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# SYMMETRIC PAIRS AND BRANCHING LAWS

PAUL-ÉMILE PARADAN

ABSTRACT. Let  $G$  be a compact connected Lie group and let  $H$  be a subgroup fixed by an involution. A classical result assures that the  $H_{\mathbb{C}}$ -action on the flag variety  $\mathcal{F}$  of  $G$  admits a finite number of orbits. In this article we propose a formula for the branching coefficients of the symmetric pair  $(G, H)$  that is parametrized by  $H_{\mathbb{C}} \backslash \mathcal{F}$ .

## CONTENTS

1. Introduction	1
2. Non abelian localization	4
3. Proof of the main theorem	9
4. Examples	16
5. Kostant multiplicity formula	23
References	28

## 1. INTRODUCTION

Let  $G$  be a compact connected Lie group equipped with an involution  $\theta$ . Let  $G^\theta := \{g \in G, \theta(g) = g\}$  be the subgroup fixed by the involution. We consider a subgroup  $H \subset G$  such that  $(G^\theta)_0 \subset H \subset G^\theta$ . The purpose of this paper is the study of the branching laws between  $G$  and  $H$ .

Let  $T$  be a maximal torus of  $G$  that we choose  $\theta$ -invariant. Let  $\mathfrak{t}$  be the Lie algebra of  $T$ . Let  $\Lambda \subset \mathfrak{t}^*$  be the lattice of weights, and let  $\mathfrak{t}_+^*$  be a Weyl chamber. The irreducible representations of  $G$  are parametrized by the semi-group  $\Lambda_+ := \Lambda \cap \mathfrak{t}_+^*$  of dominant weights.

Let  $\lambda \in \Lambda_+$ . In order to study the restriction  $V_\lambda^G|_H$  of the irreducible  $G$ -representation  $V_\lambda^G$ , we consider the  $H$ -action on the flag variety  $\mathcal{F} = G/T$  of  $G$ . An important object is the  $H$ -invariant subset

$$Z_\theta \subset \mathcal{F}$$

formed of the elements  $x \in \mathcal{F}$  for which the stabilizer subgroup  $G_x := \{g \in G, gx = x\}$  is stable under  $\theta$ . In other words,  $gT \in Z_\theta$  if and only if  $g^{-1}\theta(g)$  belongs to the normalizer subgroup  $N(T)$ . A well-known result tells us that the group  $H$  has finitely many orbits in  $Z_\theta$ , and that the map  $\mathcal{O} \in H_{\mathbb{C}} \backslash \mathcal{F} \mapsto \mathcal{O} \cap Z_\theta \in H \backslash Z_\theta$  is bijective [8, 14, 12, 9].

Let  $x \in Z_\theta$ . The stabilizer subgroup  $G_x$  is a maximal torus in  $G$ , stable under  $\theta$ , with Lie algebra  $\mathfrak{g}_x$ . We will also consider the abelian subgroup  $H_x := G_x \cap H$  (that is

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not necessarily connected). Any weight  $\mu \in \Lambda$  determines a character  $\mathbb{C}_{\mu_x}$  of the torus  $G_x$  by taking  $\mu_x = g \cdot \mu$  if  $x = gT \in \mathcal{F}$ .

We denote by  $\mathfrak{R}_x \subset \mathfrak{g}_x^*$  the set of roots relative to the action of the Cartan subalgebra  $\mathfrak{g}_x$  on  $\mathfrak{g} \otimes \mathbb{C}$ . The map  $\mu \in \mathfrak{R} \mapsto \mu_x \in \mathfrak{R}_x$  is an isomorphism, and we take  $\mathfrak{R}_x^+ \subset \mathfrak{R}_x$  as the image of  $\mathfrak{R}^+ \subset \mathfrak{R}$  through this isomorphism.

The involution  $\theta$  leaves the set  $\mathfrak{R}_x$  invariant, and  $\alpha \in \mathfrak{R}_x$  is an *imaginary root* if  $\theta(\alpha) = \alpha$ . If  $\alpha$  is imaginary, the subspace  $(\mathfrak{g} \otimes \mathbb{C})_\alpha$  is  $\theta$ -stable. There are two cases. If the action of  $\theta$  on  $(\mathfrak{g} \otimes \mathbb{C})_\alpha$  is trivial then  $\alpha$  is *compact imaginary*. If the action of  $-\theta$  on  $(\mathfrak{g} \otimes \mathbb{C})_\alpha$  is trivial, then  $\alpha$  is *non-compact imaginary*. We denote respectively by  $\mathfrak{R}_x^{\text{ci}}$  and by  $\mathfrak{R}_x^{\text{nci}}$  the subsets of compact imaginary and non-compact imaginary roots, and we introduce the following  $G_x$ -modules

$$\mathbb{E}_x^{\text{ci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} (\mathfrak{g} \otimes \mathbb{C})_\alpha, \quad \mathbb{E}_x^{\text{nci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} (\mathfrak{g} \otimes \mathbb{C})_\alpha.$$

The weight

$$\delta(x) := \frac{1}{2} \sum_{\substack{\alpha \in \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \\ \theta(\alpha) \neq \alpha}} \alpha$$

defines a character  $\mathbb{C}_{\delta(x)}$  of the abelian group  $H_x$ .

We denote by  $R(H)$  and by  $R(H_x)$  the representations rings of the compact Lie groups  $H$  and  $H_x$ . An element  $E \in R(H)$  can be represented as a *finite* sum  $E = \sum_{V \in \widehat{H}} m_V V$ , with  $m_V \in \mathbb{Z}$ . We denote by  $\widehat{R}(H)$  (resp.  $\widehat{R}(H_x)$ ) the space of  $\mathbb{Z}$ -valued functions on  $\widehat{H}$  (resp.  $\widehat{H}_x$ ). An element  $E \in \widehat{R}(H)$  can be represented as an *infinite* sum  $\sum_{V \in \widehat{H}} m_V V$ , with  $m_V \in \mathbb{Z}$ . The induction map  $\text{Ind}_{H_x}^H : \widehat{R}(H_x) \rightarrow \widehat{R}(H)$  is the dual of the restriction morphism  $R(H) \rightarrow R(H_x)$ .

Let  $m_x = \frac{1}{2} |\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + \dim \mathbb{E}_x^{\text{nci}} \in \mathbb{N}$ .

The main result of this paper is the following theorem.

**Theorem 1.1.** *Let  $\lambda \in \Lambda_+$ . We have the decomposition*

$$(1.1) \quad V_\lambda^G|_H = \sum_{Hx \in H \backslash Z_\theta} Q_{Hx}(\lambda)$$

where the terms  $Q_{Hx}(\lambda) \in \widehat{R}(H)$  are defined by the following relation :

$$Q_{Hx}(\lambda) = (-1)^{m_x} \text{Ind}_{H_x}^H \left( \mathbb{C}_{\lambda_x + \delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \text{Sym}(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge \mathbb{E}_x^{\text{ci}} \right).$$

Here  $\text{Sym}(\mathbb{E}_x^{\text{nci}})$ , which is the symmetric algebra of  $\mathbb{E}_x^{\text{nci}}$ , is an admissible representation of  $H_x$  and  $\bigwedge \mathbb{E}_x^{\text{ci}} = \bigwedge^+ \mathbb{E}_x^{\text{ci}} \ominus \bigwedge^- \mathbb{E}_x^{\text{ci}}$  is a virtual representation of  $H_x$ .

We give now another formulation for decomposition (1.1) using the (right) action of the Weyl group  $W = N(T)/T$  on the flag variety  $\mathcal{F}$ . If  $x = gT \in \mathcal{F}$  and  $w \in W$  we take  $xw := gwT$ . We notice that  $Z_\theta$  is stable under the action of  $W$  and that the quotient  $Z_\theta/W$  parametrizes the set of maximal tori of  $G$  stable under  $\theta$ .

We associate to an element  $x = gT \in Z_\theta$  the subgroup  $W_x^H \subset W$  defined by the relation  $w \in W_x^H \iff Hxw = Hx$ . We denote by  $H \backslash Z_\theta / W$  the quotient of  $Z_\theta$  by the

action of  $H \times W$ , and by  $\bar{x} \in H \backslash Z_\theta / W$  the image of  $x \in Z_\theta$  through the quotient map. We associate to  $\bar{x} \in H \backslash Z_\theta / W$  the element  $Q_{\bar{x}}(\lambda) \in \widehat{R}(H)$  defined as follows

$$Q_{\bar{x}}(\lambda) = \sum_{\bar{w} \in W_x^H \backslash W} Q_{Hxw}(\lambda).$$

The previous theorem says then that  $V_\lambda^G|_H = \sum_{\bar{x} \in H \backslash Z_\theta / W} Q_{\bar{x}}(\lambda)$ . Here is a new formulation of Theorem 1.1.

**Theorem 1.2.** *We have  $V_\lambda^G|_H = \sum_{\bar{x} \in H \backslash Z_\theta / W} Q_{\bar{x}}(\lambda)$  where  $Q_{\bar{x}}(\lambda) \in \widehat{R}(H)$  has the following description*

$$Q_{\bar{x}}(\lambda) = \text{Ind}_{H_x}^H \left( \mathbb{M}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^{\text{ci}} \right),$$

for some<sup>1</sup>  $\mathbb{M}_x(\lambda) \in \widehat{R}(H_x)$ .

