The Hartle-Hawking-Israel state on stationary black hole spacetimes

Christian Gérard

The Hartle-Hawking-Israel state on stationary black hole spacetimes

Christian GÉRARD

ABSTRACT. We consider a free quantized Klein-Gordon field in a spacetime $(M, \mathbf{g})$ containing a stationary black hole, more precisely a spacetime with a stationary bifurcate Killing horizon in the sense of Kay and Wald. We prove the existence of the *Hartle-Hawking-Israel* ground state, which is a pure state on the whole spacetime whose restriction to the exterior of the black hole is a thermal state at Hawking temperature $T_{\rm H}$.

We show that the HHI state is a Hadamard state and is the unique Hadamard extension of the above thermal state to the whole spacetime. We construct the HHI state by Wick rotation in Killing time coordinates, using the notion of the Calderón projector for elliptic boundary value problems.

1. INTRODUCTION

In this paper we consider a free quantized Klein-Gordon field in a spacetime $(M, \mathbf{g})$ containing a stationary black hole. It was conjectured by Hartle and Hawking [HH] and Israel [I] that a free Klein-Gordon quantum field admits a ground state $\omega_{\rm HHI}$, called the *Hartle-Hawking-Israel state*, whose restriction to the exterior region of the black hole is a thermal state at the Hawking temperature $T_{\rm H} = \kappa(2\pi)^{-1}$, where κ is the surface gravity of the black hole.

The physical motivation was that the stationary black hole spacetime $(M, \mathbf{g})$ describes the final state of the collapse of a massive object, and that the quantum Klein-Gordon field will eventually settle down to the ground state $\omega_{\rm HHI}$. The fact that $\omega_{\rm HHI}$ is a thermal state at Hawking temperature in the exterior of the black hole is then viewed as a justification of the Hawking radiation.

The first construction of the HHI state in the double wedge region of the Kruskal spacetime is due to Kay [K3]. This construction was valid for any temperature, the resulting state being an example of a *double KMS state*.

Later on Kay and Wald [KW] adressed the question of the extendability of the HHI state from the double wedge region to the black hole interior. In particular they introduced the definition of spacetimes with a *bifurcate Killing horizon* and gave a first rigorous definition of the notion of *Hadamard states*.

They proved that some subalgebra of the free field algebra admits at most one quasi-free state which is both invariant under the Killing isometries and Hadamard near the blackhole horizon. If such a state exists, Kay and Wald proved moreover that it is a thermal state at the Hawking temperature in the exterior region.

The first global construction of the HHI state in the whole spacetime is due to Sanders [S1], who considered spacetimes with a *static* bifurcate Killing horizon, ie such that the Killing vector field V is *static* in the exterior region. Sanders proved

Date: June 2018.

2010 *Mathematics Subject Classification.* 81T20, 35S05, 35S15.

Key words and phrases. Hartle-Hawking state, Killing horizons, Hadamard states, pseudo-differential calculus, Calderón projector.

Acknowledgments. We would like to thank Michal Wrochna and Ko Sanders for useful discussions.

in [S1] the existence of the HHI state and showed that it is a pure Hadamard state. The proof in [S1] relied on the *Wick rotation* in the Killing time coordinates, which was also the basis for the heuristic arguments in [HH, I] and which we will also use in this paper.

In [G] we gave another proof of the Hadamard property of the HHI state in the situation considered in [S1], by combining the Wick rotation with a tool which is familiar in elliptic boundary value problems, namely the *Calderón projectors*, see 1.2.2. The use of Calderón projectors allows to construct the HHI state directly on a Cauchy surface Σ and avoids to consider its behavior near the Killing horizon. In collaboration with Michal Wrochna, we have recently used Calderón projectors in [GW2] to construct *analytic Hadamard states* on general analytic spacetimes.

In the present paper we consider the more general *stationary case*, and give a construction of the HHI state for spacetimes with a stationary bifurcate Killing horizon.

1.1. **Results.** We now present more in details the result of this paper.

1.1.1. *Bifurcate Killing horizons.* Let $(M, \mathbf{g})$ a globally hyperbolic spacetime with a complete Killing vector field V. $(M, \mathbf{g})$ admits a *bifurcate Killing horizon* [KW], if the *bifurcation surface* $\mathcal{B} = \{x \in M : V(x) = 0\}$ is a compact, connected, orientable submanifold of codimension 2 and if there exists a Cauchy surface Σ containing $\mathcal{B}$. M splits then into four globally hyperbolic regions, the *right/left wedges* $\mathcal{M}^+$, $\mathcal{M}^-$ and the *future/past cones* $\mathcal{F}$, $\mathcal{P}$, each invariant under the flow of V.

The *Killing horizon* is then $\mathcal{H} = \partial(\mathcal{F} \cup \mathcal{P})$. An important object related with the Killing horizon is its *surface gravity* κ, which is a scalar, constant over all of $\mathcal{H}$.

One also assumes the existence of a *wedge reflection* $R : M \to M$ which is an isometry of $(\mathcal{M}^- \cup U \cup \mathcal{M}^+, \mathbf{g})$, where U is a neighborhood of $\mathcal{B}$ in M, such that $R \circ R = Id$, $R = Id$ on $\mathcal{B}$, R reverses the time orientation and $R^*V = V$. In concrete situations, the left wedge $\mathcal{M}^-$ is actually constructed by reflection of the right wedge $\mathcal{M}^+$, so the existence of a wedge reflection does not seem to be such a strong hypothesis.

The bifurcate Killing horizon $\mathcal{H}$ is *stationary* resp. *static* if V is *time-like* on $\Sigma \backslash \mathcal{B}$, resp. orthogonal to $\Sigma \backslash \mathcal{B}$. For technical reasons, we require V to be *uniformly time-like* near infinity on Σ, see Subsect. 2.4. This condition is imposed only far away from the bifurcation surface $\mathcal{B}$ and will hold for example if $(M, \mathbf{g})$ is asymptotically flat near spatial infinity.

We consider on (M, g) a free quantum Klein-Gordon field associated to the Klein-Gordon equation

$$-\Box_g \phi(x) + m(x)\phi(x) = 0,$$

where $m \in C^\infty(M, \mathbb{R})$ is invariant under V and R. We assume that $m(x) \geq m_0^2 > 0$ ie the Klein-Gordon field is massive.

1.1.2. *The double β-KMS state.* Since $(\mathcal{M}^+, \mathbf{g}, V)$ is a stationary spacetime, there exists (see [S2]) for any $\beta > 0$ a *thermal state* ω_β at temperature β^{-1} with respect to the group of Killing isometries of $(\mathcal{M}^+, \mathbf{g})$ generated by V.

The wedge reflection $R : \mathcal{M}^+ \xrightarrow{\sim} \mathcal{M}^-$ allows to extend ω_β to the *double β-KMS state* ω_{D} on $\mathcal{M}^+ \cup \mathcal{M}^-$. This extension exists for any $\beta > 0$ and is a pure state in $\mathcal{M}^+ \cup \mathcal{M}^-$.

We prove in this paper the following theorem.

Theorem 1.1. *Let $(M, \mathbf{g}, V)$ be a globally hyperbolic spacetime with a stationary bifurcate Killing horizon and a wedge reflection. Let $P = -\Box_g + V$ a Klein-Gordon operator invariant under the Killing vector field V and the wedge reflection R. Assume moreover that conditions (H) in Subsect. 2.4 are satisfied.*

Then there exists a state $\omega_{\rm HHI}$ *for* P *in* $(M,\mathbf{g})$ *called the* Hartle-Hawking-Israel state *such that:*

(1) $\omega_{\rm HHI}$ *is a pure Hadamard state in* M*,*
(2) *the restriction of* $\omega_{\rm HHI}$ *to* $\mathcal{M}^+\cup\mathcal{M}^-$ *is the double* β*-KMS state* $\omega_{\rm D}$ *at Hawking temperature* $T_{\rm H}=\kappa(2\pi)^{-1}$ *where* κ *is the surface gravity of the horizon,*
(3) $\omega_{\rm HHI}$ *is the unique extension of* $\omega_{\rm D}$ *such that its spacetime covariances* $\Lambda^{\pm}$ *map* $C_0^\infty(M)$ *into* $C^\infty(M)$ *continuously. In particular it is the unique Hadamard extension of* $\omega_{\rm D}$*.*

Thm. 1.1 will be proved in Sect. 9.

1.2. **Main ideas of the construction.** We now outline the construction of the HHI state $\omega_{\rm HHI}$. We look for $\omega_{\rm HHI}$ as an extension to M of the double β-KMS state $\omega_{\rm D}$ on $\mathcal{M}^-\cup\mathcal{M}^+$, where $\beta^{-1}=\kappa(2\pi)^{-1}$ is the Hawking temperature. The first step consists in understanding in sufficient details the β-KMS state in $\mathcal{M}^+$.

Writing the metric $\mathbf{g}$ in $\mathcal{M}^+$ using the Killing time coordinate associated to V and Σ, $\mathcal{M}^+$ is identified with $\mathbb{R}\times\Sigma^+$ and the metric $\mathbf{g}$ becomes

$$\mathbf{g}=-N^2(y)dt^2+\mathbf{h}_{ij}(y)(dy^i+\mathbf{w}^i(y)dt)(dy^j+\mathbf{w}^j(y)dt), \tag{1.1}$$

where N is the lapse function, $\mathbf{w}$ the shift vector field, $\mathbf{h}$ the induced metric on Σ. The Killing field V is simply $\frac{\partial}{\partial t}$. The fact that V is time-like in $\mathcal{M}^+$ is equivalent to the inequality $N^2(y)>\mathbf{w}^i(y)\cdot\mathbf{h}_{ij}(y)\mathbf{w}^j(y)$ for $y\in\Sigma^+$.

The Klein-Gordon operator P associated to $\mathbf{g}$ can be written as:

$$P=P=(\partial_t+w^*)N^{-2}(\partial_t-w)+h_0, \tag{1.2}$$

where $w=\mathbf{w}^i\cdot p_{y^i}$ and $h_0=\nabla^*\mathbf{h}^{-1}\nabla+m$ is an elliptic operator on Σ.

1.2.1. *The Wick rotation.* The *Wick rotation* consists in replacing t by is and produces the *complex metric*

$$\mathbf{g}^{\rm eucl}=N^2(y)ds^2+\mathbf{h}_{ij}(y)(dy^i+{\rm i}\mathbf{w}^i(y)ds)(dy^j+{\rm i}\mathbf{w}^j(y)ds). \tag{1.3}$$

In the static case considered in [S1, G] $\mathbf{w}$ vanishes and $\mathbf{g}^{\rm eucl}$ is Riemannian. The fact that $\mathbf{g}^{\rm eucl}$ is now a complex metric causes several new difficulties. Performing the same transformation on P yields the *Wick rotated operator*

$$K=-(\partial_s+{\rm i}w^*)N^{-2}(\partial_s+{\rm i}w)+h_0.$$

There are several different linear operators that can be associated to the formal expression K. The first one consists in working on $L^2(\mathbb{R}\times\Sigma^+)$, using the sesquilinear form

$$Q_\infty(u,u)=\|N^{-1}\partial_s u\|^2+(u|hu)-{\rm i}(N^{-1}\partial_s u|N^{-1}wu)-{\rm i}(N^{-1}wu|N^{-1}\partial_s u),$$

where $h=h_0-w^*N^{-2}w$, with ${\rm Dom}Q_\infty=C_0^\infty(\mathbb{R}\times\Sigma)$. Another possibility is to work on $L^2(\mathbb{S}_\beta\times\Sigma^+)$ where $\mathbb{S}_\beta=[-\frac{\beta}{2},\frac{\beta}{2}[$ is the circle of length β. The sesquilinear form Q_β has the same expression as Q_∞ but the domain is now ${\rm Dom}Q_\beta=C_0^\infty(\mathbb{S}_\beta\times\Sigma)$, which corresponds to imposing $\beta-$periodic boundary conditions on K.

Since we have assumed that V is *uniformly time-like* near infinity, see Subsect. 2.4, one can show that the sesquilinear forms Q_∞, Q_β are closeable and *sectorial* and hence generate *injective* linear operators K_∞, K_β. Their inverses K_∞^{-1}, K_β^{-1} are then well defined between abstract Sobolev spaces, using the Lax-Milgram theorem.

1.2.2. *Calderón projectors.* Let $\Omega_\infty =]0, +\infty[\times \Sigma^+$, $\Omega_\beta =]0, \frac{\beta}{2}[\times \Sigma^+$ and ν the exterior unit normal for $\mathbf{g}^{\rm eucl}$ to $\partial\Omega_\beta$, $\beta \in [0, +\infty]$. Note that ν is a *complex* vector field, but its imaginary part is tangent to $\partial\Omega_\beta$.

For $u \in \overline{C^\infty}(\Omega_\beta)$ such that $K_\beta u = 0$ in Ω_β, the *trace* $\gamma_\beta u$ of u on $\partial\Omega_\beta$ defined as

$$\gamma_\beta u = \begin{pmatrix} u\restriction_{\partial\Omega_\beta} \\ \partial_\nu u\restriction_{\partial\Omega_\beta} \end{pmatrix}$$

is not arbitrary, because K_β is an elliptic operator. Instead $\gamma_\beta u$ belongs to the range of a projector c_β^+, called the *Calderón projector* associated to Ω_β. The same construction with Ω_β replaced by its complement produces the complementary Calderon projector c_β^-, with $c_\beta^+ + c_\beta^- = \mathbb{1}$.

The projectors $c_\beta^\pm$ can be explicitely expressed in terms of the inverse K_β^{-1}, see Subsect. 8.7.

1.2.3. *Vacuum and double β-KMS states.* If $\beta = \infty$, the boundary $\partial\Omega_\infty$ equals Σ^+, and one can try to construct a state in $\mathcal{M}^+$ by defining its covariances on Σ^+ as

$$\lambda_\infty^\pm = \pm q \circ c_\infty^\pm,$$

where $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$ is the charge defining the symplectic structure on the space $C_0^\infty(\Sigma^+; \mathbb{C}^2)$ of Cauchy data on Σ^+. It turns out that $\lambda_\infty^\pm$ are actually the covariances of the *vacuum state* $\omega_{\rm vac}$ in $\mathcal{M}^+$.

Of course the study of the vacuum state $\omega_{\rm vac}$, corresponding to $\beta = \infty$, is not necessary for the construction of the HHI state, but gives a nice introduction to the more complicated case $\beta < \infty$.

If $\beta < \infty$, the boundary $\partial\Omega_\beta$ has two components, both isomorphic to Σ^+. The state $\omega_{\rm D}$ obtained similarly from the Calderón projectors $c_\beta^\pm$ is now the *double β-KMS state* $\omega_{\rm D}$ in $\mathcal{M}^- \cup \mathcal{M}^+$, modulo the identification of Σ^+ with Σ^- by the wedge reflection.

The proof of these facts takes up a large part of the paper. First of all we reduce ourselves to the situation $N(y) = 1$ by considering $\tilde{P} = NPN$ and $\tilde{K}_\beta = NK_\beta N$, the last identity taking a rather transparent form if we use the framework of sesquilinear forms, see Subsect. 8.6. The covariances of $\omega_{\rm vac}$, $\omega_{\rm D}$ for the Klein-Gordon operator P can similarly be deduced from those of the analogous states $\tilde{\omega}_{\rm vac}$, $\tilde{\omega}_{\rm D}$ for $\tilde{P}$.

The operator $\tilde{P}$ can be written as $(\partial_t + \tilde{w}^*)(\partial_t - \tilde{w}) + \tilde{h}_0$, and the computations of $\tilde{\omega}_{\rm vac}$, $\tilde{\omega}_{\rm D}$ can be done by reducing the Klein-Gordon equation $\tilde{P}\tilde{\phi} = 0$ to a first order system $\partial_t f - {\rm i}Hf = 0$, see Sects. 6, 7. This system is an example of a *stable symplectic dynamics*, which is studied in Sects. 4, 5.

1.2.4. *The surface gravity and the extended Euclidean metric.* All the constructions up to now are valid for any value of the inverse temperature β. The metrics $\mathbf{g}$ and $\mathbf{g}^{\rm eucl}$ are degenerate at the bifurcation surface $\mathcal{B} = \partial\Sigma^+$.

If $\beta = (2\pi)\kappa^{-1}$, ie if β^{-1} equals the *Hawking temperature* $\kappa(2\pi)^{-1}$, where κ is the *surface gravity* of the horizon, one can show that $(\mathbb{S}_\beta \times \Sigma^+, \mathbf{g}^{\rm eucl})$ has a unique extension $(M_{\rm ext}^{\rm eucl}, \mathbf{g}_{\rm ext}^{\rm eucl})$, which corresponds exactly to passing from polar to cartesian coordinates in the plane.

1.2.5. *The Hartle-Hawking-Israel state.* The open set $]0, \frac{\beta}{2}[\times \Sigma^+$ extends as an open set $\Omega_{\rm ext}$ with boundary isomorphic to the *full Cauchy surface* Σ. The Wick rotated operator K_β extends as an elliptic operator $K_{\rm ext}$ acting on $M_{\rm ext}^{\rm eucl}$, and one can consider the Calderón projectors $c_{\rm ext}^\pm$ associated to $K_{\rm ext}$ and $\Omega_{\rm ext}$.

One defines the covariances on Σ

$$\lambda^{\pm}_{\rm HHI} = \pm q \circ c^{\pm}_{\rm ext},$$

and one can rather easily show that $\lambda^{\pm}_{\rm HHI}$ are the covariances of a pure quasi-free state $\omega_{\rm HHI}$ defined on the whole of M. One uses that the restriction of $\lambda^{\pm}_{\rm HHI}$ to $C^\infty_0(\Sigma \setminus \mathcal{B})$ are precisely the covariances of the double β-KMS state $\omega_{\rm D}$, and some continuity properties of Calderón projectors and density results in Sobolev spaces, see Subsect. 9.2.

One can also prove that the HHI state $\omega_{\rm HHI}$ is a Hadamard state, by an argument already used in [G] in the static case, relying on the fact that the covariances of any Hadamard state on Σ are matrices of pseudodifferential operators.

1.3. **Notations.** We now collect some notation.

We set $\langle\lambda\rangle = (1+\lambda^2)^{\frac{1}{2}}$ for $\lambda \in \mathbb{R}$.

We write $A \Subset B$ if A is relatively compact in B.

If X, Y are sets and $f : X \to Y$ we write $f : X \xrightarrow{\sim} Y$ if f is bijective. If X, Y are equipped with topologies, we write $f : X \to Y$ if the map is continuous, and $f : X \xrightarrow{\sim} Y$ if it is a homeomorphism.

1.3.1. *Duals and antiduals.* Let $\mathcal{X}$ be a real vector space. Its dual will be denoted by $\mathcal{X}^\#$. Let $\mathcal{Y}$ be a complex vector space. We denote by $\mathcal{Y}_\mathbb{R}$ its *real form*, ie $\mathcal{Y}$ as a vector space over $\mathbb{R}$. We denote by $\mathcal{Y}^\#$ its dual, ie the space of $\mathbb{C}-$linear forms on $\mathcal{Y}$ and by $\mathcal{Y}^*$ its anti-dual, ie the space of $\mathbb{C}-$antilinear forms on $\mathcal{Y}$.

We denote by $\overline{\mathcal{Y}}$ the *conjugate vector space* to $\mathcal{Y}$, ie $\overline{\mathcal{Y}} = \mathcal{Y}_\mathbb{R}$ as a $\mathbb{R}-$vector space, equiped with the complex structure $-$i, if i $\in L(\mathcal{Y}_\mathbb{R})$ is the complex structure of $\mathcal{Y}$. The identity map $Id : \mathcal{Y} \to \overline{\mathcal{Y}}$ will be denoted by $y \mapsto \overline{y}$, ie $\overline{y}$ equals y but considered as an element of $\overline{\mathcal{Y}}$.

If $\mathcal{Y}$ is a Hilbert space, then $\overline{\mathcal{Y}}$ inherits also a Hilbert space structure by

$$(\overline{y}_1|\overline{y}_2)_{\overline{\mathcal{Y}}} := \overline{(y_1|y_2)_\mathcal{Y}}.$$

By definition we have $\mathcal{Y}^* = \overline{\mathcal{Y}}^\#$. Note that we have a $\mathbb{C}-$linear identification $\overline{\mathcal{Y}^\#} \sim \overline{\mathcal{Y}}^\#$ defined as follows: if $y \in \mathcal{Y}$ and $w \in \mathcal{Y}^\#$ then

$$\overline{w} \cdot \overline{y} := \overline{w \cdot y}$$

This identifies $\overline{w} \in \overline{\mathcal{Y}^\#}$ with an element of $\overline{\mathcal{Y}}^\#$. Similarly we have a $\mathbb{C}-$linear identification $\overline{\mathcal{Y}}^* \sim \overline{\mathcal{Y}^*}$.

1.3.2. *Linear operators.* If $\mathcal{X}_i$, $i = 1, 2$ are real or complex vector spaces and $a \in L(X_1, X_2)$ we denote by $a^\# \in L(\mathcal{X}_2^\#, \mathcal{X}_1^\#)$ its transpose. If $\mathcal{Y}_i$, $i = 1, 2$ are complex vector spaces we denote by $a^* \in L(\mathcal{Y}_2^*, \mathcal{Y}_1^*)$ its adjoint, and by $\overline{a} \in L(\overline{\mathcal{Y}}_1, \overline{\mathcal{Y}}_2)$ its *conjugate*, defined by $\overline{a}\overline{y}_1 = \overline{a y_1}$. With the above identifications we have $a^* = \overline{a}^\# = \overline{a^\#}$.

1.3.3. *Bilinear and sesquilinear forms.* If $\mathcal{X}$ is a real or complex vector space, a bilinear form on $\mathcal{X}$ is given by $a \in L(\mathcal{X}, \mathcal{X}^\#)$, its action on a couple (x_1, x_2) is denoted by $x_1 \cdot a x_2$. We denote by $L_{\rm s/a}(\mathcal{X}, \mathcal{X}^\#)$ the symmetric/antisymmetric forms on $\mathcal{X}$. a is *non-degenerate* if Ker$a = \{0\}$. An antisymmetric, non-degenerate form σ is called a *symplectic form* on $\mathcal{X}$.

Similarly if $\mathcal{Y}$ is a complex vector space, a sesquilinear form on $\mathcal{Y}$ is given by $a \in L(\mathcal{Y}, \mathcal{Y}^*)$, its action on a couple (y_1, y_2) is denoted by $\overline{y}_1 \cdot a y_2$, the last notation being a reminder that $\mathcal{Y}^* = \overline{\mathcal{Y}}^\#$. We denote by $L_{\rm h/a}(\mathcal{Y}, \mathcal{Y}^*)$ the Hermitian/antiHermitian forms on $\mathcal{Y}$. Non-degenerate forms are defined as in the real case. An antiHermitian, non-degenerate form σ is called a (complex) *symplectic form* on $\mathcal{Y}$.

If $a \in L(\mathcal{Y}, \mathcal{Y}^*)$ then $\overline{a} \in L(\overline{\mathcal{Y}}, \overline{\mathcal{Y}^*})$ and with the above identifications we have $(\overline{y}_1|\overline{ay}_2) = \overline{(y_1|ay_2)}$ for $y_1, y_2 \in \mathcal{Y}$.

1.3.4. *Linear operators on Hilbert spaces.* The domain of a closed, densely defined operator a on a Hilbert space $\mathcal{H}$ will be denoted by Dom$a$, equipped with the graph norm, its spectrum by $\sigma(a)$ and its resolvent set by res(a). We will similarly denote by Dom$Q$ the domain of a sesquilinear form Q. If Q is closeable we denote by $Q^{\rm cl}$ its closure.

If a is selfadjoint on $\mathcal{H}$, we write $a > 0$ if $a \geq 0$ and Ker$a = \{0\}$. If $a > 0$ and $s \in \mathbb{R}$ we denote by $a^s\mathcal{H}$ the completion of Doma^{-s} for the norm $\|a^{-s}u\|_{\mathcal{H}}$. Equipped with the scalar product $(u|v)_s = (a^{-s}u|a^{-s}v)_{\mathcal{H}}$, it is a Hilbert space. The spaces $a^s\mathcal{H}$ and $a^{-s}\mathcal{H}$ form a dual pair for the duality pairing $\langle u|v\rangle = (a^{-s}u|a^s v)_{\mathcal{H}}$.

We define similarly the spaces $\langle a\rangle^s\mathcal{H}$ for any selfadjoint operator on $\mathcal{H}$. We have $\langle a\rangle^{-s}\mathcal{H} = \text{Dom}|a|^s$ for $s > 0$. We have $\langle a\rangle^{-s}\mathcal{H} \subset: cH \subset \langle a\rangle^s\mathcal{H}$ for $s \geq 0$ and $\langle a\rangle^s : cH = a^s\mathcal{H}$ if $0 \notin \sigma(a)$.

If a_1, a_2 are selfadjoint on $\mathcal{H}$ with $a_1, a_2 > 0$ we write $a_1 \lesssim a_2$ if $\text{Dom}a_1^{\frac{1}{2}} \supset \text{Dom}a_2^{\frac{1}{2}}$ and $a_1 \leq ca_2$ on $\text{Dom}a_2^{\frac{1}{2}}$ for some $c > 0$. We write $a_1 \sim a_2$ if $a_1 \lesssim a_2$ and $a_2 \lesssim a_1$.

