tilde{w} - \tilde{w}^*)$, which was considered in [GGH1] in a more general situation.

6.2. Quadratic pencils. One associates to (6.3) the *quadratic pencil*

$$p(z) = z(2\tilde{k} - z) + \tilde{h} = (iz + \tilde{w}^*)(iz - \tilde{w}) + \tilde{h}_0 \in B(\langle \tilde{h}_0 \rangle^{-\frac{1}{2}}\tilde{\mathcal{H}}, \langle \tilde{h}_0 \rangle^{\frac{1}{2}}\tilde{\mathcal{H}}), z \in \mathbb{C},$$

obtained by replacing ∂_t by iz and denotes by $\rho(\tilde{h}, \tilde{k})$ the set of $z \in \mathbb{C}$ such that $p(z) : \langle \tilde{h}_0 \rangle^{-\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} \langle \tilde{h}_0 \rangle^{-\frac{1}{2}}\tilde{\mathcal{H}}$. Note that by applying the Fourier transform in t , the equation

$$(\partial_t + \tilde{w}^*)(\partial_t - \tilde{w})u + \tilde{h}_0u = v,$$

is formally equivalent to

$$p(\lambda)\hat{u}(\lambda) = \hat{v}(\lambda), \quad \lambda \in \mathbb{R}.$$

The inverse $p(z)^{-1}$ for $z \in \rho(\tilde{h}, \tilde{k})$ appears in the expression (6.16) of the resolvent $(z - \tilde{H})^{-1}$, where \tilde{H} is defined in (6.5).

Proposition 6.1. (1) *The operator $p(z)$ on $\tilde{\mathcal{H}}$ with domain $\langle \tilde{h} \rangle^{-1}\tilde{\mathcal{H}}$ is closed with*

- $p^*(z) = p(\bar{z})$;
- (2) $p(z) : \langle \tilde{h} \rangle^{-1}\tilde{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}$ iff $p(z) : \langle \tilde{h} \rangle^{-\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} \langle \tilde{h} \rangle^{\frac{1}{2}}\tilde{\mathcal{H}}$;
- (3) $p(i\lambda) : \langle \tilde{h} \rangle^{-1}\tilde{\mathcal{H}} \xrightarrow{\sim} \mathcal{H}$ for $\lambda \in \mathbb{R}^*$ and

$$\sup_{\lambda \in \mathbb{R}^*} \|(\tilde{h}_0 + \lambda^2)^{\frac{1}{2}}p(i\lambda)^{-1}(\tilde{h}_0 + \lambda^2)^{\frac{1}{2}}\|_{B(\tilde{\mathcal{H}})} < \infty.$$

Proof. Statements (1) and (2) are shown in [GGH2, Lemma 8.1], using that \tilde{k} is symmetric on $\langle \tilde{h}_0 \rangle^{-\frac{1}{2}} \tilde{\mathcal{H}} = \langle \tilde{h} \rangle^{-\frac{1}{2}} \tilde{\mathcal{H}}$ and $\tilde{k} \langle \tilde{h} \rangle^{-\frac{1}{2}} \in B(\tilde{\mathcal{H}})$. Let us now prove (3). Since $\operatorname{Re} p(i\lambda) = \tilde{h} + \lambda^2$ we have

$$\lambda^2 \|u\|^2 \leq \operatorname{Re}(u|p(i\lambda)u), \quad u \in \langle \tilde{h} \rangle^{-1} \tilde{\mathcal{H}},$$

hence $\operatorname{Ker} p(i\lambda) = \{0\}$ for $\lambda \in \mathbb{R}^*$. Since $p(i\lambda)^* = p(-i\lambda)$ we have also $\operatorname{Ker} p(i\lambda)^* = \{0\}$ hence $p(i\lambda) : \langle \tilde{h} \rangle^{-1} \tilde{\mathcal{H}} \xrightarrow{\sim} \tilde{\mathcal{H}}$ using that $p(i\lambda) : \langle \tilde{h} \rangle^{-1} \tilde{\mathcal{H}} \rightarrow \tilde{\mathcal{H}}$ is closed. We have then

$$\begin{aligned} (\tilde{h} + \lambda^2)^{-\frac{1}{2}} p(i\lambda) (\tilde{h} + \lambda^2)^{-\frac{1}{2}} &= \mathbb{1} - iA(\lambda), \\ A_\lambda &:= i(\tilde{h} + \lambda^2)^{-\frac{1}{2}} (\lambda \tilde{w} - \tilde{w}^* \lambda) (\tilde{h} + \lambda^2)^{-\frac{1}{2}}. \end{aligned}$$

From hypotheses (6.2), we obtain that $A_\lambda \in B(\tilde{\mathcal{H}})$ and $A_\lambda = A_\lambda^*$. Therefore $\|(\mathbb{1} - iA_\lambda)^{-1}\|_{B(\tilde{\mathcal{H}})} \leq 1$ and since

$$(\tilde{h} + \lambda^2)^{\frac{1}{2}} p(i\lambda)^{-1} (\tilde{h} + \lambda^2)^{\frac{1}{2}}$$

we have

$$\sup_{\lambda \in \mathbb{R}^*} \|(\tilde{h} + \lambda^2)^{\frac{1}{2}} p(i\lambda)^{-1} (\tilde{h} + \lambda^2)^{\frac{1}{2}}\|_{B(\tilde{\mathcal{H}})} \leq 1.$$

This proves (3) using that $\tilde{h} \sim \tilde{h}_0$. \square

6.3. First order system. Setting

$$(6.4) \quad \tilde{f}(t) = \tilde{\varrho}_t \tilde{\phi} := \begin{pmatrix} \tilde{\phi}(t) \\ i^{-1}(\partial_t - \tilde{w})\tilde{\phi}(t) \end{pmatrix} = \begin{pmatrix} \tilde{f}_0(t) \\ \tilde{f}_1(t) \end{pmatrix},$$

(6.1) is formally rewritten as

$$(6.5) \quad \partial_t \tilde{f} = i\tilde{H}\tilde{f}, \quad \tilde{H} = \begin{pmatrix} -i\tilde{w} & \mathbb{1} \\ \tilde{h}_0 & i\tilde{w}^* \end{pmatrix}.$$

The conserved energy is

$$(6.6) \quad \overline{\tilde{f}} \cdot \tilde{E} \tilde{f} = \|\tilde{f}_1 - i\tilde{w}\tilde{f}_0\|^2 + (\tilde{f}_0 | \tilde{h} \tilde{f}_0),$$

which is positive definite by (6.2). The Hilbert space associated to \tilde{E} will be denoted by $\tilde{\mathcal{E}}$. It equals $\tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$ as a topological vector space. We set also $\tilde{\mathcal{E}}^* := \tilde{\mathcal{H}} \oplus \tilde{h}_0^{\frac{1}{2}} \tilde{\mathcal{H}}$.

We recall from 1.4 that

$$(6.7) \quad \tilde{\mathcal{H}} \cap \tilde{h}_0^{\pm \frac{1}{2}} \tilde{\mathcal{H}} = \operatorname{Dom} \tilde{h}_0^{\mp \frac{1}{2}} \text{ hence } \tilde{\mathcal{E}} \cap \tilde{\mathcal{E}}^* = \operatorname{Dom} \tilde{h}_0^{\frac{1}{2}} \oplus \operatorname{Dom} \tilde{h}_0^{-\frac{1}{2}}.$$

The following proposition will be proved in Subsect. 6.5.

Proposition 6.2. *The operator $\tilde{H} = \begin{pmatrix} -i\tilde{w} & \mathbb{1} \\ \tilde{h}_0 & i\tilde{w}^* \end{pmatrix}$ is bounded from $\tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}^*$. It induces on $\tilde{\mathcal{E}}$ the operator \tilde{H} defined by*

$$\operatorname{Dom} \tilde{H} = \{\tilde{f} \in \tilde{\mathcal{E}} : \tilde{H}\tilde{f} \in \tilde{\mathcal{E}} \cap \tilde{\mathcal{E}}^*\}.$$

\tilde{H} is a densely defined selfadjoint operator on $\tilde{\mathcal{E}}$ with $\operatorname{res}(\tilde{H}) = \rho(\tilde{h}, \tilde{k})$.

Note that $(\tilde{\mathcal{E}}, \tilde{\mathcal{E}}^*)$ form a non degenerate dual pair for the charge

$$(6.8) \quad \overline{\tilde{f}} \cdot \tilde{q} \tilde{f}' = (\tilde{f}_1 | \tilde{f}'_0)_{\tilde{\mathcal{H}}} + (\tilde{f}_0 | \tilde{f}'_1)_{\tilde{\mathcal{H}}}, \quad \tilde{f} \in \tilde{\mathcal{E}}, \quad \tilde{f}' \in \tilde{\mathcal{E}}^*,$$

and one has

$$\overline{\tilde{f}} \cdot \tilde{E} \tilde{f} = \overline{\tilde{f}} \cdot \tilde{q} \tilde{H} \tilde{f}, \quad \tilde{f} \in \tilde{\mathcal{E}}.$$

6.4. The Wick rotated operator. Setting formally $t = is$ we obtain the formal expression

$$(6.9) \quad \tilde{K} = -(\partial_s + i\tilde{w}^*)(\partial_s - i\tilde{w}) + \tilde{h}_0.$$

To give a meaning to (6.9), we will use sesquilinear forms techniques. Let us set as in Sect. 4.1 for $0 < \beta \leq \infty$:

$$(6.10) \quad \tilde{\mathcal{H}}_\beta = L^2(\mathbb{S}_\beta) \otimes \tilde{\mathcal{H}}, \text{ for } \beta < \infty, \tilde{\mathcal{H}}_\infty = L^2(\mathbb{R}) \otimes \tilde{\mathcal{H}}.$$

We consider the sesquilinear form associated to \tilde{K} :

$$\tilde{Q}_\beta(u, u) = \|\partial_s u\|_{\tilde{\mathcal{H}}_\beta}^2 + (u|\tilde{h}u)_{\tilde{\mathcal{H}}_\beta} - i(\partial_s u|\tilde{w}u)_{\tilde{\mathcal{H}}_\beta} - i(\tilde{w}u|\partial_s u)_{\tilde{\mathcal{H}}_\beta},$$

with domain $\text{Dom}\tilde{Q}_\beta = \langle -\partial_s^2 + \tilde{h}_0 \rangle^{-\frac{1}{2}} \tilde{\mathcal{H}}_\beta$, where ∂_s is equipped with its natural domain on $\tilde{\mathcal{H}}_\beta$. From hypotheses (6.2) we obtain that

$$\text{Re}\tilde{Q}_\beta(u, u) \sim \|\partial_s u\|_{\tilde{\mathcal{H}}_\beta}^2 + (u|\tilde{h}_0 u)_{\tilde{\mathcal{H}}_\beta}, \quad |\text{Im}\tilde{Q}_\beta(u, u)| \leq C\text{Re}\tilde{Q}_\beta(u, u),$$

hence \tilde{Q}_β is a closed sectorial form.

If we apply the Lax-Milgram theorem as stated in [Gr, Lemma 12.15] to the Hilbert space $(-\partial_s^2 + \tilde{h}_0)^{-\frac{1}{2}} \tilde{\mathcal{H}}_\beta$ and note that its topological anti-dual is canonically identified with $(-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \tilde{\mathcal{H}}_\beta$, we obtain that \tilde{Q}_β induces a boundedly invertible operator

$$(6.11) \quad \tilde{K}_\beta : (-\partial_s^2 + \tilde{h}_0)^{-\frac{1}{2}} \tilde{\mathcal{H}}_\beta \xrightarrow{\sim} (-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \tilde{\mathcal{H}}_\beta.$$

We can apply the results of Subsect. 4.1 setting $\mathfrak{h} = \tilde{\mathcal{E}}$, $b = \tilde{H}$, see (4.1) for the notation used, and obtain an operator

$$(6.12) \quad \partial_s + \tilde{H} : \tilde{\mathcal{E}}_\beta \xrightarrow{\sim} (-\partial_s^2 + \tilde{H}^2)^{\frac{1}{2}} \tilde{\mathcal{E}}_\beta.$$

The relation between \tilde{K}_β^{-1} and $\partial_s + \tilde{H}$ is given by the following proposition. Below we denote by π_i the maps $\pi_i \tilde{f} = \tilde{f}_i$ for $\tilde{f} = \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \end{pmatrix}$, so that

$$\pi_1^* u = \begin{pmatrix} 0 \\ u \end{pmatrix}, \quad \pi_0^* u = \begin{pmatrix} u \\ 0 \end{pmatrix}.$$

Proposition 6.3. *One has*

- (6.13)
- i) $\pi_1^* : (-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \tilde{\mathcal{H}}_\beta \rightarrow (\partial_s^2 + \tilde{H}^2)^{\frac{1}{2}} \tilde{\mathcal{E}}_\beta$ continuously,
 - ii) $\pi_0(\partial_s + \tilde{H})^{-1} \pi_1^* : (-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \tilde{\mathcal{H}}_\beta \rightarrow (-\partial_s^2 + \tilde{h}_0)^{-\frac{1}{2}} \tilde{\mathcal{H}}_\beta$ continuously,
 - iii) $\tilde{K}_\beta^{-1} = \pi_0(\partial_s + \tilde{H})^{-1} \pi_1^*$ in $L((-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \tilde{\mathcal{H}}_\beta, (-\partial_s^2 + \tilde{h}_0)^{-\frac{1}{2}} \tilde{\mathcal{H}}_\beta)$.

6.5. Proofs of Props. 6.2 and 6.3.

6.5.1. *Preparations.* We will prove Props. 6.2, 6.3 using results in [GGH1]. There the form (6.3) of the Klein-Gordon equation is used and instead of (6.4) one sets:

$$(6.14) \quad g := \begin{pmatrix} \tilde{\phi} \\ i^{-1} \partial_t \tilde{\phi} \end{pmatrix},$$

(6.3) is formally rewritten as

$$\partial_t g = i\hat{H}g, \quad \hat{H} = \begin{pmatrix} 0 & \mathbb{1} \\ \tilde{h} & 2\tilde{k} \end{pmatrix}.$$

The conserved energy is

$$\bar{g} \cdot \hat{E}g = \|g_1\|^2 + (g_0|\tilde{h}g_0).$$

The Hilbert space $\hat{\mathcal{E}}$ naturally associated to \hat{E} equals again $\tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$.

If \tilde{f} is given by (6.4) and g by (6.14) one has

$$\tilde{f} = Ug \text{ for } U = \begin{pmatrix} \mathbb{1} & 0 \\ i\tilde{w} & \mathbb{1} \end{pmatrix},$$

and

$$(6.15) \quad U : \hat{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}, \quad (Ug|Ug)_{\tilde{\mathcal{E}}} = (g|g)_{\hat{\mathcal{E}}}.$$

Formally one has $\tilde{H} = U\hat{H}U^{-1}$, and since $U : \hat{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}$ is unitary, Prop. 6.2 follows if we prove the analogous result for \tilde{H} . One sets then

$$\hat{\mathcal{E}}^* := \tilde{\mathcal{H}} \oplus \tilde{h}_0^{\frac{1}{2}} \tilde{\mathcal{H}},$$

which forms again a dual pair with $\hat{\mathcal{E}}$ for

$$\bar{g} \cdot \hat{q}g' = (g_1 + i\tilde{w}g_0|g'_0)_{\tilde{\mathcal{H}}} + (g_0|g'_1 + i\tilde{w}g'_0), \quad g \in \hat{\mathcal{E}}, g' \in \hat{\mathcal{E}}^*.$$

We have of course $\hat{q} = U^*qU$.

6.5.2. *Proof of Prop. 6.2.* The matrix \hat{H} induces a bounded operator $\hat{H} : \hat{\mathcal{E}} \rightarrow \hat{\mathcal{E}}^*$. One denotes by \hat{H} the linear operator induced by \hat{H} on $\hat{\mathcal{E}}$. Its domain is

$$\text{Dom}\hat{H} = \{g \in \hat{\mathcal{E}} : \hat{H}g \in \hat{\mathcal{E}} \cap \hat{\mathcal{E}}^*\},$$

and as in (6.7) we have $\hat{\mathcal{E}} \cap \hat{\mathcal{E}}^* = \text{Dom}\tilde{h}_0^{\frac{1}{2}} \oplus \text{Dom}\tilde{h}_0^{-\frac{1}{2}}$.

It follows then from [GGH1, Prop. 5.8, Thm. 5.9], and the fact that there exists $z \neq 0$ in $\rho(\tilde{h}, \tilde{k})$, that \hat{H} with the domain above is a densely defined selfadjoint operator on $\hat{\mathcal{E}}$ with $\text{res}(\hat{H}) = \rho(\tilde{h}, \tilde{k})$. Setting

$$\tilde{H} := U\hat{H}U^{-1}$$

completes the proof of Prop. 6.2.

6.5.3. *Proof of Prop. 6.3.* We have seen that $\tilde{H} = U\hat{H}U^{-1}$ and $U : \hat{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}$ is unitary. Moreover $\pi_0U^{-1} = \pi_0$ and $U\pi_1^* = \pi_1^*$, so it suffices to prove the proposition with $\tilde{H}, \tilde{\mathcal{E}}$ replaced by $\hat{H}, \hat{\mathcal{E}}$.

One can express the resolvent $(\hat{H} - z)^{-1}$ using $p(z)$ as follows: if $z \in \rho(\tilde{h}, \tilde{k})$ then:

$$(6.16) \quad (\hat{H} - z)^{-1} = p(z)^{-1} \begin{pmatrix} z - 2\tilde{k} & \mathbb{1} \\ \tilde{h} & z \end{pmatrix} \in B(\hat{\mathcal{E}}, \hat{\mathcal{E}}).$$

Note that (6.16) is different from the formula found in [GGH1, Prop. 5.8], because weaker assumptions on \tilde{h}, \tilde{k} were used there. In our case using that $\tilde{k}|\tilde{h}_0|^{-\frac{1}{2}} \in B(\tilde{\mathcal{H}})$ one deduces from [GGH1, Lemma 2.2] that

$$p(z) : \tilde{\mathcal{H}} + |\tilde{h}_0|^{\frac{1}{2}}\tilde{\mathcal{H}} \rightarrow \langle \tilde{h}_0 \rangle^{-\frac{1}{2}}\tilde{\mathcal{H}} = \tilde{\mathcal{H}} \cap |\tilde{h}_0|^{-\frac{1}{2}}\tilde{\mathcal{H}}, \quad z \in \rho(\tilde{h}, \tilde{k}).$$

Using this fact it is straightforward to show that the rhs in (6.16) maps $\hat{\mathcal{E}}$ into itself.

In general we have $0 \notin \rho(\tilde{h}, \tilde{k})$ hence $0 \in \sigma(\hat{H})$ but \hat{H}^{-1} is well defined as

$$(6.17) \quad \hat{H}^{-1} = \begin{pmatrix} -2\tilde{h}^{-1}\tilde{k} & \tilde{h}^{-1} \\ \mathbb{1} & 0 \end{pmatrix} \in B(\hat{\mathcal{E}}, |\tilde{h}|^{-1}\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}})$$

which corresponds to (6.16) for $z = 0$.