We finish this section by giving two basic examples associated to the group  $SU(2)$ . Here the flag variety of  $SU(2)$  is the 2-dimensional sphere  $\mathbb{S}^2$ . For  $n \geq 0$ , we denote by  $V_n$  the irreducible representation of  $SU(2)$  of dimension  $n + 1$ .

**Example 1.**  $G = SU(2)$  and the involution  $\theta$  is the conjugation by the matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$ . The subgroup fixed by  $\theta$  is the torus  $T \simeq U(1)$  and the critical set  $Z_\theta \subset \mathbb{S}^2$  is composed of the poles  $S, N$  and the equator  $E$ , so that  $T \backslash Z_\theta$  has three terms. We take  $\lambda = n$  in  $\widehat{SU(2)} \simeq \mathbb{N}$ .

For  $Hx = E$ , we have  $\mathbb{E}_x^{\text{nci}} = \mathbb{E}_x^{\text{ci}} = \{0\}$ ,  $H_x \simeq \mathbb{Z}_2$ , and  $\mathbb{C}_{\lambda_x + \delta(x)} = \mathbb{C}_n|_{\mathbb{Z}_2}$ . The contribution of  $E$  is then  $\text{Ind}_{\mathbb{Z}_2}^{U(1)}(\mathbb{C}_n|_{\mathbb{Z}_2}) = \mathbb{C}_n \otimes \sum_{k \in \mathbb{Z}} \mathbb{C}_{2k}$ .

For  $Hx = N$ , we have  $H_x = T$ ,  $\mathbb{E}_x^{\text{nci}} = \mathbb{C}_2$ ,  $\mathbb{E}_x^{\text{ci}} = \{0\}$ , and  $\mathbb{C}_{\lambda_x + \delta(x)} = \mathbb{C}_n$ . The contribution of  $N$  is then  $-\mathbb{C}_{n+2} \otimes \text{Sym}(\mathbb{C}_2)$ .

For  $Hx = S$ , we have  $H_x = T$ ,  $\mathbb{E}_x^{\text{nci}} = \mathbb{C}_{-2}$ ,  $\mathbb{E}_x^{\text{ci}} = \{0\}$ , and  $\mathbb{C}_{\lambda_x + \delta(x)} = \mathbb{C}_{-n}$ . The contribution of  $S$  is then  $-\mathbb{C}_{-n-2} \otimes \text{Sym}(\mathbb{C}_{-2})$ .

Finally, Relations (1.1) become

$$\begin{aligned} V_n|_T &= \mathbb{C}_n \otimes \sum_{k \in \mathbb{Z}} \mathbb{C}_{2k} - \mathbb{C}_{-n-2} \otimes \text{Sym}(\mathbb{C}_{-2}) - \mathbb{C}_{n+2} \otimes \text{Sym}(\mathbb{C}_2) \\ &= \sum_{k=-n}^0 \mathbb{C}_{2k+n}. \end{aligned}$$

**Example 2.**  $G = SU(2) \times SU(2)$  and the involution  $\theta$  is the map  $(a, b) \mapsto (b, a)$ . The subgroup fixed by  $\theta$  is  $SU(2)$  embedded diagonally and the critical set  $Z_\theta \subset \mathbb{S}^2 \times \mathbb{S}^2$  is equal to the union of the orbits  $SU(2) \cdot (N, N)$  and  $SU(2) \cdot (S, N)$ . Let  $\lambda = (n, m) \in \widehat{G}$ .

For  $x = (N, N)$  or  $x = (S, N)$  we have  $\mathbb{E}_x^{\text{nci}} = \mathbb{E}_x^{\text{ci}} = \{0\}$  and  $H_x \simeq T$ . For  $x = (N, N)$  we have  $\lambda_x + \delta(x) = m + n + 2$ , and for  $x = (S, N)$  we have  $\lambda_x + \delta(x) = m - n$ . Relations (1.1) give then

$$V_n \otimes V_m = \text{Ind}_T^{SU(2)}(\mathbb{C}_{m-n}) - \text{Ind}_T^{SU(2)}(\mathbb{C}_{m+n+2}).$$

<sup>1</sup>The precise expression of  $\mathbb{M}_x(\lambda)$  is given in Proposition 3.8.

It is not difficult to see that the previous identities correspond to the classical Clebsch-Gordan relations (see Example 4.2).

Here is a brief overview of the article. Sections 2 and 3 are devoted to the proof of our main result. In Section 4, we detail the case of  $U(p) \times U(q) \subset U(n)$ : in particular, we explain the branching formula we obtain for the restriction of  $U(n)$  to  $U(n-1)$ . In the last section, we recall Kostant's branching formula and explain the formula it gives in the case of the restriction of  $U(n)$  to  $U(n-1)$ , in order to compare it with our own formula.

### Notations

Throughout the paper :

- $G$  denotes a compact connected Lie group with Lie algebra  $\mathfrak{g}$ .
- $T$  is a maximal torus in  $G$  with Lie algebra  $\mathfrak{t}$ .
- $\Lambda \subset \mathfrak{t}^*$  is the weight lattice of  $T$  : every  $\mu \in \Lambda$  defines a 1-dimensional  $T$ -representation, denoted by  $\mathbb{C}_\mu$ , where  $t = \exp(X)$  acts by  $t^\mu := e^{i\langle \mu, X \rangle}$ .
- The coadjoint action of  $g \in G$  on  $\xi \in \mathfrak{g}^*$  is denoted by  $g \cdot \xi$ .
- When a Lie group  $K$  acts on set  $X$ , the stabilizer subgroup of  $x \in X$  is denoted by  $K_x := \{k \in K \mid k \cdot x = x\}$  and the Lie algebra of  $K_x$  is denoted by  $\mathfrak{k}_x$ .
- When a Lie group  $K$  acts on a manifold  $M$ , we denote by  $X \cdot m := \frac{d}{dt} e^{tX} \cdot m|_{t=0}$ ,  $m \in M$ , the infinitesimal action of  $X \in \mathfrak{k}$  on  $M$ .

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## 2. NON ABELIAN LOCALIZATION

Our main result is obtained by means of a non-abelian localization of the Riemann-Roch character on the flag variety  $\mathcal{F}$  of  $G$ . For that purpose we will use the family  $(\Omega_r)_r$  of symplectic structure parametrized by the interior of the Weyl chamber  $\mathfrak{t}_+^*$ . The symplectic structure  $\Omega_r$  comes from the identification  $gT \rightarrow g \cdot r$  of  $\mathcal{F}$  with the coadjoint orbit  $Gr$ . The moment map  $\Phi_r : \mathcal{F} \rightarrow \mathfrak{g}^*$  associated to the action of  $G$  on  $(\mathcal{F}, \Omega_r)$  is the map  $gT \mapsto g \cdot r$ .

At the level of Lie algebras we have  $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$  where  $\mathfrak{h} = \mathfrak{g}^\theta$  and  $\mathfrak{q} = \mathfrak{g}^{-\theta}$ . For any  $\xi \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ , we denote by  $\xi^+$  his  $\mathfrak{h}$ -part and by  $\xi^-$  his  $\mathfrak{q}$ -part. We use a  $G$ -invariant scalar product  $(-, -)$  on  $\mathfrak{g}$  such that the involution  $\theta$  is an orthogonal map. It induces identifications  $\mathfrak{g}^* \simeq \mathfrak{g}$ ,  $\mathfrak{h}^* \simeq \mathfrak{h}$  and  $\mathfrak{q}^* \simeq \mathfrak{q}$ .

The moment map  $\Phi_r^H : \mathcal{F} \rightarrow \mathfrak{h}^*$  associated to the action of  $H$  on  $(\mathcal{F}, \Omega_r)$  is the map  $gT \mapsto (g \cdot r)^+$ .

**2.1. Matsuki duality.** Consider the complex reductive groups  $G_{\mathbb{C}}$  and  $H_{\mathbb{C}}$  associated to the compact Lie groups  $G$  and  $H$ . Let  $L \subset G_{\mathbb{C}}$  be the real form such that  $H \subset L$  is a maximal compact subgroup of  $L$ .

Matsuki duality is the statement that a one-to-one correspondence exists between the  $H_{\mathbb{C}}$ -orbits and the  $L$ -orbits in  $\mathcal{F}$ ; two orbits are in duality when their intersection is a single orbit of  $H$ .

Uzawa, and Mirkovic-Uzawa-Vilonen [16, 9] proved the Matsuki correspondence by showing that both  $H_{\mathbb{C}}$ -orbits and  $L$ -orbits in  $\mathcal{F}$  are parametrized by the  $H$ -orbits in the set of critical points of the function  $\|\Phi_r^H\|^2 : \mathcal{F} \rightarrow \mathbb{R}$ .

First we recall the elementary but fundamental fact that the subset  $Z_\theta$  is equal to the set of critical points of the function  $\|\Phi_r^H\|^2$  [9, 3].

**Lemma 2.1.** *Let  $x = gT \in \mathcal{F}$  and  $r \in \text{Interior}(\mathfrak{t}_+^*)$ . The following statements are equivalent:*

- i) the subalgebra  $\mathfrak{g}_x$  is invariant under  $\theta$  (i.e.  $x \in Z_\theta$ ),
- ii)  $g^{-1}\theta(g) \in N(T)$ ,
- iii)  $x$  is a critical point of the function  $\|\Phi_r^H\|^2$ ,
- iv)  $(g \cdot r)^+$  and  $(g \cdot r)^-$  commute.