If $a_1 \sim a_2$ the Kato-Heinz theorem implies that $a_2^{-1} \sim a_1^{-1}$ and that $a_1^s\mathcal{H} = a_2^s\mathcal{H}$ as Banach spaces for $s \in [-\frac{1}{2}, \frac{1}{2}]$.

1.3.5. *Quadratic forms.* Similarly if q_1, q_2 are two positive quadratic forms with $q_i(u,u) = 0 \Rightarrow u = 0$, we write $q_1 \lesssim q_2$ if Dom$q_1 \supset$ Domq_2 and $q_1 \leq cq_2$ on Domq_2 and we write $q_1 \sim q_2$ if $q_1 \lesssim q_2$ and $q_2 \lesssim q_1$.

1.3.6. *Differential operators on manifolds.* If X is a smooth manifold and a, b are differential operators on X the composition $a \circ b$ is denoted by ab. If a is a differential operator on X and $u \in C^\infty(X)$, then au denotes the composition of a with the operator of multiplication by u, while $(au) \in C^\infty(X)$ denotes the image of u under a.

1.3.7. *Spaces of distributions.* Let X a smooth manifold. Fixing a smooth density we identify distributions and distributional densities on X. If $\Omega \subset X$ is an open set with smooth boundary and $F(X) \subset \mathcal{D}'(X)$ is a vector space, we denote by $\overline{F}(\Omega) \subset \mathcal{D}'(\Omega)$ the space of *restrictions* of elements of $F(X)$ to Ω.

Any $u \in \overline{\mathcal{D}'}(\Omega)$ has a unique extension $eu \in \mathcal{D}'(X)$ with supp$eu \subset \overline{\Omega}$.

We denote by $\delta_a \in \mathcal{D}'(\mathbb{R})$ the Dirac distribution at $a \in \mathbb{R}$.

2. Spacetimes with a stationary bifurcate Killing horizon

In this section we recall the definition of spacetimes with stationary Killing horizons, following [KW, S1]. We express various natural objects, like the lapse function, shift vector field and induced Riemannian metric in Gaussian coordinates near the bifurcation surface.

We then consider the *Wick rotated metric* $\mathbf{g}^{\rm eucl}$, obtained by the Wick rotation $t \to is$ in the Killing time t, and show that if s belongs to the circle $\mathbb{S}_{(2\pi)\kappa^{-1}}$ of length $(2\pi)\kappa^{-1}$, for κ the surface gravity of the horizon, $\mathbf{g}^{\rm eucl}$ has a smooth extension up the the bifurcation surface $\mathcal{B}$. This fundamental fact, already known for static horizons, see [S1, Sect. 2.2] lies at the basis of the construction of the HHI state in later sections.

2.1. **Bifurcate Killing horizons.**

Definition 2.1. *A* spacetime with a bifurcate Killing horizon *is a triple* $(M, \mathbf{g}, V)$ *such that :*

(1) $(M, \mathbf{g})$ *is a globally hyperbolic spacetime,*

(2) *V is a smooth, complete Killing vector field on $(M, \mathbf{g})$,*
(3) *$\mathcal{B} := \{x \in M : V(x) = 0\}$ is a compact, connected, orientable submanifold of codimension 2, called the* bifurcation surface*,*
(4) *there exists a smooth, space-like Cauchy hypersurface Σ with $\mathcal{B} \subset \Sigma$.*

If n is the future directed normal vector field to Σ, one defines the *lapse function* $N \in C^\infty(\Sigma)$ and *shift vector field* $\mathbf{w}$, which is a smooth vector field tangent to Σ, by

$$V = Nn + \mathbf{w} \text{ on } \Sigma,$$

ie

$$N := -V \cdot \mathbf{g} n, \ \mathbf{w} := V - Nn \text{ on } \Sigma.$$

Let us denote by y the elements of Σ. The Cauchy surface Σ is then decomposed as

$$\Sigma = \Sigma^- \cup \mathcal{B} \cup \Sigma^+, \ \Sigma^\pm := \{y \in \Sigma : \pm N(y) > 0\},$$

ie V is future/past directed over $\Sigma^\pm$.

The spacetime M splits as

$$M = \mathcal{M}^+ \cup \mathcal{M}^- \cup \overline{\mathcal{F}} \cup \overline{\mathcal{P}},$$

where the *future cone* $\mathcal{F} := I^+(\mathcal{B})$, *the past cone* $\mathcal{P} := I^-(\mathcal{B})$, the *right/left wedges* $\mathcal{M}^\pm := D(\Sigma^\pm)$, are all globally hyperbolic when equipped with $\mathbf{g}$.

The future cone $\mathcal{F}$ may be a black hole. The *bifurcate Killing horizon* is then

$$\mathcal{H} := \partial\mathcal{F} \cup \partial\mathcal{P}.$$

The Killing vector field V is tangent to $\mathcal{H}$. In Figure 1 below the vector field V is represented by arrows.

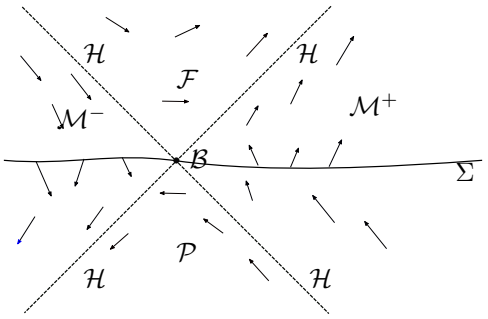

FIGURE 1.

Definition 2.2. *A triple $(M, \mathbf{g}, V)$ as in Def. 2.1 is called a spacetime with a* stationary*, resp.* static *bifurcate Killing horizon if V is time-like on $\Sigma \setminus \mathcal{B}$, resp. g-orthogonal to $\Sigma \setminus \mathcal{B}$.*

2.2. **Wedge reflection.** Additionally one assumes the existence of a *wedge reflection*, see [S1, Def. 2.6].

Definition 2.3. *A* wedge reflection *R for a spacetime (M, g, V) with a stationary Killing horizon is a diffeomorphism $R : \mathcal{M}^- \cup U \cup \mathcal{M}^+ \xrightarrow{\sim} \mathcal{M}^- \cup U \cup \mathcal{M}^+$, where U is a neighborhood of $\mathcal{B}$ in M such that:*
(1) *R is an isometry of $(\mathcal{M}^- \cup U \cup \mathcal{M}^+, \mathbf{g})$ which reverses the time orientation,*
(2) *$R \circ R = Id$, $R = Id$ on $\mathcal{B}$,*
(3) *$R^* V = V$.*

2.2.1. *Weak wedge reflection.* It is known, see [S1, Prop. 2.7] that if R is a wedge reflection, one can find a Cauchy surface Σ as in Def. 2.1 such that $R : \Sigma \xrightarrow{\sim} \Sigma$. The map $r := R_{|\Sigma}$ is called a *weak wedge reflection*. If the Riemannian metric $\mathbf{h}$ is the restriction of $\mathbf{g}$ to Σ, one has:
(1) r is an isometry of $(\Sigma, \mathbf{h})$ with $r \circ r = Id$,
(2) $r = Id$ on $\mathcal{B}$,
(3) $r^*N = -N$, $r^*\mathbf{w} = \mathbf{w}$.
By (3) above we have $r : \Sigma^{\pm} \xrightarrow{\sim} \Sigma^{\mp}$.

2.3. **Klein-Gordon operators.** We fix a real function $m \in C^\infty(M)$. As in [S1] we assume that m is stationary w.r.t. the Killing vector field V and invariant under the wedge reflection, ie:

$$V^a \nabla_a m(x) = 0,\ m \circ R(x) = m(x),\ x \in \mathcal{M}^+ \cup \mathcal{M}^- \cup U. \tag{2.1}$$

We also assume that

$$m(x) \geq m_0^2 > 0,\ x \in M, \tag{2.2}$$

ie we consider only *massive* Klein-Gordon fields. The *Klein-Gordon operator* is

$$P = -\Box_{\mathbf{g}} + m. \tag{2.3}$$

2.4. **Conditions near infinity on Σ.** It will be necessary, in order to control various energy spaces in Sect. 8, to impose conditions on the Killing vector field V near infinity on Σ.

$\exists\ U$ neighborhood of $\mathcal{B}$ in Σ such that:

(H1) $V + \delta\mathbf{w}$ is time-like on $\Sigma \setminus U$ for some $\delta > 0$,

(H2) $N^{-2}\mathbf{w}^i \cdot (\nabla^{\mathbf{h}}_i N),\ N^{-1}\nabla^{\mathbf{h}}_i \mathbf{w}^i$ are bounded on $\Sigma \setminus U$.

(H1) means that V is *uniformly time-like* near infinity on Σ. Conditions (H) are clearly satisfied if $(M, \mathbf{g})$ is for example asymptotic to the Kerr spacetime, near spatial infinity.

2.5. **The surface gravity.** The surface gravity is defined by:

$$\kappa^2 = -\frac{1}{2}(\nabla^{b(\mathbf{g})} V^a \nabla^{(\mathbf{g})}_b V_a)_{|\mathcal{B}},\ \kappa > 0.$$

It is a fundamental fact, see [KW, Sect. 2], that κ is constant on $\mathcal{B}$ and actually on the whole horizon $\mathcal{H}$.

For $\omega \in \mathcal{B}$ let $n_\omega \in T_\omega\Sigma$ the unit normal to $\mathcal{B}$ for $\mathbf{h}$ pointing towards Σ^+. We introduce Gaussian normal coordinates to $\mathcal{B}$ in $(\Sigma, \mathbf{h})$ by:

$$\chi : \begin{array}{l}]-\delta, \delta[\times \mathcal{B} \to \Sigma \\ (u, \omega) \mapsto \exp^{\mathbf{h}}_\omega(un) \end{array}$$

which is a smooth diffeomorphism from $]-\delta, \delta[\times\mathcal{B}$ to a relatively compact neighborhood U of $\mathcal{B}$ in Σ. In the next proposition we express $\mathbf{h}$, N, w and the wedge reflection r in the local coordinates (u, ω) on U. We recall that the elements of Σ are denoted by y.

Proposition 2.4. *On U one has:*

$$r(u, \omega) = (-u, \omega), \tag{2.4}$$

and

$$
\begin{aligned}
&\mathbf{h}_{ij}(y)dy^i dy^j = du^2 + \mathbf{k}_{\alpha\beta}(u,\omega)d\omega^\alpha d\omega^\beta,\\
&\mathbf{w}^i(y)\partial_{y^i} = \mathbf{w}^0(u,\omega)\partial_u + \mathbf{w}^\alpha(u,\omega)\partial_{\omega^\alpha},\\
&N(y) = N(u,\omega),\\
&m(y) = m(u,\omega),
\end{aligned}
\tag{2.5}
$$

where $\mathbf{k}_{\alpha\beta}(u,\omega)d\omega^\alpha d\omega^\beta$ *is a smooth,* $u-$*dependent Riemannian metric on* $\mathcal{B}$ *with:*

$$
\begin{aligned}
&N(u,\omega) = u(\kappa + u^2 d(u^2,\omega)),\\
&\mathbf{w}^0(u,\omega) = u^3 b(u^2,\omega),\ \ \mathbf{w}^\alpha(u,\omega) = u^2 c^\alpha(u^2,\omega),\\
&\mathbf{k}_{\alpha\beta}(u,\omega) = \mathrm{d}_{\alpha\beta}(u^2,\omega),\\
&m(u,\omega) = n(u^2,\omega)
\end{aligned}
\tag{2.6}
$$

for smooth functions $b, d, n, c^\alpha, \mathrm{d}_{\alpha\beta} :]-\epsilon,\epsilon[\times\mathcal{B} \to \mathbb{R}$ *with*

$$
n(0,\omega) \geq c > 0,\ \ c^{-1}\mathbb{1} \leq [\mathrm{d}_{\alpha\beta}(0,\omega)] \leq c\mathbb{1},\ \ \textit{for some } c > 0.
$$

The proof of Prop. 2.4 is given in Appendix A.1.

2.6. **The metric in $\mathcal{M}^+$.** Let us denote by Φ_t the flow of the Kiling vector field V. We identify $\mathbb{R}\times\Sigma^+$ with $\mathcal{M}^+$ by

$$
\chi : \mathbb{R}\times\Sigma^+ \ni (t,y) \mapsto \Phi_t(y) \in \mathcal{M}^+.
$$

We have $\chi^* V = \dfrac{\partial}{\partial t}$ and

$$
\begin{aligned}
\chi^*\mathbf{g} = &\ -N^2(y)dt^2 + (dy^i + \mathbf{w}^i(y)dt)\mathbf{h}_{ij}(y)(dy^j + \mathbf{w}^j(y)dt)\\
= &\ -v^2(y)dt^2 + w_i(y)dy^i dt + w_j(y)dt dy^j + \mathbf{h}_{ij}(y)dy^i dy^j,
\end{aligned}
$$

for $v^2(y) = (N^2(y) - \mathbf{w}^i(y)\mathbf{h}_{ij}(y)\mathbf{w}^j(y))$. Note that The fact that V is time-like in $\mathcal{M}^+$ is equivalent to

$$
N^2(y) > \mathbf{w}^i(y)\mathbf{h}_{ij}(y)\mathbf{w}^j(y), y \in \Sigma^+. \tag{2.7}
$$

The unit normal vector field to the foliation $\Sigma_t = \{t\}\times\Sigma$ is

$$
n = N^{-1}(\frac{\partial}{\partial t} - w), \tag{2.8}
$$

Denoting $\chi^*\mathbf{g}$ on $\mathbb{R}\times\Sigma^+$ simply by $\mathbf{g}$, we have $|\mathbf{g}| = N^2|\mathbf{h}|$ and

$$
\mathbf{g}^{-1} = -N^{-2}\partial_t^2 + N^{-2}(\mathbf{w}^i\partial_{y^i}\partial_t + \mathbf{w}^j\partial_t\partial_{y^j}) + (\mathbf{h}^{ij} - N^{-2}\mathbf{w}^i\mathbf{w}^j)\partial_{y^i}\partial_{y^j}. \tag{2.9}
$$

Since the potential m is invariant under the Killing vector field, we have $m = m(y)$.

2.7. **The Wick rotated metric.**

2.7.1. *Complex metrics.* If X is a smooth manifold, we denote by $\mathrm{T}^p_q(X)$ the space of smooth, real (p,q) tensors on X and by $\mathbb{C}\mathrm{T}^p_q(X)$ its complexification. An element $\mathbf{k} = \mathbf{k}_{ab}(x)dx^a dx^b$ of $\mathbb{C}\mathrm{T}^0_2(X)$ which is symmetric and non-degenerate will be called a *complex metric* on X.

2.7.2. *The Wick rotated metric.* We denote by $\mathbb{S}_\beta = [-\frac{\beta}{2}, \frac{\beta}{2}[$ with endpoints identified the circle of length β and

$$M^{\mathrm{eucl}} := \mathbb{S}_\beta \times \Sigma^+,$$

with variables (s, y). Replacing t by is we obtain the complex metric on M^{eucl}:

$$\begin{aligned} \mathbf{g}^{\mathrm{eucl}} &= N^2(y)ds^2 + (dy^j + \mathrm{i}\mathbf{w}^j(y)ds)\mathbf{h}_{jk}(y)(dy^k + \mathrm{i}\mathbf{w}^k(y)ds) \\ &= v^2(y)ds^2 + \mathrm{i}\mathbf{w}_j(y)dy^j ds + \mathrm{i}\mathbf{w}_j(y)dsdy^j + \mathbf{h}_{jk}(y)dy^j dy^k. \end{aligned} \tag{2.10}$$

We embed $\Sigma \setminus \mathcal{B}$ into $M^{\mathrm{eucl}} = \mathbb{S}_\beta \times \Sigma^+$ by the map

$$\hat{\psi} : y \mapsto \begin{array}{l} (0, y) \text{ for } y \in \Sigma^+, \\ (\frac{\beta}{2}, r(y)) \text{ for } y \in \Sigma^-, \end{array}$$

where $r : \Sigma \to \Sigma$ is the weak wedge reflection.

2.8. **The smooth extension.**

Proposition 2.5. *Assume that $\beta = (2\pi)\kappa^{-1}$. Then there exists a smooth manifold $M^{\mathrm{eucl}}_{\mathrm{ext}}$ equipped with a smooth complex metric $\mathbf{g}^{\mathrm{eucl}}_{\mathrm{ext}}$ and*

(1) *a smooth embedding $\psi : \Sigma \to M^{\mathrm{eucl}}_{\mathrm{ext}}$,*
(2) *a smooth isometric embedding $\chi : (M^{\mathrm{eucl}}, \mathbf{g}^{\mathrm{eucl}}) \to (M^{\mathrm{eucl}}_{\mathrm{ext}} \setminus \mathcal{B}_{\mathrm{ext}}, \mathbf{g}^{\mathrm{eucl}}_{\mathrm{ext}})$, where $\mathcal{B}_{\mathrm{ext}} = \psi(\mathcal{B})$,*
(3) *an open set Ω_{ext} such that $\partial\Omega_{\mathrm{ext}} = \psi(\Sigma)$ and $\chi :]0, \frac{\beta}{2}[\times \Sigma^+ \xrightarrow{\sim} \Omega_{\mathrm{ext}} \setminus \mathcal{B}_{\mathrm{ext}}$,*
(4) *a smooth function $m_{\mathrm{ext}} : M^{\mathrm{eucl}}_{\mathrm{ext}} \to \mathbb{R}$ with $m_{\mathrm{ext}} \geq m_0^2 > 0$,*

such that:

$$\psi\restriction_{\Sigma\setminus\mathcal{B}} = \chi \circ \hat{\psi},\ \chi^* m_{\mathrm{ext}} = m\restriction_{M^{\mathrm{eucl}}}.$$

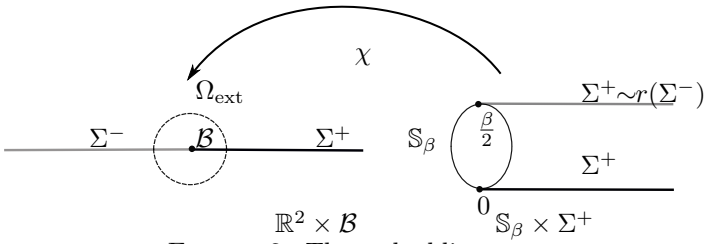

FIGURE 2. The embedding χ

The proof of Prop. 2.5 is given in Appendix A.2.

3. Free Klein-Gordon fields

In this section we briefly recall some well-known background material on free quantum Klein-Gordon fields on globally hyperbolic spacetimes. We follow the presentation in [GW1, Sect. 2] based on *charged fields*.

3.1. **Charged CCR algebra.**

3.1.1. *Charged bosonic fields.* Let $\mathcal{Y}$ a complex vector space and $q \in L_{\rm h}(\mathcal{Y}, \mathcal{Y}^*)$ a non degenerate Hermitian form on $\mathcal{Y}$.

The *CCR $^*-$algebra* $\mathrm{CCR}(\mathcal{Y}, q)$ is the complex $*$-algebra generated by symbols $\mathbb{1}, \psi(y), \psi^*(y), y \in \mathcal{Y}$ and the relations:

$$\psi(y_1 + \lambda y_2) = \psi(y_1) + \overline{\lambda}\psi(y_2),\ y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C},$$

$$\psi^*(y_1 + \lambda y_2) = \psi^*(y_1) + \lambda\psi^*(y_2),\ y_1, y_2 \in \mathcal{Y}, \lambda \in \mathbb{C},$$

$$[\psi(y_1), \psi(y_2] = [\psi^*(y_1), \psi^*(y_2)] = 0,\ [\psi(y_1), \psi^*(y_2)] = \overline{y}_1 \cdot q y_2 \mathbb{1},\ y_1, y_2 \in \mathcal{Y},$$

$$\psi(y)^* = \psi^*(y),\ y \in \mathcal{Y}.$$

A state ω on $\mathrm{CCR}(\mathcal{Y}, q)$ is *(gauge invariant) quasi-free* if

$$\omega(\prod_{i=1}^{p} \psi(y_i) \prod_{i=1}^{q} \psi^*(y_j)) = \begin{cases} 0 \text{ if } p \neq q, \\ \sum_{\sigma \in S_p} \prod_{i=1}^{p} \omega(\psi(y_i)\psi^*(y_{\sigma(i)})) \text{ if } p = q. \end{cases}$$

There is no loss of generality to restrict oneself to charged fields and gauge invariant states, see eg the discussion in [GW1, Sect. 2]. It is convenient to associate to ω its *(complex) covariances* $\lambda^{\pm} \in L_{\rm h}(\mathcal{Y}, \mathcal{Y}^*)$ defined by:

$$\begin{aligned} \omega(\psi(y_1)\psi^*(y_2)) &=: \overline{y}_1 \cdot \lambda^+ y_2, \\ \omega(\psi^*(y_2)\psi(y_1)) &=: \overline{y}_1 \cdot \lambda^- y_2, \end{aligned} \quad y_1, y_2 \in \mathcal{Y}.$$

The following results are well-known, see eg [DG, Sect. 17.1] or [GW1, Sect. 2] for Prop. 3.1 and [GOW, Prop. 7.1] for Prop. 3.2.

Proposition 3.1. *Two Hermitian forms $\lambda^{\pm} \in L_{\rm h}(\mathcal{Y}, \mathcal{Y}^*)$ are the covariances of a quasi-free state ω on $\mathrm{CCR}(\mathcal{Y}, q)$ iff*

$$\lambda^{\pm} \geq 0,\ \lambda^+ - \lambda^- = q. \tag{3.1}$$

Proposition 3.2. *Let $\mathcal{Y}_\omega$ be the completion of $\mathcal{Y}$ for the Hilbertian scalar product $\lambda^+ + \lambda^-$. Then the state ω on $\mathrm{CCR}(\mathcal{Y}, q)$ is pure iff there exist linear operators $c^{\pm} \in L(\mathcal{Y}_\omega)$ such that*

$$c^+ + c^- = \mathbb{1},\ (c^{\pm})^2 = c^{\pm},$$

(ie $c^{\pm}$ is a pair of complementary projections) and $\lambda^{\pm} = \pm q \circ c^{\pm}$.

3.2. **Free Klein-Gordon fields.** Let $P = -\Box_{\mathbf{g}} + m(x)$, $m \in C^\infty(M, \mathbb{R})$ a Klein-Gordon operator on a globally hyperbolic spacetime $(M, \mathbf{g})$ (we use the convention $(1, n-1)$ for the Lorentzian signature). Let $G_{\rm ret/adv}$ be the retarded/advanced inverses of P and $G := G_{\rm ret} - G_{\rm adv}$. We apply the above framework to

$$\mathcal{Y} = \frac{C_0^\infty(M)}{PC_0^\infty(M)},\ \overline{[u]} \cdot q[u] = \mathrm{i}(u|Gu)_M,$$

where $(u|v)_M = \int_M \overline{u} v dVol_{\mathbf{g}}$. Denoting by $\mathrm{Sol}_{\rm sc}(P)$ the space of smooth space-compact solutions of $P\phi = 0$, it is well known that

$$[G] : (\frac{C_0^\infty(M)}{PC_0^\infty(M)}, \mathrm{i}(\cdot|G\cdot)_M) \ni [u] \mapsto Gu \in (\mathrm{Sol}_{\rm sc}(P), q)$$

is unitary for

$$\overline{\phi}_1 \cdot q \phi_2 := \mathrm{i} \int_\Sigma (\nabla_\mu \overline{\phi}_1 \phi_2 - \overline{\phi}_1 \nabla_\mu \phi_2) n^\mu d\sigma, \tag{3.2}$$

where Σ is any spacelike Cauchy hypersurface, n^μ is the future directed unit normal vector field to Σ and $d\sigma$ the induced surface density. Setting

$$\varrho : C_{\rm sc}^\infty(M) \ni \phi \mapsto \begin{pmatrix} \phi\restriction_\Sigma \\ \mathrm{i}^{-1} n^\mu \partial_\mu \phi\restriction_\Sigma \end{pmatrix} = f \in C_0^\infty(\Sigma; \mathbb{C}^2)$$

the map

$$(\frac{C_0^\infty(M)}{PC_0^\infty(M)}, \mathrm{i}(\cdot|G\cdot)_M) \ni [u] \mapsto \varrho G u \in (C_0^\infty(\Sigma;\mathbb{C}^2), q)$$

is unitary for

$$\overline{f}\cdot qf := \int_\Sigma \overline{f}_1 f_0 + \overline{f}_0 f_1 d\sigma_\Sigma,\ f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}. \tag{3.3}$$

In the sequel the $*$-algebra $\mathrm{CCR}(\mathcal{Y}, q)$ where $(\mathcal{Y}, q)$ is any of the above equivalent Hermitian spaces will be denoted by $\mathrm{CCR}(P)$.