We have $\text{Ker}\hat{H} = \{0\}$ since $\hat{H}g = 0$ implies $g_1 = 0, \tilde{h}g_0 = 0$ and \tilde{h} is injective. Therefore we can apply the results of Subsect. 4.1 to construct $(\partial_s + \hat{H})^{-1}$ for $b = \hat{H}, \mathfrak{h} = \hat{\mathcal{E}}$. As before we introduce the Hilbert spaces $\tilde{\mathcal{H}}_\beta$ and $\hat{\mathcal{E}}_\beta$ for $\beta \in]0, \infty[$.

Let us now prove the proposition. We denote by \mathcal{F} the Fourier transform in s on \mathbb{R} or \mathbb{S}_β and set $\hat{v}(\lambda) = \mathcal{F}v(\lambda)$, where $\lambda \in \mathbb{R}$ or \mathbb{S}_β is the associated Fourier variable. Let

$$M := \mathcal{F} \circ (\partial_s + \hat{H})^{-1} \pi_1^* (-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \circ \mathcal{F}^{-1} = \int^{\oplus} M(\lambda) d\lambda.$$

From (6.17), (6.16) we obtain that

$$M(\lambda) = p(i\lambda)^{-1} (\lambda^2 + \tilde{h}_0)^{\frac{1}{2}} \begin{pmatrix} \mathbb{1} \\ i\lambda \mathbb{1} \end{pmatrix}, \quad \lambda \neq 0, \quad M(0) = \begin{pmatrix} \tilde{h}^{-1} \tilde{h}_0^{\frac{1}{2}} \\ 0 \end{pmatrix}.$$

From (3) in Prop. 6.1 we obtain that

$$\sup_{\lambda \in \mathbb{R}} \|M(\lambda)\|_{B(\tilde{\mathcal{H}}, \tilde{\mathcal{E}})} < \infty,$$

which proves statement *i*) and also statement *ii*) of the proposition. Next if $v \in (-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \tilde{\mathcal{H}}_\beta$ and $g = (\partial_s + \hat{H})^{-1} \pi_1^* v$ we have $g \in \tilde{\mathcal{E}}_\beta$ and

$$\partial_s g_0 + g_1 = 0, \quad \tilde{K} g_0 = v,$$

hence $\partial_s g_0 \in \tilde{\mathcal{H}}_\beta$, $\tilde{h}_0^{\frac{1}{2}} g_0 \in \tilde{\mathcal{H}}_\beta$ and $\tilde{K} g_0 = v$, which shows that $g_0 \in (-\partial_s^2 + \tilde{h}_0)^{-\frac{1}{2}} \tilde{\mathcal{H}}_\beta$ and $g_0 = \tilde{K}_\beta^{-1} v$. This completes the proof of statement *iii*). \square

7. VACUA AND KMS STATES FOR ABSTRACT KLEIN-GORDON EQUATIONS

In this section we consider vacuum and KMS states for abstract, time-independent Klein-Gordon equations, which can be reduced to the framework of Sect. 5. We will show that the covariances of the vacuum and double β -KMS states can be expressed by the Calderón projectors defined in Sect. 5.

7.1. Vacua and KMS states. Let us consider an abstract Klein-Gordon equation

$$(\partial_t + \tilde{w}^*)(\partial_t - \tilde{w})\tilde{\phi} + \tilde{h}_0\tilde{\phi} = 0,$$

as in Sect. 6, where $\tilde{\phi} : \mathbb{R} \rightarrow \tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}$ is a Hilbert space. We denote by

$$\tilde{P} = (\partial_t + \tilde{w}^*)(\partial_t - \tilde{w}) + \tilde{h}_0$$

the corresponding Klein-Gordon operator. In the sequel we use the notation introduced in Subsect. 5.1.

The assumptions corresponding to those in Subsect. 5.1 are as follows:

We assume that there exists a dense subspace $\tilde{\mathcal{D}} \subset \tilde{\mathcal{H}}$ and set

$$(7.1) \quad \tilde{\mathcal{Y}} := \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}, \quad \tilde{f} \cdot \tilde{q}\tilde{f} := (\tilde{f}_1 | \tilde{f}_0) + (\tilde{f}_0 | \tilde{f}_1), \quad \tilde{f} = \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \end{pmatrix} \in \tilde{\mathcal{Y}}.$$

We fix linear operators $\tilde{h}_0, \tilde{w}, \tilde{w}^*$ on $\tilde{\mathcal{H}}$ with domain $\tilde{\mathcal{D}}$ such that:

$$(7.2) \quad \begin{aligned} &\tilde{w}, \tilde{w}^*, \tilde{h}_0 : \tilde{\mathcal{D}} \rightarrow \tilde{\mathcal{D}}, \\ &(\tilde{w}^* u | v) = (u | \tilde{w} v), \quad u, v \in \tilde{\mathcal{D}}, \quad (u | \tilde{h}_0 u) > 0 \text{ for } u \in \tilde{\mathcal{D}}, u \neq 0, \\ &\|\tilde{w} u\|^2 \leq (1 - \delta)(u | \tilde{h}_0 u), \quad \|\tilde{w}^* u\|^2 \leq c(u | \tilde{h}_0 u), \quad u \in \tilde{\mathcal{D}} \text{ for } c > 0, 0 < \delta < 1. \end{aligned}$$

Setting $\tilde{q}_0(u, u) = (u | \tilde{h}_0 u)$ with $\text{Dom} \tilde{q}_0 = \tilde{\mathcal{D}}$, it follows that \tilde{q}_0 is closeable and we still denote by \tilde{h}_0 the operator associated to \tilde{q}_0^{cl} , i.e. the Friedrichs extension of \tilde{h}_0 on $\tilde{\mathcal{D}}$. We assume that $\text{Ker} \tilde{h}_0 = \{0\}$ and deduce then from (7.2) that hypotheses (6.2) are satisfied by $\tilde{h}_0, \tilde{w}, \tilde{w}^*$. By construction $\tilde{\mathcal{D}}$ is dense in $\tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}}$.

We set then

$$(7.3) \quad \tilde{f} \cdot \tilde{E}\tilde{f} = (\tilde{f} | \tilde{f})_{\tilde{\mathcal{E}}} = \|\tilde{f}_1 - i\tilde{w}\tilde{f}_0\|^2 + (\tilde{f}_0 | \tilde{h}\tilde{f}_0), \quad \tilde{f} \in \tilde{\mathcal{Y}},$$

and by the density of $\tilde{\mathcal{D}}$ in $\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}$ we obtain that $\tilde{\mathcal{Y}}_{\text{en}} = \tilde{\mathcal{E}}$. We set also

$$b = \tilde{H} = \begin{pmatrix} -i\tilde{w} & \mathbb{1} \\ \tilde{h}_0 & i\tilde{w}^* \end{pmatrix},$$

where \tilde{H} is defined as a selfadjoint operator on $\tilde{\mathcal{E}}$ by Prop. 6.2.

7.1.1. *Infrared condition.* To be able to apply the results of Sect. 5 we need to check the hypotheses (5.2), (5.3) and (5.10), which are used to ensure that the vacuum and KMS states are well defined as quasi-free states on $\text{CCR}(\tilde{\mathcal{Y}}, \tilde{q})$, see Remarks 5.2 and 5.4. This is done in Prop. 7.1 below. For Klein-Gordon equations on stationary spacetimes, the assumption (7.4) will be checked in Lemma 8.3.

In Subsect. 5.1 we introduced the dynamical and thermal Hilbert spaces $\tilde{\mathcal{Y}}_{\text{dyn}}, \tilde{\mathcal{Y}}_{\text{th}}$ which in the present situation equal $|\tilde{H}|^{\frac{1}{2}}\tilde{\mathcal{E}}$ and $|\tilde{H}|^{\frac{1}{2}}\tilde{\mathcal{E}} \cap |\tilde{H}|\tilde{\mathcal{E}}$ respectively.

Proposition 7.1. *Assume that*

$$(7.4) \quad \tilde{\mathcal{D}} \subset \text{Dom}\tilde{h}_0^{-\frac{1}{2}}.$$

Then $\tilde{\mathcal{Y}} = \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$ satisfies:

- (1) $\tilde{\mathcal{Y}} \subset \text{Dom}\tilde{H}^{-1}$, hence $\tilde{\mathcal{Y}} \subset \tilde{\mathcal{Y}}_{\text{th}} \subset \tilde{\mathcal{Y}}_{\text{dyn}}$,
- (2) $\tilde{f} \cdot \tilde{E}b^{-1}\tilde{f} = \tilde{f} \cdot \tilde{q}f$ for $f \in \tilde{\mathcal{Y}}$
- (3) $\tilde{\mathcal{Y}} \subset \text{Dom}\tilde{H}$.

It follows from Prop. 7.1 that hypotheses (5.2), (5.3) and (5.10) in Sect. 5 are satisfied.

Proof. To check (1) it suffices to show that

$$(7.5) \quad (\tilde{H}^{-1}\tilde{f}|\tilde{H}^{-1}\tilde{f})_{\tilde{\mathcal{E}}} < \infty, \quad \tilde{f} \in \tilde{\mathcal{Y}},$$

since $\tilde{\mathcal{Y}} \subset \tilde{\mathcal{E}}$.

Let us now check (7.5). We recall that in Subsect. 6.3 we considered also \tilde{H} as a bounded operator from $\tilde{\mathcal{E}}$ to $\tilde{\mathcal{E}}^*$, where $\tilde{\mathcal{E}}^* = \tilde{\mathcal{H}} \oplus \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}$, (see Prop. 6.2, where \tilde{H} with the above meaning was denoted by \tilde{H}). The equation $\tilde{H}g = f$ for $g \in \tilde{\mathcal{E}}$, $f \in \tilde{\mathcal{E}}^*$ is equivalent to

$$(7.6) \quad \begin{cases} (\tilde{h}_0 - \tilde{w}^*\tilde{w})g_0 = f_1 - i\tilde{w}^*f_0 \in \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}, \\ g_1 - i\tilde{w}g_0 = f_0 \in \tilde{\mathcal{H}}, \end{cases}$$

The operator $\tilde{h} = \tilde{h}_0 - \tilde{w}^*\tilde{w} \in B(\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}, \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}})$ is boundedly invertible, since from (7.2) the quadratic forms associated to \tilde{h} and \tilde{h}_0 are equivalent. Therefore (7.6) is equivalent to

$$(7.7) \quad \begin{cases} g_0 = \tilde{h}^{-1}(f_1 - i\tilde{w}^*f_0) \in \tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}, \\ g_1 = g_0 + i\tilde{w}\tilde{h}^{-1}(f_1 - i\tilde{w}^*f_0) \in \tilde{\mathcal{H}}. \end{cases}$$

It follows that $\tilde{H} : \tilde{\mathcal{E}} \rightarrow \tilde{\mathcal{E}}^*$ is boundedly invertible. Now by hypothesis (7.4), we have $\tilde{\mathcal{D}} \subset \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}$, hence $\tilde{\mathcal{Y}} \subset \tilde{\mathcal{E}}^*$. Therefore $\tilde{H}^{-1} : \tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{E}}$, which proves (7.5) and completes the proof of (1). (2) follows then by using (7.7) and (7.3).

It remains to prove (3). We have $\tilde{\mathcal{Y}} \subset \tilde{\mathcal{E}}$ and since $\tilde{\mathcal{D}} \subset \text{Dom}\tilde{h}_0^{-\frac{1}{2}}$ we have also $\tilde{\mathcal{Y}} \subset \tilde{\mathcal{E}}^* = \tilde{\mathcal{H}} \oplus \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}$. Since $\tilde{H} : \tilde{\mathcal{Y}} \rightarrow \tilde{\mathcal{Y}}$ this implies (3), by the definition of $\text{Dom}\tilde{H}$ in Prop. 6.2. \square

We can then apply Subsects. 5.2, 5.3, 5.4 and define the vacuum state $\tilde{\omega}_{\text{vac}}$, the β -KMS state $\tilde{\omega}_\beta$ and the double β -KMS state $\tilde{\omega}_d$ associated to the symplectic dynamics $r_t = e^{it\tilde{H}}$.

7.2. The Calderón projectors. In Subsect. 6.4 we defined the Wick rotated operators

$$\tilde{K}_\beta = -(\partial_s + i\tilde{w}^*)(\partial_s - i\tilde{w}) + \tilde{h}_0$$

and the Hilbert spaces $\tilde{\mathcal{H}}_\beta$ for $0 < \beta \leq \infty$ defined in (6.10). We now define Calderón projectors for \tilde{K}_β , which are similar to the Calderón projectors for the operators $B_\beta = \partial_s + \tilde{H}$, acting on the Hilbert spaces $\tilde{\mathcal{E}}_\beta$, defined in 4.1.2 and 4.1.3.

In all this subsection we will assume that (7.4) holds ie that $\tilde{\mathcal{Y}} \subset \text{Dom}\tilde{h}_0^{-\frac{1}{2}}$.

7.2.1. Calderón projectors for \tilde{K}_∞ . We follow the notation in 4.1.2, in particular $I_\infty^\pm = \pm]0, +\infty[$. We fix $\chi \in C_c^\infty(\mathbb{R})$ with $\int \chi(s)ds = 1$ and set $\chi_n = n\chi(n^{-1}s)$ for $n \in \mathbb{N}$.

$$\tilde{\gamma}_{\infty,n}^* \tilde{g} := \chi'_n(s) \otimes \tilde{g}_1 + \chi_n(s) \otimes (\tilde{g}_0 - i\tilde{w}^* \tilde{g}_1),$$

for $\tilde{g} \in \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$ so that

$$\tilde{\gamma}_{\infty,n}^* \tilde{g} \rightarrow \tilde{\gamma}_\infty^* \tilde{g} \text{ in } \mathcal{E}'(\mathbb{R}; \tilde{\mathcal{H}}), \text{ when } n \rightarrow +\infty,$$

for:

$$\tilde{\gamma}_\infty^* \tilde{g} := \delta'_0(s) \otimes \tilde{g}_1 + \delta_0(s) \otimes (\tilde{g}_0 - i\tilde{w}^* \tilde{g}_1).$$

If $\tilde{u} \in \overline{C^1(I_\infty^\pm; \tilde{\mathcal{H}})} \cap \overline{C^0(I_\infty^\pm; \tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}})}$ satisfies $\tilde{K}\tilde{u} = 0$ in I_∞^\pm we set

$$\tilde{\gamma}_\infty^\pm \tilde{u} = \begin{pmatrix} \tilde{u}(0^\pm) \\ -(\partial_s - i\tilde{w})\tilde{u}(0^\pm) \end{pmatrix},$$

and

$$\tilde{S} = \begin{pmatrix} 2i\tilde{w}^* & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Proposition 7.2. *Assume that $\tilde{f} \in \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$. Then:*

(1) *the limit*

$$\lim_{n \rightarrow +\infty} \tilde{K}_\infty^{-1} \tilde{\gamma}_{\infty,n}^* \tilde{S} \tilde{f} =: \tilde{K}_\infty^{-1} \tilde{\gamma}_\infty^* \tilde{S} \tilde{f}$$

exists in $\mathcal{D}'(I_\infty^\pm; \tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}})$ and is independent on the choice of the cutoff function

χ .

(2) *We have*

$$\tilde{K}_\infty^{-1} \tilde{\gamma}_\infty^* \tilde{S} \tilde{f} \in \overline{C^1(I_\infty^\pm; \tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}})},$$

(3)

$$\mp \tilde{\gamma}_\infty^\pm \tilde{K}_\infty^{-1} \tilde{\gamma}_\infty^* \tilde{S} \tilde{f} = C_\infty^\pm \tilde{f},$$

where C_∞^\pm are the Calderón projectors for $B_\infty = \partial_s + b$, with $b = \tilde{H}$, $\mathfrak{h} = \tilde{\mathcal{E}}$, defined in Prop. 4.1.

Definition 7.3. *The operators*

$$\tilde{c}_\infty^\pm := \mp \tilde{\gamma}_\infty^\pm \tilde{K}_\infty^{-1} \tilde{\gamma}_\infty^* \in L(\tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}; \tilde{\mathcal{E}})$$

are called the Calderón projectors for \tilde{K}_∞ .

Proof of Prop. 7.2. We prove only the + case. Note first that if $\chi \in C_c^\infty(\mathbb{R})$ and $u \in \text{Dom}\tilde{h}_0^{-\frac{1}{2}}$ then $\chi(s) \otimes u \in (-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \mathcal{H}_\infty$. In fact after Fourier transform in s , this amounts to prove that

$$\int_{\mathbb{R}} |\hat{\chi}(\lambda)|^2 |u|(\lambda^2 + \tilde{h}_0)^{-1} u_{\tilde{\mathcal{H}}} d\lambda < \infty,$$

which follows from the fact that $(u|\tilde{h}_0^{-1}u)_{\tilde{\mathcal{H}}} < \infty$. If $\tilde{f} \in \text{Dom}\tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$, then $\tilde{f}_0, \tilde{f}_1 - i\tilde{w}^* \tilde{f}_0 \in \tilde{\mathcal{D}} \subset \text{Dom}\tilde{h}_0^{-\frac{1}{2}}$ hence

$$\tilde{S} \tilde{\gamma}_{\infty,n}^* \tilde{f} = \chi'_n(s) \otimes \tilde{f}_0 + \chi_n(s) \otimes (\tilde{f}_1 - i\tilde{w}^* \tilde{f}_0)$$

belongs to $(-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}}\mathcal{H}_\infty$. Therefore we can apply Prop. 6.3 and we have

$$(7.8) \quad \tilde{K}_\infty^{-1}\tilde{\gamma}_{\infty,n}^*\tilde{S}\tilde{f} = \pi_0 B_\infty^{-1}\pi_1^*\tilde{\gamma}_{\infty,n}^*\tilde{S}\tilde{f},$$

where $B_\infty = (\partial_s + \tilde{H})$, acting on $\tilde{\mathcal{E}}_\infty$. Arguing as in the proof of Prop. 4.1, we have for $\tilde{g} \in \tilde{\mathcal{E}}$:

$$\begin{aligned} \lim_{n \rightarrow \infty} B_\infty^{-1}(\chi_n \otimes \tilde{g})(s) &= \mathbb{1}_{\mathbb{R}^+}(\tilde{H})e^{-s\tilde{H}}\tilde{g}, \\ \lim_{n \rightarrow \infty} B_\infty^{-1}(\chi'_n \otimes \tilde{g})(s) &= -\tilde{H}\mathbb{1}_{\mathbb{R}^+}(\tilde{H})e^{-s\tilde{H}}\tilde{g} \text{ in } \mathcal{D}'(I_\infty^+; \tilde{\mathcal{E}}). \end{aligned}$$

An easy computation show then that

$$\lim_{n \rightarrow +\infty} -(B_\infty^{-1}\tilde{\gamma}_{\infty,n}^*\tilde{S}\tilde{f})(s) = \mathbb{1}_{\mathbb{R}^+}(\tilde{H})e^{-s\tilde{H}}\tilde{f} =: F(s) \text{ in } \mathcal{D}'(I_\infty^+; \tilde{\mathcal{E}})$$

which using (7.8) proves (1). Next since $\tilde{f} \in \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}} \subset \text{Dom}\tilde{H}$, we have $F \in \overline{C^1}(I_\infty^+; \tilde{\mathcal{E}})$, which implies that

$$\pi_0 F(s) \in \overline{C^1}(I_\infty^+; \tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}})$$

which proves (2). Finally we check that $\tilde{\gamma}_\infty^+\pi_0 F(s) = \Gamma_\infty^+\tilde{f}$, which using the definition of C_∞^+ in Prop. 4.1 proves (3). \square

From Subsect. 5.2 we obtain the following result, expressing the covariances of the vacuum state $\tilde{\omega}_{\text{vac}}$ for P in terms of the Calderón projectors \tilde{c}_∞^\pm for the Wick rotated operator \tilde{K}_∞ .