*Proof.* Let  $n_g = g^{-1}\theta(g)$  and let  $r$  be a regular element of  $\mathfrak{t}^* \simeq \mathfrak{t}$ . Since  $\mathfrak{g}_x = \text{Ad}(g)\mathfrak{t}$  we see that

$$\begin{aligned} \theta(\mathfrak{g}_x) = \mathfrak{g}_x &\iff n_g \in N_G(T) \\ &\iff [n_g \cdot \theta(r), r] = 0 \\ &\iff [\theta(g \cdot r), g \cdot r] = 0 \\ &\iff [(g \cdot r)^+, (g \cdot r)^-] = 0. \end{aligned}$$

A small computation shows that for any  $X \in \mathfrak{g}$  the derivative of the function  $t \mapsto \|\Phi_r^H(e^{tX}x)\|^2$  at  $t = 0$  is equal to  $(X, [g \cdot r, \theta(g \cdot r)])$ . Hence  $x = gT$  is a critical point of the function  $\|\Phi_r^H\|^2$  if and only if  $[g \cdot r, \theta(g \cdot r)] = 0$ . Finally we have proved that the statements *i), ii), iii)* and *iv)* are equivalent.  $\square$

Let us check the other easy fact.

**Lemma 2.2.** *The set  $H \setminus Z_\theta$  is finite.*

*Proof.* Let  $x = gT \in Z_\theta$ . A neighborhood of  $x$  is defined by elements of the form  $e^X e^Y x$  where  $X \in \mathfrak{h}$  and  $Y \in \mathfrak{q}$ . Now we see that  $e^X e^Y gT \in Z_\theta$  if and only if  $e^{-2g^{-1}Y} \in N(T)$ . If  $Y$  is sufficiently small the former relation is equivalent to  $g^{-1}Y \in \mathfrak{t}$ , and in this case  $e^X e^Y x = e^X x$ . We have proved that any element in  $H \setminus Z_\theta$  is isolated. As  $H \setminus Z_\theta$  is compact, we can conclude that  $H \setminus Z_\theta$  is finite.  $\square$

**2.2. Borel-Weil-Bott theorem.** We first recall the Borel-Weil-Bott theorem. The flag manifold  $\mathcal{F}$  is equipped with the  $G$ -invariant complex structure such that

$$\mathbf{T}_{eT}\mathcal{F} \simeq \sum_{\alpha \in \mathfrak{R}^+} (\mathfrak{g} \otimes \mathbb{C})_\alpha$$

is an identity of  $T$ -modules. Let us consider the tangent bundle  $\mathbf{T}\mathcal{F}$  as a complex vector bundle on  $\mathcal{F}$  with the invariant Hermitian structure  $h_{\mathcal{F}}$  induced by the invariant scalar product on  $\mathfrak{g}$ .

Any weight  $\lambda \in \Lambda$  defines a line bundle  $\mathcal{L}_\lambda \simeq G \times_T \mathbb{C}_\lambda$  on  $\mathcal{F}$ .

**Definition 2.3.** *We associate to a weight  $\lambda \in \Lambda$*

- the spin-c bundle on  $\mathcal{F}$

$$\mathcal{S}_\lambda := \bigwedge_{\mathbb{C}} \mathbf{T}\mathcal{F} \otimes \mathcal{L}_\lambda,$$

• the Riemann-Roch character  $\mathrm{RR}_G(\mathcal{F}, \mathcal{L}_\lambda) \in R(G)$  which is the equivariant index of the Dirac operator  $D_\lambda$  associated to the spin-c structure  $\mathcal{S}_\lambda$ .

The Borel-Weil-Bott theorem asserts that  $V_\lambda^G = \mathrm{RR}_G(\mathcal{F}, \mathcal{L}_\lambda)$  when  $\lambda$  is dominant. Now we consider the restriction  $V_\lambda^G|_H = \mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda)$ . In the next section, we will explain how we can localize the  $H$ -equivariant Riemann-Roch character  $\mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda)$  on the critical set of the function  $\|\Phi_r^H\|^2$  [10].

**2.3. Localization of the Riemann-Roch character.** In this section, we recall how we perform the ‘‘Witten non-abelian localization’’ of the Riemann-Roch character with the help of the moment map  $\Phi_r^H : \mathcal{F} \rightarrow \mathfrak{h}^*$  attached to a regular element  $r$  of the Weyl chamber [10, 6, 11].

Let us denote by  $X \mapsto [X]_{\mathfrak{g}/\mathfrak{t}}$  the projection  $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{t}$ . The Kirwan vector field  $\kappa_r$  on  $\mathcal{F}$  is defined as follows:

$$\kappa_r(x) = -\Phi_r^H(x) \cdot x \in \mathbf{T}_x\mathcal{F}.$$

Through the identification  $\mathfrak{g}/\mathfrak{t} \simeq \mathbf{T}_x\mathcal{F}$ ,  $X \mapsto \frac{d}{dt}|_{t=0} g e^{tX} T$ , the vector  $\kappa_r(x) \in \mathbf{T}_x\mathcal{F}$  is equal to  $[g^{-1}\theta(g) \cdot r]_{\mathfrak{g}/\mathfrak{t}}$ . Hence the set  $Z_\theta \subset \mathcal{F}$  is exactly the set where  $\kappa_r$  vanishes.

Let  $D_0$  be the Dirac operator associated to the spin-c structure  $\mathcal{S}_0 = \bigwedge_{\mathbb{C}} \mathbf{T}\mathcal{F}$ . The principal symbol of the elliptic operator  $D_0$  is the bundle map  $\sigma(\mathcal{F}) \in \Gamma(\mathbf{T}^*\mathcal{F}, \mathrm{hom}(\bigwedge_{\mathbb{C}}^+ \mathbf{T}\mathcal{F}, \bigwedge_{\mathbb{C}}^- \mathbf{T}\mathcal{F}))$  defined by the Clifford action

$$\sigma(\mathcal{F})(x, \nu) = \mathbf{c}_x(\tilde{\nu}) : \bigwedge_{\mathbb{C}}^+ \mathbf{T}_x\mathcal{F} \rightarrow \bigwedge_{\mathbb{C}}^- \mathbf{T}_x\mathcal{F}.$$

Here  $\nu \in \mathbf{T}_x^*\mathcal{O} \simeq \tilde{\nu} \in \mathbf{T}_x\mathcal{O}$  is the one to one map associated to the identification  $\mathfrak{g}^* \simeq \mathfrak{g}$  (see [2]).

Now we deform the elliptic symbol  $\sigma(\mathcal{F})$  by means of the vector field  $\kappa_r$  [10, 11].

**Definition 2.4.** *The symbol  $\sigma(\mathcal{F})$  shifted by the vector field  $\kappa_r$  is the symbol on  $\mathcal{F}$  defined by*

$$\sigma_r(\mathcal{F})(x, \nu) = \mathbf{c}_x(\tilde{\nu} - \kappa_r(x))$$

for any  $(x, \nu) \in \mathbf{T}^*\mathcal{F}$ .

Consider an  $H$ -invariant open subset  $\mathcal{U} \subset \mathcal{F}$  such that  $\mathcal{U} \cap Z_\theta$  is compact in  $\mathcal{F}$ . Then the restriction  $\sigma_r(\mathcal{F})|_{\mathcal{U}}$  is a  $H$ -transversally elliptic symbol on  $\mathcal{U}$ , and so its equivariant index is a well defined element in  $\widehat{R}(H)$  (see [1, 10, 11]).

Thus we can define the following localized equivariant indices.

**Definition 2.5.** *Let  $Hx \subset Z_\theta$ . We denote by*

$$\mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx) \in \widehat{R}(H)$$

the equivariant index of  $\sigma_r(\mathcal{F}) \otimes \mathcal{L}_\lambda|_{\mathcal{U}}$  where  $\mathcal{U}$  is an invariant neighbourhood of  $Hx$  so that  $\mathcal{U} \cap Z_\theta = Hx$ .

We proved in [10] that the following decomposition holds in  $\widehat{R}(H)$ :

$$\mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda) = \sum_{Hx \in H \setminus Z_\theta} \mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx).$$

The computation of the characters  $\mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx)$  will be handle in Section 3.1. To undertake these calculations we need to describe geometrically a neighborhood of  $Hx$  in  $\mathcal{F}$ . This is the goal of the next section.

2.4. **Local model near  $Hx \subset Z_\theta$ .** Let  $x = gT \in Z_\theta$ . We need to compute a symplectic model of a neighborhood of  $Hx$  in  $(\mathcal{F}, \Omega_r)$ . Here we use the identification  $\mathfrak{g} \simeq \mathfrak{g}^*$  given by the choice of an invariant scalar product. Let  $\mu = g \cdot r$  that we write  $\mu = \mu^+ + \mu^-$  where  $\mu^+ \in \mathfrak{h}$  and  $\mu^- \in \mathfrak{q}$ .

The tangent space  $\mathbf{T}_x \mathcal{F}$  is equipped with the symplectic two form  $\Omega_r|_x$ :

$$\Omega_r|_x(X \cdot x, Y \cdot x) = (\mu, [X, Y]), \quad X, Y \in \mathfrak{g}.$$

We need to understand the structure of the symplectic vector space  $(\mathbf{T}_x \mathcal{F}, \Omega_r|_x)$ . If  $\mathfrak{a} \subset \mathfrak{g}$  is a vector subspace we denote by  $\mathfrak{a} \cdot x := \{X \cdot x, X \in \mathfrak{a}\}$  the corresponding subspace of  $\mathbf{T}_x \mathcal{F}$ . The symplectic orthogonal of  $\mathfrak{a} \cdot x$  is denoted by  $(\mathfrak{a} \cdot x)^{\perp, \Omega}$ .