3.3. **Quasi-free states.** One restricts attention to quasi-free states on $\mathrm{CCR}(P)$ whose covariances are given by distributions on $M \times M$, ie such that there exists $\Lambda^\pm \in \mathcal{D}'(M \times M)$ with

$$\begin{aligned} &\omega(\psi([u_1])\psi^*([u_2])) = (u_1|\Lambda^+ u_2)_M, \\ &\omega(\psi^*([u_2])\psi([u_1])) = (u_1|\Lambda^- u_2)_M, \end{aligned} \quad u_1, u_2 \in C_0^\infty(M). \tag{3.4}$$

In the sequel the distributions $\Lambda^\pm \in \mathcal{D}'(M \times M)$ will be called the *spacetime covariances* of the state ω.

In (3.4) we identify distributions on M with distributional densities using the density $dVol_{\mathbf{g}}$ and use the notation $(u|\varphi)_M$, $u \in C_0^\infty(M)$, $\varphi \in \mathcal{D}'(M)$ for the duality bracket. We have then

$$\begin{aligned} &P(x, \partial_x)\Lambda^\pm(x, x') = P(x', \partial_{x'})\Lambda^\pm(x, x') = 0, \\ &\Lambda^+(x, x') - \Lambda^-(x, x') = \mathrm{i}G(x, x'). \end{aligned} \tag{3.5}$$

3.4. **Cauchy surface covariances.** Using $(C_0^\infty(\Sigma;\mathbb{C}^2), q)$ instead of $(\frac{C_0^\infty(M)}{PC_0^\infty(M)}, \mathrm{i}(\cdot|G\cdot)_M)$ one can associate to a quasi-free state its *Cauchy surface covariances* $\lambda^\pm$ defined by:

$$\Lambda^\pm =: (\varrho G)^* \lambda^\pm (\varrho G). \tag{3.6}$$

Using the canonical scalar product $(f|f)_\Sigma := \int_\Sigma \overline{f}_1 f_1 + \overline{f}_0 f_0 d\sigma_\Sigma$ we identify $\lambda^\pm$ with operators, still denoted by $\lambda^\pm : C_0^\infty(\Sigma;\mathbb{C}^2) \to \mathcal{D}'(\Sigma;\mathbb{C}^2)$.

3.5. **Hadamard states.** A quasi-free state is called a *Hadamard state*, (see [R] for the neutral case and [GW1] for the complex case) if

$$\mathrm{WF}(\Lambda^\pm)' \subset \mathcal{N}^\pm \times \mathcal{N}^\pm, \tag{3.7}$$

where $\mathrm{WF}(\Lambda)'$ denotes the 'primed' wavefront set of Λ, ie $S' := \{((x,\xi),(x',-\xi')) : ((x,\xi),(x',\xi')) \in S\}$ for $S \subset T^*M \times T^*M$, and $\mathcal{N}^\pm$ are the two connected components (positive/negative energy shell) of the characteristic manifold:

$$\mathcal{N} := \{(x,\xi) \in T^*M \setminus o : \xi_\mu g^{\mu\nu}(x)\xi_\nu = 0\}. \tag{3.8}$$

We recall that $T^*X \setminus o$ denotes the cotangent bundle of X with the zero section removed.

Large classes of Hadamard states were constructed in terms of their Cauchy surface covariances in [GW1, GOW] using pseudodifferential calculus on Σ, see below for a short summary.

3.6. **Pseudodifferential operators.** We briefly recall the notion of (classical) pseudodifferential operators on a manifold, referring to [Sh, Sect. 4.3] for details.

For $m \in \mathbb{R}$ we denote by $\Psi^m(\mathbb{R}^d)$ the space of classical pseudodifferential operators on $\mathbb{R}^d$, associated with poly-homogeneous symbols of order m, see eg [Sh, Sect. 3.7].

Let X be a smooth, $d-$dimensional manifold. Let $U \subset X$ a precompact chart open set and $\psi : U \to \tilde{U}$ a chart diffeomorphism, where $\tilde{U} \subset \mathbb{R}^d$ is precompact, open. We denote by $\psi^* : C_0^\infty(\tilde{U}) \to C_0^\infty(U)$ the map $\psi^* u(x) := u \circ \psi(x)$.

Definition 3.3. *A linear continuous map $A : C_0^\infty(X) \to C^\infty(X)$ belongs to $\Psi^m(X)$ if the following condition holds:*

(C) Let $U \subset X$ be precompact open, $\psi : U \to \tilde{U}$ a chart diffeomorphism, $\chi_1, \chi_2 \in C_0^\infty(U)$ and $\tilde{\chi}_i = \chi_i \circ \psi^{-1}$. Then there exists $\tilde{A} \in \Psi^m(\mathbb{R}^d)$ such that

$$(\psi^*)^{-1} \chi_1 A \chi_2 \psi^* = \tilde{\chi}_1 \tilde{A} \tilde{\chi}_2. \tag{3.9}$$

Elements of $\Psi^m(X)$ are called (classical) pseudodifferential operators *of order m on X.*

The subspace of $\Psi^m(X)$ of pseudodifferential operators with properly supported kernels *is denoted by $\Psi_{\rm c}^m(X)$.*

Note that if $\Psi_{\rm (c)}^\infty(X) := \bigcup_{m\in\mathbb{R}} \Psi_{\rm (c)}^m(X)$, then $\Psi_{\rm c}^\infty(X)$ is an algebra, but $\Psi^\infty(X)$ is not, since without the proper support condition, pseudodifferential operators cannot in general be composed.

To $A \in \Psi^m(X)$ one can associate its *principal symbol* $\sigma_{\rm pr}(A) \in C^\infty(T^*X \setminus o)$, which is homogeneous of degree m in the fiber variable ξ in T^*M, in $\{|\xi| \geq 1\}$. A is called *elliptic* in $\Psi^m(X)$ at $(x_0, \xi_0) \in T^*X \setminus o$ if $\sigma_{\rm pr}(A)(x_0, \xi_0) \neq 0$.

If $A \in \Psi^m(X)$ there exists (many) $A_{\rm c} \in \Psi_{\rm c}^m(X)$ such that $A - A_{\rm c}$ has a smooth kernel.

3.7. **The Cauchy surface covariances of Hadamard states.** We now state a result which follows directly from a construction of Hadamard states in [GW1, Subsect. 8.2].

Theorem 3.4. *Let ω be any Hadamard state for the free Klein-Gordon field on $(M, \mathbf{g})$ and Σ a smooth space-like Cauchy surface. Then its Cauchy surface covariances $\lambda^\pm$ are 2×2 matrices with entries in $\Psi^\infty(\Sigma)$.*

We refer the reader to [G, Thm. 3.2] for the proof.

4. Green operators and Calderón projectors

In this section we collect some formulas expressing the Green operators, ie inverses for abstract operators of the form $\partial_s + b$, where s belongs either to $\mathbb{R}$ or to the circle $\mathbb{S}_\beta$. We also compute various Calderón projectors. The formulas in this section will be used later in Sect. 6 to express Calderón projectors for second order elliptic operators obtained from abstract Klein-Gordon operators by Wick rotation.

4.1. **Green operators and Calderón projectors.** Let b a selfadjoint operator on a Hilbert space $\mathfrak{h}$ with $\mathrm{Ker} b = \{0\}$. We recall that $\mathbb{S}_\beta = [-\frac{\beta}{2}, \frac{\beta}{2}[$ is the circle of length β. For $0 < \beta \leq \infty$ we set

$$\mathfrak{h}_\beta = L^2(\mathbb{S}_\beta) \otimes \mathfrak{h}, \text{ for } \beta < \infty, \ \mathfrak{h}_\infty = L^2(\mathbb{R}) \otimes \mathfrak{h}. \tag{4.10}$$

The operator ∂_s is anti-selfadjoint on $\mathfrak{h}_\beta$ with its natural domain. Denoting still by b the extension of b to $\mathfrak{h}_\beta$ we see that $B_\beta = \partial_s + b$ with domain $\mathrm{Dom}\partial_s \cap \mathrm{Dom} b$ is normal.

4.1.1. *Green operators.* If $0 \in \sigma(b)$ then $0 \in \sigma(B_\beta)$ but we can still make sense out of B_β^{-1} as

$$B_\beta^{-1} : (-\partial_s^2 + b^2)^{\frac{1}{2}}\mathfrak{h}_\beta \to \mathfrak{h}_\beta.$$

A straightforward computation shows that:

$$B_\infty^{-1} f(s) = \int_{\mathbb{R}} G_\infty(s-s')f(s')ds',\ f \in C_0^0(\mathbb{R};\mathfrak{h}), \tag{4.11}$$

for

$$G_\infty(s) := \mathrm{e}^{-sb}\left(\mathbb{1}_{\mathbb{R}^+}(s)\mathbb{1}_{\mathbb{R}^+}(b) - \mathbb{1}_{\mathbb{R}^-}(s)\mathbb{1}_{\mathbb{R}^-}(b)\right). \tag{4.12}$$

Similarly for $\beta < \infty$, we have

$$B_\beta^{-1} f(s) = \int_{\mathbb{S}_\beta} G_\beta(s-s')f(s')ds',\ f \in C^0(\mathbb{S}_\beta;\mathfrak{h}), \tag{4.13}$$

for $G_\beta(s)$ defined as follows: we set

$$G_\beta(s) := \mathrm{e}^{-sb}\left(\mathbb{1}_{\mathbb{R}^+}(s)(1-\mathrm{e}^{-\beta b})^{-1} - \mathbb{1}_{\mathbb{R}^-}(s)(1-\mathrm{e}^{\beta b})^{-1}\right),\ s \in [-\frac{\beta}{2}, \frac{\beta}{2}],$$

(note that $G_\beta(\frac{\beta}{2}) = G_\beta(-\frac{\beta}{2})$) and extend it to $s \in \mathbb{R}$ by $\beta-$periodicity. In particular we have:

$$G_\beta(s) = \mathrm{e}^{-sb}\left(\mathbb{1}_{\mathbb{R}^+}(s)(1-\mathrm{e}^{-\beta b})^{-1} - \mathbb{1}_{\mathbb{R}^-}(s)(1-\mathrm{e}^{\beta b})^{-1}\right),\ s \in [-\beta, \beta]. \tag{4.14}$$

4.1.2. *Calderón projectors for B_∞.* We set $I^\pm = \pm]0, +\infty[$. In the sequel we use the notation recalled in 1.3.7. If $F \in \overline{C^0}(I^\pm;\mathfrak{h})$ satisfies $(\partial_s + b)F = 0$ in $I^\pm$ we set

$$\Gamma_\infty^\pm F = F(0^\pm) = \lim_{s \to 0^\pm} F(s).$$

Denoting by $i_\infty^\pm F$ the extension of F by 0 in $\mathbb{R} \setminus I^\pm$ we have

$$(\partial_s + b)i_\infty^\pm F = \pm\delta_0(s) \otimes \Gamma_\infty^\pm F,$$

hence $i_\infty^\pm F = \pm B_\infty^{-1}\delta_0(s) \otimes f$ for $f = \Gamma_\infty^\pm F$. This implies formally that

$$f = \pm\Gamma_\infty^\pm \circ B_\infty^{-1}(\delta_0(s) \otimes f)$$

if $f = \Gamma_\infty^\pm F$ for F solving $(\partial_s + b)F = 0$ in $I_\infty^\pm$. This motivates the following definition:

Definition 4.1. *The* Calderón projectors $C_\infty^\pm \in B(\mathfrak{h})$ *are:*

$$C_\infty^\pm f = \pm\Gamma_\infty^\pm \circ B_\infty^{-1}(\delta_0(s) \otimes f),\ f \in \mathfrak{h}. \tag{4.15}$$

Proposition 4.2. *We have:*

$$C_\infty^\pm = \mathbb{1}_{\mathbb{R}^\pm}(b). \tag{4.16}$$

It follows that $C_\infty^\pm$ are bounded projections on $\mathfrak{h}$ with $C_\infty^+ + C_\infty^- = \mathbb{1}$.

Proof. We approximate $\delta_0(\cdot)$ by a sequence $n\chi(n\cdot)$ where $\chi \in C_0^\infty(\mathbb{R})$ with $\int \chi(s)ds = 1$ and see from (4.12) that $C_\infty^\pm$ are well defined and (4.16) follows directly from (4.12). □

4.1.3. *Calderón projectors for B_β.* For $\beta < \infty$ we set $I^\pm_\beta = \pm]0, \frac{\beta}{2}[$. If $F \in \overline{C^0}(I^\pm_\beta; \mathfrak{h})$ satisfies $(\partial_s + b)F = 0$ in $I^\pm_\beta$ we set:

$$\begin{aligned} \Gamma^+_\beta F &:= F(0^+) \oplus F(\tfrac{\beta}{2}^-) =: \Gamma^{(0)+}_\beta F \oplus \Gamma^{(\frac{\beta}{2})+}_\beta F, \\ \Gamma^-_\beta F &:= F(0^-) \oplus F(\tfrac{\beta}{2}^+) =: \Gamma^{(0)-}_\beta F \oplus \Gamma^{(\frac{\beta}{2})-}_\beta F. \end{aligned} \tag{4.17}$$

Denoting by $i^\pm_\beta F$ the extension of F by 0 in $\mathbb{S}_\beta \setminus I^\pm_\beta$, we have

$$(\partial_s + b) i^\pm_\beta F = \pm(\delta_0(s) \otimes \Gamma^{(0)\pm}_\beta F - \delta_{\frac{\beta}{2}}(s) \otimes \Gamma^{(\frac{\beta}{2})\pm}_\beta F),$$

which as before leads to the following definition:

Definition 4.3. *The* Calderón projectors $C^\pm_\beta \in B(\mathfrak{h} \oplus \mathfrak{h})$ *are:*

$$C^\pm_\beta f := \pm \Gamma^\pm_\beta \circ B^{-1}_\beta (\delta_0(s) \otimes f^{(0)} - \delta_{\frac{\beta}{2}}(s) \otimes f^{(\frac{\beta}{2})}),\ f = f^{(0)} \oplus f^{(\frac{\beta}{2})} \in \mathfrak{h} \oplus \mathfrak{h}. \tag{4.18}$$

Proposition 4.4. *We have:*

$$\begin{aligned} C^+_\beta &= \begin{pmatrix} (1 - \mathrm{e}^{-\beta b})^{-1} & (1 - \mathrm{e}^{\beta b})^{-1} \mathrm{e}^{\frac{\beta}{2} b} \\ (1 - \mathrm{e}^{-\beta b})^{-1} \mathrm{e}^{-\frac{\beta}{2} b} & (1 - \mathrm{e}^{\beta b})^{-1} \end{pmatrix}, \\ C^-_\beta &= \begin{pmatrix} (1 - \mathrm{e}^{\beta b})^{-1} & -\mathrm{e}^{\frac{\beta}{2} b}(1 - \mathrm{e}^{\beta b})^{-1} \\ -\mathrm{e}^{-\frac{\beta}{2} b}(1 - \mathrm{e}^{-\beta b})^{-1} & (1 - \mathrm{e}^{-\beta b})^{-1} \end{pmatrix}. \end{aligned} \tag{4.19}$$

On $\mathbb{1}_I(b)\mathfrak{h} \oplus \mathbb{1}_I(b)\mathfrak{h}$ for any $I \Subset \mathbb{R}^*$ one has:

$$C^\pm_\beta C^\pm_\beta = C^\pm_\beta,\ C^+_\beta + C^-_\beta = \mathbb{1}.$$

Note that if $0 \in \sigma(b)$ then $C^\pm_\beta$ are unbounded on $\mathfrak{h} \oplus \mathfrak{h}$.

Proof. The proof of (4.19) is a routine computation, using (4.14). The second statement is checked using the identity $(1 - a)^{-1} + (1 - a^{-1})^{-1} = 1$ for $a = \mathrm{e}^{-\beta b}$. □

5. Vacua and KMS states for stable symplectic dynamics

In this section we recall well-known formulas for the covariances of the vacuum and KMS states associated to a symplectic flow on a symplectic space. The symplectic flow has to be *stable*, ie generated by a positive classical energy. In concrete situations the symplectic flow is generated by a time-like Killing vector field. We also recall the definition of the *double* KMS state, due to Kay [K1, K2], which is related to the Araki-Woods representation of a KMS state.

The new result of this section is that the covariances of the vacuum and double KMS states can be expressed by the *Calderón projectors* introduced in Sect. 4. Note that only the double KMS states will be important for the construction of the HHI state later on. Nevertheless the case of vacuum state is simpler and we include it for pedagogical reasons.

5.1. **Weakly stable symplectic dynamics.** We describe now a framework for symplectic dynamics, which can be found in [DG, Sect. 18.2.1], called there a *weakly stable symplectic dynamics*.

Let $(\mathcal{Y}, q)$ a Hermitian space and $E \in L_{\mathrm{h}}(\mathcal{Y}, \mathcal{Y}^*)$ with $E > 0$, the function $\mathcal{Y} \ni y \mapsto \overline{y} \cdot Ey$ being the *classical energy*. The *energy space* $\mathcal{Y}_{\mathrm{en}}$ is the completion of $\mathcal{Y}$ for the scalar product $(y_1|y_2)_{\mathrm{en}} = \overline{y}_1 \cdot Ey_2$and is a complex Hilbert space.

Let $r_t = \mathrm{e}^{\mathrm{i}tb}$ be a strongly continuous unitary group on $\mathcal{Y}_{\mathrm{en}}$ with selfadjoint generator b. We assume that $r_t : \mathcal{Y} \to \mathcal{Y}$, $\mathcal{Y} \subset \mathrm{Dom}b$, $\mathrm{Ker}b = \{0\}$ and:

$$\overline{y}_1 \cdot Ey_2 = \overline{y}_1 \cdot qby_2,\ y_1, y_2 \in \mathcal{Y}. \tag{5.1}$$

The meaning of (5.1) is that $\{r_t\}_{t\in\mathbb{R}}$ is the symplectic evolution group associated to the classical energy $\overline{y}\cdot Ey$ and the symplectic form $\sigma = \mathrm{i}^{-1}q$.

5.1.1. *Dynamical Hilbert space.* It is convenient, in connection with the quantization of the symplectic flow $\{r_t\}_{t\in\mathbb{R}}$, to introduce the *dynamical Hilbert space*

$$\mathcal{Y}_{\rm dyn} := |b|^{\frac{1}{2}}\mathcal{Y}_{\rm en},$$

see [DG, Subsect. 18.2.1], equipped with the scalar product $(y_1|y_2)_{\rm dyn} = (y_1||b|^{-1}y_2)_{\rm en}$. The group $\{r_t\}_{t\in\mathbb{R}}$ extends obviously as a unitary group on $\mathcal{Y}_{\rm dyn}$. If we denote the generator of r_t on $\mathcal{Y}_{\rm en/dyn}$ by $b_{\rm en/dyn}$ then $b_{\rm en} = b$ and $b_{\rm dyn} = |b|^{\frac{1}{2}}b_{\rm en}|b|^{-\frac{1}{2}}$ since $|b|^{-\frac{1}{2}} : \mathcal{Y}_{\rm dyn} \to \mathcal{Y}_{\rm en}$ is unitary. Therefore we will denote both generators by the same letter b.

Moreover from (5.1) we obtain that:

$$\overline{y}_1\cdot qy_2 = (y_1|\mathrm{sgn}(b)y_2)_{\mathcal{Y}_{\rm dyn}} \tag{5.2}$$

so q is a bounded sesquilinear form on $\mathcal{Y}_{\rm dyn}$, but in general not on $\mathcal{Y}_{\rm en}$, unless $0 \notin \sigma(b)$.

5.2. **Vacuum state.** We now recall the definition of the vacuum state $\omega_{\rm vac}$ associated to the dynamics $\{r_t\}_{t\in\mathbb{R}}$.

Definition 5.1. *The* vacuum state $\omega_{\rm vac}$ *is defined by the covariances:*

$$\overline{y}_1\cdot\lambda^{\pm}_{\rm vac}y_2 = (y_1|\mathbb{1}_{\mathbb{R}^{\pm}}(b)y_2)_{\rm dyn}. \tag{5.3}$$

From (5.2) we obtain that:

$$c^{\pm}_{\rm vac} := \pm\lambda^{\pm}_{\rm vac}\circ q^{-1} = \mathbb{1}_{\mathbb{R}^{\pm}}(b). \tag{5.4}$$

It follows from Def. 4.1 that:

$$c^{+}_{\rm vac} = C^{\pm}_{\infty}, \tag{5.5}$$

where the Calderón projectors $C^{\pm}_{\infty}$ are defined in Def. 4.1, for $\mathfrak{h} = \mathcal{Y}_{\rm dyn}$.

5.3. **KMS state.** Let us now define the $\beta-$KMS state ω_β associated to the dynamics $\{r_t\}_{t\in\mathbb{R}}$.

Definition 5.2. *The* β-KMS state ω_β *is defined by the covariances:*

$$\begin{aligned}\overline{y}_1\cdot\lambda^{+}_{\beta}y_2 &= \overline{y}_1\cdot q(1-\mathrm{e}^{-\beta b})^{-1}y_2,\\ \overline{y}_1\cdot\lambda^{-}_{\beta}y_2 &= \overline{y}_1\cdot q(\mathrm{e}^{\beta b}-1)^{-1}y_2.\end{aligned} \tag{5.6}$$

5.4. **Double $\beta-$KMS states.** The *double $\beta-$KMS state* see [K1, K2] can easily be related to the *Araki-Woods* representation of ω_β, see eg [DG, Subsect. 17.1.5], that we first briefly recall. In the sequel $\mathcal{Y}_{\rm dyn}$ will be simply denoted by $\mathcal{Y}$.

5.4.1. *Araki-Woods representation.* Let us denote by $\mathcal{Z}$ the space $\mathcal{Y}_{\mathbb{R}}$ equipped with the complex structure

$$\mathrm{j} := \mathrm{i}\circ\mathrm{sgn}(b)$$

and the scalar product:

$$(z_1|z_2)_{\mathcal{Z}} := (y_{1+}|y_{2+})_{\mathcal{Y}} + (y_{2-}|y_{1-})_{\mathcal{Y}},$$

for $y_\pm := \mathbb{1}_{\mathbb{R}^{\pm}}(b)y$ and $z = y$ (considered as an element of $\mathcal{Z}$). $\mathcal{Z}$ is a Hilbert space equal to $\mathcal{Y}_+ \oplus \overline{\mathcal{Y}}_-$ for $\mathcal{Y}_\pm = \mathbb{1}_{\mathbb{R}^{\pm}}(b)\mathcal{Y}$. Note that since $[b,\mathrm{j}] = 0$, b induces a selfadjoint operator on $\mathcal{Z}$, still denoted by b. We set

$$\rho := (\mathrm{e}^{\beta|b|}-1)^{-1}, \tag{5.7}$$

which is a selfadjoint operator on $\mathcal{Z}$.

We also introduce the Hilbert space $\mathcal{Z}\oplus\overline{\mathcal{Z}}$ and the bosonic Fock space $\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$, see eg [DG, Subsect. 3.3.1]. For $(z_1,\overline{z}_2)\in\mathcal{Z}\oplus\overline{\mathcal{Z}}$ we denote by $a^{(*)}(z_1,\overline{z}_2)$ the Fock creation/annihilation operators acting on $\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$.

The *left/right Araki-Woods* creation/annihilation operators are defined by:

$$\begin{aligned} a^*_{\rm l}(z) &= a^*((1+\rho)^{\frac{1}{2}}z,0)+a(0,\overline{\rho}^{\frac{1}{2}}\overline{z}),\\ a_{\rm l}(z) &= a((1+\rho)^{\frac{1}{2}}z,0)+a^*(0,\overline{\rho}^{\frac{1}{2}}\overline{z}),\\ a^*_{\rm r}(\overline{z}) &= a(\rho^{\frac{1}{2}}z,0)+a^*(0,(1+\overline{\rho})^{\frac{1}{2}}\overline{z}),\\ a_{\rm r}(\overline{z}) &= a^*(\rho^{\frac{1}{2}}z,0)+a(0,(1+\overline{\rho})^{\frac{1}{2}}\overline{z}). \end{aligned} \tag{5.8}$$

One has:

$$[a_{\rm l}(z_1),a^*_{\rm l}(z_2)]=(z_1|z_2)_{\mathcal{Z}}\mathbb{1},\ [a_{\rm r}(\overline{z}_1),a^*_{\rm r}(\overline{z}_2)]=(\overline{z}_1|\overline{z_2})_{\overline{\mathcal{Z}}}\mathbb{1},$$

all other commutators being equal to 0. Setting $z_\pm=y_\pm$ for $y\in\mathcal{Y}$ we set

$$\begin{aligned} \psi^*_{\rm l}(y)&:=a^*_{\rm l}(z_+)+a_{\rm l}(z_-),\ \psi_{\rm l}(y):=a_{\rm l}(z_+)+a^*_{\rm l}(z_-)\\ \psi^*_{\rm r}(y)&:=a^*_{\rm r}(\overline{z}_-)+a_{\rm r}(\overline{z}_+),\ \psi_{\rm r}(y):=a_{\rm r}(\overline{z}_-)+a^*_{\rm r}(\overline{z}_+), \end{aligned}\quad y\in\mathcal{Y}. \tag{5.9}$$

An easy computation show that

$$[\psi_{\rm l}(y_1),\psi^*_{\rm l}(y_2)]=\overline{y}_1\cdot qy_2,\ [\psi_{\rm r}(y_1),\psi^*_{\rm r}(y_2)]=-\overline{y}_1\cdot qy_2, \tag{5.10}$$

all other commutators being equal to 0. Moreover $\mathcal{Y}\ni y\mapsto\psi^*_{\rm l/r}(y)$ is $\mathbb{C}$-linear. This means that $\mathcal{Y}\ni y\mapsto\psi^{(*)}_{\rm l/r}(y)$ induces two commuting representations of $\mathrm{CCR}(\mathcal{Y},\pm q)$.