We recall that from Prop. 7.1 we know that the vacuum state ω_{vac} is well defined on $\text{CCR}(\tilde{\mathcal{Y}}, \tilde{q})$.

Proposition 7.4. *The covariances of the vacuum state ω_{vac} , considered as a quasi-free state on $\text{CCR}(\tilde{\mathcal{Y}}, \tilde{q})$ are equal to:*

$$\tilde{\lambda}_{\text{vac}}^\pm = \pm\tilde{q} \circ \tilde{c}_\infty^\pm.$$

7.2.2. *Calderón projectors for \tilde{K}_β .* We follows now the construction and notation in 4.1.3, in particular $I_\beta^\pm = \pm]0, \frac{\beta}{2}[$. For χ_n as in 7.2.1, we set:

$$\begin{aligned} \tilde{\gamma}_{\beta,n}^{(0)*}\tilde{g} &= \chi'_n(s) \otimes \tilde{g}_1 + \chi_n(s) \otimes (\tilde{g}_0 - i\tilde{w}^*\tilde{g}_1), \\ \tilde{\gamma}_{\beta,n}^{(\frac{\beta}{2})*}\tilde{g} &= -\chi'_n(s - \frac{\beta}{2}) \otimes \tilde{g}_1 + \chi_n(s - \frac{\beta}{2}) \otimes (\tilde{g}_0 + i\tilde{w}^*\tilde{g}_1), \end{aligned}$$

for $\tilde{g} \in \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$, so that

$$\tilde{\gamma}_{\beta,n}^{(\sharp)*}\tilde{g} \rightarrow \tilde{\gamma}_\beta^{(\sharp)*}\tilde{g} \text{ in } \mathcal{E}'(\mathbb{R}; \tilde{\mathcal{H}}) \text{ when } n \rightarrow +\infty, \sharp = 0, \frac{\beta}{2}$$

for:

$$\begin{aligned} \tilde{\gamma}_\beta^{(0)*}\tilde{g} &= \delta'_0(s) \otimes \tilde{g}_1 + \delta_0(s) \otimes (\tilde{g}_0 - i\tilde{w}^*\tilde{g}_1), \\ \tilde{\gamma}_\beta^{(\frac{\beta}{2})*}\tilde{g} &= -\delta'_{\frac{\beta}{2}}(s) \otimes \tilde{g}_1 + \delta_{\frac{\beta}{2}}(s) \otimes (\tilde{g}_0 + i\tilde{w}^*\tilde{g}_1). \end{aligned}$$

We set also

$$\gamma_{\beta,(n)}^*(\tilde{g}^{(0)} \oplus \tilde{g}^{(\frac{\beta}{2})}) := \tilde{\gamma}_{\beta,(n)}^{(0)*}\tilde{g}^{(0)} + \tilde{\gamma}_{\beta,(n)}^{(\frac{\beta}{2})*}\tilde{g}^{(\frac{\beta}{2})}.$$

If $\tilde{u} \in \overline{C^1}(I_\beta^\pm; \tilde{\mathcal{H}}) \cap \overline{C^0}(I_\beta^\pm; \tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}})$ satisfies $\tilde{K}\tilde{u} = 0$ in I_β^\pm we set

$$\tilde{\gamma}_\beta^\pm \tilde{u} = \tilde{\gamma}_\beta^{(0)\pm} \tilde{u} \oplus \tilde{\gamma}_\beta^{(\frac{\beta}{2})\pm} \tilde{u},$$

for

$$\tilde{\gamma}_\beta^{(0)\pm} \tilde{u} = \begin{pmatrix} \tilde{u}(0^\pm) \\ -(\partial_s - i\tilde{w})\tilde{u}(0^\pm) \end{pmatrix}, \quad \tilde{\gamma}_\beta^{(\frac{\beta}{2})\pm} \tilde{u} = \begin{pmatrix} \tilde{u}(\mp\frac{\beta}{2}) \\ (\partial_s - i\tilde{w})\tilde{u}(\mp\frac{\beta}{2}) \end{pmatrix}.$$

Note the change of sign in the second component of $\tilde{\gamma}^{(\frac{\beta}{2})\pm}\tilde{u}$, which corresponds to choosing the *exterior* normal derivative to I_β^+ . We set also:

$$(7.9) \quad \tilde{S}^{(0)} = \begin{pmatrix} 2i\tilde{w}^* & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad \tilde{S}^{(\frac{\beta}{2})} = \begin{pmatrix} -2i\tilde{w}^* & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Proposition 7.5. *Assume that $\tilde{f} = \tilde{f}^{(0)} \oplus \tilde{f}^{(\frac{\beta}{2})}$ with $\tilde{f}^{(0/\frac{\beta}{2})} \in \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$. Then:*

(1) *the limit*

$$\lim_{n \rightarrow +\infty} \tilde{K}_\beta^{-1} \tilde{\gamma}_{\beta,n}^* (\tilde{S}^{(0)} \tilde{f}^{(0)} \oplus \tilde{S}^{(\frac{\beta}{2})} \tilde{f}^{(\frac{\beta}{2})}) =: \tilde{K}_\beta^{-1} \tilde{\gamma}_\beta^* (\tilde{S}^{(0)} \tilde{f}^{(0)} \oplus \tilde{S}^{(\frac{\beta}{2})} \tilde{f}^{(\frac{\beta}{2})})$$

exists in $\mathcal{D}'(I_\beta^\pm; \tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}})$ and is independent on the choice of the cutoff function

χ .

(2) *We have*

$$\tilde{K}_\beta^{-1} \tilde{\gamma}_\beta^* (\tilde{S}^{(0)} \tilde{f}^{(0)} \oplus \tilde{S}^{(\frac{\beta}{2})} \tilde{f}^{(\frac{\beta}{2})}) \in \overline{C^1}(I_\beta^\pm; \tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}}).$$

(3)

$$\mp \tilde{\gamma}_\beta^\pm \tilde{K}_\beta^{-1} \tilde{\gamma}_\beta^* (\tilde{S}^{(0)} \tilde{f}^{(0)} \oplus \tilde{S}^{(\frac{\beta}{2})} \tilde{f}^{(\frac{\beta}{2})}) = (\mathbb{1} \oplus T) \circ C_\beta^\pm \circ (\mathbb{1} \oplus T)^{-1} \tilde{f},$$

where C_β^\pm are the Calderón projectors for $B_\beta = \partial_s + b$, with $b = \tilde{H}$, $\mathfrak{h} = \tilde{\mathcal{E}}$, defined in Prop. 4.2.

Again this leads to the following definition.

Definition 7.6. *The operators*

$$\tilde{c}_\beta^\pm := \mp \tilde{\gamma}_\beta^\pm \tilde{K}_\beta^{-1} \tilde{\gamma}_\beta^* (\tilde{S}^{(0)} \pi^{(0)} \oplus \tilde{S}^{(\frac{\beta}{2})} \pi^{(\frac{\beta}{2})}) \in L((\tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}})^2; \tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}}),$$

where $\pi^{(0/\frac{\beta}{2})} \tilde{g} = \tilde{g}^{(0/\frac{\beta}{2})}$ are called Calderón projectors for \tilde{K}_β .

Proof of Prop. 7.5. We prove only the + case. Let $\tilde{f} \in \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$. Then:

$$\tilde{\gamma}_{\beta,n}^{(0)*} \tilde{S}^{(0)} \tilde{f} = \chi'_n(s) \otimes \tilde{f}_0 + \chi_n(s) \otimes (i\tilde{w}^* \tilde{f}_0 - \tilde{f}_1),$$

$$\tilde{\gamma}_{\beta,n}^{(\frac{\beta}{2})*} \tilde{S}^{(\frac{\beta}{2})} \tilde{f} = -\chi'_n(s - \frac{\beta}{2}) \otimes \tilde{f}_0 - \chi_n(s - \frac{\beta}{2}) \otimes (\tilde{f}_1 + i\tilde{w}^* \tilde{f}_0),$$

and as in the proof of Prop. 7.2 we see that $\tilde{\gamma}_{\beta,n}^{(0)*} \tilde{S}^{(0)} \tilde{f}$ and $\tilde{\gamma}_{\beta,n}^{(\frac{\beta}{2})*} \tilde{S}^{(\frac{\beta}{2})} \tilde{f}$ belong to $(-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}} \tilde{\mathcal{H}}_\beta$. Therefore we can apply Prop. 6.3 and obtain

$$(7.10) \quad \tilde{K}_\beta^{-1} \tilde{\gamma}_{\beta,n}^{(0/\frac{\beta}{2})*} \tilde{S}^{(0/\frac{\beta}{2})} \tilde{f} = \pi_0 B_\beta^{-1} \pi_1^* \tilde{\gamma}_{\beta,n}^{(0/\frac{\beta}{2})*} \tilde{S}^{(0/\frac{\beta}{2})} \tilde{f},$$

where $B_\beta = (\partial_s + \tilde{H})$, acting on $\tilde{\mathcal{E}}_\beta$. If $u \in \tilde{\mathcal{D}}$ then $\pi_1^* u \in \text{Dom} \tilde{H} \cap \text{Dom} \tilde{H}^{-1}$ by Prop. 7.1. Therefore as in the proof of Prop. 4.1 we obtain for $\tilde{g} = \pi_1^* u$ and $u \in \tilde{\mathcal{D}}$:

$$\lim_{n \rightarrow \infty} B_\beta^{-1} (\chi_n \otimes \tilde{g})(s) = e^{-s\tilde{H}} (\mathbb{1} - e^{-\beta\tilde{H}})^{-1} \tilde{g},$$

$$\lim_{n \rightarrow \infty} B_\beta^{-1} (\chi'_n \otimes \tilde{g})(s) = -\tilde{H} e^{-s\tilde{H}} (\mathbb{1} - e^{-\beta\tilde{H}})^{-1} \tilde{g},$$

$$\lim_{n \rightarrow \infty} B_\beta^{-1} (\chi_n(\cdot - \frac{\beta}{2}) \otimes \tilde{g})(s) = -e^{-(s-\frac{\beta}{2})\tilde{H}} (\mathbb{1} - e^{\beta\tilde{H}})^{-1} \tilde{g},$$

$$\lim_{n \rightarrow \infty} B_\beta^{-1} (\chi'_n(\cdot - \frac{\beta}{2}) \otimes \tilde{g})(s) = \tilde{H} e^{-(s-\frac{\beta}{2})\tilde{H}} (\mathbb{1} - e^{\beta\tilde{H}})^{-1} \tilde{g}, \text{ in } \mathcal{D}'(I_\beta^+; \tilde{\mathcal{E}}).$$

An easy computation shows that for $\tilde{f} \in \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$ we have:

$$\lim_{n \rightarrow +\infty} -(B_\beta^{-1} \pi_1^* \tilde{\gamma}_{\beta,n}^{(0)*} \tilde{S}^{(0)} \tilde{f})(s) = e^{-s\tilde{H}} (\mathbb{1} - e^{-\beta\tilde{H}})^{-1} \tilde{f} =: F^{(0)}(s),$$

$$\lim_{n \rightarrow +\infty} -(B_\beta^{-1} \pi_1^* \tilde{\gamma}_{\beta,n}^{(\frac{\beta}{2})*} \tilde{S}^{(\frac{\beta}{2})} \tilde{f})(s) = e^{-(s-\frac{\beta}{2})\tilde{H}} (\mathbb{1} - e^{\beta\tilde{H}})^{-1} T \tilde{f} =: F^{(\frac{\beta}{2})}(s), \text{ in } \mathcal{D}'(I_\beta^+; \tilde{\mathcal{E}}),$$

which using (7.10) proves (1).

Next since $\tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}} \subset \text{Dom} \tilde{H}^{-1}$, we have $(1 - e^{\pm\beta\tilde{H}})^{-1} \tilde{f} \in \tilde{\mathcal{E}}$, and since \tilde{H} preserves $\tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}}$, we have also $\tilde{H}(1 - e^{\pm\beta\tilde{H}})^{-1} \tilde{f} \in \tilde{\mathcal{E}}$. Therefore $F^{(0)}, F^{(\frac{\beta}{2})} \in \overline{C^1}(I_\beta^+; \tilde{\mathcal{E}})$ which shows that

$$\pi_0 F^{(0/\frac{\beta}{2})}(s) \in \overline{C^1}(I_\beta^+; \tilde{h}_0^{-\frac{1}{2}} \tilde{\mathcal{H}})$$

and proves (2). Finally we check that for $F(s) \in \overline{C^1}(I_\beta^+; \tilde{\mathcal{E}})$ with $(\partial_s + \tilde{H})F(s) = 0$ in I_β^+ we have:

$$\tilde{\gamma}_\beta^{(0)+} \pi_0 F = \Gamma_\beta^{(0)+} F, \quad \tilde{\gamma}_\beta^{(\frac{\beta}{2})+} \pi_0 F = T \Gamma_\beta^{(\frac{\beta}{2})+} F,$$

which proves (3). \square

As in Prop. 7.4 we can using Subsect. 5.4 express the covariances of the double β -KMS state $\tilde{\omega}_d$ for \tilde{P} in terms of the Calderón projectors \tilde{c}_β^\pm for the Wick rotated operator \tilde{K}_β .

We recall that from Prop. 7.1 we know that the double β -KMS state $\tilde{\omega}_d$ is well defined on $\text{CCR}(\tilde{\mathcal{Y}} \oplus \tilde{\mathcal{Y}}, \tilde{q} \oplus -\tilde{q})$.

Proposition 7.7. *The covariances of the double β -KMS state for \tilde{P} , considered as a state on $\text{CCR}(\tilde{\mathcal{Y}} \oplus \tilde{\mathcal{Y}}, \tilde{q} \oplus -\tilde{q})$ are equal to*

$$\tilde{\lambda}_d^\pm = \pm \tilde{Q} \circ (\mathbf{1} \oplus T)^{-1} \tilde{c}_\beta^\pm (\mathbf{1} \oplus T), \quad \text{for } \tilde{Q} = \tilde{q} \oplus -\tilde{q}.$$

8. KLEIN-GORDON EQUATIONS ON STATIONARY SPACETIMES

In this section we consider Klein-Gordon equations on stationary spacetimes. If the lapse function N associated to the Killing vector field \mathbf{w} is equal to 1, one can directly reduce oneself to the situation of Sect. 7. In general one has to replace the Klein-Gordon operator P by $\tilde{P} = NPN$, which has the same purpose as a conformal transformation.

As an application we consider the Klein-Gordon operator P in \mathcal{M}^+ and express the covariances of the double β -KMS state in $\mathcal{M}^- \cup \mathcal{M}^+$ using the Calderón projectors for the elliptic operator K_β obtained from P by Wick rotation in the Killing time coordinate t .

8.1. Klein-Gordon equations on stationary spacetimes.

8.1.1. *Stationary metrics.* Let (S, \mathbf{h}) be a Riemannian manifold, $N \in C^\infty(S)$, $N > 0$ and \mathbf{w}^i a vector field on S . Let us denote by y the elements of S . We define the Lorentzian metric \mathbf{g} on $M = \mathbb{R} \times S$:

$$\mathbf{g} = -N^2(y) dt^2 + \mathbf{h}_{ij}(y) (dy^i + \mathbf{w}^i(y) dt) (dy^j + \mathbf{w}^j(y) dt).$$

We assume that $\{0\} \times S$ is a Cauchy surface for (M, \mathbf{g}) . Such spacetimes are called *standard stationary spacetimes* in the terminology of [S2].

The vector field $\frac{\partial}{\partial t}$ is Killing for \mathbf{g} and is time-like iff

$$(8.1) \quad N^2(y) > \mathbf{w}^i(y) \mathbf{h}_{ij}(y) \mathbf{w}^j(y), \quad y \in S.$$

We will need later to impose the following stronger condition:

Definition 8.1. *The Killing vector field $\frac{\partial}{\partial t}$ is uniformly time-like if there exists $0 < \delta < 1$ such that:*

$$(1 - \delta) N^2(y) \geq \mathbf{w}^i(y) \mathbf{h}_{ij}(y) \mathbf{w}^j(y), \quad x \in S.$$

We have:

$$(8.2) \quad |\mathbf{g}| = N^2 |\mathbf{h}|, \quad n = N^{-1} \left(\frac{\partial}{\partial t} - w \right),$$

where n is the future directed unit normal to the foliation $S_t = \{t\} \times S$.

8.1.2. *Stationary Klein-Gordon operators.* We consider a stationary Klein-Gordon operator on (M, \mathbf{g}) :

$$(8.3) \quad P = -\square_{\mathbf{g}} + m(y), \quad m \in C^\infty(S; \mathbb{R}).$$

We will always assume that

$$(8.4) \quad m(y) \geq m_0^2, \quad m_0 > 0,$$

ie that the Klein-Gordon equation is *massive*. Setting

$$(8.5) \quad h_0 := \nabla^* \mathbf{h}^{-1} \nabla + m, \quad w := \mathbf{w}^i \cdot \partial_{y^i},$$

we have

$$(8.6) \quad P = (\partial_t + w^*) N^{-2} (\partial_t - w) + h_0,$$

where in (8.5), (8.6) the adjoints are computed with respect to the scalar product

$$(u|v)_M = \int_M \bar{u}v N |\mathbf{h}|^{\frac{1}{2}} dt dy.$$

8.1.3. *Hilbert spaces.* We denote by $L^2(M)$ the Hilbert space associated to the scalar product $(\cdot|\cdot)_M$ and by $\mathcal{H} = L^2(S, |\mathbf{h}|^{\frac{1}{2}} dy)$ the Hilbert space associated to the scalar product

$$(u|v)_{\mathcal{H}} = \int_S \bar{u}v |\mathbf{h}|^{\frac{1}{2}} dy.$$

We will also need the Hilbert space $\tilde{\mathcal{H}} = L^2(S, N |\mathbf{h}|^{\frac{1}{2}} dy)$ associated to the scalar product

$$(u|v)_{\tilde{\mathcal{H}}} = \int_S \bar{u}v N |\mathbf{h}|^{\frac{1}{2}} dy,$$

so that $L^2(M) = L^2(\mathbb{R}, dt; \tilde{\mathcal{H}})$.

8.1.4. *An operator inequality.* The inequality in Lemma 8.2 below is understood as an operator inequality on $\tilde{\mathcal{H}}$.