If  $\mathfrak{a}, \mathfrak{b}$  are two subspaces, a small computation gives that

$$(2.2) \quad (\mathfrak{a} \cdot x)^{\perp, \Omega} \cap \mathfrak{b} \cdot x \simeq \mathfrak{a}^\perp \cap [\mathfrak{b}, \mu],$$

where  $\mathfrak{a}^\perp \subset \mathfrak{g}$  is the orthogonal of  $\mathfrak{a}$  relatively to the scalar product.

We denote by  $\mathfrak{g}_{\mu^+} = \mathfrak{h}_{\mu^+} \oplus \mathfrak{q}_{\mu^+}$  the subspaces fixed by  $ad(\mu^+)$ . Notice that  $\mathfrak{g}_\mu = \mathfrak{g}_x$  is an abelian subalgebra containing  $\mu^+$  since  $[\mu^+, \mu^-] = 0$ . It follows that  $\mathfrak{g}_x \subset \mathfrak{g}_{\mu^+}$ .

**Lemma 2.6.**  $[\mathfrak{g}, \mu^+] \cdot x, \mathfrak{g}_{\mu^+} \cdot x$  and  $[\mathfrak{h}, \mu^+] \cdot x$  are symplectic subspaces of  $\mathbf{T}_x \mathcal{F}$ .

*Proof.* It is a direct consequence of (2.2).  $\square$

We consider now the symplectic subspace  $V_x \subset \mathbf{T}_x \mathcal{F}$  defined by the relation

$$(2.3) \quad V_x = ([\mathfrak{h}, \mu^+] \cdot x)^{\perp, \Omega} \cap [\mathfrak{g}, \mu^+] \cdot x.$$

A small computation shows that  $X \cdot x \in V_x$  if and only if  $[X, \mu] \in [\mathfrak{q}, \mu^+]$ .

We have the following important Lemma.

**Lemma 2.7.** • We have the following decomposition

$$(2.4) \quad \mathbf{T}_x \mathcal{F} = \mathfrak{g}_{\mu^+} \cdot x \overset{\perp}{\oplus} [\mathfrak{h}, \mu^+] \cdot x \overset{\perp}{\oplus} V_x$$

where  $\perp$  stands for the orthogonal relative to  $\Omega_r|_x$ .

- $\mathfrak{g}_{\mu^+} \cdot x$  is symplectomorphic to  $\mathfrak{h}_{\mu^+}/\mathfrak{h}_x \oplus (\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^*$ .
- $[\mathfrak{h}, \mu^+] \cdot x$  is symplectomorphic to  $\mathfrak{h}/\mathfrak{h}_{\mu^+}$  equipped with the symplectic structure  $\Omega_{\mu^+}(\bar{u}, \bar{v}) = (\mu^+, [u, v])$ .
- $V_x$  is symplectomorphic to  $(\mathfrak{h} \cdot x)^{\perp, \Omega} / [(\mathfrak{h} \cdot x)^{\perp, \Omega} \cap \mathfrak{h} \cdot x]$ .

*Proof.* If we use the decomposition  $\mathfrak{g} = \mathfrak{g}_{\mu^+} \oplus [\mathfrak{g}, \mu^+]$  and the fact that the abelian subalgebra  $\mathfrak{g}_x$  is contained in  $\mathfrak{g}_{\mu^+}$  we obtain

$$\mathbf{T}_x \mathcal{F} = \mathfrak{g}_{\mu^+} \cdot x \oplus [\mathfrak{g}, \mu^+] \cdot x.$$

It is obvious to check that the subspaces  $[\mathfrak{g}, \mu^+] \cdot x$  and  $\mathfrak{g}_{\mu^+} \cdot x$  are orthogonal relatively to the symplectic form  $\Omega_r|_x$ . Since  $[\mathfrak{h}, \mu^+] \cdot x$  is a symplectic subspace we have  $[\mathfrak{g}, \mu^+] \cdot x = [\mathfrak{h}, \mu^+] \cdot x \overset{\perp}{\oplus} V_x$  where  $V_x$  is defined by (2.3). The first point is proved.

The identities  $\mathfrak{g}_x = \theta(\mathfrak{g}_x) = \mathfrak{g}_{\theta(x)}$  imply the decompositions  $\mathfrak{g}_x = \mathfrak{h}_x \oplus \mathfrak{q}_x$  and  $\mathfrak{g}_{\mu^+} \cdot x = \mathfrak{q}_{\mu^+} \cdot x \oplus \mathfrak{h}_{\mu^+} \cdot x$ . The vector subspace  $\mathfrak{h}_{\mu^+} \cdot x$  is isomorphic to  $\mathfrak{h}_{\mu^+}/\mathfrak{h}_x$ , and the map  $v \mapsto \Omega_r|_x(v, -)$  defines an isomorphism between  $\mathfrak{q}_{\mu^+} \cdot x$  and the dual of  $\mathfrak{h}_{\mu^+} \cdot x$ . The second point is proved.



For the third point we use the isomorphism  $j : [\mathfrak{h}, \mu^+] \rightarrow \mathfrak{h}/\mathfrak{h}_{\mu^+}$  induced by the projection  $\mathfrak{h} \rightarrow \mathfrak{h}/\mathfrak{h}_{\mu^+}$ . Then the map  $\bar{u} \mapsto j(\bar{u}) \cdot x$  defines a symplectomorphism between  $(\mathfrak{h}/\mathfrak{h}_{\mu^+}, \Omega_{\mu^+})$  and  $[\mathfrak{h}, \mu^+] \cdot x$ .

Now we see that (2.4) together with the decomposition  $\mathfrak{h} \cdot x = \mathfrak{h}_{\mu^+} \cdot x \oplus [\mathfrak{h}, \mu^+] \cdot x$  leads to

$$\begin{aligned} (\mathfrak{h} \cdot x)^{\perp, \Omega} &= ([\mathfrak{h}, \mu^+] \cdot x)^{\perp, \Omega} \cap (\mathfrak{h}_{\mu^+} \cdot x)^{\perp, \Omega} \\ &= \mathfrak{h}_{\mu^+} \cdot x \oplus V_x \\ &= [(\mathfrak{h} \cdot x)^{\perp, \Omega} \cap \mathfrak{h} \cdot x] \oplus V_x. \end{aligned}$$

The last point follows.  $\square$

We denote by  $\Omega_{V_x}$  the restriction of  $\Omega_r|_x$  on the symplectic vector subspace  $V_x$ . The action of  $H_x$  on  $(V_x, \Omega_{V_x})$  is Hamiltonian, with moment map  $\Phi_{V_x} : V_x \rightarrow \mathfrak{h}_x^*$  defined by the relation

$$\langle \Phi_{V_x}(v), A \rangle = \frac{1}{2} \Omega_{V_x}(v, Av), \quad v \in V_x, \quad A \in \mathfrak{h}_x.$$

Thanks to Lemma 2.7, we know that the  $H_x$ -symplectic vector space  $(\mathbf{T}_x \mathcal{F}, \Omega_r|_x)$  admits the following decomposition

$$\mathbf{T}_x \mathcal{F} \simeq \mathfrak{h}_{\mu^+}/\mathfrak{h}_x \oplus (\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \oplus \mathfrak{h}/\mathfrak{h}_{\mu^+} \oplus V_x.$$

Thanks to the normal form Theorem of Marle [7] and Guillemin-Sternberg [5], we get the following result.

**Corollary 2.8.** *An  $H$ -equivariant symplectic model of a neighborhood of  $Hx$  in  $\mathcal{F}$  is  $\mathcal{F}_x := H \times_{H_{\mu^+}} Y_x$  where*

$$Y_x = H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x).$$

The corresponding moment map on  $\mathcal{F}_x$  is

$$\Phi_{\mathcal{F}_x}([h; \eta, v]) = h(\eta + \mu^+ + \Phi_{V_x}(v))$$

for  $[h; \eta, v] \in H \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x)$ .

We finish this section by computing a compatible complex structure on  $V_x$ .

By definition, the map that sends  $X \cdot x$  to  $[X, \mu]$  defines an isomorphism  $i : V_x \rightarrow [\mathfrak{q}, \mu^+]$ . The adjoint map  $ad(\mu)$  defines also an automorphism of  $[\mathfrak{g}, \mu^+]$ : for any  $X \in [\mathfrak{g}, \mu^+]$  we denote by  $\tilde{X} \in [\mathfrak{g}, \mu^+]$  the unique element such that  $ad(\mu)\tilde{X} = X$ .

The symplectic structure  $\Omega_{\mu} := (i^{-1})^* \Omega_{V_x}$  satisfies the relations

$$\Omega_{\mu}(X, Y) = (\mu, [\tilde{X}, \tilde{Y}]) = (X, \tilde{Y}) = -(\tilde{X}, Y), \quad \forall X, Y \in [\mathfrak{q}, \mu^+].$$

We work with the following  $H_x$ -equivariant maps

- the one to one map  $T_{\mu} := -ad(\mu)ad(\theta(\mu)) : [\mathfrak{g}, \mu^+] \rightarrow [\mathfrak{g}, \mu^+]$ ,
- the complex structure  $J_{\mu^+} = ad(\mu^+)(-ad(\mu^+)^2)^{-1/2}$  on  $[\mathfrak{g}, \mu^+]$ .