From (5.9) we obtain that:

$$\begin{aligned} (\Omega|\psi_{\rm l}(y_1)\psi^*_{\rm l}(y_2)\Omega)_{\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})}&=\overline{y}_1\cdot\lambda^+_\beta y_2,\\ (\Omega|\psi^*_{\rm l}(y_2)\psi_{\rm l}(y_1)\Omega)_{\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})}&=\overline{y}_1\cdot\lambda^-_\beta y_2,,\\ (\Omega|\psi_{\rm l}(y_2)\psi_{\rm l}(y_1)\Omega)_{\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})}&=(\Omega|\psi^*_{\rm l}(y_2)\psi^*_{\rm l}(y_1)\Omega)_{\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})}=0, \end{aligned}$$

where Ω is the vacuum vector in $\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})$. If $\pi_{\rm AW,l}$ is the representation of $\mathrm{CCR}(\mathcal{Y},q)$ defined by $\pi_{\rm AW,l}(\psi^{(*)})(y)=\psi^{(*)}_{\rm l}(y)$, then $(\pi_{\rm AW,l},\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}}),\Omega)$ is the GNS representation associated to the $\beta-$KMS state ω_β.

5.4.2. *The double $\beta-$KMS state.* To define the double $\beta-$KMS state associated to ω_β we set

$$(\mathcal{X},Q):=(\mathcal{Y}\oplus\mathcal{Y},q\oplus-q).$$

Recalling that $\sigma=\mathrm{i}^{-1}q$, this corresponds to add to the real symplectic space $(\mathcal{Y}_{\mathbb{R}},\mathrm{Re}\sigma)$ its anti-symplectic copy $(\mathcal{Y}_{\mathbb{R}},-\mathrm{Re}\sigma)$. From (5.10) we see that $\mathcal{X}\ni x\mapsto\Psi^{(*)}_{\rm AW}(x)$ for

$$\Psi^{(*)}_{\rm AW}(x):=\psi^{(*)}_{\rm l}(y)+\psi^{(*)}_{\rm r}(y'),\ x=(y,y')\in\mathcal{X} \tag{5.11}$$

induces a representation of $\mathrm{CCR}(\mathcal{X},Q)$.

Definition 5.3. *The* double $\beta-$KMS state $\omega_{\rm d}$ *is the quasi-free state on* $\mathrm{CCR}(\mathcal{X},Q)$ *defined by*

$$\omega_{\rm d}(\Psi^{(*)}(x_1)\Psi^{(*)}(x_2)):=(\Omega|\Psi^{(*)}_{\rm AW}(x_1)\psi^{(*)}_{\rm AW}(x_2)\Omega)_{\Gamma_{\rm s}(\mathcal{Z}\oplus\overline{\mathcal{Z}})},\ x_1,x_2\in\mathcal{X}.$$

Proposition 5.4. *$\omega_{\rm d}$ is a pure, gauge invariant quasi-free state on* $\mathrm{CCR}(\mathcal{X},Q)$. *If $\lambda^\pm_{\rm d}$ are the covariances of $\omega_{\rm d}$ we have*

$$\overline{x}_1\cdot\lambda^\pm_{\rm d}x_2=\pm\overline{x}_1\cdot QC^\pm_\beta x_2,\ x_1,x_2\in\mathcal{X}.$$

where $C^\pm_\beta$ is the Calderón projectors for B_β defined in Def. 4.3.

Remark 5.5. *Let us denote $(\mathcal{Y}, q)$ by $(\mathcal{Y}_1, q_1)$ and let $(\mathcal{Y}_2, q_2)$ another Hermitian space with $I : (\mathcal{Y}_2, q_2) \to (\mathcal{Y}_1, -q_1)$ unitary. Then $\omega_{\rm d}$ induces a quasi-free state on $\mathrm{CCR}(\mathcal{X}_1 \oplus \mathcal{X}_2, q_1 \oplus q_2)$. Its covariances are*

$$\begin{pmatrix} \mathbb{1} & 0 \\ 0 & I^* \end{pmatrix} \lambda_{\rm d}^{\pm} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & I \end{pmatrix} = \pm \begin{pmatrix} q_1 & 0 \\ 0 & q_2 \end{pmatrix} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & I^{-1} \end{pmatrix} C_\beta^{\pm} \begin{pmatrix} \mathbb{1} & 0 \\ 0 & I \end{pmatrix}.$$

Proof of Prop. 5.4. We obtain from (5.9), (5.8):

$$\begin{aligned} & \Psi_{\rm AW}^*(x)\Omega = a_{\rm l}^*(z_+)\Omega + a_{\rm l}(z_-)\Omega + a_{\rm r}^*(\overline{z}'_-)\Omega + a_{\rm r}(\overline{z}'_+)\Omega \\ = \; & ((\rho+1)^{\frac{1}{2}} z_+ + \rho^{\frac{1}{2}} z'_+, \overline{\rho}^{\frac{1}{2}} \overline{z}_- + (\overline{\rho}+1)^{\frac{1}{2}} \overline{z}'_-), \\ & \Psi_{\rm AW}(x)\Omega = a_{\rm l}(z_+)\Omega + a_{\rm l}^*(z_-)\Omega + a_{\rm r}(\overline{z}'_-)\Omega + a_{\rm r}^*(\overline{z}'_+)\Omega \\ = \; & ((\rho+1)^{\frac{1}{2}} z_- + \rho^{\frac{1}{2}} z'_-, \overline{\rho}^{\frac{1}{2}} \overline{z}_+ + (\overline{\rho}+1)^{\frac{1}{2}} \overline{z}'_+), \end{aligned} \tag{5.12}$$

as elements of $\mathcal{Z} \oplus \overline{\mathcal{Z}}$. From (5.12) we immediately obtain that

$$\omega_{\rm d}(\Psi_{\rm AW}(x_1)\Psi_{\rm AW}(x_2)) = \omega_d(\Psi_{\rm AW}^*(x_1)\Psi_{\rm AW}^*(x_2)) = 0,$$

ie ω_d is gauge invariant for the complex structure $\mathrm{i} \oplus \mathrm{i}$ of $\mathcal{X}$. We have next

$$\begin{aligned} & \omega_{\rm d}(\Psi_{\rm AW}(x_1)\Psi_{\rm AW}^*(x_2)) = (\Psi_{\rm AW}^*(x_1)\Omega | \Psi_{\rm AW}^*(x_2)\Omega)_{\mathcal{Z}\oplus\overline{\mathcal{Z}}} \\ = \; & \Big((\rho+1)^{\frac{1}{2}} y_{1+} + \rho^{\frac{1}{2}} y'_{1+} | (\rho+1)^{\frac{1}{2}} y_{2+} + \rho^{\frac{1}{2}} y'_{2+}\Big)_{\mathcal{Y}} \\ & + \Big(\rho^{\frac{1}{2}} y_{1-} + (\rho+1)^{\frac{1}{2}} y'_{1-} | \rho^{\frac{1}{2}} y_{2-} + (\rho+1)^{\frac{1}{2}} y'_{2-}\Big)_{\mathcal{Y}} \end{aligned}$$

If $\lambda_{\rm d}^+ =: Q \circ C_{\rm d}^+$, where $Q = q \oplus -q$, we obtain from (5.2) that:

$$C_{\rm d}^+ = \begin{pmatrix} (\rho+1)\mathbb{1}_+ - \rho\mathbb{1}_- & -\rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}\mathbb{1}_+ + \rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}\mathbb{1}_- \\ \rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}\mathbb{1}_+ - \rho^{\frac{1}{2}}(\rho+1)^{\frac{1}{2}}\mathbb{1}_- & -\rho\mathbb{1}_+ + (\rho+1)\mathbb{1}_- \end{pmatrix},$$

for $\mathbb{1}_\pm = \mathbb{1}_{\mathbb{R}^\pm}(b)$. We compute:

$$\begin{aligned} & (1+\rho)\mathbb{1}_+ - \rho\mathbb{1}_- \\ = \; & (1-\mathrm{e}^{-\beta b})^{-1}\mathbb{1}_+ - \mathrm{e}^{\beta b}(1-\mathrm{e}^{\beta b})^{-1}\mathbb{1}_- = (1-\mathrm{e}^{-\beta b})^{-1}; \\ & -\rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\mathbb{1}_+ + \rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\mathbb{1}_- \\ = \; & -\mathrm{e}^{-\beta b/2}(1-\mathrm{e}^{-\beta b})^{-1} + \mathrm{e}^{\beta b/2}(1-\mathrm{e}^{\beta b})^{-1} = \mathrm{e}^{\beta b/2}(1-\mathrm{e}^{\beta b})^{-1}; \\ & \rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\mathbb{1}_+ - \rho^{\frac{1}{2}}(1+\rho)^{\frac{1}{2}}\mathbb{1}_- \\ = \; & -\mathrm{e}^{\beta b/2}(1-\mathrm{e}^{\beta b})^{-1} = \mathrm{e}^{-\beta b/2}(1-\mathrm{e}^{-\beta b})^{-1}; \\ & -\rho\mathbb{1}_+ + (1+\rho)\mathbb{1}_- \\ = \; & -\mathrm{e}^{-\beta b}(1-\mathrm{e}^{-\beta b})^{-1}\mathbb{1}_+ + (1-\mathrm{e}^{\beta b})^{-1}\mathbb{1}_- = (1-\mathrm{e}^{\beta b})^{-1}. \end{aligned}$$

Therefore $C_{\rm d}^+ = C_\beta^+$. Since $C_{\rm d}^+ + C_{\rm d}^- = \mathbb{1}$, we have also $C_{\rm d}^- = C_\beta^-$.

To see that $\omega_{\rm d}$ is pure, we have to check that the representation of the Weyl algebra $\mathrm{CCR}^{\rm Weyl}(\mathcal{X}, Q)$ associated to $\Psi_{\rm AW}^{(*)}(x)$, $x \in \mathcal{X}$ is irreducible. This follows from the definition (5.11) of $\Psi_{\rm AW}^{(*)}$ and statements (5), (7) in [DG, Thm. 17.24]. □

6. Abstract Klein-Gordon equations

In this section we collect some results about abstract Klein-Gordon equations of the form

$$(\partial_t + \tilde{w}^*)(\partial_t - \tilde{w})\tilde{\phi} + \tilde{h}_0\tilde{\phi} = 0, \tag{6.1}$$

where $\tilde{\phi} : \mathbb{R} \to \tilde{\mathcal{H}}$, $\tilde{\mathcal{H}}$ is some Hilbert space and $\tilde{h}_0, \tilde{w}$ are linear operators on $\tilde{\mathcal{H}}$. Such Klein-Gordon equations arise from stationary metrics on a spacetime $M = \mathbb{R} \times S$,

with Killing vector field equal to $\frac{\partial}{\partial t}$, when $\tilde{w}$ represent the shift vector field and the lapse function is equal to 1. The case of general stationary Klein-Gordon operators will be considered later in Sect. 8.

We will also consider the Wick rotated operator $\tilde{K}_\beta$ obtained by setting $t = \mathrm{i}s$, where s belongs either to $\mathbb{R}$ or to $\mathbb{S}_\beta$. Using sesquilinear form techniques we give a rigorous meaning to its inverse $\tilde{K}_\beta^{-1}$ and relate it to the Green operators in Sect. 4.

6.1. **Hypotheses.** We will assume the following hypotheses:

$$\begin{aligned} &i) \quad \tilde{h}_0 \text{ is selfadjoint on } \tilde{\mathcal{H}} \text{ and } \tilde{h}_0 > 0, \\ &ii) \quad \tilde{w}|\tilde{h}_0|^{-\frac{1}{2}}, \tilde{w}^*|\tilde{h}_0|^{-\frac{1}{2}} \in B(\tilde{\mathcal{H}}), \\ &iii) \quad \text{if } \tilde{h} := \tilde{h}_0 - \tilde{w}^*\tilde{w} \text{ then } \tilde{h} \sim \tilde{h}_0. \end{aligned} \tag{6.2}$$

We can rewrite (6.1) as

$$\partial_t^2\tilde{\phi} - 2\mathrm{i}\tilde{k}\partial_t\tilde{\phi} + \tilde{h}\tilde{\phi} = 0, \tag{6.3}$$

where $\tilde{k} = (2\mathrm{i})^{-1}(\tilde{w} - \tilde{w}^*)$, which was considered in [GGH] in a more general situation.

6.2. **Quadratic pencils.** One associates to (6.3) the *quadratic pencil*

$$p(z) = z(2\tilde{k} - z) + \tilde{h} = (\mathrm{i}z + \tilde{w}^*)(\mathrm{i}z - \tilde{w}) + \tilde{h}_0 \in B(\langle\tilde{h}_0\rangle^{-\frac{1}{2}}\tilde{\mathcal{H}}, \langle\tilde{h}_0\rangle^{\frac{1}{2}}\tilde{\mathcal{H}}), z \in \mathbb{C},$$

obtained by replacing ∂_t by $\mathrm{i}z$, and denotes by $\rho(\tilde{h}, \tilde{k})$ the set of $z \in \mathbb{C}$ such that $p(z) : \langle\tilde{h}_0\rangle^{-\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} \langle\tilde{h}_0\rangle^{-\frac{1}{2}}\tilde{\mathcal{H}}$. Since $\tilde{h} > 0$ it follows from [GGH, Prop. 2.3] that $\{z : |\mathrm{Im}z| \geq |\mathrm{Re}z| + c_0\} \subset \rho(\tilde{h}, \tilde{k})$ for some $c_0 > 0$.

6.3. **First order system.** Setting

$$\tilde{f}(t) = \tilde{\varrho}_t\tilde{\phi} := \begin{pmatrix} \tilde{\phi}(t) \\ \mathrm{i}^{-1}(\partial_t - \tilde{w})\tilde{\phi}(t) \end{pmatrix} = \begin{pmatrix} \tilde{f}_0(t) \\ \tilde{f}_1(t) \end{pmatrix}, \tag{6.4}$$

(6.1) is formally rewritten as

$$\partial_t\tilde{f} = \mathrm{i}\tilde{H}\tilde{f}, \ \tilde{H} = \begin{pmatrix} -\mathrm{i}\tilde{w} & \mathbb{1} \\ \tilde{h}_0 & \mathrm{i}\tilde{w}^* \end{pmatrix}. \tag{6.5}$$

The conserved energy is

$$\overline{\tilde{f}}\cdot\tilde{E}\tilde{f} = \|\tilde{f}_1 - \mathrm{i}\tilde{w}\tilde{f}_0\|^2 + (\tilde{f}_0|\tilde{h}\tilde{f}_0), \tag{6.6}$$

which is positive definite by (6.2). The Hilbert space associated to $\tilde{E}$ will be denoted by $\tilde{\mathcal{E}}$. It equals $\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$ as a topological vector space. We set also $\tilde{\mathcal{E}}^* := \tilde{\mathcal{H}} \oplus \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}$.

The following proposition will be proved in Subsect. 6.5.

Proposition 6.1. *The operator* $\tilde{\mathrm{H}} = \begin{pmatrix} -\mathrm{i}\tilde{w} & \mathbb{1} \\ \tilde{h}_0 & \mathrm{i}\tilde{w}^* \end{pmatrix}$ *is bounded from* $\tilde{\mathcal{E}}$ *to* $\tilde{\mathcal{E}}^*$. *It induces on* $\tilde{\mathcal{E}}$ *the operator* $\tilde{H}$ *defined by*

$$\mathrm{Dom}\tilde{H} = \{\tilde{f} \in \tilde{\mathcal{E}} : \tilde{\mathrm{H}}\tilde{f} \in \tilde{\mathcal{E}} \cap \tilde{\mathcal{E}}^*\}.$$

$\tilde{H}$ *is a densely defined selfadjoint operator on* $\tilde{\mathcal{E}}$ *with* $\mathrm{res}(\tilde{H}) = \rho(\tilde{h}, \tilde{k})$.

Note that $(\tilde{\mathcal{E}}, \tilde{\mathcal{E}}^*)$ form a non degenerate dual pair for the charge

$$\overline{\tilde{f}}\cdot\tilde{q}\tilde{f}' = (\tilde{f}_1|\tilde{f}_0')_{\tilde{\mathcal{H}}} + (\tilde{f}_0|\tilde{f}_1')_{\tilde{\mathcal{H}}}, \ \tilde{f} \in \tilde{\mathcal{E}}, \ \tilde{f}' \in \tilde{\mathcal{E}}^*, \tag{6.7}$$

and one has

$$\overline{\tilde{f}}\cdot\tilde{E}\tilde{f} = \overline{\tilde{f}}\cdot\tilde{q}\tilde{\mathrm{H}}\tilde{f}, \tilde{f} \in \tilde{\mathcal{E}}.$$

6.4. **The Wick rotated operator.** Setting formally $t = \mathrm{i}s$ we obtain the formal expression

$$\tilde{K} = -(\partial_s + \mathrm{i}\tilde{w}^*)(\partial_s - \mathrm{i}\tilde{w}) + \tilde{h}_0. \tag{6.8}$$

To give a meaning to (6.8), we will use sesquilinear forms techniques. Let us set as in Sect. 4.1 for $0 < \beta \leq \infty$:

$$\tilde{\mathcal{H}}_\beta = L^2(\mathbb{S}_\beta) \otimes \tilde{\mathcal{H}}, \text{ for } \beta < \infty, \ \tilde{\mathcal{H}}_\infty = L^2(\mathbb{R}) \otimes \tilde{\mathcal{H}}. \tag{6.9}$$

We consider the sesquilinear form associated to $\tilde{K}$:

$$\tilde{Q}_\beta(u,u) = \|\partial_s u\|^2_{\tilde{\mathcal{H}}_\beta} + (u|\tilde{h}u)_{\tilde{\mathcal{H}}_\beta} - \mathrm{i}(\partial_s u|\tilde{w}u)_{\tilde{\mathcal{H}}_\beta} - \mathrm{i}(\tilde{w}u|\partial_s u)_{\tilde{\mathcal{H}}_\beta},$$

with domain $\mathrm{Dom}\tilde{Q}_\beta = \langle -\partial_s^2 + \tilde{h}_0\rangle^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta$, where ∂_s is equipped with its natural domain on $\tilde{\mathcal{H}}_\beta$. From hypotheses (6.2) we obtain that

$$\mathrm{Re}\tilde{Q}_\beta(u,u) \sim \|\partial_s u\|^2_{\tilde{\mathcal{H}}_\beta} + (u|\tilde{h}_0 u)_{\tilde{\mathcal{H}}_\beta}, \ |\mathrm{Im}\tilde{Q}_\beta(u,u)| \leq C\mathrm{Re}\tilde{Q}_\beta(u,u),$$

hence $\tilde{Q}_\beta$ is a closed sectorial form. By Lax-Milgram theorem $\tilde{Q}_\beta$ induces a boundedly invertible operator

$$\tilde{K}_\beta : (-\partial_s^2 + \tilde{h}_0)^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta \xrightarrow{\sim} (-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta. \tag{6.10}$$

We can apply the results of Subsect. 4.1 setting $\mathfrak{h} = \tilde{\mathcal{E}}$, $b = \tilde{H}$, see (4.10) for the notation used, and obtain an operator

$$\partial_s + \tilde{H} : \tilde{\mathcal{E}}_\beta \xrightarrow{\sim} (-\partial_s^2 + \tilde{H}^2)^{\frac{1}{2}}\tilde{\mathcal{E}}_\beta.$$

The relation between $\tilde{K}_\beta^{-1}$ and $\partial_s + \tilde{H}$ is given by the following proposition. Below we denote by π_i the maps $\pi_i \tilde{f} = \tilde{f}_i$ for $\tilde{f} = \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \end{pmatrix}$.

Proposition 6.2. *One has*

$$\tilde{K}_\beta^{-1} = \pi_0(\partial_s + \tilde{H})^{-1}\pi_1^*. \tag{6.11}$$

6.5. **Proofs of Props. 6.1 and 6.2.**

6.5.1. *Preparations.* We will prove Props. 6.1, 6.2 using results in [GGH]. There the form (6.3) of the Klein-Gordon equation is used and instead of (6.4) one sets:

$$g := \begin{pmatrix} \tilde{\phi} \\ \mathrm{i}^{-1}\partial_t\tilde{\phi} \end{pmatrix}, \tag{6.12}$$

(6.3) is formally rewritten as

$$\partial_t g = \mathrm{i}\hat{H}g, \ \hat{H} = \begin{pmatrix} 0 & \mathbb{1} \\ \tilde{h} & 2\tilde{k} \end{pmatrix}.$$

The conserved energy is

$$\overline{g}\cdot\hat{E}g = \|g_1\|^2 + (g_0|\tilde{h}g_0).$$

The Hilbert space $\hat{\mathcal{E}}$ naturally associated to $\hat{E}$ equals again $\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$.

If $\tilde{f}$ is given by (6.4) and g by (6.12) one has

$$\tilde{f} = Ug \text{ for } U = \begin{pmatrix} \mathbb{1} & 0 \\ \mathrm{i}\tilde{w} & \mathbb{1} \end{pmatrix},$$

and

$$U : \hat{\mathcal{E}} \xrightarrow{\sim} \tilde{\mathcal{E}}, \ (Ug|Ug)_{\tilde{\mathcal{E}}} = (g|g)_{\hat{\mathcal{E}}}. \tag{6.13}$$

Formally one has $\tilde{H} = U\hat{H}U^{-1}$, and since $U : \hat{\mathcal{E}} \to \tilde{\mathcal{E}}$ is unitary, Prop. 6.1 follows if we prove the analogous result for $\hat{H}$. One sets then

$$\hat{\mathcal{E}}^* := \tilde{\mathcal{H}} \oplus \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}},$$

which forms again a dual pair with $\hat{\mathcal{E}}$ for

$$\overline{g}\cdot\hat{q}g' = (g_1 + \mathrm{i}\tilde{w}g_0|g_0')_{\tilde{\mathcal{H}}} + (g_0|g_1' + \mathrm{i}\tilde{w}g_0'),\ g \in \hat{\mathcal{E}},, g' \in \hat{\mathcal{E}}^*.$$

We have of course $\hat{q} = U^* q U$.

6.5.2. *Proof of Prop. 6.1.* The matrix $\hat{H}$ induces a bounded operator $\hat{\mathrm{H}} : \hat{\mathcal{E}} \to \hat{\mathcal{E}}^*$. One denotes by $\hat{H}$ the linear operator induced by $\hat{\mathrm{H}}$ on $\hat{\mathcal{E}}$. Its domain is

$$\mathrm{Dom}\hat{H} = \{g \in \hat{\mathcal{E}} : \hat{\mathrm{H}}g \in \hat{\mathcal{E}} \cap \hat{\mathcal{E}}^*\}.$$

Note that although in general $\hat{\mathcal{E}}$ is not included in $\hat{\mathcal{E}}^*$, the intersection $\hat{\mathcal{E}} \cap \hat{\mathcal{E}}^*$ is well defined. In fact the intersections $\tilde{\mathcal{H}} \cap \tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}} \cap \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}$ are well defined, as follows easily from the spectral theorem. In concrete applications, where $\tilde{h}_0, \tilde{w}$ are differential operators on some manifold N, one can also consider these intersections inside $\mathcal{D}'(N)$.

It follows then from [GGH, Prop. 5.8, Thm. 5.9], and the fact that there exists $z \neq 0$ in $\rho(\tilde{h}, \tilde{k})$, that $\hat{H}$ with the domain above is a densely defined selfadjoint operator on $\hat{\mathcal{E}}$ with $\mathrm{res}(\hat{H}) = \rho(\tilde{h}, \tilde{k})$. Setting

$$\tilde{H} := U\hat{H}U^{-1}$$

completes the proof of Prop. 6.1.

6.5.3. *Proof of Prop. 6.2.* One can express the resolvent $(\hat{H} - z)^{-1}$ using $p(z)$ as follows: if $z \in \rho(\tilde{h}, \tilde{k})$ then:

$$(\hat{H} - z)^{-1} = p(z)^{-1} \begin{pmatrix} z - 2\tilde{k} & \mathbb{1} \\ \tilde{h} & z \end{pmatrix} \in B(\hat{\mathcal{E}}, \hat{\mathcal{E}}). \tag{6.14}$$

Note that (6.14) is different from the formula found in [GGH, Prop. 5.8], because weaker assumptions on $\tilde{h}, \tilde{k}$ were used there. In our case using that $\tilde{k}|\tilde{h}_0|^{-\frac{1}{2}} \in B(\tilde{\mathcal{H}})$ one deduces from [GGH, Lemma 2.2] that

$$p(z) : \tilde{\mathcal{H}} + |\tilde{h}_0|^{\frac{1}{2}}\tilde{\mathcal{H}} \to \langle\tilde{h}_0\rangle^{-\frac{1}{2}}\tilde{\mathcal{H}} \subset \tilde{\mathcal{H}} \cap |\tilde{h}_0|^{-\frac{1}{2}}\tilde{\mathcal{H}},\ z \in \rho(\tilde{h}, \tilde{k}).$$

Using this fact it is straightforward to show that the rhs in (6.14) maps $\hat{\mathcal{E}}$ into itself.