Lemma 8.2. *Assume that $\frac{\partial}{\partial t}$ is uniformly time-like. Then*

$$(1 - \delta)h_0 \geq w^* N^{-2} w \text{ on } C_c^\infty(S).$$

Proof. Let \mathcal{X} be a real vector space, $\mathbf{k} \in L_s(\mathcal{X}, \mathcal{X}^\#)$ strictly positive and $c \in \mathcal{X}$. Then for $\gamma = \mathbf{k}c \in \mathcal{X}^\#$ and $\xi \in \mathbb{C}\mathcal{X}^\#$ we have

$$\begin{aligned} & (\bar{\xi} - \langle \bar{\xi} | c \rangle \gamma) \cdot \mathbf{k}^{-1} (\xi - \langle \xi | c \rangle \gamma) \\ &= \bar{\xi} \cdot \mathbf{k}^{-1} \xi - 2 \operatorname{Re} (\langle \bar{\xi} | c \rangle \gamma \cdot \mathbf{k}^{-1} \xi) + |\langle \xi | c \rangle|^2 \gamma \cdot \mathbf{k}^{-1} \gamma \\ &= \bar{\xi} \cdot \mathbf{k}^{-1} \xi - (2 - c \cdot \mathbf{k}c) |\langle \xi | c \rangle|^2, \end{aligned}$$

hence

$$\mathbf{k}^{-1} - |c\rangle\langle c| \geq (1 - c \cdot \mathbf{k}c) |c\rangle\langle c|.$$

Replacing \mathbf{k} by $(1 - \delta)^{-1} \mathbf{k}$ shows that if $(1 - \delta) \geq c \cdot \mathbf{k}c$ we have

$$(8.7) \quad (1 - \delta) \mathbf{k}^{-1} \geq |c\rangle\langle c|.$$

For $u \in C_c^\infty(S)$ we write

$$\begin{aligned} & (u|((1 - \delta)h_0 - w^* N^{-2} w)u)_{\tilde{\mathcal{H}}} \\ & \geq \int_S \partial_{y^i} \bar{u} ((1 - \delta) \mathbf{h}^{ij}(y) - \mathbf{w}^i(y) N^{-2} \mathbf{w}^j(y)) \partial_{y^j} u(y) N |\mathbf{h}|^{\frac{1}{2}} dy. \end{aligned}$$

Applying (8.7) under the integral sign for $\mathbf{k} = \mathbf{h}_{ij}(y)$, $c = N^{-1}(y) \mathbf{w}^i(y)$ we obtain the lemma. \square

8.2. Selfadjoint operators. In the rest of this section we will assume that $\frac{\partial}{\partial t}$ is uniformly time-like.

Let $q_0(u, u) = (u|h_0u)_{\tilde{\mathcal{H}}}$ with $\text{Dom}q_0 = C_c^\infty(S)$. The form q_0 is closeable and we denote still denote by h_0 the selfadjoint operator on $\tilde{\mathcal{H}}$ associated to q_0^{cl} , i.e. the Friedrichs extension of h_0 on $C_c^\infty(S)$. We have:

$$h_0 : h_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} h_0^{\frac{1}{2}}\tilde{\mathcal{H}}.$$

Note that $h_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \subset \tilde{\mathcal{H}}$ since $h_0 \geq m_0^2$. We set also

$$(8.8) \quad \tilde{q}_0(u, u) = q_0(Nu, Nu)_{\tilde{\mathcal{H}}}, \quad \text{Dom}\tilde{q}_0 = C_c^\infty(S),$$

and denote by \tilde{h}_0 the selfadjoint operator on $\tilde{\mathcal{H}}$ associated to \tilde{q}_0^{cl} , which formally equals Nh_0N . From (8.8) we obtain that

$$(8.9) \quad N : \tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} h_0^{-\frac{1}{2}}\tilde{\mathcal{H}}, \quad N : h_0^{\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}},$$

and we have:

$$(8.10) \quad \tilde{h}_0 = Nh_0N \text{ as an identity in } B(\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}, \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}).$$

We also set

$$(8.11) \quad \begin{aligned} \tilde{w} &= N^{-1}wN = N^{-1}\mathbf{w}^i \cdot \partial_{y^i}N, \\ \tilde{w}^* &= Nw^*N^{-1} = -|\mathbf{h}|^{-\frac{1}{2}}\partial_{y^i}\mathbf{w}^i|\mathbf{h}|^{\frac{1}{2}}, \end{aligned}$$

with domain $C_c^\infty(S)$.

Let us introduce the assumption

$$(8.12) \quad N^{-2}\mathbf{w}^i \cdot (\nabla_i^{\mathbf{h}}N), \quad N^{-1}\nabla_i^{\mathbf{h}}\mathbf{w}^i \text{ are bounded on } S.$$

Lemma 8.3. *Assume (8.12). Then $\tilde{h}_0, \tilde{w}, \tilde{w}^*$ satisfy the conditions (7.2) and (7.4) for $\tilde{\mathcal{D}} = C_c^\infty(S)$.*

Proof. We have seen in Lemma 8.2 that $w^*N^{-2}w \leq (1-\delta)h_0$ on $C_c^\infty(S)$, which implies $\tilde{w}^*\tilde{w} \leq (1-\delta)\tilde{h}_0$ on $C_c^\infty(S)$. Let (y^1, \dots, y^d) be local coordinates on S . We have

$$\begin{aligned} \tilde{w} &= \mathbf{w}^i \cdot \partial_{y^i} + N^{-1}\mathbf{w}^i \cdot (\partial_{y^i}N), \\ \tilde{w}^* &= -\mathbf{w}^i \cdot \partial_{y^i} - |\mathbf{h}|^{-\frac{1}{2}}(\partial_{y^i}\mathbf{w}^i|\mathbf{h}|^{\frac{1}{2}}) = -\mathbf{w}^i \cdot \partial_{y^i} - \nabla_i^{\mathbf{h}}\mathbf{w}^i. \end{aligned}$$

Condition (8.12) implies that $\tilde{w}^* = -\tilde{w} + r$, where $r \in C^\infty(S)$ and rN^{-1} is bounded on $\tilde{\mathcal{H}}$. The inequality $\tilde{w}\tilde{w}^* \leq C\tilde{h}_0$ follows from $\tilde{w}^*\tilde{w} \leq (1-\delta)\tilde{h}_0$ and $m_0^2N^2 \leq \tilde{h}_0$.

Applying the Kato-Heinz theorem to this last inequality we obtain that $\tilde{h}_0^{-1} \leq m_0^{-2}N^{-2}$, i.e. $\text{Dom}N^{-1} \subset \text{Dom}\tilde{h}_0^{-\frac{1}{2}}$. Since $N > 0$, $C_c^\infty(S) \subset \text{Dom}N^{-1}$, which proves (7.4). \square

8.3. Associated first order system. We set:

$$(8.13) \quad \varrho_t\phi = \begin{pmatrix} \phi(t) \\ i^{-1}N^{-1}(\partial_t - w)\phi(t) \end{pmatrix} = f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix},$$

and rewrite $P\phi = 0$ as:

$$(8.14) \quad N^{-1}\partial_t f = iHf, \quad H = \begin{pmatrix} -iN^{-1}w & \mathbb{1} \\ h_0 & iw^*N^{-1} \end{pmatrix}, \quad f \in C_c^\infty(S; \mathbb{C}^2).$$

The conserved energy is

$$(8.15) \quad \bar{f} \cdot Ef = \|f_1 - iN^{-1}wf_0\|_{\tilde{\mathcal{H}}}^2 + (f_0|hf_0)_{\tilde{\mathcal{H}}}, \quad h = h_0 - w^*N^{-2}w,$$

and the conserved charge is

$$\bar{f} \cdot qf = (f_1|f_0)_{\mathcal{H}} + (f_0|f_1)_{\mathcal{H}}.$$

The energy space \mathcal{E} associated to E equals $h_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$ as topological vector spaces.

8.4. Reduction. We now introduce the Klein-Gordon operator

$$\tilde{P} = NPN = (\partial_t + \tilde{w}^*)(\partial_t - \tilde{w}) + \tilde{h}_0,$$

which is of the form considered in Sects. 6, 7. The operators $\tilde{\varrho}_t, \tilde{H}$, the energy \tilde{E} and charge \tilde{q} are defined as in Subsect. 6.3:

$$\begin{aligned} \tilde{\varrho}_t \tilde{\phi} &= \begin{pmatrix} \tilde{\phi}(t) \\ i^{-1}(\partial_t - \tilde{w})\tilde{\phi}(t) \end{pmatrix}, \quad \tilde{H} = \begin{pmatrix} -i\tilde{w} & \mathbb{1} \\ \tilde{h}_0 & i\tilde{w}^* \end{pmatrix}, \\ \tilde{f} \cdot \tilde{E} \tilde{f} &= \|\tilde{f}_1 - i\tilde{w}\tilde{f}_0\|_{\tilde{\mathcal{H}}}^2 + (\tilde{f}_0 | \tilde{h}\tilde{f}_0)_{\tilde{\mathcal{H}}}, \\ \tilde{f} \cdot \tilde{q} \tilde{f} &= (\tilde{f}_1 | \tilde{f}_0)_{\tilde{\mathcal{H}}} + (\tilde{f}_0 | \tilde{f}_1)_{\tilde{\mathcal{H}}}, \quad \tilde{f} \in C_c^\infty(S; \mathbb{C}^2). \end{aligned}$$

Setting

$$(8.16) \quad Z =: \begin{pmatrix} N & 0 \\ 0 & \mathbb{1} \end{pmatrix}, \quad Z' := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & N^{-1} \end{pmatrix},$$

we have:

$$(8.17) \quad \varrho_t N = Z \tilde{\varrho}_t, \quad N^{-1} \partial_t - iH = Z'(\partial_t - i\tilde{H})Z^{-1},$$

$$(8.18) \quad Z^* E Z = \tilde{E}, \quad Z^* q Z = \tilde{q} \text{ on } C_c^\infty(S; \mathbb{C}^2).$$

We saw that the energy space $\tilde{\mathcal{E}}$ associated to \tilde{E} equals $\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$, and from (8.9) we obtain that:

$$(8.19) \quad Z : \tilde{\mathcal{E}} \xrightarrow{\sim} \mathcal{E}.$$

8.5. Vacuum and KMS states. In Subsect. 7.1 we defined vacuum and β -KMS states for \tilde{P} . We obtain the corresponding vacuum and β -KMS states for P by conjugation by the map Z .

In Def. 8.4 below, ω_{vac} and ω_β are states on $\text{CCR}(C_c^\infty(S; \mathbb{C}^2), \tilde{q})$, while ω_d is a state on $\text{CCR}(C_c^\infty(S; \mathbb{C}^2) \oplus C_c^\infty(S; \mathbb{C}^2)^2, \tilde{q} \oplus -\tilde{q})$.

Definition 8.4. We define the vacuum state ω_{vac} , the β -KMS state ω_β and the double β -KMS state ω_d , by their covariances:

$$\begin{aligned} \lambda_{\text{vac}}^\pm &= (Z^{-1})^* \tilde{\lambda}_{\text{vac}}^\pm Z^{-1}, \quad \lambda_\beta^\pm = (Z^{-1})^* \tilde{\lambda}_\beta^\pm Z^{-1}, \\ \lambda_d^\pm &= (Z^{-1} \oplus Z^{-1})^* \tilde{\lambda}_d^\pm (Z^{-1} \oplus Z^{-1}), \end{aligned}$$

where the covariances $\tilde{\lambda}_{\text{vac}}^\pm, \tilde{\lambda}_\beta^\pm$ and $\tilde{\lambda}_d^\pm$ are defined in Defs. 5.1, 5.3 and Prop. 5.6 for $b = \tilde{H}$.

8.6. The Wick rotated operator.

8.6.1. The Wick rotated metric. Let us denote by \mathbf{g}^{eucl} the complex metric on $\mathbb{R} \times S$ obtained from \mathbf{g} by the substitution $t = is$. We have:

$$\mathbf{g}^{\text{eucl}} = N^2(y)ds^2 + \mathbf{h}_{ij}(y)(dy^i + i\mathbf{w}^i(y)ds)(dy^j + i\mathbf{w}^j(y)ds),$$

Using that $\frac{\partial}{\partial t}$ is uniformly time-like we obtain that there exists $C > 0$ such that

$$(8.20) \quad |\text{Im}(\bar{\eta} \cdot \mathbf{g}^{\text{eucl}}(y)\eta)| \leq C \text{Re}(\bar{\eta} \cdot \mathbf{g}^{\text{eucl}}(y)\eta), \quad y \in S, \eta \in \mathbb{C}T_{(s,y)}(\mathbb{R} \times S),$$

Moreover we have

$$(8.21) \quad |\mathbf{g}^{\text{eucl}}|(y) \text{ is real valued, positive, and } |\mathbf{g}^{\text{eucl}}|^{\frac{1}{2}}(y) = N(y)|\mathbf{h}|^{\frac{1}{2}}(y).$$

With the terminology in Def. 9.1 this means that the complex metric \mathbf{g}^{eucl} is *uniformly sectorial*.

If $\Omega =]0, +\infty[\times S$, the outer unit normal vector field to Ω for \mathbf{g}^{eucl} , see 9.1.2, is

$$(8.22) \quad \nu = -N^{-1} \left(\frac{\partial}{\partial s} - i\mathbf{w} \right),$$

while if $\Omega =]0, \frac{\beta}{2}[\times S$ it equals

$$(8.23) \quad \nu^{(0/\frac{\beta}{2})} = \mp N^{-1} \left(\frac{\partial}{\partial s} - i\mathbf{w} \right) \text{ on } \{0/\frac{\beta}{2}\} \times S.$$

The real vectors $\text{Im}\nu$, $\text{Im}\nu^{(0/\frac{\beta}{2})}$ are tangent to S , i.e. condition (9.4) below is satisfied.

8.6.2. *The Wick rotated operator.* We consider now the Wick rotated operator K obtained from P by the substitution $t = is$. We have:

$$(8.24) \quad K = -\Delta_{\mathbf{g}^{\text{eucl}}} + m(y) = -(\partial_s + iw^*)N^{-2}(\partial_s - iw) + h_0,$$

acting on the Hilbert spaces $\tilde{\mathcal{H}}_\beta$ for $0 < \beta \leq \infty$ defined in (6.10). We refer the reader to 9.1.1 for the Laplacian $\Delta_{\mathbf{g}^{\text{eucl}}}$ associated to \mathbf{g}^{eucl} . We recall that

$$\begin{aligned} \tilde{\mathcal{H}}_\beta &= L^2(\mathbb{S}_\beta \times S, N(y)|\mathbf{h}|^{\frac{1}{2}}(y)dyds), \text{ for } 0 < \beta < \infty, \\ \tilde{\mathcal{H}}_\infty &= L^2(\mathbb{R} \times S, N(y)|\mathbf{h}|^{\frac{1}{2}}(y)dyds). \end{aligned}$$

It follows from Lemma 8.2 that if $h = h_0 - w^*N^{-2}w$ we have:

$$h \sim h_0, \quad w^*N^{-2}w \lesssim h, \text{ on } C_c^\infty(\mathbb{R} \times S),$$

where we use the scalar product of $\tilde{\mathcal{H}}_\beta$ in the operator inequalities. We have:

$$(8.25) \quad \begin{aligned} (u|Ku)_{\tilde{\mathcal{H}}_\beta} &= \|N^{-1}\partial_s u\|_{\tilde{\mathcal{H}}_\beta}^2 + (u|hu)_{\tilde{\mathcal{H}}_\beta} \\ &\quad - i(N^{-1}\partial_s u|N^{-1}wu)_{\tilde{\mathcal{H}}_\beta} - i(N^{-1}wu|N^{-1}\partial_s u)_{\tilde{\mathcal{H}}_\beta}, \end{aligned}$$

for $u \in C_c^\infty(\mathbb{R} \times S)$ if $\beta = \infty$ and $u \in C_c^\infty(\mathbb{S}_\beta \times S)$ if $\beta < \infty$.

The sesquilinear form associated to the realization K_β of K is

$$\begin{aligned} Q_\beta(u, u) &= \|N^{-1}\partial_s u\|_{\tilde{\mathcal{H}}_\beta}^2 + (u|hu)_{\tilde{\mathcal{H}}_\beta} \\ &\quad - i(N^{-1}\partial_s u|N^{-1}wu)_{\tilde{\mathcal{H}}_\beta} - i(N^{-1}wu|N^{-1}\partial_s u)_{\tilde{\mathcal{H}}_\beta}, \end{aligned}$$

with domain $\text{Dom}Q_\beta = \langle K_0 \rangle^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta$ and $K_0 = -N^{-2}\partial_s^2 + h_0$ with its natural domain on $\tilde{\mathcal{H}}_\beta$. From (8.9) we obtain that

$$N^{-1} : \text{Dom}Q_\beta \xrightarrow{\sim} \text{Dom}\tilde{Q}_\beta, \quad Q_\beta(Nu, Nu) = \tilde{Q}_\beta(u, u),$$

where \tilde{Q}_β is defined in Subsect. 6.4. It follows that Q_β is a closed sectorial form and we denote as before by K_β :

$$K_\beta : K_0^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta \xrightarrow{\sim} K_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta$$

the induced boundedly invertible operator. We have

$$(8.26) \quad \begin{aligned} i) \quad N : \tilde{K}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta &\xrightarrow{\sim} K_0^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta, \quad N : K_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta \xrightarrow{\sim} \tilde{K}_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta \\ ii) \quad \tilde{K}_\beta &= NK_\beta N, \text{ as elements of } B(\tilde{K}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta, \tilde{K}_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta), \end{aligned}$$

where \tilde{K}_β is the operator defined in Subsect. 6.4.

8.7. **Calderón projectors.** We now define the Calderón projectors for K_β and relate them to those for \tilde{K}_β defined in Subsect. 7.2. We use the notation I_β^\pm , i_β^\pm , $\tilde{\gamma}_\beta$, $\tilde{\gamma}_\beta^\pm$ introduced in Subsect. 7.2. We recall that $\chi_n \in C_c^\infty(\mathbb{R})$ is a sequence of cutoff functions converging to δ_0 .

8.7.1. *Calderón projectors for K_∞ .* We set

$$\gamma_\infty^* g := \delta'_0(s) \otimes g_1 + \delta_0(s) \otimes (N^{-1}g_0 - iw^*N^{-2}g_1) \quad g \in C_c^\infty(S; \mathbb{C}^2),$$

and

$$S := \begin{pmatrix} 2iNw^*N^{-2} & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

so that

$$(8.27) \quad N^{-1}\tilde{\gamma}_\infty^* \tilde{S} Z^{-1} = \gamma_\infty^* S.$$

If $N^{-1}u \in \overline{C^0}(I_\infty^\pm; \tilde{\mathcal{H}}) \cap \overline{C^0}(I_\infty^\pm; \tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}})$ and $Ku = 0$ in I_∞^\pm we set:

$$\gamma_\infty^\pm u = \begin{pmatrix} u(0^\pm) \\ -N^{-1}(\partial_s - iw)u(0^\pm) \end{pmatrix},$$

so that

$$(8.28) \quad \gamma_\infty^\pm N = Z\gamma_\infty^\pm.$$

From Prop. 7.2 and (8.19), (8.26) *ii*), (8.27) and (8.28), we see that the operators

$$(8.29) \quad \mp \gamma_\infty^\pm K_\infty^{-1} \gamma_\infty^* S = \mp Z \circ (\tilde{\gamma}_\infty^\pm \tilde{K}_\infty^{-1} \tilde{\gamma}_\infty^* \tilde{S}) \circ Z^{-1}$$

are well defined as linear operators from $C_c^\infty(S; \mathbb{C}^2)$ to \mathcal{E} . This leads to the following definition.