The map  $T_\mu$  restricts to a one to one map  $T_x : [\mathfrak{q}, \mu^+] \rightarrow [\mathfrak{q}, \mu^+]$  and  $J_{\mu^+}$  defines a complex structure on  $[\mathfrak{q}, \mu^+]$  (still denoted by  $J_{\mu^+}$ ).

Let  $S_x := (T_x^2)^{-1/2} T_x$ . The map  $J_{V_x} := J_{\mu^+} \circ S_x$  defines a  $H_x$ -invariant complex structure on  $[\mathfrak{q}, \mu^+]$ .

**Lemma 2.9.** *The  $H_x$ -symplectic space  $(V_x, \Omega_{V_x})$  is isomorphic to  $[\mathfrak{q}, \mu^+]$  equipped with the symplectic form  $\Omega_\mu^1(v, w) = (J_{V_x} v, w)$ .*

*Proof.* We know already that  $(V_x, \Omega_{V_x}) \simeq ([\mathfrak{q}, \mu^+], \Omega_\mu)$ . If one takes  $L = T_x \circ (-ad(\mu^+)^2)^{-1/4} \circ (T_x^2)^{-1/4}$ , we check easily that  $\Omega_\mu(L(v), L(w)) = (J_{V_x} v, w)$ .  $\square$

### 3. PROOF OF THE MAIN THEOREM

We start with the following lemma.

**Lemma 3.1.** *The quantity  $\text{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx)$  does not depend on the choice of the regular element  $r$  in the Weyl chamber. In the following we will denote it by  $Q_{Hx}(\lambda) \in \widehat{R}(H)$ .*

*Proof.* Let  $r_0, r_1$  be two regular elements of the Weyl chamber. For  $t \in [0, 1]$ , we consider the regular element  $r(t) = tr_1 + (1-t)r_0$ : the Kirwan vector field  $\kappa_{r(t)}$  vanishes exactly on  $Z_\theta$  for any  $t \in [0, 1]$ . If  $\mathcal{U}$  is an invariant neighbourhood of  $Hx$  so that  $\mathcal{U} \cap Z_\theta = Hx$ , then  $t \in [0, 1] \mapsto \sigma_{r(t)}(\mathcal{F}) \otimes \mathcal{L}_\lambda|_{\mathcal{U}}$  defines an homotopy of transversally elliptic symbols. Accordingly, the equivariant index of  $\sigma_{r_0}(\mathcal{F}) \otimes \mathcal{L}_\lambda|_{\mathcal{U}}$  and  $\sigma_{r_1}(\mathcal{F}) \otimes \mathcal{L}_\lambda|_{\mathcal{U}}$  are equal.  $\square$

**3.1. Computation of  $Q_{Hx}(\lambda)$ .** The computation of  $Q_{Hx}(\lambda)$  is done in three steps.

**3.1.1. Step 1: holomorphic induction.** Let  $H_{\mu^+} \subset H$  be the stabilizer subgroup of  $\mu^+ := \Phi_r^H(x)$ . By Corollary 2.8, a symplectic  $H$ -equivariant model of a neighborhood of  $Hx$  in  $\mathcal{F}$  is the manifold  $H \times_{H_{\mu^+}} Y_x$  where

$$Y_x = H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x).$$

The symplectic two form on  $Y_x$  is built from the canonical symplectic structure on  $H_{\mu^+} \times_{H_x} (\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \simeq \mathbf{T}^*(H_{\mu^+}/H_x)$  and the symplectic structure on  $V_x$ . The moment map relative to the action of  $H_{\mu^+}$  on  $Y_x$  is

$$\Phi_{Y_x}([h; \eta, v]) = h(\eta + \mu^+ + \Phi_{V_x}(v)) \in \mathfrak{h}_{\mu^+}^*,$$

for  $[h; \eta, v] \in H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x)$ .

Let  $\kappa_{Y_x}$  the Kirwan vector field on  $Y_x$ . It is immediate to check that  $[h; \eta, v] \in \{\kappa_{Y_x} = 0\}$  if and only if  $\eta = 0$  and  $(\mu^+ + \Phi_{V_x}(v)) \cdot v = 0$ . The map  $v \in V_x \mapsto \mu^+ \cdot v \in V_x$  is bijective and  $v \mapsto \Phi_{V_x}(v) \cdot v$  is homogeneous of degree equal to 3. Then there exists  $\epsilon > 0$  such that

$$(\mu^+ + \Phi_{V_x}(v)) \cdot v = 0 \quad \text{and} \quad \|v\| \leq \epsilon \implies v = 0.$$

In  $Y_x$ , we still denote by  $x$  the point  $[e, 0, 0]$ . We equip  $Y_x$  with an invariant almost complex structure that is compatible with the symplectic structure, and we denote by  $\text{RR}_{H_{\mu^+}}(Y_x, \mathcal{L}_\lambda|_{Y_x}, \Phi_{Y_x}, H_{\mu^+}x)$  the Riemann-Roch character on  $Y_x$  localized on the component  $H_{\mu^+}x \subset \{\kappa_{Y_x} = 0\}$ .

The quotient  $\mathfrak{h}/\mathfrak{h}_{\mu^+}$ , which is equipped with the invariant complex structure  $J_{\mu^+} := ad(\mu^+)(-ad(\mu^+)^2)^{-1/2}$ , is a complex  $H_{\mu^+}$ -module.

In [10][Theorem 7.5], we proved that  $Q_{H_x}(\lambda) = \text{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx)$  is equal to

$$(3.5) \quad \text{Ind}_{H_{\mu^+}}^H \left( \text{RR}_{H_{\mu^+}}(Y_x, \mathcal{L}_\lambda|_{Y_x}, \Phi_{Y_x}, H_{\mu^+}x) \otimes \bigwedge \mathfrak{h}/\mathfrak{h}_{\mu^+} \right).$$

3.1.2. *Step 2: cotangent induction.* The map  $\Phi_x(v) := \mu^+ + \Phi_{V_x}(v)$  is a moment map for the Hamiltonian action of  $H_x$  on  $V_x$ . The moment map on the  $H_{\mu^+}$ -manifold

$$Y_x = H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x)$$

is  $\Phi_{Y_x}([h; \eta, v]) = h(\eta + \Phi_x(v)) \in \mathfrak{h}_\mu^*$ .

Let  $\kappa_{V_x}(v) = -\Phi_x(v) \cdot v$  be the Kirwan vector field on  $V_x$ . We are interested in the connected component  $\{0\}$  of  $\{\kappa_{V_x} = 0\}$ . We choose a compatible almost complex structure on the symplectic vector space  $V_x$  and we denote by  $\text{RR}_{H_x}(V_x, \Phi_x, \{0\}) \in \widehat{\text{R}}(H_x)$  the Riemann-Roch character localized on  $\{0\} \subset \{\kappa_{V_x} = 0\}$ .

In Section 3.3 of [11] we have proved that

$$(3.6) \quad \text{RR}_{H_{\mu^+}}(Y_x, \mathcal{L}_\lambda|_{Y_x}, \Phi_{Y_x}, H_{\mu^+}x) = \text{Ind}_{H_x}^{H_{\mu^+}} (\text{RR}_{H_x}(V_x, \Phi_x, \{0\}) \otimes \mathcal{L}_\lambda|_x).$$

3.1.3. *Step 3: linear case.* We write  $\mathfrak{q}/\mathfrak{q}_{\mu^+}$  for the vector space  $[\mathfrak{q}, \mu^+]$  equipped with the complex structure  $J_{\mu^+}$ . So  $\mathfrak{q}/\mathfrak{q}_{\mu^+}$  is a  $H_{\mu^+}$ -module and we denote by  $\text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+})$  the corresponding symmetric algebra.

We need to compare the virtual  $H_x$ -modules  $\bigwedge_{J_{V_x}} V_x$  and  $\bigwedge_{-J_{\mu^+}} V_x$ . The weight

$$\delta(x) := \frac{1}{2} \sum_{\substack{\alpha \in \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \\ \theta(\alpha) \neq \alpha}} \alpha$$

defines a character  $\mathbb{C}_{\delta(x)}$  of the abelian group  $H_x$ . Recall that  $m_x \in \mathbb{N}$  corresponds to the quantity  $\frac{1}{2} |\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + \dim \mathbb{E}_x^{\text{nci}}$ .

The following lemma will be proved in Section 3.2.

**Lemma 3.2.** *The following identity holds :*

$$\bigwedge_{J_{V_x}} V_x \simeq (-1)^{m_x} \mathbb{C}_{\delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge_{-J_{\mu^+}} V_x.$$

On the vector space  $V_x$ , we can work with two localized Riemann-Roch characters:

- $\text{RR}_{H_x}(V_x, \Phi_x, \{0\})$  is defined with the complex structure  $J_{V_x}$ ,
- $\widetilde{\text{RR}}_{H_x}(V_x, \Phi_x, \{0\})$  is defined with the complex structure  $-J_{\mu^+}$ .

Thanks to the previous Lemma we know that  $\text{RR}_{H_x}(V_x, \Phi_x, \{0\})$  is equal to  $(-1)^{m_x} \mathbb{C}_{\delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \widetilde{\text{RR}}_{H_x}(V_x, \Phi_x, \{0\})$ .