In general we have $0 \notin \rho(\tilde{h}, \tilde{k})$ hence $0 \in \sigma(\hat{H})$ but $\hat{H}^{-1}$ is well defined as

$$\hat{H}^{-1} = \begin{pmatrix} -2\tilde{h}^{-1}\tilde{k} & \tilde{h}^{-1} \\ \mathbb{1} & 0 \end{pmatrix} \in B(\hat{\mathcal{E}}, |\tilde{h}|^{-1}\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}) \tag{6.15}$$

which corresponds to (6.14) for $z = 0$.

We have $\mathrm{Ker}\hat{H} = \{0\}$ since $\hat{H}g = 0$ implies $g_1 = 0, \tilde{h}g_0 = 0$ and $\tilde{h}$ is injective. Therefore we can apply the results of Subsect. 4.1 to construct $(\partial_s + \hat{H})^{-1}$ for $b = \hat{H}$, $\mathfrak{h} = \hat{\mathcal{E}}$. As before we introduce the Hilbert spaces $\tilde{\mathcal{H}}_\beta$ and $\hat{\mathcal{E}}_\beta$ for $\beta \in]0, \infty]$.

Using Fourier transform in s either on $\mathbb{R}$ for $\beta = \infty$ or on $\mathbb{S}_\beta$ for $\beta < \infty$ we can express $(\partial_s + \hat{H})^{-1}$ using (6.14), replacing z by $-\partial_s$. We claim that for $\tilde{K}_\beta$ defined in Subsect. 6.4 we have

$$\tilde{K}_\beta^{-1} = \pi_0(\partial_s + \hat{H})^{-1}\pi_1^*, \tag{6.16}$$

which will prove Prop. 6.2 since $\pi_0 U^{-1} = \pi_0$ and $U\pi_1^* = \pi_1^*$.

Let us prove (6.16). We have:

$$(-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta = |\partial_s|\tilde{\mathcal{H}}_\beta + \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta,$$
$$(\partial_s^2 + \hat{H}^2)^{\frac{1}{2}}\tilde{\mathcal{E}}_\beta = |\partial_s|\tilde{\mathcal{E}}_\beta + |\hat{H}|\tilde{\mathcal{E}}_\beta.$$

If $v = \partial_s u \in |\partial_s|\tilde{\mathcal{H}}_\beta$ then $\pi_1^* v = \partial_s \pi_1^* u \in \partial_s \tilde{\mathcal{E}}_\beta$. Similarly if $v = \tilde{h}_0^{\frac{1}{2}} u \in |\tilde{h}_0|^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta$ then $\pi_1^* v = \hat{H}\begin{pmatrix} \tilde{h}^{-1}\tilde{h}_0^{\frac{1}{2}} u \\ 0 \end{pmatrix} \in \hat{H}\tilde{\mathcal{E}}_\beta$. In conclusion we have shown that

$$\pi_1^*(-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta \to (\partial_s^2 + \hat{H}^2)^{\frac{1}{2}}\tilde{\mathcal{E}}_\beta \text{ continuously.}$$

Next if $g \in \hat{\mathcal{E}}_\beta$ and $(\partial_s + \hat{H})g = \begin{pmatrix} 0 \\ v \end{pmatrix}$ for $v \in (-\partial_s^2 + \tilde{h}_0)^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta$ we have

$$\partial_s g_0 + g_1 = 0,\ \tilde{K} g_0 = v,$$

hence $\partial_s g_0 \in \tilde{\mathcal{H}}_\beta$, $\tilde{h}_0^{\frac{1}{2}} g_0 \in \tilde{\mathcal{H}}_\beta$ and $\tilde{K} g_0 = v$, which shows that $\pi_0(\partial_s + \hat{H})^{-1}\pi_1^* v = \tilde{K}_\beta^{-1} v$. This completes the proof of (6.16). □

7. Vacua and KMS states for abstract Klein-Gordon equations

In this section we consider vacuum and KMS states for abstract, time-independent Klein-Gordon equations, which can be reduced to the framework of Sect. 5. We will show that the covariances of the vacuum and double β-KMS states can be expressed by the Calderón projectors defined in Sect. 5.

7.1. **Vacua and KMS states.** Let us consider an abstract Klein-Gordon equation

$$(\partial_t + \tilde{w}^*)(\partial_t - \tilde{w})\tilde{\phi} + \tilde{h}_0\tilde{\phi} = 0,$$

as in Sect. 6, where $\tilde{\phi} : \mathbb{R} \to \tilde{\mathcal{H}}$ and $\tilde{\mathcal{H}}$ is a Hilbert space. We denote by

$$\tilde{P} = (\partial_t + \tilde{w}^*)(\partial_t - \tilde{w}) + \tilde{h}_0$$

the corresponding Klein-Gordon operator. In the sequel we use the notation introduced in Subsect. 5.1.

The assumptions corresponding to those in Subsect. 5.1 are as follows:

We assume that there exists a dense subspace $\tilde{\mathcal{D}} \subset \tilde{\mathcal{H}}$ and set

$$\tilde{\mathcal{Y}} := \tilde{\mathcal{D}} \oplus \tilde{\mathcal{D}},\ \overline{\tilde{f}}\cdot\tilde{q}\tilde{f} := (\tilde{f}_1|\tilde{f}_0) + (\tilde{f}_0|\tilde{f}_1),\ \tilde{f} = \begin{pmatrix} \tilde{f}_0 \\ \tilde{f}_1 \end{pmatrix} \in \tilde{\mathcal{Y}}. \tag{7.1}$$

We fix linear operators $\tilde{h}_0, \tilde{w}, \tilde{w}^*$ on $\tilde{\mathcal{H}}$ with domain $\tilde{\mathcal{D}}$ such that:

$$\begin{aligned} &(u|\tilde{h}_0 u) \geq 0,\ (\tilde{w}^* u|v) = (u|\tilde{w}v),\ u, v \in \tilde{\mathcal{D}}, \\ &\|\tilde{w}u\|^2 \leq (1-\delta)(u|\tilde{h}_0 u),\ \|\tilde{w}^* u\|^2 \leq c(u|\tilde{h}_0 u),\ u \in \tilde{\mathcal{D}} \text{ for } c > 0, 0 < \delta < 1. \end{aligned} \tag{7.2}$$

Setting $\tilde{q}_0(u,u) = (u|\tilde{h}_0 u)$ with $\mathrm{Dom}\tilde{q}_0 = \tilde{\mathcal{D}}$, it follows that $\tilde{q}_0$ is closeable and we still denote by $\tilde{h}_0$ the operator associated to $\tilde{q}_0^{\mathrm{cl}}$, ie the Friedrichs extension of $\tilde{h}_0$ on $\mathcal{D}$. We assume that $\mathrm{Ker}\tilde{h}_0 = \{0\}$ and deduce then from (7.2) that hypotheses (6.2) are satisfied by $\tilde{h}_0, \tilde{w}, \tilde{w}^*$. By construction $\tilde{\mathcal{D}}$ is dense in $\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}$.

We set then

$$\overline{\tilde{f}}\cdot\tilde{E}\tilde{f} = (\tilde{f}|\tilde{f})_{\tilde{\mathcal{E}}} = \|\tilde{f}_1 - \mathrm{i}\tilde{w}\tilde{f}_0\|^2 + (\tilde{f}_0|\tilde{h}\tilde{f}_0),\ \tilde{f} \in \tilde{\mathcal{Y}}, \tag{7.3}$$

and by the density of $\tilde{\mathcal{D}}$ in $\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}$ we obtain that $\tilde{\mathcal{Y}}_{\mathrm{en}} = \tilde{\mathcal{E}}$. Setting then

$$b = \tilde{H} = \begin{pmatrix} -\mathrm{i}\tilde{w} & \mathbb{1} \\ \tilde{h}_0 & \mathrm{i}\tilde{w}^* \end{pmatrix},$$

where $\tilde{H}$ is defined as a selfadjoint operator on $\tilde{\mathcal{E}}$ by Prop. 6.1, we see that the identity (5.1) follows from (7.1), (7.3).

We can then apply Subsects. 5.2, 5.3, 5.4 and define the vacuum state $\tilde{\omega}_{\mathrm{vac}}$, the β-KMS state $\tilde{\omega}_\beta$ and the double β-KMS state $\tilde{\omega}_{\mathrm{d}}$ associated to the symplectic dynamics $r_t = \mathrm{e}^{\mathrm{i}t\tilde{H}}$.

7.2. **The Calderón projectors.** In Subsect. 6.4 we defined the Wick rotated operators

$$\tilde{K}_\beta = -(\partial_s + \mathrm{i}\tilde{w}^*)(\partial_s - \mathrm{i}\tilde{w}) + \tilde{h}_0$$

and the Hilbert spaces $\tilde{\mathcal{H}}_\beta$ for $0 < \beta \leq \infty$ defined in (6.9). We now define Calderón projectors for $\tilde{K}_\beta$, which are similar to the Calderón projectors for the operators $B_\beta = \partial_s + \tilde{H}$, acting on the Hilbert spaces $\tilde{\mathcal{E}}_\beta$, defined in 4.1.2 and 4.1.3.

7.2.1. *Calderón projectors for $\tilde{K}_\infty$.* We follow the construction and notation in 4.1.2, in particular $I^\pm_\infty = \pm]0, +\infty[$, and $i^\pm_\infty$ is the extension by 0 in $\mathbb{R} \setminus I^\pm_\infty$. If $\tilde{u} \in C^0(\mathbb{R}; \tilde{\mathcal{H}})$ we denote by $\tilde{\gamma}_\infty \tilde{u}$ the trace of $\tilde{u}$ at $s = 0$:

$$\tilde{\gamma}_\infty \tilde{u} = \begin{pmatrix} u(0) \\ -(\partial_s - \mathrm{i}\tilde{w})\tilde{u}(0) \end{pmatrix},$$

whose formal adjoint $\tilde{\gamma}^*_\infty$ is given by:

$$\tilde{\gamma}^*_\infty \tilde{g} = \delta'_0(s) \otimes \tilde{g}_1 + \delta_0(s) \otimes (\tilde{g}_0 - \mathrm{i}\tilde{w}^* \tilde{g}_1).$$

If $\tilde{u} \in \overline{C^0}(I^\pm_\infty; \tilde{\mathcal{H}})$ satisfies $\tilde{K}\tilde{u} = 0$ in $I^\pm_\infty$ we set

$$\tilde{\gamma}^\pm_\infty \tilde{u} = \begin{pmatrix} \tilde{u}(0^\pm) \\ -(\partial_s - \mathrm{i}\tilde{w})\tilde{u}(0^\pm) \end{pmatrix} = \begin{pmatrix} \tilde{g}_0 \\ \tilde{g}_1 \end{pmatrix}.$$

We have formally

$$\begin{aligned} \tilde{K}_\infty i^\pm_\infty \tilde{u}(s) = &\ \pm(-\delta'_0(s) \otimes \tilde{g}_0 + \delta_0(s) \otimes (\tilde{g}_1 - \mathrm{i}\tilde{w}^* \tilde{g}_0)) \\ = &\ \mp \tilde{\gamma}^*_\infty \tilde{S} \tilde{g}, \end{aligned}$$

for

$$\tilde{S} = \begin{pmatrix} 2\mathrm{i}\tilde{w}^* & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

It follows that $i^\pm_\infty \tilde{u} = \mp \tilde{K}^{-1}_\infty \tilde{\gamma}^*_\infty \tilde{S} \tilde{g}$ for $\tilde{g} = \tilde{\gamma}^\pm_\infty \tilde{u}$. This implies formally that

$$\tilde{g} = \mp \tilde{\gamma}^\pm_\infty \tilde{K}^{-1}_\infty \tilde{\gamma}^*_\infty \tilde{S} \tilde{g}$$

if $\tilde{g} = \tilde{\gamma}^\pm_\infty \tilde{u}$ for $\tilde{u}$ solving $\tilde{K}\tilde{u} = 0$ in $I^\pm_\infty$. This leads to the following definition.

Definition 7.1. *The* Calderón projectors $\tilde{c}^\pm_\infty$ *are*

$$\tilde{c}^\pm_\infty = \mp \tilde{\gamma}^\pm_\infty \tilde{K}^{-1}_\infty \tilde{\gamma}^*_\infty \tilde{S}. \tag{7.4}$$

Proposition 7.2. *We have:*

$$\tilde{c}^\pm_\infty = C^\pm_\infty,$$

where $C^\pm_\infty$ are the Calderón projectors for $B_\infty = \partial_s + b$, with $b = \tilde{H}$, $\mathfrak{h} = \tilde{\mathcal{E}}$, defined in Def. 4.1.

Proof. We prove only the + case. Let $F \in \overline{C^0}(I^+_\infty; \tilde{\mathcal{E}})$ with $(\partial_s + \tilde{H})F(s) = 0$ in I^+_∞. If $\tilde{u}(s) = F_0(s)$ we have $\tilde{K}\tilde{u}(s) = 0$ and $F_1(s) = -(\partial_s - \mathrm{i}\tilde{w})\tilde{u}(s)$ in I^+_∞ which implies that $\Gamma^+_\infty F = \tilde{\gamma}^+_\infty \tilde{u}$. We have $(\partial_s + \tilde{H}) i^+_\infty F(s) = \delta_0(s) \otimes \tilde{g}$ for $\tilde{g} = \Gamma^+_\infty F$ which implies that

$$\tilde{K} i^+_\infty F_0(s) = -\delta'_0(s) \otimes \tilde{g}_0 + \delta_0(s) \otimes (\tilde{g}_1 - \mathrm{i}\tilde{w}^* \tilde{g}_0).$$

This implies using the relation between $\tilde{K}^{-1}_\infty$ and $(\partial_s + \tilde{H})^{-1}$ in Prop. 6.2 that:

$$\begin{aligned} C^+_\infty \tilde{g} = &\ \Gamma^+_\infty (\partial_s + \tilde{H})^{-1} (\delta_0 \otimes \tilde{g}) \\ = &\ \tilde{\gamma}^+_\infty \tilde{K}^{-1} (-\delta'_0(s) \otimes \tilde{g}_0 + \delta_0(s) \otimes (\tilde{g}_1 - \mathrm{i}\tilde{w}^* \tilde{g}_0)) = \tilde{c}^+_\infty \tilde{g}. \ \square \end{aligned}$$

From Subsect. 5.2 we obtain the following result, expressing the covariances of the vacuum state $\tilde{\omega}_{\mathrm{vac}}$ for P in terms of the Calderón projectors $\tilde{c}^\pm_\infty$ for the Wick rotated operator $\tilde{K}_\infty$.

Proposition 7.3. *The covariances of the vacuum state $\omega_{\rm vac}$ are equal to:*

$$\tilde\lambda^{\pm}_{\rm vac} = \pm\tilde q \circ \tilde c^{\pm}_{\infty}.$$

7.2.2. *Calderón projectors for $\tilde K_\beta$.* We follows now the construction and notation in 4.1.3, in particular $I^{\pm}_\beta = \pm]0, \frac{\beta}{2}[$ and $i^{\pm}_\beta$ is the extension by 0 in $\mathbb{S}_\beta \setminus I^{\pm}_\beta$. If $\tilde u \in \overline{C^0}(\mathbb{S}_\beta; \tilde{\mathcal{H}})$ we denote by $\tilde\gamma_\beta\tilde u$ the vector obtained from its traces at $s = 0$ and $s = \frac{\beta}{2}$:

$$\tilde\gamma_\beta\tilde u = \tilde\gamma^{(0)}_\beta\tilde u \oplus \tilde\gamma^{(\frac{\beta}{2})}_\beta\tilde u,$$

for

$$\tilde\gamma^{(0)}_\beta\tilde u = \begin{pmatrix} \tilde u(0) \\ -(\partial_s - \mathrm{i}\tilde w)\tilde u(0)\end{pmatrix},\ \tilde\gamma^{(\frac{\beta}{2})}_\beta\tilde u = \begin{pmatrix} \tilde u(\frac{\beta}{2}) \\ (\partial_s - \mathrm{i}\tilde w)\tilde u(\frac{\beta}{2})\end{pmatrix}.$$

Note the change of sign in the second component of $\tilde\gamma^{(\frac{\beta}{2})}\tilde u$, which corresponds to choosing the *exterior* normal derivative to I^+_β. We have

$$\tilde\gamma^*_\beta = \tilde\gamma^{(0)*}_\beta + \tilde\gamma^{(\frac{\beta}{2})*}_\beta,$$

for

$$\begin{aligned}\tilde\gamma^{(0)*}_\beta f^{(0)} &= \delta_0'(s)\otimes f^{(0)}_1 + \delta_0(s)\otimes(f^{(0)}_0 - \mathrm{i}\tilde w^* f^{(0)}_1),\\ \tilde\gamma^{(\frac{\beta}{2})*}_\beta f^{(\frac{\beta}{2})} &= -\delta'_{\frac{\beta}{2}}(s)\otimes f^{(\frac{\beta}{2})}_1 + \delta_{\frac{\beta}{2}}(s)\otimes(f^{(\frac{\beta}{2})}_0 + \mathrm{i}\tilde w^* f^{(\frac{\beta}{2})}_1).\end{aligned}$$

If $\tilde u \in \overline{C^0}(I^{\pm}_\beta; \tilde{\mathcal{H}})$ satisfies $\tilde K\tilde u = 0$ in $I^{\pm}_\beta$ we set:

$$\tilde\gamma^{\pm}_\beta\tilde u = \tilde\gamma^{(0)\pm}_\beta\tilde u \oplus \tilde\gamma^{(\frac{\beta}{2})\pm}_\beta\tilde u,$$

for

$$\tilde\gamma^{(0)\pm}_\beta\tilde u = \begin{pmatrix} \tilde u(0^{\pm}) \\ -(\partial_s - \mathrm{i}\tilde w)\tilde u(0^{\pm})\end{pmatrix},\ \tilde\gamma^{(\frac{\beta}{2})\pm}_\beta\tilde u = \begin{pmatrix} \tilde u(\mp\frac{\beta}{2}) \\ (\partial_s - \mathrm{i}\tilde w)\tilde u(\mp\frac{\beta}{2})\end{pmatrix}.$$

The same computation as before shows that if

$$\tilde\gamma^{(s)}_\beta\tilde u = \tilde g^{(s)} = \begin{pmatrix}\tilde g^{(s)}_0 \\ \tilde g^{(s)}_1\end{pmatrix},\ \text{for } s = 0, \frac{\beta}{2},$$

one has:

$$\begin{aligned}\tilde K i^{\pm}_\beta\tilde u = &\ \pm\Big(-\delta_0'(s)\otimes\tilde g^{(0)}_0 + \delta_0(s)\otimes(\tilde g^{(0)}_1 - \mathrm{i}\tilde w^*\tilde g^{(0)}_0)\Big)\\ &\pm\Big(\delta'_{\frac{\beta}{2}}(s)\otimes\tilde g^{(\frac{\beta}{2})}_0 + \delta_{\frac{\beta}{2}}(s)\otimes(\tilde g^{(\frac{\beta}{2})}_1 + \mathrm{i}\tilde w^*\tilde g^{(\frac{\beta}{2})}_0)\Big)\\ = &\ \mp\Big(\tilde\gamma^{(0)*}_\beta\tilde S^{(0)}\tilde g^{(0)} + \tilde\gamma^{(\frac{\beta}{2})*}_\beta\tilde S^{(\frac{\beta}{2})}\tilde g^{(\frac{\beta}{2})}\Big),\end{aligned} \tag{7.5}$$

for

$$\tilde S^{(0)} = \begin{pmatrix} 2\mathrm{i}\tilde w^* & -\mathbb{1}\\ \mathbb{1} & 0\end{pmatrix},\ \tilde S^{(\frac{\beta}{2})} = \begin{pmatrix} -2\mathrm{i}\tilde w^* & -\mathbb{1}\\ \mathbb{1} & 0\end{pmatrix}. \tag{7.6}$$

Again this leads to the following definition.

Definition 7.4. *The* Calderón projectors $\tilde c^{\pm}_\beta$ *are*

$$\tilde c^{\pm}_\beta = \mp\tilde\gamma^{\pm}_\beta\tilde K^{-1}_\beta\tilde\gamma^*_\beta(\tilde S^{(0)}\pi^{(0)} + \tilde S^{(\frac{\beta}{2})}\pi^{(\frac{\beta}{2})}),$$

where $\pi^{(0/\frac{\beta}{2})}\tilde g = \tilde g^{(0/\frac{\beta}{2})}$.

Proposition 7.5. *On has*

$$\tilde{c}^+_\beta = (\mathbb{1} \oplus T) \circ C^+_\beta \circ (\mathbb{1} \oplus T)^{-1}, \text{ on } \tilde{\mathcal{E}} \oplus \tilde{\mathcal{E}},$$

where $C^\pm_\beta$ are the Calderón projectors for $B_\beta = \partial_s + b$, with $b = \tilde{H}$, $\mathfrak{h} = \tilde{\mathcal{E}}$, defined in Def. 4.3, and $T = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$.

Proof. We prove only the + case. Let $F \in \overline{C^0}(I^+_\beta; \tilde{\mathcal{E}})$ with $(\partial_s + \tilde{H})F(s) = 0$ in I^+_β. If $\tilde{u}(s) = F_0(s)$ we have $\tilde{K}\tilde{u}(s) = 0$ and $F_1(s) = -(\partial_s - \mathrm{i}\tilde{w})\tilde{u}(s)$ in I^+_β. This implies that

$$\tilde{\gamma}^{(0)+}_\beta \tilde{u} = \Gamma^{(0)+}_\beta F, \ \tilde{\gamma}^{(\frac{\beta}{2})+}_\beta \tilde{u} = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix} \Gamma^{(\frac{\beta}{2})+}_\beta F,$$

where $\Gamma^{(0)+}_\beta, \Gamma^{(\frac{\beta}{2})+}_\beta$ are defined in (4.17). Setting

$$\tilde{g} = \tilde{\gamma}^+_\beta \tilde{u}, \ f = \Gamma^+_\beta F,$$

we can rewrite this identity as

$$\tilde{g} = (\mathbb{1} \oplus T) f, \text{ for } T = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}. \tag{7.7}$$

Next we have

$$(\partial_s + \tilde{H}) i^+_\beta F = \delta_0(s) \otimes f^{(0)} - \delta_{\frac{\beta}{2}}(s) \otimes f^{(\frac{\beta}{2})}$$

where $f = f^{(0)} \oplus f^{(\frac{\beta}{2})}$, which implies that:

$$\begin{aligned} \tilde{K} i^+_\beta \tilde{u} = \ & (-\delta'_0(s) \otimes f^{(0)}_0 + \delta_0(s) \otimes (f^{(0)}_1 - \mathrm{i}\tilde{w}^* f^{(0)}_0)) \\ & + (\delta'_{\frac{\beta}{2}}(s) \otimes f^{(\frac{\beta}{2})}_0 + \delta_{\frac{\beta}{2}}(s) \otimes (-f^{(\frac{\beta}{2})}_1 + \mathrm{i}\tilde{w}^* f^{(\frac{\beta}{2})}_0)). \end{aligned}$$

If we compare this with the first line in (7.5) and use also the relation between $\tilde{K}^{-1}_\beta$ and $(\partial_s + \tilde{H})^{-1}$ in Prop. 6.2 we obtain that $\tilde{c}^+_\beta = (\mathbb{1} \oplus T) \circ C^+_\beta \circ (\mathbb{1} \oplus T)^{-1}$ as claimed. □

As in Prop. 7.3 we can using Subsect. 5.4 express the covariances of the double β-KMS state $\tilde{\omega}_{\mathrm{d}}$ for $\tilde{P}$ in terms of the Calderón projectors $\tilde{c}^\pm_\beta$ for the Wick rotated operator $\tilde{K}_\beta$.

Proposition 7.6. *The covariances of the double β-KMS state for $\tilde{P}$ are equal to*

$$\tilde{\lambda}^\pm_{\mathrm{d}} = \pm \tilde{Q} \circ (\mathbb{1} \oplus T)^{-1} \tilde{c}^\pm_\beta (\mathbb{1} \oplus T), \text{ for } \tilde{Q} = \tilde{q} \oplus -\tilde{q}.$$

8. Klein-Gordon equations on stationary spacetimes

In this section we consider Klein-Gordon equations on stationary spacetimes. If the lapse function N associated to the Killing vector field $\mathbf{w}$ is equal to 1, one can directly reduce oneself to the situation of Sect. 7. In general one has to replace the Klein-Gordon operator P by $\tilde{P} = NPN$, which has the same purpose as a conformal transformation.

As an application we consider the Klein-Gordon operator P in $\mathcal{M}^+$ and express the covariances of the double β-KMS state in $\mathcal{M}^- \cup \mathcal{M}^+$ using the Calderón projectors for the elliptic operator K_β obtained from P by Wick rotation in the Killing time coordinate t.