Definition 8.5. *The Calderón projectors c_∞^\pm for K_∞ are:*

$$c_\infty^\pm := \mp \gamma_\infty^\pm K_\infty^{-1} \gamma_\infty^* S \in L(C_c^\infty(S; \mathbb{C}^2); \mathcal{E}).$$

Proposition 8.6. *The covariances of the vacuum state ω_{vac} are equal to:*

$$\lambda_{\text{vac}}^\pm = \pm q \circ c_\infty^\pm.$$

Proof. This follows from the identities:

$$\begin{aligned} i) \quad & \tilde{\lambda}_{\text{vac}}^\pm = Z^* \lambda_{\text{vac}}^\pm Z, \quad \tilde{q} = Z^* q Z, \\ ii) \quad & \tilde{\lambda}_{\text{vac}}^\pm = \pm \tilde{q} \tilde{c}_\infty^\pm, \quad c_\infty^\pm = Z \tilde{c}_\infty^\pm Z^{-1}. \end{aligned}$$

The identities in *i*) are obvious, the first identity in *ii*) is shown in Prop. 7.4, the second follows from (8.29). \square

8.7.2. *Calderón projectors for K_β .* We set

$$\gamma_\beta^{(0)*} g = \delta'_0(s) \otimes N^{-2}g_1 + \delta_0(s) \otimes (N^{-1}g_0 - iw^*N^{-2}g_1),$$

$$\gamma_\beta^{(\frac{\beta}{2})*} g = -\delta'_{\frac{\beta}{2}}(s) \otimes N^{-2}g_1 + \delta_{\frac{\beta}{2}}(s) \otimes (N^{-1}g_0 + iw^*N^{-2}g_1), \quad g \in C_c^\infty(S; \mathbb{C}^2)$$

$$S^{(0)} := \begin{pmatrix} 2iNw^*N^{-2} & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}, \quad S^{(\frac{\beta}{2})} := \begin{pmatrix} -2iNw^*N^{-2} & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

$$\gamma_\beta^*(g^{(0)} \oplus g^{(\frac{\beta}{2})}) := \gamma_\beta^{(0)*} g^{(0)} + \gamma_\beta^{(\frac{\beta}{2})*} g^{(\frac{\beta}{2})}.$$

As before we have:

$$(8.30) \quad N^{-1}\tilde{\gamma}_\beta^{(0/\frac{\beta}{2})*} \tilde{S}^{(0/\frac{\beta}{2})} Z^{-1} = \gamma_\beta^{(0/\frac{\beta}{2})*} S^{(0/\frac{\beta}{2})}.$$

If $N^{-1}u \in \overline{C^0}(I_\beta^\pm; \tilde{h}^{-1}\tilde{\mathcal{H}})$ and $\partial_s N^{-1}u \in \overline{C^0}(I_\beta^\pm; \tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}})$ with $Ku = 0$ in I_β^\pm , we set

$$\gamma_\beta^\pm u = \gamma_\beta^{(0)\pm} u \oplus \gamma_\beta^{(\frac{\beta}{2})\pm} u,$$

for

$$\gamma_\beta^{(0)\pm} u = \begin{pmatrix} u(0^\pm) \\ -N^{-1}(\partial_s - iw)u(0^\pm) \end{pmatrix}, \quad \gamma_\beta^{(\frac{\beta}{2})\pm} u = \begin{pmatrix} u(\mp \frac{\beta}{2}) \\ N^{-1}(\partial_s - iw)u(\mp \frac{\beta}{2}) \end{pmatrix},$$

and we have:

$$(8.31) \quad \gamma_\beta^\pm N = (Z \oplus Z) \tilde{\gamma}_\beta^\pm.$$

Again from Prop. 7.5 and (8.19), (8.26) *ii*), (8.30) and (8.31), we see that the operators

$$(8.32) \quad \begin{aligned} & \pm \gamma_\beta^\pm K_\beta^{-1} \gamma_\beta^{(0)*} S^{(0)} \pi^{(0)} + \gamma_\beta^{(\frac{\beta}{2})*} S^{(\frac{\beta}{2})} \pi^{(\frac{\beta}{2})} \\ &= \pm (Z \oplus Z) \circ (\tilde{\gamma}_\beta^\pm \tilde{K}_\beta^{-1} \tilde{\gamma}_\beta^{(0)*} \tilde{S}^{(0)} \pi^{(0)} + \tilde{\gamma}_\beta^{(\frac{\beta}{2})*} \tilde{S}^{(\frac{\beta}{2})} \pi^{(\frac{\beta}{2})}) \circ (Z \oplus Z)^{-1} \end{aligned}$$

are well defined as linear operators from $C_c^\infty(S; \mathbb{C}^2) \oplus C_c^\infty(S; \mathbb{C}^2)$ to $\mathcal{E} \oplus \mathcal{E}$.

Again this leads to the following definition.

Definition 8.7. *The Calderón projectors c_β^\pm for K_β are:*

$$c_\beta^\pm := \mp \gamma_\beta^\pm K_\beta^{-1} (\gamma_\beta^{(0)*} S^{(0)} \pi^{(0)} + \gamma_\beta^{(\frac{\beta}{2})*} S^{(\frac{\beta}{2})} \pi^{(\frac{\beta}{2})}),$$

where $\pi^{(0/\frac{\beta}{2})} g = g^{(0/\frac{\beta}{2})}$.

Using now Prop. 7.7, the same argument as in 8.7.1 gives the following proposition.

Proposition 8.8. *The covariances of the double β -KMS state ω_d are equal to:*

$$\lambda_d^\pm = \pm Q \circ (\mathbb{1} \oplus T)^{-1} c_\beta^\pm (\mathbb{1} \oplus T), \quad Q = q \oplus q,$$

$$\text{where } T = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}.$$

8.8. The double β -KMS state in $\mathcal{M}^+ \cup \mathcal{M}^-$. We now apply the computations of the previous subsections to $S = \Sigma^+$. In fact if ϕ_t is the flow of the Killing vector field V , the map

$$\chi : \mathbb{R} \times \Sigma^+ \ni (t, y) \mapsto \phi_t(y) \in \mathcal{M}^+$$

is a diffeomorphism such that $\chi^* \mathbf{g}$ is as in Subsect. 8.1.

We first claim that $\frac{\partial}{\partial t}$ is uniformly time-like and (8.12) holds on Σ^+ .

Proposition 8.9. *Assume that hypothesis (H3) holds. Then $\frac{\partial}{\partial t}$ is uniformly time-like and (8.12) holds on Σ^+ .*

Proof. By (H3) $\frac{\partial}{\partial t}$ is uniformly time-like and (8.12) holds on $\Sigma^+ \setminus U$, where U is any small neighborhood of \mathcal{B} in Σ^+ , by hypotheses (H). To check the conditions on U , we use Prop. 2.4. Recalling that (u, ω) are Gaussian normal coordinates to \mathcal{B} in (Σ, \mathbf{h}) , we obtain

$$\mathbf{w} \cdot \mathbf{h}\mathbf{w} \in O(u^4), \quad \mathbf{w} \cdot (\nabla N) \in O(u^3), \quad (\nabla \cdot \mathbf{w}) \in O(u^2), \quad \mathbf{w} \cdot \nabla (|\mathbf{h}|^{\frac{1}{2}}) \in O(u^2),$$

from which our claim follows, since $N(y) = \kappa u + O(u^3)$. \square

8.8.1. The double β -KMS state in $\mathcal{M}^+ \cup \mathcal{M}^-$. Let us now define the double β -KMS state in $\mathcal{M}^+ \cup \mathcal{M}^-$.

The wedge reflection R is an isometric involution from $(\mathcal{M}^-, \mathbf{g})$ to $(\mathcal{M}^+, \mathbf{g})$. It induces on Σ the weak wedge reflection r , which equals the identity on \mathcal{B} and maps Σ^- bijectively on Σ^+ .

R reverses the time orientation, hence induces a unitary involution:

$$\mathcal{R} : \left(\frac{C_c^\infty(\mathcal{M}^-)}{PC_c^\infty(\mathcal{M}^-)}, iG \right) \in [u] \mapsto [u \circ R] \in \left(\frac{C_c^\infty(\mathcal{M}^+)}{PC_c^\infty(\mathcal{M}^+)}, -iG \right).$$

In a more familiar language, \mathcal{R} is anti-symplectic. Since

$$\varrho_{\Sigma^\pm} \circ G : \left(\frac{C_c^\infty(\mathcal{M}^\pm)}{PC_c^\infty(\mathcal{M}^\pm)}, i(\cdot|G \cdot) \right) \xrightarrow{\sim} (C_c^\infty(\Sigma^\pm; \mathbb{C}^2), q)$$

is unitary, \mathcal{R} induces the unitary involution

$$\mathcal{R}_\Sigma : (C_c^\infty(\Sigma^-; \mathbb{C}^2), q) \xrightarrow{\sim} (C_c^\infty(\Sigma^+; \mathbb{C}^2), -q).$$

The following expression for \mathcal{R}_Σ follows from the fact that R reverses the time orientation.

Lemma 8.10. *One has*

$$\mathcal{R}_\Sigma f = Tr^* f,$$

where $T = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ and $r^* f(y) = f(r(y))$.

We have defined in Subsect. 8.5 the double β -KMS state ω_d through its Cauchy surface covariances λ_d^\pm . The associated Hermitian space is

$$(C_c^\infty(\Sigma^+; \mathbb{C}^2), q) \oplus (C_c^\infty(\Sigma^+; \mathbb{C}^2), -q).$$

$(\mathcal{M}^+ \cup \mathcal{M}^-, \mathbf{g})$ is a (disconnected) globally hyperbolic spacetime with Cauchy surface $\Sigma^+ \cup \Sigma^-$ and we denote a Cauchy data on $\Sigma^+ \cup \Sigma^-$ as

$$f = f^+ \oplus f^-, \quad f^\pm \in C_c^\infty(\Sigma^\pm; \mathbb{C}^2).$$

Using Remark 5.7, we obtain from ω_d a pure, quasi-free state ω_D in $\mathcal{M}^+ \cup \mathcal{M}^-$ as follows:

Definition 8.11. *The double β -KMS state ω_D in $\mathcal{M}^+ \cup \mathcal{M}^-$ is defined by the Cauchy surface covariances:*

$$\bar{f} \cdot \lambda_D^\pm f := \overline{(\mathbb{1} \oplus \mathcal{R}_\Sigma) f} \cdot \lambda_d^\pm (\mathbb{1} \oplus \mathcal{R}_\Sigma) f, \quad f = f^+ \oplus f^- \in C_c^\infty(\Sigma^+ \cup \Sigma^-; \mathbb{C}^2).$$

From Prop. 8.8 and Lemma 8.10 we obtain the following expression for λ_D^\pm .

Proposition 8.12. *One has:*

$$\lambda_D^\pm = \pm Q \circ (\mathbb{1} \oplus r^*)^{-1} c_\beta^\pm (\mathbb{1} \oplus r^*),$$

where c_β^\pm are the Calderón projectors for K_β defined in Def. 8.7 and $Q = q \oplus q$.

9. THE HHI STATE

In this section we construct the HHI state ω_{HHI} in M and prove that it is a pure Hadamard state, extending the double β -KMS state ω_D in $\mathcal{M}^- \cup \mathcal{M}^+$ for $\beta = (2\pi)\kappa^{-1}$. We use the expression of ω_D by Calderón projectors for the Wick rotated operator K_β , see Subsect. 8.8.

Since K_β is a Laplace operator for the complex metric \mathbf{g}^{eucl} on $M^{\text{eucl}} = \mathbb{S}_\beta \times \Sigma^+$, one can if $\beta = (2\pi)\kappa^{-1}$ extend it to a Laplace operator K_{ext} on the smooth extension $(M_{\text{ext}}^{\text{eucl}}, \mathbf{g}_{\text{ext}}^{\text{eucl}})$.

The boundary of the open set Ω_{ext} extending $\Omega_\beta =]0, \frac{\beta}{2}[\times \Sigma^+$ is diffeomorphic to the full Cauchy surface Σ , and we can use the Calderón projectors for $K_{\text{ext}}, \Omega_{\text{ext}}$ to define a pair of covariances λ_{HHI}^\pm . The fact that they define a state is actually quite easy, using some standard continuity properties of the Calderón projectors and density results in Sobolev spaces. The proof of the Hadamard property of ω_{HHI} relies also on an easy argument using pseudodifferential calculus, taken from [G].

9.1. Laplacians for complex metrics. We recall that complex metrics on a manifold X are defined in 2.7.1.

Definition 9.1. *A complex metric \mathbf{k} on a manifold X is called uniformly sectorial if*

(1) *there exists $C > 0$ such that*

$$(9.1) \quad |\text{Im}(\bar{v}^a \mathbf{k}_{ab}(x) v^b)| \leq C \text{Re}(\bar{v}^a \mathbf{k}_{ab}(x) v^b), \quad \forall x \in X, \quad v \in \mathcal{C}T_x X;$$

(2) $|\mathbf{k}(x)| = \det(\mathbf{k}_{ab}(x)) > 0 \forall x \in X$.

Note that if \mathbf{k} is uniformly sectorial, then

$$(9.2) \quad |\operatorname{Im}(\bar{\xi}_a k^{ab}(x) \xi_b)| \leq C \operatorname{Re}(\bar{\xi}_a k^{ab}(x) \xi_b) \forall x \in X, \xi \in \mathbb{C}T_x^* X,$$

i.e. \mathbf{k}^{-1} is also uniformly sectorial. In fact if $\xi = \mathbf{k}v$ we have $\bar{\xi} \cdot \mathbf{k}^{-1} \xi = \overline{\mathbf{k}v} \cdot v = \overline{v} \cdot \overline{\mathbf{k}v}$ and (9.2) follows from (9.1).

9.1.1. *Laplacians for complex metrics.* If \mathbf{k} is a complex metric on X , one defines the Christoffel symbols:

$$\Gamma_{ab}^c := \frac{1}{2} \mathbf{k}^{cd} (\partial_a \mathbf{k}_{cd} + \partial_b \mathbf{k}_{ad} - \partial_d \mathbf{k}_{ab}),$$

the covariant derivative:

$$\nabla_a^{(\mathbf{k})} T^b = \partial_a T^b + \Gamma_{ac}^b T^c,$$

and the Laplacian associated to \mathbf{k} , acting on $C_c^\infty(X)$:

$$\Delta_{\mathbf{k}} := \nabla_a^{(\mathbf{k})} \mathbf{k}^{ab} \nabla_b^{(\mathbf{k})}$$

as for real metrics. For $m \in C^\infty(X, \mathbb{R})$, we set:

$$K := -\Delta_{\mathbf{k}} + m,$$

and equip $C_c^\infty(X)$ with the scalar product:

$$(u|v)_X := \int_X \bar{u}v |\mathbf{k}|^{\frac{1}{2}} dx.$$

We have

$$(9.3) \quad \nabla_a^{(\mathbf{k})} T^a = |\mathbf{k}|^{-\frac{1}{2}} \partial_a (|\mathbf{k}|^{\frac{1}{2}} T^a),$$

see Subsect. A.3 and hence

$$K = -|\mathbf{k}|^{-\frac{1}{2}} \partial_a \mathbf{k}^{ab} |\mathbf{k}|^{\frac{1}{2}} \partial_a + m.$$

The formal adjoint of K for the scalar product $(\cdot|\cdot)_X$ equals

$$K^* = -|\mathbf{k}|^{-\frac{1}{2}} \partial_a \bar{\mathbf{k}}^{ba} |\mathbf{k}|^{\frac{1}{2}} \partial_b + m = -\Delta_{\mathbf{k}^*} + m.$$

Proposition 9.2. *Assume that \mathbf{k} is uniformly sectorial and that $m_0 \leq m(x)$ for $m_0 > 0$. Let*

$$Q(u, u) = (u|Ku)_X = \int_X (\partial_a \bar{u} \mathbf{k}^{ab} \partial_b u + m \bar{u}u) |\mathbf{k}|^{\frac{1}{2}} dx,$$

$$Q^*(u, u) = (u|K^*u)_X = \int_X (\partial_a \bar{u} \bar{\mathbf{k}}^{ba} \partial_b u + m \bar{u}u) |\mathbf{k}|^{\frac{1}{2}} dx,$$

with $\operatorname{Dom} Q = \operatorname{Dom} Q^* = C_c^\infty(X)$. Then:

- (1) Q, Q^* are closeable with $\operatorname{Dom} Q^{\operatorname{cl}} = \operatorname{Dom} Q^{*\operatorname{cl}} = H_{\mathbf{k}}^1(X)$, equal to the completion of $C_c^\infty(X)$ for the norm

$$\|u\|_1^2 = \int_X (\overline{\partial_a u} \operatorname{Re} \mathbf{k}^{ab} \partial_b u + m \bar{u}u) |\mathbf{k}|^{\frac{1}{2}} dx.$$

- (2) $Q^{\operatorname{cl}}, Q^{*\operatorname{cl}}$ are sectorial and induce isomorphisms:

$$K^{\operatorname{cl}}, K^{*\operatorname{cl}} : H_{\mathbf{k}}^1(X) \xrightarrow{\sim} H_{\mathbf{k}}^1(X)^*,$$

with $K^{\operatorname{cl}} = K, K^{*\operatorname{cl}} = K^*$ on $C_c^\infty(X)$.

- (3) Let us denote also by $K^{\operatorname{cl}}, K^{*\operatorname{cl}}$ the associated linear operators on $L^2(X, |\mathbf{k}|^{\frac{1}{2}} dx)$ with domains

$$\operatorname{Dom} K^{\operatorname{cl}} = H_{\mathbf{k}}^1(X) \cap (K^{\operatorname{cl}})^{-1} L^2(X, |\mathbf{k}|^{\frac{1}{2}} dx),$$

$$\operatorname{Dom} K^{*\operatorname{cl}} = H_{\mathbf{k}}^1(X) \cap (K^{*\operatorname{cl}})^{-1} L^2(X, |\mathbf{k}|^{\frac{1}{2}} dx).$$

Then $0 \notin \sigma(K^{\operatorname{cl}}), 0 \notin \sigma(K^{*\operatorname{cl}})$ and $(K^{*\operatorname{cl}})^{-1} = ((K^{\operatorname{cl}})^{-1})^*$.