**Proposition 3.3.** *We have*

$$(3.7) \quad \text{RR}_{H_x}(V_x, \Phi_x, \{0\}) = (-1)^{m_x} \mathbb{C}_{\delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+}).$$

*Proof.* For  $s \in [0, 1]$ , we consider the  $H_x$ -equivariant map  $\Phi^s : V_x \rightarrow \mathfrak{h}_x^*$  defined by the relation  $\Phi^s(v) = \mu^+ + s\Phi_{V_x}(v)$ . The corresponding Kirwan vector field on  $V_x$  is  $\kappa^s(v) = -\Phi^s(v) \cdot v$ . It is not difficult to see that there exists  $\epsilon > 0$  such that

$\{\kappa^s = 0\} \cap \{\|v\| \leq \epsilon\} = \{0\}$  for any  $s \in [0, 1]$ . Then a simple deformation argument gives that  $\widetilde{\text{RR}}_{H_x}(V_x, \Phi^s, \{0\})$  does not depend on  $s \in [0, 1]$ . We have proved that

$$\widetilde{\text{RR}}_{H_x}(V_x, \Phi_x, \{0\}) = \widetilde{\text{RR}}_{H_x}(V_x, \mu^+, \{0\})$$

where  $\mu^+$  denotes the constant map  $\Phi^0$ . Standard computations give  $\widetilde{\text{RR}}_{H_x}(V_x, \mu^+, \{0\}) = \text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+})$  (see [10][Proposition 5.4]). Our proof is completed.  $\square$

3.1.4. *Conclusion.* If we use the formulas (3.5), (3.6) and (3.7) we obtain the following expression

$$Q_{H_x}(\lambda) = (-1)^{m_x} \text{Ind}_{H_x}^H \left( \mathbb{C}_{\lambda_x + \delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+}) \otimes \bigwedge_{\mathbb{C}} \mathfrak{h}/\mathfrak{h}_{\mu^+} \right)$$

in  $\widehat{R}(H)$ . Here  $\mathbb{C}_{\lambda_x}$  is the character of  $G_x$  associated to the weight  $\lambda_x = g\lambda$ .

The previous formula depends on a choice of a regular element  $r$  in the Weyl chamber. In the next section we will propose another expression for  $Q_{H_x}(\lambda)$  that does not depend on this choice.

3.2. **Another expression for  $Q_{H_x}(\lambda)$ .** Let  $\mathfrak{R}_x \subset \mathfrak{g}_x^*$  be the roots for the action of the torus  $G_x$  on  $\mathfrak{g} \otimes \mathbb{C}$ . The involution  $\theta : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$  leaves the set  $\mathfrak{R}_x$  invariant and a root  $\alpha \in \mathfrak{R}_x$  is called *imaginary* if  $\theta(\alpha) = \alpha$ . We denote respectively by  $\mathfrak{R}_x^{\text{ci}}$  and by  $\mathfrak{R}_x^{\text{nci}}$  the subsets of *compact imaginary* and *non-compact imaginary* roots.

We choose a generic element  $r \in \mathfrak{t}_+^*$  such that  $\mu^+ = (g \cdot r)^+$  satisfies the following relation : for any  $\alpha \in \mathfrak{R}_x$ , we have

$$(\alpha, \mu^+) = 0 \iff \theta(\alpha) = -\alpha.$$

Notice that an imaginary roots  $\alpha$  is positive if and only if  $(\alpha, \mu^+) > 0$ .

**Definition 3.4.** We consider the subset  $\mathfrak{A}_x \subset \mathfrak{R}_x$  defined by the following relations:

$$\alpha \in \mathfrak{A}_x \iff \alpha(\mu^+) > 0 \quad \text{and} \quad \theta(\alpha) \neq \alpha.$$

The involution  $\theta$  defines a free action of  $\mathbb{Z}_2$  on the set  $\mathfrak{A}_x$ . We denote by  $\mathfrak{A}_x/\mathbb{Z}_2$  its quotient. For any  $\alpha \in \mathfrak{R}_x$ , we denote by  $\mathbb{C}_\alpha$  the corresponding 1-dimensional representation of  $G_x$ , and  $\mathbb{C}_\alpha|_{H_x}$  its restriction to the subgroup  $H_x$ . We have a natural map  $[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2 \mapsto \mathbb{C}_\alpha|_{H_x} \in R(H_x)$ .

For any  $\alpha \in \mathfrak{R}_x$  we define

$$\tilde{\alpha} = \pm \alpha$$

where  $\pm$  is the sign of  $\alpha(\mu)\alpha(\theta(\mu))$ .

We consider the  $H_x$ -modules  $\mathfrak{h}/\mathfrak{h}_{\mu^+} := ([\mathfrak{h}, \mu^+], J_{\mu^+})$ ,  $\mathfrak{q}/\mathfrak{q}_{\mu^+} := (V_x, J_{\mu^+})$  and  $(V_x, J_{V_x})$ .

**Lemma 3.5.** We have the following isomorphisms of  $H_x$ -modules

$$\begin{aligned} \mathfrak{h}/\mathfrak{h}_{\mu^+} &\simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha|_{H_x} & [A], \\ \mathfrak{q}/\mathfrak{q}_{\mu^+} &\simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha|_{H_x} & [B], \\ (V_x, J_{V_x}) &\simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_{\tilde{\alpha}}|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_{\tilde{\alpha}}|_{H_x} & [C]. \end{aligned}$$

*Proof.* Thanks to Lemma 2.9, we know that the  $H_x$ -module  $(V_x, J_{V_x})$  is isomorphic to the vector space  $[\mathfrak{q}, \mu^+]$  equipped with the complex structure  $J_{V_x} := J_{\mu^+} \circ S_x$ . We consider the vector spaces  $[\mathfrak{q}, \mu^+]$  and  $[\mathfrak{g}, \mu^+]$  equipped with the complex structure  $J_{\mu^+}$ . The projection (taking the real part)  $\mathbf{r} : \mathfrak{g} \otimes \mathbb{C} \rightarrow \mathfrak{g}$  induces an isomorphism of  $G_x$ -modules

$$\mathbf{r} : \bigoplus_{\alpha(\mu^+) > 0} (\mathfrak{g} \otimes \mathbb{C})_\alpha \longrightarrow [\mathfrak{g}, \mu^+].$$

The orthogonal projections  $\mathbf{p}_1 : [\mathfrak{g}, \mu^+] \rightarrow [\mathfrak{q}, \mu^+]$  and  $\mathbf{p}_2 : [\mathfrak{g}, \mu^+] \rightarrow [\mathfrak{h}, \mu^+]$  commute with the  $H_x$ -action, so the maps

$$\begin{aligned} \mathbf{p}_1 \circ \mathbf{r} &: \bigoplus_{\alpha(\mu^+) > 0} (\mathfrak{g} \otimes \mathbb{C})_\alpha \longrightarrow [\mathfrak{q}, \mu^+], \\ \mathbf{p}_2 \circ \mathbf{r} &: \bigoplus_{\alpha(\mu^+) > 0} (\mathfrak{g} \otimes \mathbb{C})_\alpha \longrightarrow [\mathfrak{h}, \mu^+] \end{aligned}$$

are surjective morphisms of  $H_x$ -modules.

Let  $V_x^1(\alpha) = \mathbf{p}_1 \circ \mathbf{r}((\mathfrak{g} \otimes \mathbb{C})_\alpha)$ . We notice that  $\dim_{\mathbb{C}} V_x^1(\alpha) \in \{0, 1\}$ :  $V_x^1(\alpha) = \{0\}$  only if  $\alpha$  is a non-compact imaginary root and  $V_x^1(\alpha) \simeq \mathbb{C}_\alpha|_{H_x}$  when  $V_x^1(\alpha) \neq \{0\}$ . We notice also that  $V_x^1(\alpha) = V_x^1(\theta(\alpha))$ , hence

$$\mathfrak{q}/\mathfrak{q}_{\mu^+} = ([\mathfrak{q}, \mu^+], J_{\mu^+}) \simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} V_x^1(\alpha) \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} V_x^1(\alpha).$$

The identity [B] is proved.

Similarly we consider  $V_x^2(\alpha) = \mathbf{p}_2 \circ \mathbf{r}((\mathfrak{g} \otimes \mathbb{C})_\alpha)$ . We notice that  $\dim_{\mathbb{C}} V_x^2(\alpha) \in \{0, 1\}$ :  $V_x^2(\alpha) = \{0\}$  only if  $\alpha$  is a compact imaginary root and  $V_x^2(\alpha) \simeq \mathbb{C}_\alpha|_{H_x}$  when  $V_x^2(\alpha) \neq \{0\}$ . We notice also that  $V_x^2(\alpha) = V_x^2(\theta(\alpha))$ , hence

$$\mathfrak{h}/\mathfrak{h}_{\mu^+} = ([\mathfrak{h}, \mu^+], J_{\mu^+}) \simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} V_x^2(\alpha) \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} V_x^2(\alpha).$$

The identity [A] is proved.

Finally we check that the complex structures  $J_{\mu^+}$  and  $J_{V_x}$  preserve each  $V_x^1(\alpha)$  and that  $(V_x(\alpha), J_{V_x}) \simeq \mathbb{C}_{\tilde{\alpha}}|_{H_x}$  when  $(\alpha, \mu^+) > 0$ . The identity [C] follows.  $\square$

We consider the  $H_x$ -module  $\mathbb{V}_x := \sum_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x}$ , and the  $G_x$ -modules  $\mathbb{E}_x^{\text{nci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha$  and  $\mathbb{E}_x^{\text{ci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha$ . In the previous lemma we have proved that  $H_x$ -modules  $\mathfrak{h}/\mathfrak{h}_{\mu^+}$  and  $\mathfrak{q}/\mathfrak{q}_{\mu^+}$  are respectively isomorphic to  $\mathbb{V}_x \oplus \mathbb{E}_x^{\text{ci}}$  and  $\mathbb{V}_x \oplus \mathbb{E}_x^{\text{nci}}$ . If we use the fact that  $\text{Sym}(\mathbb{V}_x) \otimes \bigwedge \mathbb{V}_x = 1$ , we get the following corollary.