8.1. **Klein-Gordon equations on stationary spacetimes.**

8.1.1. *Stationary metrics.* Let $(S, \mathbf{h})$ a Riemannian manifold, $N \in C^\infty(S)$, $N > 0$ and $\mathbf{w}^i$ a vector field on S. Let us denote by y the elements of S. We define the Lorentzian metric $\mathbf{g}$ on $M = \mathbb{R} \times S$:

$$\mathbf{g} = \quad -N^2(y)dt^2 + \mathbf{h}_{ij}(y)(dy^i + \mathbf{w}^i(y)dt)(dy^j + \mathbf{w}^j(y)dt).$$

We assume that $\{0\} \times S$ is a Cauchy surface for $(M, \mathbf{g})$. Such spacetimes are called *standard stationary spacetimes* in the terminology of [S2].

The vector field $\dfrac{\partial}{\partial t}$ is Killing for $\mathbf{g}$ and is time-like iff

$$N^2(y) > \mathbf{w}^i(y)\mathbf{h}_{ij}(y)\mathbf{w}^j(y),\ y \in S. \tag{8.1}$$

We will need later to impose the following stronger condition:

Definition 8.1. *The Killing vector field* $\dfrac{\partial}{\partial t}$ *is* uniformly time-like *if there exists* $0 < \delta < 1$ *such that:*

$$(1-\delta)N^2(y) \geq \mathbf{w}^i(y)\mathbf{h}_{ij}(y)\mathbf{w}^j(y),\ x \in S.$$

We have:

$$|\mathbf{g}| = N^2|\mathbf{h}|, n = N^{-1}(\frac{\partial}{\partial t} - w), \tag{8.2}$$

where n is the future directed unit normal to the foliation $S_t = \{t\} \times S$.

8.1.2. *Stationary Klein-Gordon operators.* We consider a stationary Klein-Gordon operator on $(M, \mathbf{g})$:

$$P = -\Box_{\mathbf{g}} + m(y),\ m \in C^\infty(S; \mathbb{R}). \tag{8.3}$$

We will always assume that

$$m(y) \geq m_0^2,\ m_0 > 0, \tag{8.4}$$

ie that the Klein-Gordon equation is *massive*. Setting

$$h_0 := \nabla^* \mathbf{h}^{-1} \nabla + m,\ w := \mathbf{w}^i \cdot \partial_{y^i}, \tag{8.5}$$

we have

$$P = (\partial_t + w^*)N^{-2}(\partial_t - w) + h_0, \tag{8.6}$$

where in (8.5), (8.6) the adjoints are computed with respect to the scalar product

$$(u|v)_M = \int_M \overline{u} v N |\mathbf{h}|^{\frac{1}{2}} dt dy.$$

8.1.3. *Hilbert spaces.* We denote by $L^2(M)$ the Hilbert space associated to the scalar product $(\cdot|\cdot)_M$ and by $\mathcal{H} = L^2(S, |\mathbf{h}|^{\frac{1}{2}}dy)$ the Hilbert space associated to the scalar product

$$(u|v)_{\mathcal{H}} = \int_S \overline{u} v |\mathbf{h}|^{\frac{1}{2}} dy.$$

We will also need the Hilbert space $\tilde{\mathcal{H}} = L^2(S, N|\mathbf{h}|^{\frac{1}{2}}dy)$ associated to the scalar product

$$(u|v)_{\tilde{\mathcal{H}}} = \int_S \overline{u} v N |\mathbf{h}|^{\frac{1}{2}} dy,$$

so that $L^2(M) = L^2(\mathbb{R}, dt; \tilde{\mathcal{H}})$.

8.1.4. *An operator inequality.* The inequality in Lemma 8.2 below is understood as an operator inequality on $\tilde{\mathcal{H}}$.

Lemma 8.2. *Assume that* $\dfrac{\partial}{\partial t}$ *is uniformly time-like. Then*

$$(1-\delta)h_0 \geq w^* N^{-2} w \text{ on } C_0^\infty(S).$$

Proof. Let $\mathcal{X}$ a real vector space, $\mathbf{k} \in L_{\rm s}(\mathcal{X}, \mathcal{X}^\#)$ strictly positive and $c \in \mathcal{X}$. Then for $\gamma = \mathbf{k}c \in \mathcal{X}^\#$ and $\xi \in \mathbb{C}\mathcal{X}^\#$ we have

$$\begin{aligned} & (\overline{\xi} - \langle\overline{\xi}|c\rangle\gamma)\cdot\mathbf{k}^{-1}(\xi - \langle\xi|c\rangle\gamma) \\ = \;& \overline{\xi}\cdot\mathbf{k}^{-1}\xi - 2\mathrm{Re}(\langle\overline{\xi}|c\rangle\gamma\cdot\mathbf{k}^{-1}\xi) + |\langle\xi|c\rangle|^2\gamma\cdot\mathbf{k}^{-1}\gamma \\ = \;& \overline{\xi}\cdot\mathbf{k}^{-1}\xi - (2 - c\cdot\mathbf{k}c)|\langle\xi|c\rangle|^2, \end{aligned}$$

hence

$$\mathbf{k}^{-1} - |c\rangle\langle c| \geq (1 - c\cdot\mathbf{k}c)|c\rangle\langle c|.$$

Replacing $\mathbf{k}$ by $(1-\delta)^{-1}\mathbf{k}$ shows that if $(1-\delta) \geq c\cdot\mathbf{k}c$ we have

$$(1-\delta)\mathbf{k}^{-1} \geq |c\rangle\langle c|. \tag{8.7}$$

For $u \in C_0^\infty(S)$ we write

$$\begin{aligned} & (u|((1-\delta)h_0 - w^*N^{-2}w)u)_{\tilde{\mathcal{H}}} \\ = \;& \int_S \partial_{y^i}\overline{u}((1-\delta)\mathbf{h}^{ij}(y) - \mathbf{w}^i(y)N^{-2}\mathbf{w}^j(y))\partial_{y^j}u(y)N|\mathbf{h}|^{\frac{1}{2}}dy. \end{aligned}$$

Applying (8.7) under the integral sign for $\mathbf{k} = \mathbf{h}_{ij}(y)$, $c = N^{-1}(y)\mathbf{w}^i(y)$ we obtain the lemma. □

8.2. **Selfadjoint operators.** In the rest of this section we will assume that $\frac{\partial}{\partial t}$ is uniformly time-like.

Let $q_0(u,u) = (u|h_0u)_{\tilde{\mathcal{H}}}$ with $\mathrm{Dom}q_0 = C_0^\infty(S)$. The form q_0 is closeable and we denote still denote by h_0 the selfadjoint operator on $\tilde{\mathcal{H}}$ associated to $q_0^{\rm cl}$, ie the Friedrichs extension of h_0 on $C_0^\infty(S)$. We have:

$$h_0 : h_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} h_0^{\frac{1}{2}}\tilde{\mathcal{H}}.$$

Note that $h_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \subset \tilde{\mathcal{H}}$ since $h_0 \geq m_0^2$. We set also

$$\tilde{q}_0(u,u) = q_0(Nu, Nu)_{\tilde{\mathcal{H}}},\ \mathrm{Dom}\tilde{q}_0 = C_0^\infty(S), \tag{8.8}$$

and denote by $\tilde{h}_0$ the selfadjoint operator on $\tilde{\mathcal{H}}$ associated to $\tilde{q}_0$, which formally equals Nh_0N. From (8.8) we obtain that

$$N : \tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} h_0^{-\frac{1}{2}}\tilde{\mathcal{H}},\ N : h_0^{\frac{1}{2}}\tilde{\mathcal{H}} \xrightarrow{\sim} \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}, \tag{8.9}$$

and we have:

$$\tilde{h}_0 = Nh_0N \text{ as an identity in } B(\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}, \tilde{h}_0^{\frac{1}{2}}\tilde{\mathcal{H}}). \tag{8.10}$$

We also set

$$\begin{aligned} & \tilde{w} = N^{-1}wN = N^{-1}\mathbf{w}^i\cdot\partial_{y^i}N, \\ & \tilde{w}^* = Nw^*N^{-1} = -|\mathbf{h}|^{-\frac{1}{2}}\partial_{y^i}\cdot\mathbf{w}^i|\mathbf{h}|^{\frac{1}{2}}, \end{aligned} \tag{8.11}$$

with domain $C_0^\infty(S)$.

Let us introduce the assumption

$$N^{-2}\mathbf{w}^i\cdot(\nabla_i^{\mathbf{h}}N),\ N^{-1}\nabla_i^{\mathbf{h}}\mathbf{w}^i \text{ are bounded on } S. \tag{8.12}$$

Lemma 8.3. *Assume* (8.12). *Then* $\tilde{h}_0, \tilde{w}, \tilde{w}^*$ *satisfy the conditions* (7.2) *for* $\tilde{\mathcal{D}} = C_0^\infty(S)$.

Proof. We have seen in Lemma 8.2 that $w^*N^{-2}w \leq (1-\delta)h_0$ on $C_0^\infty(S)$, which implies $\tilde{w}^*\tilde{w} \leq (1-\delta)\tilde{h}_0$ on $C_0^\infty(S)$. Let $(y^1,\ldots,y^d)$ be local coordinates on S. We have

$$\tilde{w} = \mathbf{w}^i\cdot\partial_{y^i} + N^{-1}\mathbf{w}^i\cdot(\partial_{y^i}N),$$
$$\tilde{w}^* = -\mathbf{w}^i\cdot\partial_{y^i} - |\mathbf{h}|^{-\frac{1}{2}}(\partial_{y^i}\mathbf{w}^i|\mathbf{h}|^{\frac{1}{2}}) = -\mathbf{w}^i\cdot\partial_{y^i} - \nabla_i^{\mathbf{h}}\mathbf{w}^i.$$

Condition (8.12) implies that $\tilde{w}^* = -\tilde{w} + r$, where $r \in C^\infty(S,\, rN^{-1}$ is bounded on $\tilde{\mathcal{H}}$. The inequality $\tilde{w}\tilde{w}^* \leq C\tilde{h}_0$ follows from $\tilde{w}^*\tilde{w} \leq (1-\delta)\tilde{h}_0$ and $m_0^2N^2 \leq \tilde{h}_0$. □

8.3. **Associated first order system.** We set:

$$\varrho_t\phi = \begin{pmatrix} \phi(t) \\ \mathrm{i}^{-1}N^{-1}(\partial_t - w)\phi(t) \end{pmatrix} = f = \begin{pmatrix} f_0 \\ f_1 \end{pmatrix}, \tag{8.13}$$

and rewrite $P\phi = 0$ as:

$$N^{-1}\partial_t f = \mathrm{i}Hf, H = \begin{pmatrix} -\mathrm{i}N^{-1}w & \mathbb{1} \\ h_0 & \mathrm{i}w^*N^{-1} \end{pmatrix},\ f \in C_0^\infty(S;\mathbb{C}^2). \tag{8.14}$$

The conserved energy is

$$\overline{f}\cdot Ef = \|f_1 - \mathrm{i}N^{-1}wf_0\|_{\tilde{\mathcal{H}}}^2 + (f_0|hf_0)_{\tilde{\mathcal{H}}},\ h = h_0 - w^*N^{-2}w, \tag{8.15}$$

and the conserved charge is

$$\overline{f}\cdot qf = (f_1|f_0)_{\mathcal{H}} + (f_0|f_1)_{\mathcal{H}}.$$

The energy space $\mathcal{E}$ associated to E equals $h_0^{\frac{1}{2}}\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$ as topological vector spaces.

8.4. **Reduction.** We now introduce the Klein-Gordon operator

$$\tilde{P} = NPN = (\partial_t + \tilde{w}^*)(\partial_t - \tilde{w}) + \tilde{h}_0,$$

which is of the form considered in Sects. 6, 7. The operators $\tilde{\varrho}_t, \tilde{H}$, the energy $\tilde{E}$ and charge $\tilde{q}$ are defined as in Subsect. 6.3:

$$\tilde{\varrho}_t\tilde{\phi} = \begin{pmatrix} \tilde{\phi}(t) \\ \mathrm{i}^{-1}(\partial_t - \tilde{w})\tilde{\phi}(t) \end{pmatrix},\ \tilde{H} = \begin{pmatrix} -\mathrm{i}\tilde{w} & \mathbb{1} \\ \tilde{h}_0 & \mathrm{i}\tilde{w}^* \end{pmatrix},$$

$$\overline{\tilde{f}}\cdot\tilde{E}\tilde{f} = \|\tilde{f}_1 - \mathrm{i}\tilde{w}\tilde{f}_0\|_{\tilde{\mathcal{H}}}^2 + (\tilde{f}_0|\tilde{h}\tilde{f}_0)_{\tilde{\mathcal{H}}},$$
$$\overline{\tilde{f}}\cdot\tilde{q}\tilde{f} = (\tilde{f}_1|\tilde{f}_0)_{\tilde{\mathcal{H}}} + (\tilde{f}_0|\tilde{f}_1)_{\tilde{\mathcal{H}}},\ \tilde{f} \in C_0^\infty(S;\mathbb{C}^2).$$

Setting

$$Z =: \begin{pmatrix} N & 0 \\ 0 & \mathbb{1} \end{pmatrix},\ Z' := \begin{pmatrix} \mathbb{1} & 0 \\ 0 & N^{-1} \end{pmatrix}, \tag{8.16}$$

we have:

$$\varrho_t N = Z\tilde{\varrho}_t,\ N^{-1}\partial_t - \mathrm{i}H = Z'(\partial_t - \mathrm{i}\tilde{H})Z^{-1}, \tag{8.17}$$

$$Z^*EZ = \tilde{E},\ Z^*qZ = \tilde{q} \text{ on } C_0^\infty(S;\mathbb{C}^2). \tag{8.18}$$

We saw that the energy space $\tilde{\mathcal{E}}$ associated to $\tilde{E}$ equals $\tilde{h}_0^{-\frac{1}{2}}\tilde{\mathcal{H}} \oplus \tilde{\mathcal{H}}$, and from (8.9) we obtain that:

$$Z : \tilde{\mathcal{E}} \xrightarrow{\sim} \mathcal{E}.$$

8.5. **Vacuum and KMS states.** In Subsect. 7.1 we defined vacuum and $\beta-$KMS states for $\tilde{P}$. We obtain the corresponding vacuum and $\beta-$KMS states for P by conjugation by the map Z.

Definition 8.4. *We define the* vacuum state $\omega_{\rm vac}$*, the* β-KMS state ω_β *and the* double β-KMS state $\omega_{\rm d}$ *by their covariances:*

$$\lambda^{\pm}_{\rm vac} = (Z^{-1})^* \tilde{\lambda}^{\pm}_{\rm vac} Z^{-1},\ \lambda^{\pm}_{\beta} = (Z^{-1})^* \tilde{\lambda}^{\pm}_{\beta} Z^{-1},$$
$$\lambda^{\pm}_{\rm d} = (Z^{-1} \oplus Z^{-1})^* \tilde{\lambda}^{\pm}_{\rm d} (Z^{-1} \oplus Z^{-1}),$$

where the covariances $\tilde{\lambda}^{\pm}_{\rm vac}$*,* $\tilde{\lambda}^{\pm}_{\beta}$ *and* $\tilde{\lambda}^{\pm}_{\rm d}$ *are defined in Defs. 5.1, 5.2 and Prop. 5.4 for* $b = \tilde{H}$*.*

8.6. **The Wick rotated operator.**

8.6.1. *The Wick rotated metric.* Let us denote by $\mathbf{k}$ the complex metric on $\mathbb{R} \times S$ obtained from $\mathbf{g}$ by the substitution $t = {\rm i}s$. We have:

$$\mathbf{k} = N^2(y)ds^2 + \mathbf{h}_{ij}(y)(dy^i + {\rm i}\mathbf{w}^i(y)ds)(dy^j + {\rm i}\mathbf{w}^j(y)ds),$$

Using that $\dfrac{\partial}{\partial t}$ is uniformly time-like we obtain that there exists $C > 0$ such that

$$|\overline{\eta} \cdot {\rm Im}\mathbf{k}(y)\eta| \leq C\overline{\eta} \cdot {\rm Re}\eta(y)\eta,\ y \in S, \eta \in \mathbb{C}T_yS, \tag{8.19}$$

Moreover we have

$$|\mathbf{k}|(y) \text{ is real valued and } |\mathbf{k}|^{\frac{1}{2}}(y) = N(y)|\mathbf{h}|^{\frac{1}{2}}(y). \tag{8.20}$$

With the terminology in Def. 9.1 this means that the complex metric $\mathbf{k}$ is *uniformly sectorial*.

If $\Omega =]0, +\infty[\times S$, the outer unit normal vector field to Ω for $\mathbf{k}$, see 9.1.2, is

$$\nu = -N^{-1}(\frac{\partial}{\partial s} - {\rm i}\mathbf{w}), \tag{8.21}$$

while if $\Omega =]0, \frac{\beta}{2}[\times S$ it equals

$$\nu^{(0/\frac{\beta}{2})} = \mp N^{-1}(\frac{\partial}{\partial s} - {\rm i}\mathbf{w}) \text{ on } \{0/\frac{\beta}{2}\} \times S. \tag{8.22}$$

The real vectors ${\rm Im}\nu$, ${\rm Im}\nu^{(0/\frac{\beta}{2})}$ are tangent to S, ie condition (9.5) below is satisfied.

8.6.2. *The Wick rotated operator.* We consider now the Wick rotated operator K obtained from P by the substitution $t = {\rm i}s$. We have:

$$K = -\Delta_{\mathbf{k}} + m(y) = -(\partial_s + {\rm i}w^*)N^{-2}(\partial_s - {\rm i}w) + h_0, \tag{8.23}$$

acting on the Hilbert spaces $\tilde{\mathcal{H}}_\beta$ for $0 < \beta \leq \infty$ defined in (6.9). We refer the reader to 9.1.1 fo the Laplacian $\Delta_{\mathbf{k}}$ associated to $\mathbf{k}$. We recall that

$$\tilde{\mathcal{H}}_\beta = L^2(\mathbb{S}_\beta \times S, N(y)|\mathbf{h}|^{\frac{1}{2}}(y)dyds),\ \text{for } 0 < \beta < \infty,$$
$$\tilde{\mathcal{H}}_\infty = L^2(\mathbb{R} \times S, N(y)|\mathbf{h}|^{\frac{1}{2}}(y)dyds).$$

It follows from Lemma 8.2 that if $h = h_0 - w^*N^{-2}w$ we have:

$$h \sim h_0,\ w^*N^{-2}w \lesssim h,\ \text{on } C_0^\infty(\mathbb{R} \times S),$$

where we use the scalar product of $\tilde{\mathcal{H}}_\beta$ in the operator inequalities. We have:

$$\begin{aligned}(u|Ku)_{\tilde{\mathcal{H}}_\beta} = \ & \|N^{-1}\partial_s u\|^2_{\tilde{\mathcal{H}}_\beta} + (u|hu)_{\tilde{\mathcal{H}}_\beta} \\ & -{\rm i}(N^{-1}\partial_s u|N^{-1}wu)_{\tilde{\mathcal{H}}_\beta} - {\rm i}(N^{-1}wu|N^{-1}\partial_s u)_{\tilde{\mathcal{H}}_\beta},\ u \in C_0^\infty(\mathbb{R} \times S).\end{aligned} \tag{8.24}$$

The sesquilinear form associated to the realization K_β of K is

$$\begin{aligned} Q_\beta(u,u) = \quad & \|N^{-1}\partial_s u\|^2_{\tilde{\mathcal{H}}_\beta} + (u|hu)_{\tilde{\mathcal{H}}_\beta} \\ & -\mathrm{i}(N^{-1}\partial_s u|N^{-1}wu)_{\tilde{\mathcal{H}}_\beta} - \mathrm{i}(N^{-1}wu|N^{-1}\partial_s u)_{\tilde{\mathcal{H}}_\beta}, \end{aligned}$$

with domain $\mathrm{Dom}Q_\beta = \langle K_0\rangle^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta$ and $K_0 = -N^{-2}\partial_s^2 + h_0$ with its natural domain on $\tilde{\mathcal{H}}_\beta$. From (8.9) we obtain that

$$N^{-1} : \mathrm{Dom}Q_\beta \xrightarrow{\sim} \mathrm{Dom}\tilde{Q}_\beta,\ Q_\beta(Nu,Nu) = \tilde{Q}_\beta(u,u),$$

where $\tilde{Q}_\beta$ is defined in Subsect. 6.4. It follows that Q_β is a closed sectorial form and we denote as before by K_β:

$$K_\beta : K_0^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta \xrightarrow{\sim} K_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta$$

the induced operator. We have

$$\begin{aligned} & N : \tilde{K}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta \xrightarrow{\sim} K_0^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta,\ N : K_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta \to \tilde{K}_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta \\ & \tilde{K}_\beta = NK_\beta N, \text{ as elements of } B(\tilde{K}_0^{-\frac{1}{2}}\tilde{\mathcal{H}}_\beta, \tilde{K}_0^{\frac{1}{2}}\tilde{\mathcal{H}}_\beta), \end{aligned}$$

where $\tilde{K}_\beta$ is the operator defined in Subsect. 6.4.

8.7. **Calderón projectors.** We now define the Calderón projectors for K_β and relate them to those for $\tilde{K}_\beta$ defined in Subsect. 7.2. We use the notation $I_\beta^\pm$, $i_\beta^\pm$, $\tilde{\gamma}_\beta$, $\tilde{\gamma}_\beta^\pm$ introduced in Subsect. 7.2.

8.7.1. *Calderón projectors for K_∞.* If $u \in \overline{C^0}(\mathbb{R};\tilde{\mathcal{H}})$ the trace $\gamma_\infty u$ of u on $s=0$ is

$$\gamma_\infty u = \begin{pmatrix} u(0) \\ -N^{-1}(\partial_s - \mathrm{i}w)u(0) \end{pmatrix} = \begin{pmatrix} u(0) \\ \nu u(0) \end{pmatrix},$$

see (8.21), and we have

$$\gamma_\infty N = Z\tilde{\gamma}_\infty,$$

where Z is defined in (8.16). We denote by γ_∞^* the formal adjoint of γ_∞ from $L^2(\mathbb{R}\times S; N|\mathbf{h}|^{\frac{1}{2}}dsdy) = \tilde{\mathcal{H}}_\infty$ to $L^2(S,|\mathbf{h}|^{\frac{1}{2}}dy;\mathbb{C}^2) = \mathcal{H}\otimes\mathbb{C}^2$. We have:

$$\gamma_\infty^* g = \delta_0'(s)\otimes N^{-2}g_1 + \delta_0(s)\otimes(N^{-1}g_0 - \mathrm{i}w^*N^{-2}g_1). \tag{8.25}$$

If $u(s) \in \overline{C^0}(I_\infty^\pm;\tilde{\mathcal{H}})$ satisfies $Ku = 0$ in $I_\infty^\pm$ we set

$$\gamma_\infty^\pm u = \begin{pmatrix} u(0^\pm) \\ -N^{-1}(\partial_s - \mathrm{i}w)u(0^\pm) \end{pmatrix},$$

so that

$$\gamma_\infty^\pm N = Z\tilde{\gamma}_\infty^\pm.$$

Setting $g = \gamma_\infty u$ and $u = N\tilde{u}$, $\tilde{g} = \tilde{\gamma}_\infty^\pm\tilde{u} = Z^{-1}g$, we obtain from 7.2.1 that

$$\tilde{g} = \mp\tilde{\gamma}_\infty^\pm\tilde{K}_\infty^{-1}\tilde{\gamma}_\infty^*\tilde{S}\tilde{g},$$

hence using $\gamma_\infty^\pm N = Z\gamma_\infty^\pm$ and $\tilde{K}_\infty = NK_\infty N$:

$$g = \mp\gamma_\infty^\pm K_\infty^{-1}N^{-1}\tilde{\gamma}_\infty^*\tilde{S}Z^{-1}g,$$

where $\tilde{S} = \begin{pmatrix} 2\mathrm{i}\tilde{w}^* & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}$. A tedious computation shows that

$$N^{-1}N^{-1}\tilde{\gamma}_\infty^*\tilde{S}Z^{-1} = \gamma_\infty^* S,\ S := \begin{pmatrix} 2\mathrm{i}Nw^*N^{-2} & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix}.$$

Note that the imaginary part of ν equals $N^{-1}w$ and its adjoint on $L^2(S,|\mathbf{h}|^{\frac{1}{2}}dy)$ equals Nw^*N^{-2} (recall that w^* is the adjoint of w for the scalar product of $L^2(S,N|\mathbf{h}|^{\frac{1}{2}}dy)$).

This leads to the following definition.