Proof. Using (9.2) under the integral sign defining Q or Q^* , we obtain $|\operatorname{Im}Q^*(u, u)| \leq C\operatorname{Re}Q(u, u)$ and that Q, Q^* are closeable. The domain of their closures equals $H_{\mathbf{k}}^1(X)$ by definition and their closures are sectorial. By the Lax-Milgram theorem we obtain isomorphisms $K^{\operatorname{cl}}, K^{*\operatorname{cl}}$, which completes the proof of (1) and (2). The first properties of the linear operators $K^{\operatorname{cl}}, K^{*\operatorname{cl}}$ acting on $L^2(X, |\mathbf{k}|^{\frac{1}{2}} dx)$ are immediate. From [K, Thm. VI.2.5] we have $K^{*\operatorname{cl}} = (K^{\operatorname{cl}})^*$ hence $(K^{*\operatorname{cl}})^{-1} = ((K^{\operatorname{cl}})^{-1})^*$. \square

9.1.2. *Outer unit normal.* Let $\Omega \subset X$ be an open set with a smooth boundary $\partial\Omega$. We set

$$\Omega^+ := \Omega, \quad \Omega^- := X \setminus \Omega^{\operatorname{cl}}.$$

We can define the outer unit normal vector field to $\partial\Omega$, denoted by $n \in \mathcal{C}TX$ by the following conditions:

- i) $n(x) \cdot \mathbf{k}(x)v = 0, \quad \forall v \in T_x\partial\Omega,$
- ii) $n(x) \cdot \mathbf{k}(x)n(x) = 1,$
- iii) $\operatorname{Re} n(x)$ is outwards pointing.

If Ω is locally equal to $\{f > 0\}$ for $f \in C^\infty(X, \mathbb{R})$ with $df \neq 0$ on $\{f = 0\}$, we have:

$$n^a = \frac{-\mathbf{k}^{ab}\nabla_b f}{(\nabla_a f \mathbf{k}^{ab} \nabla_b f)^{\frac{1}{2}}},$$

where in the denominator we take the usual determination of $z^{\frac{1}{2}}$.

We also assume the following condition:

$$(9.4) \quad \operatorname{Im} n(x) \in T_x\partial\Omega, \quad x \in \partial\Omega,$$

which is equivalent to $\nabla_a f \mathbf{k}^{ab} \nabla_b f \in \mathbb{R}$ on $\partial\Omega$, if $\Omega = \{f > 0\}$.

The volume form $d\operatorname{Vol}_{\mathbf{k}} = |\mathbf{k}|^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$ associated to \mathbf{k} is real, as is the associated density $d\mu_{\mathbf{k}} = |d\operatorname{Vol}_{\mathbf{k}}| = |\mathbf{k}|^{\frac{1}{2}} dx$. It is easy to see from (9.4) that the induced density $d\sigma_{\mathbf{h}} = |d\operatorname{Vol}_{\mathbf{h}}|$ associated to the induced metric \mathbf{h} on $\partial\Omega$ is also real valued.

9.1.3. *Gauss-Green formula.* One obtains by the same arguments as in the real case the Gauss-Green formula:

$$(9.5) \quad \int_{\Omega} \nabla_a^{(\mathbf{k})} w^a d\mu_{\mathbf{k}} = \int_{\partial\Omega} w^a n_a d\sigma_{\mathbf{h}},$$

for w a smooth vector field on X .

Lemma 9.3. *We have:*

$$(9.6) \quad \int_{\Omega} (\bar{v}Ku - \overline{K^*v}u) d\mu_{\mathbf{k}} = \int_{\partial\Omega} (n^a \nabla_a^{(\mathbf{k})} \bar{v}u - \bar{v}n^a \nabla_a^{(\mathbf{k})} u) d\sigma_{\mathbf{h}}, \quad u \in \overline{C_c^\infty}(\Omega), \quad v \in \overline{C^\infty}(\Omega).$$

Proof. For $X^a = \bar{v}\mathbf{k}^{ab}\nabla_b^{(\mathbf{k})}u$ we obtain $\nabla_a^{(\mathbf{k})}X^a = \nabla_a^{(\mathbf{k})}\bar{v}\mathbf{k}^{ab}\nabla_b^{(\mathbf{k})}u - \bar{v}Ku$, hence by (9.5):

$$(9.7) \quad \int_{\Omega} \bar{v}Kud\mu_{\mathbf{k}} = \int_{\Omega} (\nabla_a^{(\mathbf{k})}\bar{v}\mathbf{k}^{ab}\nabla_b^{(\mathbf{k})}u + m\bar{v}u)d\mu_{\mathbf{k}} - \int_{\partial\Omega} \bar{v}n^a \nabla_a^{(\mathbf{k})}u d\sigma_{\mathbf{h}}.$$

The same identity replacing \mathbf{k} by $\bar{\mathbf{k}}$ gives:

$$\int_{\Omega} \bar{u}K^*vd\mu_{\bar{\mathbf{k}}} = \int_{\Omega} (\nabla_a^{(\bar{\mathbf{k}})}\bar{u}\bar{\mathbf{k}}^{ab}\nabla_b^{(\bar{\mathbf{k}})}v + m\bar{u}v)d\mu_{\bar{\mathbf{k}}} - \int_{\partial\Omega} \bar{u}n^a \nabla_a^{(\bar{\mathbf{k}})}v d\sigma_{\bar{\mathbf{h}}},$$

which by taking complex conjugates gives:

$$(9.8) \quad \int_{\Omega} u\overline{K^*v}d\mu_{\mathbf{k}} = \int_{\Omega} (\nabla_a^{(\mathbf{k})}u\mathbf{k}^{ab}\nabla_b^{(\mathbf{k})}\bar{v} + m\bar{u}v)d\mu_{\mathbf{k}} - \int_{\partial\Omega} un^a \nabla_a^{(\mathbf{k})}\bar{v} d\sigma_{\mathbf{h}},$$

since $\overline{\nabla_a^{(\mathbf{k})}u} = \nabla_a^{(\overline{\mathbf{k}})}\overline{u} = \nabla_a^{(\mathbf{k})}\overline{u}$. Subtracting (9.8) from (9.7) gives (9.6). \square

9.1.4. *Trace operators.* For $u \in C^\infty(X)$ we set:

$$\gamma u := \begin{pmatrix} u|_{\partial\Omega} \\ \partial_n u|_{\partial\Omega} \end{pmatrix} \in C^\infty(\partial\Omega; \mathbb{C}^2).$$

The formal adjoint γ^* of γ is given by:

$$(9.9) \quad \gamma^* f = (d\mu_{\mathbf{k}})^{-1} f_0 d\partial\Omega + (d\mu_{\mathbf{k}})^{-1} (n^\mu \partial_\mu)^* f_1 d\partial\Omega,$$

where if $g \in C^\infty(\partial\Omega)$, $gd\partial\Omega$ is the distributional density defined as

$$\langle u | gd\partial\Omega \rangle = \int_{\partial\Omega} u g d\sigma_{\mathbf{h}}, \quad u \in C_c^\infty(X),$$

and

$$\langle u | (n^\mu \partial_\mu)^* g d\partial\Omega \rangle = \langle n^\mu \partial_\mu u | g d\partial\Omega \rangle.$$

Similarly for $u \in \overline{C^\infty}(\Omega^\pm)$ we set:

$$\gamma^\pm u := \begin{pmatrix} u|_{\partial\Omega} \\ \partial_n u|_{\partial\Omega} \end{pmatrix},$$

where the trace is taken from Ω^\pm .

In the rest of this subsection, we assume that \mathbf{k} is uniformly sectorial and that (9.4) holds.

9.1.5. *Calderón projectors.*

Definition 9.4. *The Calderón projectors c^\pm associated to (K, Ω) are defined as*

$$c^\pm = \mp \gamma^\pm \circ (K^{\text{cl}})^{-1} \circ \gamma^* \circ S,$$

where

$$S = \begin{pmatrix} 2ib^* & -\mathbf{1} \\ \mathbf{1} & 0 \end{pmatrix},$$

$b = \text{Im } n^a \nabla_a$ and b^* is the formal adjoint of b in $L^2(\partial\Omega, d\sigma_{\mathbf{h}})$.

Note that the operator S is well defined on $C_c^\infty(\partial\Omega; \mathbb{C}^2)$, since $\text{Im } n$ is tangent to $\partial\Omega$.

It is not a priori clear that c^\pm are well defined, since even for $f \in C_c^\infty(\partial\Omega; \mathbb{C}^2)$, $\gamma^* S f$ does not belong to $H_{\mathbf{k}}^1(X)^*$.

To show that c^\pm make sense, one can apply the following proposition. We denote by $H_c^s(\partial\Omega)$ resp. $H_{\text{loc}}^s(\partial\Omega)$ for $s \in \mathbb{R}$, the compactly supported, resp. local Sobolev spaces on $\partial\Omega$ and set:

$$(9.10) \quad \mathcal{H}_{c/\text{loc}}^s(\partial\Omega) = H_{c/\text{loc}}^{s-\frac{1}{2}}(\partial\Omega) \oplus H_{c/\text{loc}}^{s-1-\frac{1}{2}}(\partial\Omega), \quad s \in \mathbb{R}.$$

Proposition 9.5. (1) $c^\pm : \mathcal{H}_c^s(\partial\Omega) \rightarrow \mathcal{H}_{\text{loc}}^s(\partial\Omega)$ continuously for any $s \in \mathbb{R}$,
(2) c^\pm are 2×2 matrices with entries in $\Psi^\infty(\partial\Omega)$.

Proof. The differential operator K is elliptic, since its principal symbol equals $\xi \cdot \mathbf{k}^{-1}(x)\xi$. K admits hence a properly supported parametrix $Q \in \Psi_c^{-2}(X)$. We claim that $(K^{\text{cl}})^{-1} - Q$ is a smoothing operator, i.e. has a smooth distributional kernel. In fact by the usual argument of commuting derivatives with K , we first obtain that $(K^{\text{cl}})^{-1} : H_c^s(\partial\Omega) \rightarrow H_{\text{loc}}^{s+2}(\partial\Omega)$ is continuous for all $s \geq -1$. Next since $KQ = \mathbf{1} + R_{-\infty}$, where R is smoothing and properly supported, we have

$$K^{\text{cl}}((K^{\text{cl}})^{-1} - Q)u = Ru, \quad u \in C_c^\infty(\partial\Omega),$$

hence $(K^{\text{cl}})^{-1} = Q + (K^{\text{cl}})^{-1}R_{-\infty}$ on $C_c^\infty(\partial\Omega)$. This implies that $(K^{\text{cl}})^{-1}$ extends as a continuous map from $H_c^s(\partial\Omega)$ to $H_{\text{loc}}^{s+2}(\partial\Omega)$ for all $s \in \mathbb{R}$ and that $(K^{\text{cl}})^{-1} - Q$ is smoothing.

It suffices hence to prove the proposition with $(K^{\text{cl}})^{-1}$ replaced by Q in the definition of c^\pm . Note that Q is properly supported, and the support of its distributional kernel can be assumed to lie in an arbitrary small neighborhood of the diagonal in $X \times X$.

We claim that to prove the proposition, we can reduce ourselves to the case when X and $\partial\Omega$ are compact, which is the situation considered in [Gr, Sect. 11.1]. Let us now explain this reduction.

Let $(U_i)_{i \in \mathbb{N}}$ be an open cover of a neighborhood of $\partial\Omega$ in X and $\chi_i : U_i \rightarrow B_{n-1}(0, 1) \times]-1, 1[$, an associated chart diffeomorphism with $\chi_i(U_i \cap \partial\Omega) = B_{n-1}(0, 1) \times \{0\}$, where $n = \dim X$ and $B_d(0, 1)$ is the open unit ball in \mathbb{R}^d . If $V_i = U_i \cap \partial\Omega$ and $\psi_i = \chi_i|_{U_i \cap \partial\Omega}$, then $(V_i, \psi_i)_{i \in \mathbb{N}}$ is an atlas of $\partial\Omega$. Refining if necessary the cover $(U_i)_{i \in \mathbb{N}}$, we can assume that if $V_i \cap V_j \neq \emptyset$, then there exists an chart open set $U \subset X$ with $U_i, U_j \Subset U$ and a chart diffeomorphism $\chi : U \rightarrow B_{n-1}(0, 1) \times]-1, 1[$ with $\chi(U \cap \partial\Omega) = B_{n-1}(0, 1) \times \{0\}$.

Let $1 = \sum_{i \in \mathbb{N}} \varphi_i$ with $\varphi_i \in C_c^\infty(V_i)$ a subordinate partition of unity of $\partial\Omega$. From the topologies $H_c^s(\partial\Omega)$ and $H_{\text{loc}}^s(\partial\Omega)$, we see that the continuity of a map $A : H_c^s(\partial\Omega) \rightarrow H_{\text{loc}}^{s'}(\partial\Omega)$ is equivalent to the continuity of $\varphi_i A \varphi_j : H_c^s(\partial\Omega) \rightarrow H_{\text{loc}}^{s'}(\partial\Omega)$ for all $i, j \in \mathbb{N}$.

If $V_i \cap V_j = \emptyset$ then $\varphi_i c^\pm \varphi_j$ is smoothing, since the kernel of Q is smooth outside of the diagonal. If $V_i \cap V_j \neq \emptyset$, we use the open set U and diffeomorphism χ introduced above to reduce ourselves to the situation when $X = B_{n-1}(0, 1) \times]-1, 1[$, $\partial\Omega = B_{n-1}(0, 1) \times \{0\}$ and $V_i, V_j \Subset B_{n-1}(0, 1)$.

We can then embed $B_{n-1}(0, 1) \times]-1, 1[$ in a compact manifold \tilde{X} , $\partial\Omega$ in a smooth, compact submanifold $\tilde{\partial\Omega}$ of \tilde{X} , and extend Q as a properly supported pseudodifferential operator $\tilde{Q} \in \Psi_c^{-2}(\tilde{X})$ such that $\varphi_i(Q - \tilde{Q})\varphi_j$ is a smoothing operator. Therefore we can replace $X, \partial\Omega, Q$ by $\tilde{X}, \tilde{\partial\Omega}, \tilde{Q}$ as claimed.

A neighborhood V of $\partial\Omega$ in X is then diffeomorphic to $] - \delta, \delta[\times \partial\Omega$, and one can use coordinates (s, y) on $] - \delta, \delta[\times \partial\Omega$. In [Gr, Sect. 11.1] the trace operator is defined as

$$\tilde{\gamma}u = \begin{pmatrix} u(0, y) \\ \mathbf{i}^{-1} \partial_s u(0, y) \end{pmatrix}.$$

Clearly we have $\gamma = L \circ \tilde{\gamma}$, where $L = \begin{pmatrix} \mathbb{1} & 0 \\ r & \mathbb{1} \end{pmatrix}$ and r is a first order differential operator on $\partial\Omega$. This implies that $L : \mathcal{H}_{c/\text{loc}}^s(\partial\Omega) \xrightarrow{\sim} \mathcal{H}_{c/\text{loc}}^s(\partial\Omega)$. The Calderón projectors \tilde{C}^\pm in [Gr, Sect. 11.1] are equal to $L^{-1} \circ c^\pm \circ L$, and [Gr, Prop. 11.7] implies that $\tilde{C}^\pm : \mathcal{H}_c^s(\partial\Omega) \rightarrow \mathcal{H}_{\text{loc}}^s(\partial\Omega)$ for all $s \in \mathbb{R}$, which implies (1).

Property (2) is a standard fact, see [Gr, Sect. 11.1]. \square

9.2. Construction of the HHI state.

9.2.1. The Laplacian on $M_{\text{ext}}^{\text{eucl}}$. We now apply the above framework to $(X, \mathbf{k}) = (M_{\text{ext}}^{\text{eucl}}, \mathbf{g}_{\text{ext}}^{\text{eucl}})$, the smooth extension of $(M^{\text{eucl}}, \mathbf{g}^{\text{eucl}})$ constructed in Prop. 2.5, for $\beta = (2\pi)\kappa^{-1}$. We assume that hypothesis (H3) in Subsect. 2.4 holds. By Prop. 8.9 the Wick rotated metric \mathbf{g}^{eucl} satisfies the conditions in 8.6.1, i.e. is uniformly sectorial. By continuity the same is true of its extension $\mathbf{g}_{\text{ext}}^{\text{eucl}}$. We denote by

$$K_{\text{ext}} = \Delta_{\mathbf{g}_{\text{ext}}^{\text{eucl}}} + m_{\text{ext}},$$

the associated Laplacian. We choose the open set $\Omega_{\text{ext}} \subset M_{\text{ext}}^{\text{eucl}}$, whose boundary $\partial\Omega_{\text{ext}}$ is diffeomorphic to Σ , see Prop. 2.5. We saw in 8.6.1 that if ν is the unit outer normal to $]0, \frac{\beta}{2}[\times \Sigma^+$, then $\text{Im } \nu$ is tangent to $\partial(]0, \frac{\beta}{2}[\times \Sigma^+)$. Again by continuity, the same is true of the unit outer normal to Ω_{ext} , i.e. condition (9.4) is satisfied. Therefore we can apply the results of Subsect. 9.1 to K_{ext} and Ω_{ext} .

We need one more result, which states that K_{ext} is the unique extension of $K_{(2\pi)\kappa^{-1}}$ to $L^2(M_{\text{ext}}^{\text{eucl}})$.

Proposition 9.6. *Let $U : C_c^\infty(M^{\text{eucl}}) \rightarrow C_c^\infty(M_{\text{ext}}^{\text{eucl}} \setminus \mathcal{B}_{\text{ext}})$ defined by:*

$$Uu = u \circ \chi^{-1},$$

where $\chi : M^{\text{eucl}} \xrightarrow{\sim} M_{\text{ext}}^{\text{eucl}} \setminus \mathcal{B}_{\text{ext}}$ is the diffeomorphism constructed in Prop. 2.5. Then U extends as a unitary operator

$$U : L^2(M^{\text{eucl}}, N(y)|\mathbf{h}|^{\frac{1}{2}}(y)dyds) \rightarrow L^2(M_{\text{ext}}^{\text{eucl}}, |g_{\text{ext}}^{\text{eucl}}|^{\frac{1}{2}}dx),$$

with $K_{\text{ext}} = UK_{(2\pi)\kappa^{-1}}U^*$.

Proof. U clearly extends as a unitary operator. Let us check the second statement.

As a differential operator, $K_{(2\pi)\kappa^{-1}}$ equals $-\Delta_{\mathbf{g}^{\text{eucl}}} + m$. As an unbounded operator, $K_{(2\pi)\kappa^{-1}}$ is defined in 8.6.2 using the sesquilinear form $Q_{(2\pi)\kappa^{-1}}$, while K_{ext} is defined with the sesquilinear form Q_{ext} for $\mathbf{k} = \mathbf{g}_{\text{ext}}^{\text{eucl}}$ and $m = m_{\text{ext}}$, see Prop. 9.2.

$Q_{(2\pi)\kappa^{-1}}$ is the closure of its restriction to $C_c^\infty(M^{\text{eucl}})$, while Q_{ext} is the closure of its restriction to $C_c^\infty(M_{\text{ext}}^{\text{eucl}})$.

Taking into account the isometry $\chi : M^{\text{eucl}} \xrightarrow{\sim} M_{\text{ext}}^{\text{eucl}} \setminus \mathcal{B}_{\text{ext}}$, it suffices to check that $C_c^\infty(M_{\text{ext}}^{\text{eucl}} \setminus \mathcal{B}_{\text{ext}})$ is a form core for Q_{ext} , i.e. that this space is dense in the space $H_{\mathbf{k}}^1(X)$ for $(X, \mathbf{k}) = (M_{\text{ext}}^{\text{eucl}}, \mathbf{g}_{\text{ext}}^{\text{eucl}})$, see Prop. 9.2.