**Corollary 3.6.** *We have the following identity of virtual  $H_x$ -modules:*

$$\text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+}) \otimes \bigwedge \mathfrak{h}/\mathfrak{h}_{\mu^+} \simeq \text{Sym}(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge \mathbb{E}_x^{\text{ci}}.$$

*Proof of Lemma 3.2.* Let  $\mathcal{B} := \mathfrak{A}_x/\mathbb{Z}_2 \cup (\mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+)$ . We see that  $\bigwedge_{J_{V_x}} V_x = \prod_{\alpha \in \mathcal{B}} (1 - t^{\tilde{\alpha}})$  whereas  $\bigwedge_{-J_{\mu^+}} V_x = \prod_{\alpha \in \mathcal{B}} (1 - t^{-\alpha})$ . Accordingly we get  $\bigwedge_{J_{V_x}} V_x \simeq (-1)^{|\mathcal{B}'|} \mathbb{C}_\eta \otimes \bigwedge_{-J_{\mu^+}} V_x$  where  $\mathcal{B}' = \{\alpha \in \mathcal{B}, \tilde{\alpha} = \alpha\}$  and  $\eta = \sum_{\alpha \in \mathcal{B}'} \alpha$ . Now it is easy

to check that an element  $\alpha \in \mathcal{B}$  belongs to  $\mathcal{B}'$  if and only if  $\alpha$  and  $\theta(\alpha)$  both belong to  $\mathfrak{R}_x^+$ . In other words

$$\mathcal{B}' = \{\alpha \in \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+), \theta(\alpha) \neq \alpha\} / \mathbb{Z}_2 \bigcup \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+.$$

We have proved that

$$\bigwedge_{J_{V_x}} V_x \simeq (-1)^{m_x} \mathbb{C}_{\delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge_{-J_{\mu^+}} V_x.$$

□

Finally, thanks to Lemma 3.2 and Corollary 3.6, we obtain the final formula for  $Q_{H_x}(\lambda)$  (that does not depend on the choice of  $r$ ):

$$Q_{H_x}(\lambda) = (-1)^{m_x} \text{Ind}_{H_x}^H \left( \mathbb{C}_{\lambda_x + \delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \text{Sym}(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge \mathbb{E}_x^{\text{ci}} \right).$$

**3.3. Computation of the virtual module  $\mathbb{M}_x(\lambda)$ .** According to Theorem 1.1, we have the decomposition  $V_\lambda^G|_H = \sum_{\bar{x}} Q_{\bar{x}}(\lambda)$  where  $Q_{\bar{x}}(\lambda) = \text{Ind}_{H_x}^H (\mathbb{A}_x(\lambda))$ , and  $\mathbb{A}_x(\lambda) \in \widehat{R}(H_x)$  has the following description

$$\mathbb{A}_x(\lambda) = \frac{1}{|W_x^H|} \sum_{w \in W} (-1)^{m_{xw}} \mathbb{C}_{\lambda_{xw} + \delta(xw)} \otimes \det(\mathbb{E}_{xw}^{\text{nci}}) \otimes \text{Sym}(\mathbb{E}_{xw}^{\text{nci}}) \otimes \bigwedge \mathbb{E}_{xw}^{\text{ci}}.$$

The aim of this section is to simplify the expression of the virtual  $H_x$ -module  $\mathbb{A}_x(\lambda)$ . We start by comparing the  $G_x$ -modules  $\mathbb{E}_{xw}^{\text{nci}}$  and  $\mathbb{E}_x^{\text{nci}}$ . We use the decomposition  $\mathbb{E}_x^{\text{nci}} = (\mathbb{E}_x^{\text{nci}})_w^+ \oplus (\mathbb{E}_x^{\text{nci}})_w^-$  where

$$(\mathbb{E}_x^{\text{nci}})_w^+ := \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+} \mathbb{C}_\alpha, \quad \text{and} \quad (\mathbb{E}_x^{\text{nci}})_w^- = \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+ \cap \neg \mathfrak{R}_{xw}^+} \mathbb{C}_\alpha.$$

We have the following basic lemma (see Lemma 3.10).

**Lemma 3.7.** *The  $G_x$ -module  $|\mathbb{E}_x^{\text{nci}}|_w := (\mathbb{E}_x^{\text{nci}})_w^+ \oplus \overline{(\mathbb{E}_x^{\text{nci}})_w^-}$  is isomorphic to  $\mathbb{E}_{xw}^{\text{nci}}$ .*

Let  $\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}^+} \alpha$ . We denote by  $w \bullet \lambda = w(\lambda + \rho) - \rho$  the affine action of the Weyl group on the lattice  $\Lambda$ .

The main result of this section is the following proposition.

**Proposition 3.8.** *Let  $x \in Z_\theta$ . We have*

$$\mathbb{A}_x(\lambda) = \mathbb{M}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^{\text{ci}}$$

where  $\mathbb{M}_x(\lambda) \in \widehat{R}(H_x)$  is defined by the following expression

$$\mathbb{M}_x(\lambda) = \frac{(-1)^{n_x}}{|W_x^H|} \sum_{w \in W} (-1)^{k_{x,w}} \mathbb{C}_{(w \bullet \lambda)_x} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+) \otimes \text{Sym}(|\mathbb{E}_x^{\text{nci}}|_w),$$

and

- $k_{x,w} = |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq \pm \alpha\}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{ci}}|,$
- $n_x := |\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_x^+| - \frac{1}{2} |\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_x^+ \cap \{\theta(\alpha) \neq \alpha\}|.$

**Remark 3.9.** We can describe  $Q_{\bar{x}}(\lambda)$  differently by taking  $\{w_1, \dots, w_p\} \subset W$  such that  $W_x^H \setminus W \simeq \{\bar{w}_1, \dots, \bar{w}_p\}$ . We have  $Q_{\bar{x}}(\lambda) = \text{Ind}_{H_x}^H \left( \tilde{\mathbb{A}}_x(\lambda) \right)$  with

$$\tilde{\mathbb{A}}_x(\lambda) = \tilde{\mathbb{M}}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^{\text{ci}}$$

and where  $\tilde{\mathbb{M}}_x(\lambda) \in \widehat{R}(H_x)$  is defined by the following expression

$$\tilde{\mathbb{M}}_x(\lambda) = (-1)^{n_x} \sum_{k=1}^p (-1)^{k_{x,w_k}} \mathbb{C}_{(w_k \bullet \lambda)_x} \otimes \det((\mathbb{E}_x^{\text{nci}})_{w_k}^+) \otimes \text{Sym}(|\mathbb{E}_x^{\text{nci}}|_{w_k}).$$

We need to introduce some notations. To  $x \in Z_\theta$ , we associate :

- The polarized roots : to  $\alpha \in \mathfrak{R}_x$  and  $w \in W$ , we associate  $|\alpha|_w \in \mathfrak{R}_x$  defined as follows

$$|\alpha|_w = \begin{cases} \alpha & \text{if } \alpha \in \mathfrak{R}_{xw}^+, \\ -\alpha & \text{if } \alpha \notin \mathfrak{R}_{xw}^+. \end{cases}$$

- The following  $G_x$ -weights :

$$\gamma_{x,w}^{\text{ci}} := \sum_{\substack{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+ \\ |\alpha|_w \neq \alpha}} \alpha, \quad \gamma_{x,w}^{\text{nci}} := \sum_{\substack{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+ \\ |\alpha|_w \neq \alpha}} \alpha, \quad \gamma_{x,w} := \sum_{\substack{\alpha \in \mathfrak{R}_x^+ \\ |\alpha|_w \neq \alpha}} \alpha.$$

The proof of Proposition 3.8 is based on the following Lemma.

**Lemma 3.10.** Let  $x \in Z_\theta$  and  $w \in W$ . Let  $d_{x,w}^{\text{ci}}$  be the cardinal of the set  $\{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+, |\alpha|_w \neq \alpha\}$ . We have the following relations

- (1)  $\mathbb{E}_{xw}^{\text{nci}} \simeq \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_{|\alpha|_w}$  and  $\mathbb{E}_{xw}^{\text{ci}} \simeq \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \mathbb{C}_{|\alpha|_w}$ ,
- (2)  $\det(\mathbb{E}_{xw}^{\text{nci}}) = \mathbb{C}_{-\gamma_{x,w}^{\text{nci}}} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+)$ ,
- (3)  $\bigwedge \mathbb{E}_{xw}^{\text{ci}} = (-1)^{d_{x,w}^{\text{ci}}} \mathbb{C}_{-\gamma_{x,w}^{\text{ci}}} \otimes \bigwedge \mathbb{E}_x^{\text{ci}}$ .
- (4) The  $H_x$ -weight  $\delta(xw) - \delta(x)$  is equal to the restriction of  $G_x$ -weight  $\gamma_{x,w}^{\text{nci}} + \gamma_{x,w}^{\text{ci}} - \gamma_{x,w}$  to  $H_x$ .