Definition 8.5. *The* Calderón projectors $c_\infty^\pm$ *for* K_∞ *are:*

$$c_\infty^\pm := \pm\gamma_\infty^\pm K_\infty^{-1}\gamma_\infty^* S.$$

Proposition 8.6. *The covariances of the vacuum state* $\omega_{\rm vac}$ *are equal to:*

$$\lambda_{\rm vac}^\pm = \pm q \circ c_\infty^\pm.$$

Proof. This follows from the identities:

$$i)\quad \tilde\lambda_{\rm vac}^\pm = Z^*\lambda_{\rm vac}^\pm Z,\ \tilde q = Z^* q Z,$$
$$ii)\quad \tilde\lambda_{\rm vac}^\pm = \pm\tilde q\tilde c_\infty^\pm,\ c_\infty^\pm = Z\tilde c_\infty^\pm Z^{-1}.$$

The identities in *i)* are obvious, the first identity in *ii)* is shown in Prop. 7.3, the second follows from the computations before Def. 8.5. □

8.7.2. *Calderón projectors for* K_β. If $u \in \overline{C^0}(\mathbb{S}_\beta;\tilde{\mathcal{H}})$ we denote by $\gamma_\beta u$ the vector obtained from its traces at $s=0$ and $s=\frac{\beta}{2}$:

$$\gamma_\beta u = \gamma_\beta^{(0)} u \oplus \gamma_\beta^{(\frac{\beta}{2})} u,$$

for

$$\gamma_\beta^{(0)} u = \begin{pmatrix} u(0) \\ -N^{-1}(\partial_s - \mathrm{i}w)u(0)\end{pmatrix} = \begin{pmatrix} u(0) \\ \nu^{(0)}u(0)\end{pmatrix}$$
$$\gamma_\beta^{(\frac{\beta}{2})} u = \begin{pmatrix} u(\frac{\beta}{2}) \\ (N^{-1}(\partial_s - \mathrm{i}w)u(\frac{\beta}{2})\end{pmatrix} = \begin{pmatrix} u(\frac{\beta}{2}) \\ \nu^{(\frac{\beta}{2})}u(\frac{\beta}{2})\end{pmatrix},$$

see (8.22), and we have:

$$\gamma_\beta N = (Z\oplus Z)\tilde\gamma_\beta,$$

where Z is defined in (8.16). Again we denote by γ_β^* the formal adjoint of γ_β from $L^2(\mathbb{R}\times S; N|\mathbf{h}|^{\frac12}dsdy) = \tilde{\mathcal{H}}_\infty$ to $L^2(S,|\mathbf{h}|^{\frac12}dy;\mathbb{C}^2)\oplus L^2(S,|\mathbf{h}|^{\frac12}dy;\mathbb{C}^2)$. We have:

$$\gamma_\beta^* = \gamma_\beta^{(0)*} + \gamma_\beta^{(\frac{\beta}{2})*},$$

for

$$\gamma_\beta^{(0)*} g^{(0)} = \delta_0'(s)\otimes N^{-2}g_1^{(0)} + \delta_0(s)\otimes(N^{-1}g_0^{(0)} - \mathrm{i}w^*N^{-2}g_1^{(0)}),$$
$$\gamma_\beta^{(\frac{\beta}{2})*} g^{(0)} = -\delta_{\frac{\beta}{2}}'(s)\otimes N^{-2}g_1^{(\frac{\beta}{2})} + \delta_{\frac{\beta}{2}}(s)\otimes(N^{-1}g_0^{(\frac{\beta}{2})} + \mathrm{i}w^*N^{-2}g_1^{(\frac{\beta}{2})}).$$

If $u\in\overline{C^0}(I_\beta^\pm;\tilde{\mathcal{H}})$ satisfies $Ku=0$ in $I_\beta^\pm$ we set:

$$\gamma_\beta^\pm u = \gamma_\beta^{(0)\pm} u \oplus \gamma_\beta^{(\frac{\beta}{2})\pm}\langle u\rangle,$$

for

$$\gamma_\beta^{(0)\pm} u = \begin{pmatrix} u(0^\pm) \\ -N^{-1}(\partial_s - \mathrm{i}w)u(0^\pm)\end{pmatrix},\ \gamma_\beta^{(\frac{\beta}{2})\pm} u = \begin{pmatrix} u(\mp\frac{\beta}{2}) \\ N^{-1}(\partial_s - \mathrm{i}w)u(\mp\frac{\beta}{2})\end{pmatrix}.$$

Setting $g=\gamma_\beta^\pm u$ and $u = N\tilde u$, $\tilde g = \tilde\gamma_\beta^\pm\tilde u = (Z\oplus Z)^{-1}g$, we obtain that

$$\tilde g = \mp\tilde\gamma_\beta^\pm \tilde K_\beta^{-1}(\tilde\gamma_\beta^{(0)*}\tilde S^{(0)}\tilde g^{(0)} + \tilde\gamma_\beta^{(\frac{\beta}{2})*}\tilde S^{(\frac{\beta}{2})}\tilde g^{(\frac{\beta}{2})}),$$

where $\tilde S^{(0)}$, $\tilde S^{(\frac{\beta}{2})}$ are defined in (7.6). The same computation as in 8.7.1 gives that

$$g = \mp\gamma_\beta^\pm K_\beta^{-1}(\gamma_\beta^{(0)*}\mathcal{S}^{(0)}g^{(0)} + \gamma_\beta^{(\frac{\beta}{2})*}\mathcal{S}^{(\frac{\beta}{2})}g^{(\frac{\beta}{2})}),$$

for

$$S^{(0)} := \begin{pmatrix} 2\mathrm{i}Nw^*N^{-2} & -\mathbb{1} \\ \mathbb{1} & 0\end{pmatrix},\ S^{(\frac{\beta}{2})} := \begin{pmatrix} -2\mathrm{i}Nw^*N^{-2} & -\mathbb{1} \\ \mathbb{1} & 0\end{pmatrix}.$$

Again this leads to the following definition.

Definition 8.7. *The* Calderón projectors $c_\beta^\pm$ *for* K_β *are:*

$$c_\beta^\pm := \mp\gamma_\beta^\pm K_\beta^{-1}(\gamma_\beta^{(0)*}\mathcal{S}^{(0)}\pi^{(0)} + \gamma_\beta^{(\frac{\beta}{2})*}\mathcal{S}^{(\frac{\beta}{2})}\pi^{(\frac{\beta}{2})}),$$

where $\pi^{(0/\frac{\beta}{2})}g = g^{(0/\frac{\beta}{2})}$.

Using now Prop. 7.6, the same argument as in 8.7.1 gives the following proposition.

Proposition 8.8. *The covariances of the double* β*-KMS state* $\omega_{\rm d}$ *are equal to:*

$$\lambda_{\rm d}^\pm = \pm Q \circ (\mathbb{1} \oplus T)^{-1} c_\beta^\pm (\mathbb{1} \oplus T),\ Q = q \oplus q,$$

where $T = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$.

8.8. **The double β-KMS state in $\mathcal{M}^+ \cup \mathcal{M}^-$.** We now apply the computations of the previous subsections to $S = \Sigma^+$. In fact if ϕ_t is the flow of the Killing vector field V, the map

$$\chi : \mathbb{R} \times \Sigma^+ \ni (t, y) \mapsto \phi_t(y) \in \mathcal{M}^+$$

is a diffeormorphism such that $\chi^*\mathbf{g}$ is as in Subsect. 8.1.

We first claim that $\frac{\partial}{\partial t}$ is uniformly time-like and (8.12) holds on Σ^+.

Proposition 8.9. *Assume that hypothesis (H) holds. Then* $\frac{\partial}{\partial t}$ *is uniformly time-like and* (8.12) *holds on* Σ^+.

Proof. We first check that $\frac{\partial}{\partial t}$ is uniformly time-like and that (8.12) holds on $\Sigma^+ \setminus U$, where U is any small neighborhood of $\mathcal{B}$ in Σ^+, by hypotheses (H). To check the conditions on U, we use Prop. 2.4. Recalling that (u, ω) are Gaussian normal coordinates to $\mathcal{B}$ in $(\Sigma, \mathbf{h})$, we obtain

$$w \cdot \mathbf{h} w \in O(u^4), w \cdot (\nabla N) \in O(u^3), (\nabla \cdot w) \in O(u^2), w \cdot \nabla(|\mathbf{h}|^{\frac{1}{2}}) \in O(u^2),$$

from which our claim follows, since $N(y) = \kappa u + O(u^3)$. □

8.8.1. *The double* β*-KMS state in* $\mathcal{M}^+ \cup \mathcal{M}^-$. Let us now define the double β-KMS state in $\mathcal{M}^+ \cup \mathcal{M}^-$.

The wedge reflection R is an isometric involution from $(\mathcal{M}^-, \mathbf{g})$ to $(\mathcal{M}^+, \mathbf{g})$. It induces on Σ the weak wedge reflection r, which equals the identity on $\mathcal{B}$ and maps Σ^- bijectively on Σ^+.

R reverses the time orientation, hence induces a unitary involution:

$$\mathcal{R} : (\frac{C_0^\infty(\mathcal{M}^-)}{PC_0^\infty(\mathcal{M}^-)}, {\rm i}G) \in [u] \mapsto [u \circ R] \in (\frac{C_0^\infty(\mathcal{M}^+)}{PC_0^\infty(\mathcal{M}^+)}, -{\rm i}G).$$

In a more familiar language, $\mathcal{R}$ is anti-symplectic. Since

$$\varrho_{\Sigma^\pm} \circ G : (\frac{C_0^\infty(\mathcal{M}^\pm)}{PC_0^\infty(\mathcal{M}^\pm)} \xrightarrow{\sim} (C_0^\infty(\Sigma^\pm), q)$$

is unitary, $\mathcal{R}$ induces the unitary involution

$$\mathcal{R}_\Sigma : (C_0^\infty(\Sigma^-), q) \xrightarrow{\sim} (C_0^\infty(\Sigma^+), -q).$$

The following expression for $\mathcal{R}_\Sigma$ follows from the fact that R reverses the time orientation.

Lemma 8.10. *One has*

$$\mathcal{R}_\Sigma f = T r^* f,$$

where $T = \begin{pmatrix} \mathbb{1} & 0 \\ 0 & -\mathbb{1} \end{pmatrix}$ *and* $r^* f(y) = f(r(y))$.

We have defined in Subsect. 8.5 the double β-KMS state $\omega_{\rm d}$ through its Cauchy surface covariances $\lambda_{\rm d}^{\pm}$. The associated Hermitian space is

$$C_0^\infty(\Sigma^+;\mathbb{C}^2),q)\oplus C_0^\infty(\Sigma^+;\mathbb{C}^2),-q).$$

$(\mathcal{M}^+\cup\mathcal{M}^-,\mathbf{g})$ is a (disconnected) globally hyperbolic spacetime with Cauchy surface $\Sigma^+\cup\Sigma^-$ and we denote a Cauchy data on $\Sigma^+\cup\Sigma^-$ as

$$f=f^+\oplus f^-,\ f^\pm\in C_0^\infty(\Sigma^\pm;\mathbb{C}^2).$$

Using Remark 5.5, we obtain from $\omega_{\rm d}$ a pure, quasi-free state $\omega_{\rm D}$ in $\mathcal{M}^+\cup\mathcal{M}^-$ as follows:

Definition 8.11. *The double β-KMS state $\omega_{\rm D}$ in $\mathcal{M}^+\cup\mathcal{M}^-$ is defined by the Cauchy surface covariances:*

$$\overline{f}\cdot\lambda_{\rm D}^{\pm}f:=\overline{(\mathbb{1}\oplus\mathcal{R}_\Sigma)f}\cdot\lambda_{\rm d}^{\pm}(\mathbb{1}\oplus\mathcal{R}_\Sigma)f,\ f=f^+\oplus f^-\in C_0^\infty(\Sigma^+\cup\Sigma^-;\mathbb{C}^2).$$

From Prop. 8.8 and Lemma 8.10 we obtain the following expresssion for $\lambda_{\rm D}^{\pm}$.

Proposition 8.12. *One has:*

$$\lambda_{\rm D}^{\pm}=\pm Q\circ(\mathbb{1}\oplus r^*)^{-1}c_\beta^{\pm}(\mathbb{1}\oplus r^*),$$

where $c_\beta^{\pm}$ are the Calderón projectors for K_β defined in Def. 8.7 and $Q=q\oplus q$.

9. The HHI state

In this section we construct the HHI state $\omega_{\rm HHI}$ in M and prove that it is a pure Hadamard state, extending the double β-KMS state $\omega_{\rm D}$ in $\mathcal{M}^-\cup\mathcal{M}^+$ for $\beta=(2\pi)\kappa^{-1}$. We use the expression of $\omega_{\rm D}$ by Calderón projectors for the Wick rotated operator K_β, see Subsect. 8.8.

Since K_β is a Laplace operator for the complex metric $\mathbf{g}^{\rm eucl}$ on $M^{\rm eucl}=\mathbb{S}_\beta\times\Sigma^+$, one can if $\beta=(2\pi)\kappa^{-1}$ extend it to a Laplace operator $K_{\rm ext}$ on the smooth extension $(M_{\rm ext}^{\rm eucl},\mathbf{g}_{\rm ext}^{\rm eucl})$.

The boundary of the open set $\Omega_{\rm ext}$ extending $\Omega_\beta=]0,\frac{\beta}{2}[\times\Sigma^+$ is diffeomorphic to the full Cauchy surface Σ, and we can use the Calderón projectors for $K_{\rm ext},\Omega_{\rm ext}$ to define a pair of covariances $\lambda_{\rm HHI}^{\pm}$. The fact that they define a pure state is actually quite easy, using some standard continuity properties of the Calderón projectors and density results in Sobolev spaces. The proof of the Hadamard property of $\omega_{\rm HHI}$ relies also on an easy argument using pseudodifferential calculus, taken from [G].

9.1. **Laplacians for complex metrics.** We recall that complex metrics on a manifold X are defined in 2.7.1.

Definition 9.1. *A complex metric* $\mathbf{k}$ *on a manifold X is called* uniformly sectorial *if*

(1) *there exists $C>0$ such that*

$$|{\rm Im}(\overline{v}^a\mathbf{k}_{ab}(x)v^b)|\leq C{\rm Re}(\overline{v}^a\mathbf{k}_{ab}(x)v^b),\ \forall\, x\in X,\ v\in\mathbb{C}T_xX;\tag{9.1}$$

(2) $|\mathbf{k}(x)|=\det(\mathbf{k}_{ab}(x))>0\ \forall x\in X$.

Note that if $\mathbf{k}$ is uniformly sectorial, then

$$|{\rm Im}(\overline{\xi}_a k^{ab}(x)\xi_b)|\leq C{\rm Re}(\overline{\xi}_a k^{ab}(x)\xi_b)\ \forall\, x\in X,\ \xi\in\mathbb{C}T_x^*X,\tag{9.2}$$

ie $\mathbf{k}^{-1}$ is also uniformly sectorial. In fact if $\xi=\mathbf{k}v$ we have $\overline{\xi}\cdot\mathbf{k}^{-1}\xi=\overline{\mathbf{k}v}\cdot v=\overline{\overline{v}\cdot\mathbf{k}v}$ and (9.2) follows from (9.1).

9.1.1. *Laplacians for complex metrics.* If $\mathbf{k}$ is a complex metric on X, one defines the Christoffel symbols:

$$\Gamma^c_{ab} := \frac{1}{2}\mathbf{k}^{cd}(\partial_a \mathbf{k}_{cd} + \partial_b \mathbf{k}_{ad} - \partial_d \mathbf{k}_{ab}),$$

the covariant derivative:

$$\nabla^{(\mathbf{k})}_a T^b = \partial_a T^b + \Gamma^b_{ac} T^c,$$

and the Laplacian associated to $\mathbf{k}$, acting on $C^\infty_0(X)$:

$$\Delta_{\mathbf{k}} := \nabla^{(\mathbf{k})}_a \mathbf{k}^{ab} \nabla^{(\mathbf{k})}_b$$

as for real metrics.

For $m \in C^\infty(X, \mathbb{R})$, we set:

$$K := -\Delta_{\mathbf{k}} + m,$$

and equip $C^\infty_0(X)$ with the scalar product:

$$(u|v) := \int_X \overline{u} v |\mathbf{k}|^{\frac{1}{2}} dx.$$

Proposition 9.2. *Assume that* $\mathbf{k}$ *is uniformly sectorial and that* $m_0^2 \leq m(x)$ *for* $m_0 > 0$. *Let*

$$Q(u,u) = (u|Ku), \mathrm{Dom} Q = C^\infty_0(X).$$

Then Q *is closeable, the domain* $\mathrm{Dom} Q^{\mathrm{cl}}$ *of its closure* Q^{cl} *is the space* $H^1_{\mathbf{k}}(X)$ *equal to the completion of* $C^\infty_0(X)$ *for the norm*

$$\|u\|^2_1 = \int_X (\overline{\partial_a u} \mathrm{Re}\mathbf{k}^{ab} \partial_b u + m(x)\overline{u}u)|\mathbf{k}|^{\frac{1}{2}} dx.$$

Moreover Q^{cl} *is sectorial and induces an isomorphism:*

$$K^{\mathrm{cl}} : H^1_{\mathbf{k}}(X) \xrightarrow{\sim} H^1_{\mathbf{k}}(X)^*,$$

with $K^{\mathrm{cl}} = K$ *on* $C^\infty_0(X)$.

Proof. We have

$$\nabla^{(\mathbf{k})}_a T^a = |\mathbf{k}|^{-\frac{1}{2}} \partial_a (|\mathbf{k}|^{\frac{1}{2}} T^a), \tag{9.3}$$

which is proved in Subsect. A.3. Therefore $K = -|\mathbf{k}|^{-\frac{1}{2}} \partial_a \mathbf{k}^{ab} |\mathbf{k}|^{\frac{1}{2}} \partial_a + m$ and

$$Q(u,u) = (u|Ku) = \int_X (\overline{\partial_a u} \mathbf{k}^{ab} \partial_b u + m(x)\overline{u}u)|\mathbf{k}|^{\frac{1}{2}} dx. \tag{9.4}$$

Using (9.2) under the integral sign, we obtain $|\mathrm{Im} Q(u,u)| \leq C \mathrm{Re} Q(u,u)$ and that Q is closeable. The domain of its closure Q^{cl} equals $H^1_{\mathbf{k}}(X)$. The statement about K^{cl} follows from the Lax-Milgram theorem. □

9.1.2. *Outer unit normal.* Let $\Omega \subset X$ with a smooth boundary $\partial\Omega$ denoted by Σ in the sequel. We set

$$\Omega^+ := \Omega, \ \Omega^- := X \backslash \Omega^{\mathrm{cl}}.$$

We can define the outer unit normal vector field to Σ, denoted by $n \in \mathbb{C}TX$ by the following conditions:

$$\begin{aligned} &i) \quad n(x) \cdot \mathbf{k}(x) v = 0, \ \forall v \in T_x \Sigma, \\ &ii) \quad n(x) \cdot \mathbf{k}(x) n(x) = 1, \\ &iii) \quad \mathrm{Re} n(x) \text{ is outwards pointing.} \end{aligned}$$

If Ω is locally equal to $\{f > 0\}$ for $f \in C^\infty(X, \mathbb{R})$ with $df \neq 0$ on $\{f = 0\}$, we have:

$$n^a = \frac{-\mathbf{k}^{ab} \nabla_b f}{(\nabla_a f \mathbf{k}^{ab} \nabla_b f)^{\frac{1}{2}}},$$

where in the denominator we take the usual determination of $z^{\frac{1}{2}}$.

We also assume the following condition:

$$\mathrm{Im} n(x) \in T_x\Sigma,\ x \in \Sigma, \tag{9.5}$$

which is equivalent to $\nabla_a f \mathbf{k}^{ab} \nabla_b f \in \mathbb{R}$ on Σ, if $\Omega = \{f > 0\}$.

The volume form $d\mathrm{Vol}_{\mathbf{k}} = |\mathbf{k}|^{\frac{1}{2}} dx^1 \wedge \cdots \wedge dx^n$ associated to $\mathbf{k}$ is real, as is the associated density $d\mu_{\mathbf{k}} = |d\mathrm{Vol}_{\mathbf{k}}| = |\mathbf{k}|^{\frac{1}{2}} dx$. It is easy to see from (9.5) that the induced density $d\sigma_{\mathbf{h}} = |d\mathrm{Vol}_{\mathbf{h}}|$ associated to the induced metric $\mathbf{h}$ on Σ is also real valued.

9.1.3. *Trace operators.* For $u \in C^\infty(X)$ we set:

$$\gamma u := \begin{pmatrix} u\restriction_\Sigma \\ \partial_n u\restriction_\Sigma \end{pmatrix} \in C^\infty(\Sigma; \mathbb{C}^2).$$

We denote by γ^* the formal adjoint of

$$\gamma : L^2(X, d\mu_{\mathbf{k}}) \to L^2(\Sigma, d\sigma_{\mathbf{h}}) \otimes \mathbb{C}^2.$$

We have

$$\gamma^* f = (d\mu_{\mathbf{k}})^{-1} f_0 d\Sigma + (d\mu_{\mathbf{k}})^{-1} (n^\mu \partial_\mu)^* f_1 d\Sigma,$$

where if $g \in C^\infty(\Sigma)$, $g d\Sigma$ is the distributional density defined as

$$\langle u | g d\Sigma \rangle = \int_\Sigma \overline{u} g d\sigma_{\mathbf{h}},\ u \in C_0^\infty(X),$$

and

$$\langle u | (n^\mu \partial_\mu)^* g d\Sigma \rangle = \langle n^\mu \partial_\mu u | g d\Sigma \rangle.$$

Similarly for $u \in \overline{C^\infty}(\Omega^\pm)$ we set:

$$\gamma^\pm u := \begin{pmatrix} u\restriction_\Sigma \\ \partial_n u\restriction_\Sigma \end{pmatrix},$$

where the trace is taken from $\Omega^\pm$.

In the rest of this subsection, we assume that $\mathbf{k}$ is uniformly sectorial and that (9.5) holds.

9.1.4. *Calderón projectors.*

Definition 9.3. *The* Calderón projectors $c^\pm$ *associated to* (K, Ω) *are defined as*

$$c^\pm = \mp \gamma^\pm \circ K^{-1} \circ \gamma^* \circ S,$$

where

$$S = \begin{pmatrix} 2\mathrm{i} b^* & -\mathbb{1} \\ \mathbb{1} & 0 \end{pmatrix},$$

$b = \mathrm{Im} n^a \nabla_a$ *and* b^* *is the adjoint of* b *in* $L^2(\Sigma, d\sigma_{\mathbf{h}})$.

Note that the operator S is well defined on $C_0^\infty(\Sigma; \mathbb{C}^2)$, since $\mathrm{Im} n$ is tangent to Σ.

It is not a priori clear that $c^\pm$ are well defined, since even for $f \in C_0^\infty(\Sigma; \mathbb{C}^2)$, $\gamma^* S f$ does not belong to $H^1_{\mathbf{k}}(X)^*$.

To show that $c^\pm$ make sense, one can apply the following proposition. We denote by $H^s_{\mathrm{c}}(\Sigma)$ resp. $H^s_{\mathrm{loc}}(\Sigma)$ for $s \in \mathbb{R}$, the compactly supported, resp. local Sobolev spaces on Σ and set:

$$\begin{aligned} \mathcal{H}^s_{\mathrm{c/loc}}(\Sigma) &= H^{s-\frac{1}{2}}_{\mathrm{c/loc}}(\Sigma) \oplus H^{s-1-\frac{1}{2}}_{\mathrm{c/loc}}(\Sigma), \\ \tilde{\mathcal{H}}^s_{\mathrm{c/loc}}(\Sigma) &= H^{s+\frac{1}{2}}_{\mathrm{c/loc}}(\Sigma) \oplus H^{s+1+\frac{1}{2}}_{\mathrm{c/loc}}(\Sigma),\ s \in \mathbb{R}. \end{aligned} \tag{9.6}$$

Proposition 9.4. (1) $c^\pm : \mathcal{H}^s_{\mathrm{c}}(\Sigma) \to \mathcal{H}^s_{\mathrm{loc}}(\Sigma)$ *continuously for any* $s \in \mathbb{R}$,
(2) $c^\pm$ *are* 2×2 *matrices with entries in* $\Psi^\infty(\Sigma)$.

Proof. The differential operator K is elliptic, since its principal symbol equals $\xi\cdot\mathbf{k}^{-1}(x)\xi$. K admits hence a properly supported parametrix $Q\in\Psi_{\rm c}^{-2}(X)$, and $K^{-1}-Q$ is a smoothing operator, ie has a smooth distributional kernel.

It suffices hence to check the proposition with K^{-1} replaced by Q in the definition of $c^{\pm}$. From the topology of $\mathcal{H}^s_{\rm c/loc}(\Sigma)$, we see that we can assume that Σ is compact, and since Q is properly supported, that X is compact, which is the situation considered in [Gr, Sect. 11.1].

A neighborhood V of Σ in X is then diffeomorphic to $]-\delta,\delta[\times\Sigma$, and one can use coordinates (s,y) on $]-\delta,\delta[\times\Sigma$. In [Gr, Sect. 11.1] the trace operator is defined as

$$\tilde\gamma u=\begin{pmatrix}u(0,y)\\ \mathrm{i}^{-1}\partial_s u(0,y)\end{pmatrix}.$$

Clearly we have $\gamma=L\circ\tilde\gamma$, where $L=\begin{pmatrix}\mathbb{1}&0\\ r&\mathbb{1}\end{pmatrix}$ and r is a first order differential operator on Σ. This implies that $L:\mathcal{H}^s_{\rm c/loc}(\Sigma)\xrightarrow{\sim}\mathcal{H}^s_{\rm c/loc}(\Sigma)$. The Calderón projectors $\tilde C^{\pm}$ in [Gr, Sect. 11.1] are equal to $L^{-1}\circ c^{\pm}\circ L$, and [Gr, Prop. 11.7] implies that $\tilde C^{\pm}:\mathcal{H}^s_{\rm c}(\Sigma)\to\mathcal{H}^s_{\rm loc}(\Sigma)$ for all $s\in\mathbb{R}$, which implies (1).