Using the coordinates (X, Y, ω) near $\mathcal{B}_{\text{ext}} \sim \{0\} \times \mathcal{B}$, this follows from the fact that $C_c^\infty(\mathbb{R}^2 \setminus \{0\})$ is dense in $H^1(\mathbb{R}^2)$, see eg [A, Thm. 3.23]. \square

9.2.2. *The HHI state.* Let us denote by c_{ext}^\pm the Calderón projectors for $(K_{\text{ext}}, \Omega_{\text{ext}})$, defined as in Def. 9.4. The following theorem is a slightly more precise version of Thm. 1.1.

Theorem 9.7. *Assume conditions (H1), (H2), (H3). Then:*

- (1) $\lambda_{\text{HHI}}^\pm = \pm q \circ c_{\text{ext}}^\pm$ are the Cauchy surface covariances of a quasi-free state ω_{HHI} for P in M , called the HHI state.
- (2) The restriction of ω_{HHI} to $\mathcal{M}^+ \cup \mathcal{M}^-$ is the double β -KMS state ω_{D} for $\beta = (2\pi)\kappa^{-1}$.
- (3) ω_{HHI} is a Hadamard state in M .
- (4) Let ω be a quasi-free state for P in M whose restriction to $\mathcal{M}^+ \cup \mathcal{M}^-$ equals ω_{D} and such that its space-time covariances map $C_c^\infty(M)$ into $C^\infty(M)$ continuously. Then $\omega = \omega_{\text{HHI}}$.
- (5) Assume moreover that (H4) holds. Then ω_{HHI} is a pure state.

Note that it follows from (4) above that ω_{HHI} is the unique Hadamard state in M whose restriction to $\mathcal{M}^+ \cup \mathcal{M}^-$ equals ω_{D} .

Remark 9.8. *If we know that ω_{D} is a pure state on $\text{CCR}(C_c^\infty(\Sigma \setminus \mathcal{B}; \mathbb{C}^2), q)$ then ω_{HHI} is also a pure state on $\text{CCR}(C_c^\infty(\Sigma; \mathbb{C}^2), q)$. In fact let \mathcal{Y}^{cl} be the completion of $C_c^\infty(\Sigma \setminus \mathcal{B}; \mathbb{C}^2)$ for the scalar product $\lambda_{\text{D}}^+ + \lambda_{\text{D}}^-$. In the proof of Thm. 9.7 we show that \mathcal{Y}^{cl} is also the completion of $C_c^\infty(\Sigma; \mathbb{C}^2)$ for $\lambda_{\text{D}}^+ + \lambda_{\text{D}}^- = \lambda_{\text{HHI}}^+ + \lambda_{\text{HHI}}^-$. Applying then Prop. 3.4 we obtain that ω_{HHI} is also a pure state on $\text{CCR}(C_c^\infty(\Sigma; \mathbb{C}^2), q)$.*

However it seems difficult to prove directly that ω_{D} is pure on $\text{CCR}(C_c^\infty(\Sigma \setminus \mathcal{B}; \mathbb{C}^2), q)$, because of the infrared problem caused by the fact that the lapse function N vanishes at \mathcal{B} . Therefore we will prove directly that ω_{HHI} is a pure state on $\text{CCR}(C_c^\infty(\Sigma; \mathbb{C}^2), q)$, using arguments from [GW2].

Proof of Thm. 9.7. We start by checking some useful claims.

Claim 1. We claim that

$$(9.11) \quad \lambda_{\text{HHI}}^\pm = \lambda_{\text{D}}^\pm \text{ on } C_c^\infty(\Sigma \setminus \mathcal{B}; \mathbb{C}^2).$$

We note that the map $(\mathbb{1} \oplus r^*)$ in Prop. 8.12 corresponds to the embedding of $C_c^\infty(\Sigma^+ \cup \Sigma^-; \mathbb{C}^2)$ into $C_c^\infty(\Sigma \setminus \mathcal{B})$ obtained from $\psi : \Sigma \rightarrow M_{\text{ext}}^{\text{eucl}}$ in Prop. 2.5. The exterior normal to Ω_{ext} is a smooth extension of the image under χ of the exterior normal to $]0, \pi \kappa^{-1}[\times \Sigma^+$ defined in (8.23). Therefore using also Prop. 9.6 we obtain that

$$(9.12) \quad (\mathbb{1} \oplus r^*)^{-1} c_{(2\pi/\kappa)}^\pm (\mathbb{1} \oplus r^*) = c_{\text{ext}}^\pm,$$

on $C_c^\infty(\Sigma^+ \cup \Sigma^-; \mathbb{C}^2)$. This implies (9.11).

Claim 2. We claim that

$$(9.13) \quad \lambda_{\text{HHI}}^\pm, q \text{ are continuous sesquilinear forms on } \mathcal{H}_c^1(\Sigma),$$

where the spaces $\mathcal{H}_{c/\text{loc}}^s(\Sigma)$ are defined in (9.10). Let us denote by \mathbf{h}_{ext} the metric induced by $\mathbf{g}_{\text{ext}}^{\text{eucl}}$ on Σ and use the scalar product of $L^2(\Sigma, |\mathbf{h}_{\text{ext}}|^{\frac{1}{2}} dy) \otimes \mathbb{C}^2$ to identify sesquilinear forms with operators, so that $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$. Let us set

$$\tilde{\mathcal{H}}_{c/\text{loc}}^s(\Sigma) = H_{c/\text{loc}}^{s+\frac{1}{2}}(\Sigma) \oplus H_{c/\text{loc}}^{s+1+\frac{1}{2}}(\Sigma), \quad s \in \mathbb{R}.$$

Using that $H_c^s(\Sigma)$ and $H_{\text{loc}}^{-s}(\Sigma)$ form a dual pair, we see that $\mathcal{H}_c^s(\Sigma)$ and $\tilde{\mathcal{H}}_{\text{loc}}^{-s}(\Sigma)$ form a dual pair. Moreover $q : \mathcal{H}_{\text{loc}}^s(\Sigma) \rightarrow \tilde{\mathcal{H}}_{\text{loc}}^{s-2}(\Sigma)$ continuously, and by Prop. 9.5 (1) the same is true of $\lambda_{\text{HHI}}^\pm = \pm q \circ c_{\text{ext}}^\pm$. For $s = 1$ we have $s - 2 = -s$, which proves (9.13).

Proof of (1) and (2). Since ω_{D} is a state we deduce from (9.11) that

$$(9.14) \quad \lambda_{\text{HHI}}^+ - \lambda_{\text{HHI}}^- = q, \quad \lambda_{\text{HHI}}^\pm \geq 0 \text{ on } C_c^\infty(\Sigma \setminus \mathcal{B}; \mathbb{C}^2).$$

It is a well-known fact that since $\mathcal{B} \subset \Sigma$ is of codimension 1, $C_c^\infty(\Sigma \setminus \mathcal{B})$ is dense in $H_c^{\pm\frac{1}{2}}(\Sigma)$. Applying then (9.13) we deduce from (9.14) that:

$$\lambda_{\text{HHI}}^+ - \lambda_{\text{HHI}}^- = q, \quad \lambda_{\text{HHI}}^\pm \geq 0 \text{ on } \mathcal{H}_c^1(\Sigma) = H_c^{\frac{1}{2}}(\Sigma) \oplus H_c^{-\frac{1}{2}}(\Sigma),$$

hence

$$\lambda_{\text{HHI}}^+ - \lambda_{\text{HHI}}^- = q, \quad \lambda_{\text{HHI}}^\pm \geq 0 \text{ on } C_c^\infty(\Sigma; \mathbb{C}^2),$$

which proves (1). Statement (2) follows from (9.11).

Proof of (3). By Thm. 3.8 there exists a reference Hadamard state ω_{ref} for P in M whose Cauchy surface covariances on Σ , denoted by λ_{ref}^\pm are 2×2 matrices with entries in $\Psi^\infty(\Sigma)$. By Prop. 9.5 the same is true for λ_{HHI}^\pm .

The restriction of ω_{HHI} to \mathcal{M}^+ is a Hadamard state for P , since it is a $(2\pi)\kappa^{-1}$ -KMS state for a time-like, complete Killing vector field. The restriction of ω_{HHI} to \mathcal{M}^- is also a Hadamard state for P .

In fact by Prop. 8.12, its Cauchy surface covariances on Σ^- are the images of those of ω_{D} on Σ^+ by the weak wedge reflection r . Since $r^*\mathbf{h} = \mathbf{h}$, $r^*N = -N$ and $r^*w = w$, see 2.2.1, the expression (8.6) of P in $\mathbb{R} \times \Sigma^-$ shows that the restriction of ω_{D} to \mathcal{M}^- is also a Hadamard state.

This implies that the restriction of ω_{HHI} to $\mathcal{M}^+ \cup \mathcal{M}^-$ is a Hadamard state. The same is true of the restriction of the reference Hadamard state ω_{ref} to $\mathcal{M}^+ \cup \mathcal{M}^-$. Denoting by $\Lambda_{\text{HHI}/\text{ref}}^\pm$ the spacetime covariances of $\omega_{\text{HHI}/\text{ref}}$, this implies that $\Lambda_{\text{HHI}}^\pm - \Lambda_{\text{ref}}^\pm \in C^\infty((\mathcal{M}^+ \cup \mathcal{M}^-)^2)$, since the spacetime covariances of two Hadamard states differ by smooth kernels. Passing to Cauchy surface covariances on $\Sigma^+ \cup \Sigma^-$, this implies using (3.10) that if $\chi \in C_c^\infty(\Sigma \setminus \mathcal{B})$, then

$$\chi \circ (\lambda_{\text{HHI}}^\pm - \lambda_{\text{ref}}^\pm) \circ \chi \text{ is a smoothing operator on } \Sigma.$$

We claim that this implies that $\lambda_{\text{HHI}}^\pm - \lambda_{\text{ref}}^\pm$ is smoothing, which will imply that ω_{HHI} is a Hadamard state.

In fact let a be one of the entries of $\lambda_{\text{HHI}}^\pm - \lambda_{\text{ref}}^\pm$, which is a scalar pseudodifferential operator belonging to $\Psi^m(\Sigma)$ for some $m \in \mathbb{R}$. We know that $\chi \circ a \circ \chi$ is smoothing for any $\chi \in C_c^\infty(\Sigma \setminus \mathcal{B})$. Then its principal symbol $\sigma_{\text{pr}}(a)$ vanishes on $T^*(\Sigma \setminus \mathcal{B})$ hence on $T^*\Sigma$ by continuity, so $a \in \Psi^{m-1}(\Sigma)$. Iterating this argument we obtain that a is smoothing, which completes the proof of (3).

Proof of (4). The proof of (4) is identical to [G, Prop. 7.4].

Proof of (5). The proof of (5) will be given in Sect. 10. \square

10. PURITY OF THE HARTLE-HAWKING-ISRAEL STATE

To prove that ω_{HHI} is a pure state, we will follow the method in [GW2], by using the criterion for purity recalled in Prop. 3.6. We start by some preparations.

10.1. Preparations. In all this section the manifold $M_{\text{eucl}}^{\text{ext}}$ will be denoted by X , the complex metric $\mathbf{g}_{\text{ext}}^{\text{eucl}}$ by \mathbf{k} , the elliptic operator K_{eucl} by K , the open set Ω_{ext} defined in Prop. 2.5 by Ω , and the Calderón projectors c_{ext}^\pm by c^\pm , in accordance with the notation used in Subsect. 9.1. We recall that

$$\|u\|_{H_{\mathbf{k}}^1(X)}^2 = \int_X \partial_a \bar{u} \text{Re } \mathbf{k}^{ab} \partial_b u + m \bar{u} u |\mathbf{k}|^{\frac{1}{2}} dx,$$

and will set

$$(10.1) \quad (u|v)_\Omega = \int_\Omega \bar{u} v |\mathbf{k}|^{\frac{1}{2}} dx, \quad \|u\|_{H_{\mathbf{k}}^1(\Omega)}^2 = \int_\Omega \partial_a \bar{u} \text{Re } \mathbf{k}^{ab} \partial_b u + m \bar{u} u |\mathbf{k}|^{\frac{1}{2}} dx.$$

Note that the closure of $\overline{C_c^\infty}(\Omega)$ for the norm $\|\cdot\|_{H_{\mathbf{k}}^1(\Omega)}$ coincide with the space $\overline{H_{\mathbf{k}}^1}(\Omega)$ of restrictions to Ω of elements of $H_{\mathbf{k}}^1(\Omega)$.

From the construction of the manifold $M_{\text{eucl}}^{\text{ext}}$ in Subsect. A.2, we see that there exist $U \Subset X$ and $V \subset \Sigma$ which are compact neighborhoods of $\{0\} \times \mathcal{B}$, such that $X \setminus U$ equals $\mathbb{S}_\beta \times (\Sigma^+ \setminus V)$. Recall also that the coordinates on $\mathbb{S}_\beta \times \Sigma^+$ are denoted by (s, y) .

Lemma 10.1. *Assume hypothesis (H4). Then there exists a family $\chi_n \in C_c^\infty(M_{\text{eucl}}^{\text{ext}})$ such that:*

- i) $\chi_n = 1$ on U ,
- ii) $\chi_n = \chi_n(y)$ on $X \setminus U$,
- iii) $\chi_n \rightarrow 1$ locally uniformly on X ,
- iv) $\|d\chi\|_\infty = \sup_X |d\chi_n| \in O(n^{-1})$,

where we set $|d\chi_n| = (d\chi_n \cdot \mathbf{h}^{-1} d\chi_n)^{\frac{1}{2}}$.

Proof. If Σ is compact it suffices to take $\chi_n = 1$. If Σ is not compact, then (Σ, \mathbf{h}) is complete by hypothesis (H4). It follows that if we fix a point $y^0 \in \Sigma$, (for example $y^0 \in \mathcal{B}$), the geodesic balls $B_{\mathbf{h}}(y^0, n)$ are compact and $\bigcup_{n \in \mathbb{N}} B_{\mathbf{h}}(y^0, n) = \Sigma$. By [AFLR, Thm. 1], there exists $r \in C^\infty(\Sigma)$ such that:

$$\frac{1}{2} d(y^0, y) \leq r(y) \leq 2d(y^0, y) \|\nabla r\|_\infty \leq 2.$$

We choose $F \in C_c^\infty(\mathbb{R})$ with $F(s) = 1$ in $s \leq 1$, $F(s) = 0$ in $s \geq 2$, and $\psi \in C_c^\infty(X)$ equal to 1 near the neighborhood U of $\{0\} \times \mathcal{B}$. We choose

$$\chi_n(x) = \psi(x) + (1 - \psi)(x) F(n^{-1} r(y)),$$

where we recall that outside U , $X = \mathbb{S}_\beta \times \Sigma^+$ with coordinates (s, y) . It is easy to check that χ_n has the required properties. \square

Lemma 10.2. *Let $\chi \in C_c^\infty(X)$ with $\chi = 1$ near $\{0\} \times \mathcal{B}$ and $\chi(x) = \chi(y)$ outside a neighborhood of $\{0\} \times \mathcal{B}$. Then*

$$|(v|[K, \chi]u)_\Omega| \leq C \|d\chi\|_\infty \|v\|_{H_{\mathbf{k}}^1(\Omega)} \|u\|_{H_{\mathbf{k}}^1(\Omega)},$$

where $\|d\chi\|_\infty = \sup_\Sigma (d\chi \cdot \mathbf{h}^{-1} d\chi)^{\frac{1}{2}}$.

Proof. Let U a neighborhood of $\{0\} \times \mathcal{B}$ such that $\chi = 1$ on U . We have $K = (-\partial_s + iw^*)N^{-2}(\partial_s - iw)$ on $\text{supp } \nabla \chi \subset X \setminus U$, see (8.24), and

$$\|u\|_{H_{\mathbf{k}}^1(X)}^2 \sim \int_{\mathbb{S}_\beta \times \Sigma^+} (|N^{-1}\partial_s u|^2 + \nabla \bar{u} \mathbf{h}^{-1} \nabla u + m \bar{u} u) N |\mathbf{h}|^{\frac{1}{2}} ds dy, \quad u \in C_c^\infty(X \setminus U).$$

Therefore

$$(10.3) \quad [K, \chi] = 2iN^{-1}w \cdot d\chi N^{-1}\partial_s + \nabla^* \cdot \mathbf{h}^{-1} d\chi - d\chi \cdot \mathbf{h}^{-1} \nabla.$$

We have

$$|w \cdot d\chi| \leq (w \cdot \mathbf{h}w)^{\frac{1}{2}} (d\chi \cdot \mathbf{h}^{-1} d\chi)^{\frac{1}{2}},$$

hence

$$N^{-1}|w \cdot d\chi| \leq (N^{-2}w \cdot \mathbf{h}w)^{\frac{1}{2}} (d\chi \cdot \mathbf{h}^{-1} d\chi)^{\frac{1}{2}} \leq (d\chi \cdot \mathbf{h}^{-1} d\chi)^{\frac{1}{2}} \leq \|d\chi\|_\infty,$$

since $w \cdot \mathbf{h}w \leq N^2$ by (2.5). Combining (10.2) and (10.3) we obtain the lemma. \square

Proposition 10.3. *Let $f \in C_c^\infty(\Sigma)^2$. Then:*

$$(10.4) \quad \begin{aligned} i) \quad & \chi_n c^+ f - c^+ \chi_n c^+ f \rightarrow 0, \\ ii) \quad & \chi_n c^+ \chi_n c^+ f - c^+ \chi_n c^+ \chi_n c^+ f \rightarrow 0, \end{aligned}$$

in $\mathcal{D}'(\Sigma)^2$ when $n \rightarrow \infty$.

Proof. Let us note that the identity (9.5) is of course still valid if $u \in \overline{C_c^\infty}(\Omega)$ and $v \in \overline{C^\infty}(\Omega)$. It is also valid if $u \in \overline{C_c^\infty}(\Omega)$ and $v \in \overline{H_{\text{loc}}^2}(\Omega)$, since all the terms in the identity are still well defined.

For $f \in C_c^\infty(\Sigma)^2$ we set

$$V^+ f := -r_\Omega (K^{\text{cl}})^{-1} \gamma^* S f,$$

where r_Ω is the operator of restriction to Ω and we recall that S is defined in Def. 9.4. Let $w \in \overline{C_c^\infty}(\Omega)$ and

$$v = (K^{*\text{cl}})^{-1} e_\Omega w = (K^{\text{cl}})^{-1*} e_\Omega w,$$

where e_Ω is the operator of extension by 0 in $X \setminus \Omega$.