Property (2) is a standard fact, see [Gr, Sect. 11.1]. □

9.2. **Construction of the HHI state.**

9.2.1. *The Laplacian on $M^{\rm eucl}_{\rm ext}$.* We now apply the above framework to $(X,\mathbf{k})=(M^{\rm eucl}_{\rm ext},\mathbf{g}^{\rm eucl}_{\rm ext})$, the smooth extension of $(M^{\rm eucl},\mathbf{g}^{\rm eucl})$ constructed in Prop. 2.5, for $\beta=(2\pi)\kappa^{-1}$. We assume that hypothesis (H) in Subsect. 2.4 holds. By Prop. 8.9 the Wick rotated metric $\mathbf{g}^{\rm eucl}$ satisfies the conditions in 8.6.1, ie is uniformly sectorial. By continuity the same is true of its extension $\mathbf{g}^{\rm eucl}_{\rm ext}$. We denote by

$$K_{\rm ext}=\Delta_{\mathbf{g}^{\rm eucl}_{\rm ext}}+m_{\rm ext},$$

the associated Laplacian. We choose the open set $\Omega_{\rm ext}\subset M^{\rm eucl}_{\rm ext}$, whose boundary $\partial\Omega_{\rm ext}$ is diffeomorphic to Σ, see Prop. 2.5. We saw in 8.6.1 that if ν is the unit outer normal to $]0,\frac{\beta}{2}[\times\Sigma^+$, then $\mathrm{Im}\nu$ is tangent to $\partial(]0,\frac{\beta}{2}[\times\Sigma^+)$. Again by continuity, the same is true of the unit outer normal to $\Omega_{\rm ext}$, ie condition (9.5) is satisfied. Therefore we can apply the results of Subsect. 9.1 to $K_{\rm ext}$ and $\Omega_{\rm ext}$.

We need one more result, which states that $K_{\rm ext}$ is the unique extension of $K_{(2\pi)\kappa^{-1}}$ to $L^2(M^{\rm eucl}_{\rm ext})$.

Proposition 9.5. *Let $U:C_0^\infty(M^{\rm eucl})\to C_0^\infty(M^{\rm eucl}_{\rm ext}\setminus\mathcal{B}_{\rm ext})$ defined by:*

$$Uu=u\circ\chi^{-1}.$$

Then U extends as a unitary operator

$$U:L^2(M^{\rm eucl},N(y)|\mathbf{h}|^{\frac12}(y)dyds)\to L^2(M^{\rm eucl}_{\rm ext},|g^{\rm eucl}_{\rm ext}|^{\frac12}dx),$$

with $K_{\rm ext}=UK_{(2\pi)\kappa^{-1}}U^*$.

Proof. U clearly extends as a unitary operator. Let us check the second statement.

As a differential operator, $K_{(2\pi)\kappa^{-1}}$ equals $-\Delta_{\mathbf{g}^{\rm eucl}}+m$. As an unbounded operator, $K_{(2\pi)\kappa^{-1}}$ is defined in 8.6.2 using the sesquilinear form $Q_{(2\pi)\kappa^{-1}}$, while $K_{\rm ext}$ is defined with the sesquilinear form $Q_{\rm ext}$ for $\mathbf{k}=\mathbf{g}^{\rm eucl}_{\rm ext}$ and $m=m_{\rm ext}$, see Prop. 9.2.

$Q_{(2\pi)\kappa^{-1}}$ is the closure of its restriction to $C_0^\infty(M^{\rm eucl})$, while $Q_{\rm ext}$ is the closure of its restriction to $C_0^\infty(M^{\rm eucl}_{\rm ext})$.

Taking into account the isometry $\chi:M^{\rm eucl}\xrightarrow{\sim}M^{\rm eucl}_{\rm ext}\setminus\mathcal{B}_{\rm ext}$, it suffices to check that $C_0^\infty(M^{\rm eucl}_{\rm ext}\setminus\mathcal{B}_{\rm ext})$ is a form core for $Q_{\rm ext}$, ie that this space is dense in the space $H^1_{\mathbf{k}}(X)$ for $(X,\mathbf{k})=(M^{\rm eucl}_{\rm ext},\mathbf{g}^{\rm eucl}_{\rm ext})$, see Prop. 9.2.

Using the coordinates (X, Y, ω) near $\mathcal{B}_{\rm ext} \sim \{0\} \times \mathcal{B}$, this follows from the fact that $C_0^\infty(\mathbb{R}^2 \backslash \{0\})$ is dense in $H^1(\mathbb{R}^2)$, see eg [A, Thm. 3.23]. □

9.2.2. *The HHI state.* Let us denote by $c^\pm_{\rm ext}$ the Calderón projectors for $(K_{\rm ext}, \Omega_{\rm ext})$, defined as in Def. 9.3.

The following theorem is a slightly more precise version of Thm. 1.1.

Theorem 9.6. (1) $\lambda^\pm_{\rm HHI} = \pm q \circ c^\pm_{\rm ext}$ *are the Cauchy surface covariances of a pure quasi-free state* $\omega_{\rm HHI}$ *for* P *in* M*, called the* HHI state.
(2) *The restriction of* $\omega_{\rm HHI}$ *to* $\mathcal{M}^+ \cup \mathcal{M}^-$ *is the double* $\beta-$*KMS state* $\omega_{\rm D}$ *for* $\beta = (2\pi)\kappa^{-1}$.
(3) $\omega_{\rm HHI}$ *is a* Hadamard state *in* M.
(4) *Let* ω *a quasi-free state for* P *in* M *whose restriction to* $\mathcal{M}^+ \cup \mathcal{M}^-$ *equals* $\omega_{\rm D}$ *and such that its space-time covariances map* $C_0^\infty(M)$ *into* $C^\infty(M)$*. Then* $\omega = \omega_{\rm HHI}$.

Note that it follows from (4) above that $\omega_{\rm HHI}$ is the unique Hadamard state in M whose restriction to $\mathcal{M}^+ \cup \mathcal{M}^-$ equals $\omega_{\rm D}$.

Proof. We first prove (2). We note that the map $(\mathbb{1} \oplus r^*)$ in Prop. 8.12 corresponds to the embedding of $C_0^\infty(\Sigma^+ \cup \Sigma^-; \mathbb{C}^2)$ into $C_0^\infty(\Sigma \backslash \mathcal{B})$ obtained from $\psi : \Sigma \to M^{\rm eucl}_{\rm ext}$ in Prop. 2.5. The exterior normal to $\Omega_{\rm ext}$ is the image under χ of the exterior normal to $]0, \pi\kappa^{-1}[\times \Sigma^+$ for defined in (8.22). Therefore using also Prop. 9.5 we obtain that

$$(\mathbb{1} \oplus r^*)^{-1} c^\pm_{(2\pi/\kappa)} (\mathbb{1} \oplus r^*) = c^\pm_{\rm ext}, \tag{9.7}$$

on $C_0^\infty(\Sigma^+ \cup \Sigma^-)$. This implies (2).

Let us now prove (1). Let us denote by $\mathbf{h}_{\rm ext}$ the metric induced by $\mathbf{g}^{\rm eucl}_{\rm ext}$ on Σ and use the scalar product of $L^2(\Sigma, |\mathbf{h}_{\rm ext}|^{\frac{1}{2}} dy) \otimes \mathbb{C}^2$ to identify sesquilinear forms with operators, so that $q = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$.

We recall that the spaces $\mathcal{H}^s_{\rm loc}(\Sigma)$, $\tilde{\mathcal{H}}^s_{\rm loc}(\Sigma)$ are defined in (9.6). We note that $q : \mathcal{H}^s_{\rm loc}(\Sigma) \to \tilde{\mathcal{H}}^{s-2}_{\rm loc}(\Sigma)$, and that $\mathcal{H}^s_{\rm c}(\Sigma), \tilde{\mathcal{H}}^s_{\rm loc}(\Sigma)$ form a dual pair for the above scalar product. It follows then from Prop. 9.4 that $\lambda^\pm, q$ are continuous sesquilinear forms for the topology of $\mathcal{H}^1_{\rm c}(\Sigma) = H_c^{\frac{1}{2}}(\Sigma) \oplus H_{\rm c}^{-\frac{1}{2}}(\Sigma)$. Now it is a well-known fact that since $\mathcal{B} \subset \Sigma$ is of codimension 1, $C_0^\infty(\Sigma \setminus \mathcal{B})$ is dense in $H_{\rm c}^{\pm\frac{1}{2}}(\Sigma)$.

By (2) and the fact that $\omega_{\rm D}$ is a state, we obtain that

$$\lambda^\pm_{\rm HHI} \geq 0, \ \lambda^+_{\rm HHI} - \lambda^-_{\rm HHI} = q,$$

on $C_0^\infty(\Sigma \setminus \mathcal{B}; \mathbb{C}^2)$. By the continuity and density result shown above, this extends to $C_0^\infty(\Sigma; \mathbb{C}^2)$, which proves that $\lambda^\pm_{\rm HHI}$ are the Cauchy surface covariances of a state $\omega_{\rm HHI}$.

Let us now prove that $\omega_{\rm HHI}$ is pure. Let us set for simplicity of notation $\mathcal{Y} = C_0^\infty(\Sigma; \mathbb{C}^2)$, $\mathcal{Y}_0 = C_0^\infty(\Sigma \setminus \mathcal{B}; \mathbb{C}^2)$ and denote by $\mathcal{Y}^{\rm cpl}$, $\mathcal{Y}_0^{\rm cpl}$ the completion of $\mathcal{Y}$, $\mathcal{Y}_0$ for the norm $\|f\|^2_\omega = \overline{f} \cdot (\lambda^+_{\rm HHI} + \lambda^-_{\rm HHI}) f$. The density and continuity result above shows that $\mathcal{Y}^{\rm cpl} = \mathcal{Y}_0^{\rm cpl}$. The purity of $\omega_{\rm HHI}$ follows then from the purity of $\omega_{\rm D}$ and Prop. 3.2.

Let us now prove (3). By Thm. 3.4 there exists a reference Hadamard state $\omega_{\rm ref}$ for P in M whose Cauchy surface covariances on Σ $\lambda^\pm_{\rm ref}$ are 2×2 matrices with entries in $\Psi^\infty(\Sigma)$. By Prop. 9.4 the same is true for $\lambda^\pm_{\rm HHI}$.

The restriction of $\omega_{\rm HHI}$ to $\mathcal{M}^+$ is a Hadamard state for P, since it is a $(2\pi)\kappa^{-1}$-KMS state for a time-like, complete Killing vector field. The restriction of $\omega_{\rm HHI}$ to $\mathcal{M}^-$ is also a Hadamard state for P.

In fact by Prop. 8.12, its Cauchy surface covariances on Σ^- are the images of those of $\omega_{\rm D}$ on Σ^+ by the weak wedge reflection r. Since $r^*\mathbf{h} = \mathbf{h}$, $r^*N = -N$ and $r^*w = w$, see 2.2.1, the expression (8.6) of P in $\mathbb{R} \times \Sigma^-$ shows that the restriction of $\omega_{\rm D}$ to $\mathcal{M}^-$ is also a Hadamard state.

This implies that the restriction of $\omega_{\rm HHI}$ to $\mathcal{M}^+ \cup \mathcal{M}^-$ is a Hadamard state. The same is true of the restriction of the reference Hadamard state $\omega_{\rm ref}$ to $\mathcal{M}^+ \cup \mathcal{M}^-$. Passing to Cauchy surface covariances on $\Sigma^+ \cup \Sigma^-$, this implies that if $\chi \in C_0^\infty(\Sigma^\pm)$, then

$$\chi \circ (\lambda^\pm_{\rm HHI} - \lambda^\pm_{\rm ref}) \circ \chi \text{ is a smoothing operator on } \Sigma.$$

We claim that this implies that $\lambda^\pm_{\rm HHI} - \lambda^\pm_{\rm ref}$ is smoothing, which will imply that $\omega_{\rm HHI}$ is a Hadamard state.

If fact let a be one of the entries of $\lambda^\pm_{\rm HHI} - \lambda^\pm_{\rm ref}$, which is a scalar pseudodifferential operator belonging to $\Psi^m(\Sigma)$ for some $m \in \mathbb{R}$. We know that $\chi \circ a \circ \chi$ is smoothing for any $\chi \in C_0^\infty(\Sigma \backslash \mathcal{B})$. Then its principal symbol $\sigma_{\rm pr}(a)$ vanishes on $T^*(\Sigma \backslash \mathcal{B})$ hence on $T^*\Sigma$ by continuity, so $a \in \Psi^{m-1}(\Sigma)$. Iterating this argument we obtain that a is smoothing, which completes the proof of (3).

The proof of (4) is identical to [G, Prop. 7.4]. □

Appendix A

A.1. **Proof of Prop. 2.4.** Since r is an isometry of $(\Sigma, \mathbf{h})$, $r_{|\mathcal{B}} = Id$ and $r : \Sigma^+ \to \Sigma^-$ we obtain (2.4). The first identity in (2.5) follows from the fact that (u, ω) are normal Gaussian coordinates to $\mathcal{B}$ for $\mathbf{h}$, the other are tautologies.

We obtain from (2.4) and 2.2.1 that v, $\mathbf{w}^0$ are odd in u, $\mathbf{w}^\alpha$, $\mathbf{k}_{\alpha\beta}$ are even in u with $\mathbf{w}^\alpha(0, \omega) = 0$. The function m is even in u by invariance under r. We now use Killing's equation

$$\nabla_a V_b + \nabla_b V_a = 0, \tag{A.1}$$

noting that since $V = 0$ on $\mathcal{B}$ we have

$$\nabla_a V_b = \partial_a V_b \text{ on } \mathcal{B}. \tag{A.2}$$

If we work in Gaussian normal coordinates to Σ for $\mathbf{g}$, so that

$$\mathbf{g} = -dt^2 + \mathbf{h}_{ij}(t, y) dy^i dy^j, \ V = -N(t, y)\partial_t + \mathbf{w}^0(t, y)\partial_u + \mathbf{w}^\alpha(t, y)\partial_{\omega^\alpha}$$

and $y = (u, \omega)$, we obtain from (A.1), (A.2) that:

$$\partial_u V_0(0, \omega) = 0 \Rightarrow \partial_u \mathbf{w}^0(0, \omega) = 0.$$

Summarizing we have:

$$\begin{aligned} &N(u, \omega) = u a(u^2, \omega), \\ &\mathbf{w}^0(u, \omega) = u^3 b(u^2, \omega), \ \mathbf{w}^\alpha(u, \omega) = u^2 c^\alpha(u^2, \omega), \\ &\mathbf{k}_{\alpha\beta}(u, \omega) = \mathrm{d}_{\alpha\beta}(u^2, \omega), \ m(u, \omega) = n(u^2, \omega), \end{aligned} \tag{A.3}$$

for smooth functions $a, b, c^\alpha, \mathrm{d}_{\alpha\beta} :]-\epsilon, \epsilon[\times \mathcal{B} \to \mathbb{R}$ with

$$n(0, \omega) \geq c > 0, c^{-1}\mathbb{1} \leq [\mathrm{d}_{\alpha\beta}(0, \omega)] \leq c\mathbb{1}, \text{ for some } c > 0.$$

To complete the proof of the proposition it remains to show that $\kappa = a(0, \omega)$.

To do this we reexpress the surface gravity κ. By [S1, Lemma 2.5] we have:

$$\kappa^2 = (\mathbf{h}^{ij}\partial_i N \partial_j N)_{|\mathcal{B}} - \frac{1}{2}(\mathbf{h}^{ij}\mathbf{h}^{kl}\nabla^{(\mathbf{h})}_i \mathbf{w}_l \nabla^{(\mathbf{h})}_j \mathbf{w}_k)_{|\mathcal{B}},$$

which using (A.3) gives $\kappa = a(0, \omega)$. □

A.2. **Proof of Prop. 2.5.** We recall that we defined the coordinates $(u,\omega) \in]-\delta,\delta[\times\mathcal{B}$ on a small neighborhood U of $\mathcal{B}$ in Σ. $U\cap\Sigma^+$ is diffeomorphic to $]0,\delta[\times\mathcal{B}$ using the coordinates (u,ω). If

$$X = u\cos(\kappa s), Y = u\sin(\kappa s),$$

we have:

$$du = u^{-1}(XdX + YdY),\ ds = \kappa^{-1}u^{-2}(XdY - YdX).$$

By Prop. 2.4 we obtain:

$$\begin{aligned}
&\mathbf{k}_{\alpha\beta}(u,\omega)d\omega^\alpha d\omega^\beta = d_{\alpha\beta}(X^2+Y^2,\omega)d\omega^\alpha d\omega^\beta,\\
&\mathrm{i}\mathbf{w}_\alpha(u,\omega)d\omega^\alpha ds = \mathrm{i}\kappa^{-1}b_\alpha(X^2+Y^2,\omega)(XdY-YdX)d\omega^\alpha,\\
&\mathrm{i}\mathbf{w}_0(u,\omega)duds = \mathrm{i}\kappa^{-1}b_0(X^2+Y^2,\omega)(XdY-YdX)(XdX+YdY),\\
&v^2(u,\omega)ds^2+du^2\\
&= u^2\kappa^2(1+u^2d(u^2,\omega))\kappa^{-2}u^{-4}(XdY-YdX)^2 + u^{-2}(XdX+YdY)^2\\
&= dX^2+dY^2+d(X^2+Y^2,\omega)(XdY-YdX)^2.
\end{aligned}$$

Let us denote by $B_2(0,\delta)=\{(X,Y)\in\mathbb{R}^2 : X^2+Y^2\leq\delta^2\}$ the open disk of center 0 and radius δ in $\mathbb{R}^2$. If $\beta=(2\pi)\kappa^{-1}$, then $(u,\kappa s)\in]0,\delta[\times\mathbb{S}_{2\pi}$ are polar coordinates on $B_2(0,\delta)\setminus\{0\}$. The expression (2.10) for $\mathbf{g}^{\mathrm{eucl}}$ and the estimates above show that $\mathbf{g}^{\mathrm{eucl}}$ extends as a smooth complex metric on $B_2(0,\delta)\otimes\mathcal{B}$.

We then construct $M^{\mathrm{eucl}}_{\mathrm{ext}}$ by gluing $B_2(0,\delta)\times\mathcal{B}$ with $M^{\mathrm{eucl}}=\mathbb{S}_\beta\times\Sigma^+$ over $\{(X,Y)\in\mathbb{R}^2 : \frac{1}{2}\delta^2<X^2+Y^2<\delta^2\}\times\mathcal{B}$ using the map:

$$\begin{aligned}
&\mathbb{S}_\beta\times]0,\delta[\times\mathcal{B}\to B_2(0,\delta)\times\mathcal{B}\\
&(s,u,\omega)\mapsto(u\cos(\kappa s),u\sin(\kappa s),\omega).
\end{aligned}\tag{A.4}$$

The complex metric $\mathbf{g}^{\mathrm{eucl}}$ defined on $\mathbb{S}_\beta\times\Sigma^+$ extends to a smooth complex metric $\mathbf{g}^{\mathrm{eucl}}_{\mathrm{ext}}$ on $M^{\mathrm{eucl}}_{\mathrm{ext}}$. By Prop. 2.4 we have $m=n(X^2+Y^2,\omega)$, hence m extends as a smooth function on $M^{\mathrm{eucl}}_{\mathrm{ext}}$.

Let us now embed Σ isometrically into $M^{\mathrm{eucl}}_{\mathrm{ext}}$. In the coordinates (u,ω) on Σ near $\mathcal{B}$ the embedding $\hat{\psi}$ becomes

$$(u,\omega)\mapsto\begin{cases}(0,u,\omega)\text{ for }0<u<\delta,\\ (\frac{\beta}{2},-u,\omega)\text{ for }-\delta<u<0,\end{cases}$$

which smoothly extends to $u=0$, the image of Σ under this extension being locally equal to $\{Y=0\}$.

The open set Ω_{ext} is obtained by gluing $\{Y>0\}$ with $]0,\frac{\beta}{2}[\times\Sigma^+$ using the map (A.4). This completes the proof.

A.3. **Proof of** (9.3)**.** A mechanical computation gives:

$$\begin{aligned}
\textstyle\sum_i\nabla_iT^i &= \textstyle\sum_i\partial_iT^i+\frac{1}{2}\sum_{i,k,l}\mathbf{k}^{il}(\partial_i\mathbf{k}_{kl}+\partial_k\mathbf{k}_{il}-\partial_l\mathbf{k}_{ik})T^k\\
&= \textstyle\sum_i\partial_iT^i+\frac{1}{2}\sum_{i,k,l}\mathbf{k}^{il}\partial_k\mathbf{k}_{il}T^k =: I,
\end{aligned}$$

using that $\mathbf{k}^{li}=\mathbf{k}^{il}$, $\mathbf{k}_{kl}=\mathbf{k}_{lk}$. Next

$$\sum_i|\mathbf{k}|^{-\frac{1}{2}}\partial_i(|\mathbf{k}|^{\frac{1}{2}}T^i)=\sum_i\partial_iT^i+\frac{1}{2}\sum_i|\mathbf{k}|^{-1}\partial_i|\mathbf{k}|T^i =: II.$$

Since

$$\det A(t)^{-1}\frac{d}{dt}\det A(t)=\mathrm{Tr}(A(t)^{-1}\frac{d}{dt}A(t)),$$

we get that $\partial_i|\mathbf{k}|=|\mathbf{k}|\mathrm{Tr}(\mathbf{k}^{-1}\partial_i\mathbf{k})$. Next we compute:

$$(\mathbf{k}^{-1}\partial_i\mathbf{k})^j_k=\sum_l\mathbf{k}^{jl}\partial_i\mathbf{k}_{lk},\ \mathrm{Tr}(\mathbf{k}^{-1}\partial_i\mathbf{k})=\sum_{k,l}\mathbf{k}^{kl}\partial_i\mathbf{k}_{lk},$$

which shows that $I = II$. □

References

[A] Adams, R.: *Sobolev Spaces*, Academic Press, (1975).

[DG] Derezinski, J., Gérard, C.: *Mathematics of Quantization and Quantum Fields*, Cambridge Monographs in Mathematical Physics, Cambridge University Press, 2013.

[GGH] Georgescu V., Gérard C., Häfner D.: Resolvent and propagation estimates for Klein-Gordon equations with non-positive energy, J. Spectr. Theory **5** (2015) 113-192.

[G] Gérard, C.: On the Hartle-Hawking-Israel states for spacetimes with static bifurcate Killing horizons, preprint arXiv:1608.06739 (2016).

[GOW] Gérard, C., Oulghazi, O., Wrochna, M.: Hadamard states for the Klein-Gordon equation on Lorentzian manifolds of bounded geometry, Commun. Math. Phys. **352** (2017) 352–519.

[GW1] Gérard C., Wrochna, M.: Construction of Hadamard states by pseudo-differential calculus, Comm. Math. Phys. **325** (2014) 713–755.

[GW2] Gérard C., Wrochna, M.: Analytic Hadamard states, Calderón projectors and Wick rotation near analytic Cauchy surfaces, (2017) ArXiv: 1706.08942.

[Gr] Grubb, G.: *Distributions and Operators*, Graduate Texts in Mathematics, Springer (2009)

[HH] Hartle, J., Hawking, S.: Path-integral derivation of black-hole radiance, Phys. Rev. D **13** (1976) 2188–2203.

[I] Israel, W.: Thermo-field dynamics of black holes, Phys. Lett. **57** (1976) 107-110.

[K1] Kay, B. S.: A uniqueness result for quasifree KMS states, Helv. Phys. Acta **58** (1985), 1017–1029.

[K2] Kay, B. S.: Purification of KMS states, Helv. Phys. Acta **58** (1985), 1030–1040.

[K3] Kay, B. S.: The double-wedge algebra for quantum fields on Schwarzschild and Minkowski spacetimes, Commun. Math. Phys. **100** (1985) 57–81.

[KW] Kay, B. S., Wald, R. M.: Theorems on the uniqueness and thermal properties of stationary, nonsingular, quasifree states on spacetimes with a bifurcate Killing horizon, Phys. Rep. **207** (1991) 49–136.

[R] Radzikowski, M.: Micro-local approach to the Hadamard condition in quantum field theory on curved spacetime, Comm. Math. Phys. **179** (1996) 529–553.

[S1] Sanders, K.: On the construction of Hartle-Hawking-Israel states across a static bifurcate Killing horizon, Letters in Math. Phys. **105** (2015) 575–640.

[S2] Sanders, K.: Thermal equilibrium states of a linear scalar quantum field in stationary spacetimes, Int. J. of Mod. Phys. A **28** (2013) 1330010.

[Sh] Shubin, M.: *Pseudodifferential Operators and Spectral Theory*, Springer 2001.

Université Paris-Sud XI, Département de Mathématiques, 91405 Orsay Cedex, France

E-mail address: christian.gerard@math.u-psud.fr