We know that $v \in H_{\mathbf{k}}^1(X)$ by Prop. 9.2, and $v \in H_{\text{loc}}^2(X)$ using that $e_\Omega w \in L_{\text{loc}}^2(X)$ and elliptic regularity. For $u \in \overline{C_c^\infty}(\Omega^+)$ we obtain from (9.6):

$$(10.5) \quad (v|Ku)_\Omega - (K^*v|u)_\Omega = (\gamma^+ v | S \gamma^+ u)_\Sigma = (\gamma v | S \gamma^+ u)_\Sigma,$$

where in the last equality we use that $\gamma^\pm v = \gamma v$. In fact γv is well defined as an element of $H_{\text{loc}}^{3/2}(\Sigma) \oplus H_{\text{loc}}^{1/2}(\Sigma)$ since $v \in H_{\text{loc}}^2(X)$. Next, we obtain:

$$(10.6) \quad \begin{aligned} (\gamma v | S \gamma^+ u)_\Sigma &= (v | \gamma^* S \gamma^+ u)_X = ((K^{\text{cl}})^{-1*} e_\Omega w | \gamma^* S \gamma^+ u)_X \\ &= (e_\Omega w | (K^{\text{cl}})^{-1} \gamma^* S \gamma^+ u)_X = -(w | V^+ \gamma^+ u)_\Omega. \end{aligned}$$

From (10.5), (10.6) we obtain:

$$(10.7) \quad (w|u - V^+ \gamma^+ u)_\Omega = (v|Ku)_\Omega, \quad u, w \in \overline{C_c^\infty}(\Omega).$$

We now fix $f \in C_c^\infty(\Sigma)^2$ and $u = V^+ f$. By the same argument as in [GW2, Lemma A.1] we know that $u \in \overline{H_{\mathbf{k}}^1}(\Omega) \cap \overline{C^\infty}(\Omega)$. We now apply (10.7) replacing u by $u_n = \chi_n u$, which belongs to $\overline{C_c^\infty}(\Omega)$.

Since $Ku = 0$ in Ω , we have $K\chi_n u = [K, \chi_n]u$ and by Lemma 10.2 we obtain:

$$(10.8) \quad |(v|[K, \chi_n]u)_\Omega| \leq Cn^{-1}\|v\|_{H_{\mathbf{k}}^1(\Omega)}\|u\|_{H_{\mathbf{k}}^1(\Omega)}.$$

Using (10.7) this yields:

$$(10.9) \quad |(w|\chi_n u - V^+\gamma^+\chi_n u)_\Omega| \leq Cn^{-1}\|v\|_{H_{\mathbf{k}}^1(\Omega)}\|u\|_{H_{\mathbf{k}}^1(\Omega)} \leq Cn^{-1}\|e_\Omega w\|_{H_{\mathbf{k}}^1(X)^*}\|u\|_{H_{\mathbf{k}}^1(\Omega)},$$

since by Prop. 9.2 $K^{*\text{cl}} : H_{\mathbf{k}}^1(X) \xrightarrow{\sim} H_{\mathbf{k}}^1(X)^*$.

We recall that the space $\overline{H_{\mathbf{k}}^1(\Omega)}$ is the space of restrictions to Ω of elements in $H_{\mathbf{k}}^1(X)$. Now for $g \in C_c^\infty(X)$ we have

$$|(e_\Omega w|g)_\Omega| = (w|r_\Omega g)_\Omega \leq C\|w\|_{\overline{H_{\mathbf{k}}^1(\Omega)^*}}\|r_\Omega g\|_{\overline{H_{\mathbf{k}}^1(\Omega)}} \leq C\|w\|_{\overline{H_{\mathbf{k}}^1(\Omega)^*}}\|g\|_{H_{\mathbf{k}}^1(X)},$$

which implies that

$$(10.10) \quad \|e_\Omega w\|_{H_{\mathbf{k}}^1(X)^*} \leq C\|w\|_{\overline{H_{\mathbf{k}}^1(\Omega)^*}}.$$

Therefore we deduce from (10.9) by duality that if

$$r_{1,n} := \chi_n u - V^+\gamma^+\chi_n u,$$

we have

$$(10.11) \quad \|r_{1,n}\|_{\overline{H_{\mathbf{k}}^1(\Omega)}} \leq Cn^{-1}\|u\|_{\overline{H_{\mathbf{k}}^1(\Omega)}}.$$

Hence,

$$(10.12) \quad r_{1,n} \rightarrow 0 \text{ in } \overline{H_{\mathbf{k}}^1(\Omega)} \text{ as } n \rightarrow \infty.$$

We now apply (10.7) once again replacing u by $v_n = \chi_n V^+\gamma^+\chi_n u$. Note that $V^+\gamma^+\chi_n u \in \overline{C_c^\infty(\Omega)}$, so $v_n \in \overline{C_c^\infty(\Omega)}$. We obtain since $KV^+\gamma^+\chi_n u = 0$ in Ω :

$$(w|v_n - V^+\gamma^+v_n)_\Omega = (v|[K, \chi_n]V^+\gamma^+\chi_n u)_\Omega,$$

hence by (10.8) and (10.10)

$$|(w|v_n - V^+\gamma^+v_n)_\Omega| \leq Cn^{-1}\|w\|_{\overline{H_{\mathbf{k}}^1(\Omega)^*}}\|V^+\gamma^+\chi_n u\|_{H_{\mathbf{k}}^1(\Omega)} \leq Cn^{-1}\|\chi_n u\|_{\overline{H_{\mathbf{k}}^1(\Omega)}},$$

where we use (10.11) in the last inequality. Since

$$\|\chi_n u\|_{\overline{H_{\mathbf{k}}^1(\Omega)}} \leq C(\|\chi\|_\infty + \|d\chi\|_\infty)\|u\|_{\overline{H_{\mathbf{k}}^1(\Omega)}} \leq c\|u\|_{\overline{H_{\mathbf{k}}^1(\Omega)}},$$

we obtain finally that if

$$r_{2,n} := \chi_n V_0^+\gamma^+\chi_n u - V_0^+\gamma^+\chi_n V_0^+\gamma^+\chi_n u$$

we have:

$$(10.13) \quad r_{2,n} \rightarrow 0 \text{ in } \overline{H_{\mathbf{k}}^1(\Omega)} \text{ as } n \rightarrow \infty.$$

We note then that if $V \Subset X$ and $V^+ = V \cap \Omega$, we have $K\chi_n u = 0$ in V^+ for $n \gg 1$, hence $Kr_{1,n} = Kr_{2,n} = 0$ in V^+ for $n \gg 1$.

We can introduce local coordinates on Σ near $y^0 \in \Sigma$, and map V to a neighborhood \tilde{V} of $(0,0)$ in \mathbb{R}^{1+d} for $d = \dim \Sigma$. Denoting by $H^{m,\ell}(\mathbb{R}^{1+d})$ the space of $u \in \mathcal{S}'(\mathbb{R}^{1+d})$ such that $\langle D_x \rangle^m \langle D_y \rangle^\ell u \in L^2(\mathbb{R}^{1+d})$, and then using the coordinates we define the spaces $H_{\text{loc}}^{1+k,-k}(X)$ and $\overline{H_{\text{loc}}^{1+k,-k}(\Omega)}$ (the definition depends on the choice of coordinates, but this is not important here). Then one deduces, using that $Kr_{i,n} = 0$ in V^+ that $r_{i,n} \rightarrow 0$ also in $\overline{H_{\text{loc}}^{1+k,-k}(\Omega)}$ for any $k \in \mathbb{N}$, see eg [Gr, page 311].

We can then safely apply the boundary value operator γ^+ and deduce from (10.12), (10.13) that:

$$\gamma^+ r_{i,n} \rightarrow 0 \text{ in } \mathcal{D}'(\Sigma; \mathbb{C}^2).$$

Since $c^+f = \gamma^+V^+f = \gamma^+u$ this yields:

$$\begin{aligned}\chi_n c^+ f - c^+ \chi_n c^+ f &\rightarrow 0, \\ \chi_n c^+ \chi_n c^+ f - c^+ \chi_n c^+ \chi_n c^+ f &\rightarrow 0,\end{aligned}$$

in $\mathcal{D}'(\Sigma; \mathbb{C}^2)^2$ when $n \rightarrow \infty$, which completes the proof. \square

10.2. Purity of the Hartle-Hawking-Israel state. We will follow the arguments in [GW2, Subsect. 4.4].

By Prop. 3.6 it suffices to find, for each $f \in C_c^\infty(\Sigma; \mathbb{C}^2)$, a sequence $f_n \in C_c^\infty(\Sigma; \mathbb{C}^2)$ such that:

$$(10.14) \quad \lim_{n \rightarrow +\infty} \frac{|\bar{f} \cdot q f_n|^2}{\bar{f}_n \cdot (\lambda^+ + \lambda^-) f_n} = \bar{f} \cdot (\lambda^+ + \lambda^-) f.$$

We take

$$f_n = \chi_n (c^+ - c^-) f,$$

which belongs to $C_c^\infty(\Sigma; \mathbb{C}^2)$ since $c^\pm : C_c^\infty(\Sigma; \mathbb{C}^2) \rightarrow C_c^\infty(\Sigma; \mathbb{C}^2)$. Since $\chi_n f = f$ for $n \gg 1$ we have

$$(10.15) \quad \bar{f} \cdot q f_n = \bar{f} \cdot (\lambda^+ + \lambda^-) f \text{ for } n \gg 1.$$

Let us now compute the limit of the denominator in (10.14). We have $f_n = \chi_n (2c^+ - 1)f = (2\chi_n c^+ - 1)f$ for $n \gg 1$ and using that $q c^+$ is Hermitian by Thm. 9.7, we obtain that $\bar{f}_n \cdot (\lambda^+ + \lambda^-) f_n = \bar{f} q g_n$ for $n \gg 1$ for

$$\begin{aligned}g_n &= (2c^+ \chi_n - 1)(2c^+ - 1)(2\chi_n c^+ - 1)f \\ &= (8c^+ \chi_n c^+ \chi_n c^+ - 8c^+ \chi_n c^+ - 4c \chi_n^2 c^+ + 6c - 1)f \text{ for } n \gg 1.\end{aligned}$$

We apply Prop. 10.3 *ii*) and *i*) (if necessary replacing χ_n by χ_n^2) and obtain that

$$\begin{aligned}\bar{f} \cdot q g_n &= \bar{f} \cdot q (8\chi_n c^+ \chi_n - 8\chi_n c^+ - 4\chi_n^2 c^+ + 6c^+ - 1)f + o(1) \\ &= \bar{f} \cdot q (2c^+ - 1)f + o(1) = \bar{f} \cdot (\lambda^+ + \lambda^-) f + o(1)\end{aligned}$$

when $n \rightarrow \infty$. Using (10.15) this completes the proof of (10.14). \square

APPENDIX A

A.1. Proof of Prop. 2.4. Since r is an isometry of (Σ, \mathbf{h}) , $r|_{\mathcal{B}} = Id$ and $r : \Sigma^+ \rightarrow \Sigma^-$ we obtain (2.2). The first identity in (2.3) follows from the fact that (u, ω) are normal Gaussian coordinates to \mathcal{B} for \mathbf{h} , the other are tautologies.

We obtain from (2.2) and 2.2.1 that N, \mathbf{w}^0 are odd in u , $\mathbf{w}^\alpha, \mathbf{k}_{\alpha\beta}$ are even in u with $\mathbf{w}^\alpha(0, \omega) = 0$. The function m is even in u by invariance under r . We now use Killing's equation

$$(A.1) \quad \nabla_a V_b + \nabla_b V_a = 0,$$

noting that since $V = 0$ on \mathcal{B} we have

$$(A.2) \quad \nabla_a V_b = \partial_a V_b \text{ on } \mathcal{B}.$$

If we work in Gaussian normal coordinates to Σ for \mathbf{g} , so that

$$\mathbf{g} = -dt^2 + \mathbf{h}_{ij}(t, y) dy^i dy^j, \quad V = -N(t, y) \partial_t + \mathbf{w}^0(t, y) \partial_u + \mathbf{w}^\alpha(t, y) \partial_{\omega^\alpha}$$

and $y = (u, \omega)$, we obtain from (A.1), (A.2) that:

$$\partial_u V_0(0, \omega) = 0 \Rightarrow \partial_u \mathbf{w}^0(0, \omega) = 0.$$

Summarizing we have:

$$(A.3) \quad \begin{aligned} N(u, \omega) &= ua(u^2, \omega), \\ \mathbf{w}^0(u, \omega) &= u^3b(u^2, \omega), \quad \mathbf{w}^\alpha(u, \omega) = u^2c^\alpha(u^2, \omega), \\ \mathbf{k}_{\alpha\beta}(u, \omega) &= d_{\alpha\beta}(u^2, \omega), \quad m(u, \omega) = n(u^2, \omega), \end{aligned}$$

for smooth functions $a, b, c^\alpha, d_{\alpha\beta} :] - \epsilon, \epsilon[\times \mathcal{B} \rightarrow \mathbb{R}$ with

$$n(0, \omega) \geq c > 0, c^{-1}\mathbb{1} \leq [d_{\alpha\beta}(0, \omega)] \leq c\mathbb{1}, \text{ for some } c > 0.$$

To complete the proof of the proposition it remains to show that $\kappa = a(0, \omega)$.

To do this we reexpress the surface gravity κ . By [S1, Lemma 2.5] we have:

$$\kappa^2 = (\mathbf{h}^{ij}\partial_i N \partial_j N)|_{\mathcal{B}} - \frac{1}{2}(\mathbf{h}^{ij}\mathbf{h}^{kl}\nabla_i^{(\mathbf{h})}\mathbf{w}_l \nabla_j^{(\mathbf{h})}\mathbf{w}_k)|_{\mathcal{B}},$$

which using (A.3) gives $\kappa = a(0, \omega)$. \square

A.2. Proof of Prop. 2.5. We recall that we defined the coordinates $(u, \omega) \in] - \delta, \delta[\times \mathcal{B}$ on a small neighborhood U of \mathcal{B} in Σ . $U \cap \Sigma^+$ is diffeomorphic to $]0, \delta[\times \mathcal{B}$ using the coordinates (u, ω) . If

$$X = u \cos(\kappa s), Y = u \sin(\kappa s),$$

we have:

$$du = u^{-1}(XdX + YdY), \quad ds = \kappa^{-1}u^{-2}(XdY - YdX).$$

By Prop. 2.4 we obtain:

$$\begin{aligned} \mathbf{k}_{\alpha\beta}(u, \omega)d\omega^\alpha d\omega^\beta &= d_{\alpha\beta}(X^2 + Y^2, \omega)d\omega^\alpha d\omega^\beta, \\ i\mathbf{w}_\alpha(u, \omega)d\omega^\alpha ds &= i\kappa^{-1}b_\alpha(X^2 + Y^2, \omega)(XdY - YdX)d\omega^\alpha, \\ i\mathbf{w}_0(u, \omega)duds &= i\kappa^{-1}b_0(X^2 + Y^2, \omega)(XdY - YdX)(XdX + YdY), \\ v^2(u, \omega)ds^2 + du^2 &= u^2\kappa^2(1 + u^2d(u^2, \omega))\kappa^{-2}u^{-4}(XdY - YdX)^2 + u^{-2}(XdX + YdY)^2 \\ &= dX^2 + dY^2 + d(X^2 + Y^2, \omega)(XdY - YdX)^2. \end{aligned}$$

Let us denote by $B_2(0, \delta) = \{(X, Y) \in \mathbb{R}^2 : X^2 + Y^2 \leq \delta^2\}$ the open disk of center 0 and radius δ in \mathbb{R}^2 . If $\beta = (2\pi)\kappa^{-1}$, then $(u, \kappa s) \in]0, \delta[\times \mathbb{S}_{2\pi}$ are polar coordinates on $B_2(0, \delta) \setminus \{0\}$. The expression (2.8) for \mathbf{g}^{eucl} and the estimates above show that \mathbf{g}^{eucl} extends as a smooth complex metric on $B_2(0, \delta) \times \mathcal{B}$.

We then construct $M_{\text{ext}}^{\text{eucl}}$ by gluing $B_2(0, \delta) \times \mathcal{B}$ with $M^{\text{eucl}} = \mathbb{S}_\beta \times \Sigma^+$ over $\{(X, Y) \in \mathbb{R}^2 : \frac{1}{2}\delta^2 < X^2 + Y^2 < \delta^2\} \times \mathcal{B}$ using the map:

$$(A.4) \quad \begin{aligned} \mathbb{S}_\beta \times]0, \delta[\times \mathcal{B} &\rightarrow B_2(0, \delta) \times \mathcal{B} \\ (s, u, \omega) &\mapsto (u \cos(\kappa s), u \sin(\kappa s), \omega). \end{aligned}$$

The complex metric \mathbf{g}^{eucl} defined on $\mathbb{S}_\beta \times \Sigma^+$ extends to a smooth complex metric $\mathbf{g}_{\text{ext}}^{\text{eucl}}$ on $M_{\text{ext}}^{\text{eucl}}$. By Prop. 2.4 we have $m = n(X^2 + Y^2, \omega)$, hence m extends as a smooth function on $M_{\text{ext}}^{\text{eucl}}$.

Let us now embed Σ isometrically into $M_{\text{ext}}^{\text{eucl}}$. In the coordinates (u, ω) on Σ near \mathcal{B} the embedding $\hat{\psi}$ becomes

$$(u, \omega) \mapsto \begin{cases} (0, u, \omega) & \text{for } 0 < u < \delta, \\ (\frac{\beta}{2}, -u, \omega) & \text{for } -\delta < u < 0, \end{cases}$$

which smoothly extends to $u = 0$, the image of Σ under this extension being locally equal to $\{Y = 0\}$.

The open set Ω_{ext} is obtained by gluing $\{Y > 0\}$ with $]0, \frac{\beta}{2}[\times \Sigma^+$ using the map (A.4). This completes the proof.

A.3. **Proof of (9.3).** A mechanical computation gives:

$$\begin{aligned}\sum_i \nabla_i T^i &= \sum_i \partial_i T^i + \frac{1}{2} \sum_{i,k,l} \mathbf{k}^{il} (\partial_i \mathbf{k}_{kl} + \partial_k \mathbf{k}_{il} - \partial_l \mathbf{k}_{ik}) T^k \\ &= \sum_i \partial_i T^i + \frac{1}{2} \sum_{i,k,l} \mathbf{k}^{il} \partial_k \mathbf{k}_{il} T^k =: I,\end{aligned}$$

using that $\mathbf{k}^{li} = \mathbf{k}^{il}$, $\mathbf{k}_{kl} = \mathbf{k}_{lk}$. Next

$$\sum_i |\mathbf{k}|^{-\frac{1}{2}} \partial_i (|\mathbf{k}|^{\frac{1}{2}} T^i) = \sum_i \partial_i T^i + \frac{1}{2} \sum_i |\mathbf{k}|^{-1} \partial_i |\mathbf{k}| T^i =: II.$$

Since

$$\det A(t)^{-1} \frac{d}{dt} \det A(t) = \text{Tr}(A(t)^{-1} \frac{d}{dt} A(t)),$$

we get that $\partial_i |\mathbf{k}| = |\mathbf{k}| \text{Tr}(\mathbf{k}^{-1} \partial_i \mathbf{k})$. Next we compute:

$$(\mathbf{k}^{-1} \partial_i \mathbf{k})_k^j = \sum_l \mathbf{k}^{jl} \partial_i \mathbf{k}_{lk}, \quad \text{Tr}(\mathbf{k}^{-1} \partial_i \mathbf{k}) = \sum_{k,l} \mathbf{k}^{kl} \partial_i \mathbf{k}_{lk},$$

which shows that $I = II$. \square

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