*Proof.* We remark that  $\mathfrak{R}_x = \mathfrak{R}_{xw}$ ,  $\mathfrak{R}_x^+ = g(\mathfrak{R}^+)$  and  $\mathfrak{R}_{xw}^+ = g(w\mathfrak{R}^+)$ . The first point follows and points (ii) and (iii) derive from the first.

Let us check the last point. The term  $\rho_x := \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_x^+} \alpha$  is the image of  $\rho := \frac{1}{2} \sum_{\alpha' \in \mathfrak{R}^+} \alpha'$  through the map  $\mu \mapsto \mu_x$ . We see that

$$\rho_x + \theta(\rho_x) = \sum_{\alpha \in \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+)} \alpha = 2\delta(x) + 2\rho_x^{\text{nci}} + 2\rho_x^{\text{ci}}$$

where  $\rho_x^{\text{nci}} = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \alpha$  and  $\rho_x^{\text{ci}} = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \alpha$ . Similarly we have

$$\rho_{xw} + \theta(\rho_{xw}) = 2\delta(xw) + 2\rho_{xw}^{\text{nci}} + 2\rho_{xw}^{\text{ci}}.$$

Thus the  $H_x$ -weight  $\delta(xw) - \delta(x)$  is equal to the restriction to  $H_x$  of the  $G_x$ -weight

$$\beta(x, w) := \rho_{xw} - \rho_x + (\rho_x^{\text{nci}} - \rho_{xw}^{\text{nci}}) + (\rho_x^{\text{ci}} - \rho_{xw}^{\text{ci}}).$$

We notice that  $\rho_{xw} - \rho_x = (w\rho - \rho)_x = -\gamma_{x,w}$ . Furthermore, small computations give that  $\rho_x^{\text{nci}} - \rho_{xw}^{\text{nci}} = \gamma_{x,w}^{\text{nci}}$  and  $\rho_x^{\text{ci}} - \rho_{xw}^{\text{ci}} = \gamma_{x,w}^{\text{ci}}$ . We have proved that  $\beta(x, w) = \gamma_{x,w}^{\text{nci}} + \gamma_{x,w}^{\text{ci}} - \gamma_{x,w}$ . The last point follows.  $\square$

Now, we can finish the proof of the Proposition 3.8. We must check that the virtual  $H_x$ -module

$$\mathbf{A} := (-1)^{m_{xw}} \mathbb{C}_{\lambda_{xw} + \delta(xw)} \otimes \det(\mathbb{E}_{xw}^{\text{nci}}) \otimes \bigwedge \mathbb{E}_{xw}^{\text{ci}}$$

is equal to the virtual  $H_x$ -module

$$\mathbf{B} := (-1)^{n_x + k_{x,w}} \mathbb{C}_{(w \bullet \lambda)_x + \delta(x)} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+) \otimes \bigwedge \mathbb{E}_x^{\text{ci}}.$$

If we use Lemma 3.10, we get

$$\mathbf{A} = (-1)^{m_{xw} + d_{x,w}^{\text{ci}}} \mathbb{C}_{(w(\lambda + \rho) - \rho)_x + \delta(x)} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+) \otimes \bigwedge \mathbb{E}_x^{\text{ci}}.$$

Thus the equality  $\mathbf{A} = \mathbf{B}$  follows from the following lemma.

**Lemma 3.11.** *For any  $x \in Z_\theta$  and  $w \in W$ , we have  $n_x + k_{x,w} = m_{xw} + d_{x,w}^{\text{ci}} \pmod{2}$ .*

*Proof.* In order to simplify our notations, we write  $a \equiv b$  for  $a = b \pmod{2}$ .

We have  $\dim \mathbb{E}_x^{\text{nci}} = \dim \mathbb{E}_{xw}^{\text{nci}}$  and  $\dim \mathbb{E}_x^{\text{ci}} = \dim \mathbb{E}_{xw}^{\text{ci}}$ , then

$$\begin{aligned} m_{xw} - m_x &= \frac{1}{2} (|\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) \cap \{\theta(\alpha) \neq \alpha\}| - |\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}|) \\ &= \frac{1}{2} (|\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)| - |\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+)|). \end{aligned}$$

We remark now that

$$\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) = A_{++} \cup A_{--} \cup A_{+-} \cup A_{-+}$$

with  $A_{++} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$ ,  $A_{--} = -\mathfrak{R}_x^+ \cap -\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$ ,  $A_{+-} = \mathfrak{R}_x^+ \cap \theta(-\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$  and  $A_{-+} = -\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$ .

Similarly we have

$$\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) = B_{++} \cup B_{--} \cup B_{+-} \cup B_{-+}$$

with  $B_{++} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$ ,  $B_{--} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap -\mathfrak{R}_{xw}^+ \cap \theta(-\mathfrak{R}_{xw}^+)$ ,  $B_{+-} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(-\mathfrak{R}_{xw}^+)$  and  $B_{-+} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap -\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$ .

We have the obvious relations :  $A_{++} = B_{++}$ ,  $A_{--} = -B_{--}$ ,  $\theta(A_{+-}) = A_{-+}$ ,  $\theta(B_{+-}) = B_{-+}$  and  $A_{++} = B_{++}$ . So we get  $m_{xw} - m_x \equiv |A_{+-}| + |B_{+-}|$ .

Let consider  $\mathcal{A} := \mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+$  and  $\mathcal{B} := \mathfrak{R}_x^+ \cap -\mathfrak{R}_{xw}^+$ . We have

$$\begin{aligned} m_{xw} - m_x &\equiv |\mathcal{A} \cap \theta(\mathcal{B})| + |\mathcal{A} \cap -\theta(\mathcal{B})| \\ &\equiv |\mathcal{A}| + |\mathcal{A} \cap \theta(\mathcal{A})| + |\mathcal{A} \cap -\theta(\mathcal{A})|. \end{aligned}$$

Now we remark that

$$\begin{aligned} |\mathcal{A} \cap \theta(\mathcal{A})| &\equiv |\mathcal{A} \cap \theta(\mathcal{A}) \cap \{\theta(\alpha) = \alpha\}| \\ &\equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}|. \end{aligned}$$

Similarly

$$\begin{aligned} |\mathcal{A} \cap -\theta(\mathcal{A})| &\equiv |\mathcal{A} \cap -\theta(\mathcal{A}) \cap \{\theta(\alpha) = -\alpha\}| \\ &\equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = -\alpha\}|. \end{aligned}$$



At this stage we have proved that

$$\begin{aligned} m_{xw} - m_x &\equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = -\alpha\}| \\ &\equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq -\alpha\}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}|. \end{aligned}$$

As  $d_{x,w}^{\text{ci}} = |\mathfrak{R}_x^+ \cap -\mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{ci}}|$ , we have  $|\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}| + d_{x,w}^{\text{ci}}$  is equal to  $\dim \mathbb{E}_x^{\text{ci}} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{nci}}|$ . This implies that  $m_{xw} + d_{x,w}^{\text{ci}}$  is equal, modulo 2, to

$$\begin{aligned} &m_x + \dim \mathbb{E}_x^{\text{ci}} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{nci}}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq -\alpha\}| \\ &\equiv m_x + \dim \mathbb{E}_x^{\text{ci}} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{ci}}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq \pm\alpha\}|. \end{aligned}$$

By definition  $m_x = \frac{1}{2}|\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + \dim \mathbb{E}_x^{\text{nci}}$  and then

$$\begin{aligned} m_x + \dim \mathbb{E}_x^{\text{ci}} &\equiv \frac{1}{2}|\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + |\mathfrak{R}_x^+ \cap \{\theta(\alpha) = \alpha\}| \\ &\equiv n_x. \end{aligned}$$

Finally we have proved that  $m_{xw} + d_{x,w}^{\text{ci}}$  is equal, modulo 2, to  $n_x + k_{x,w}$ . □

#### 4. EXAMPLES

In this section we will study in details some examples of our formula

$$V_\lambda^G|_H = \sum_{\bar{x} \in H \backslash Z_\theta / W} Q_{\bar{x}}(\lambda)$$

where  $Q_{\bar{x}}(\lambda) = \text{Ind}_{H_x}^H (\mathbb{M}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^{\text{ci}})$  and

$$\mathbb{M}_x(\lambda) = \frac{(-1)^{n_x}}{|W_x^H|} \sum_{w \in W} (-1)^{k_{x,w}} \mathbb{C}_{(w \bullet \lambda)_x} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+) \otimes \text{Sym}(|\mathbb{E}_x^{\text{nci}}|_w).$$

Here the integers  $k_{x,w}$  and  $n_x$  are defined as follows:

- $k_{x,w} = |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq \pm\alpha\}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{ci}}|$ ,
- $n_x = |\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_x^+| - \frac{1}{2}|\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_x^+ \cap \{\theta(\alpha) \neq \alpha\}|$ .

4.1.  $K \subset K \times K$ . Let  $K$  be a connected compact Lie group. Here we work with the Lie group  $G = K \times K$  and the involution  $\theta(k_1, k_2) = (k_2, k_1)$ . The subgroup  $H = G^\theta$  is the group  $K$  embedded diagonally in  $G$ .

Let  $T$  be a maximal torus of  $K$  and let  $W_K = N_K(T)/T$  be the Weyl group of  $K$ . We denote by  $\mathfrak{R}_K$  the set of roots for  $(K, T)$ , and we make the choice of a set  $\mathfrak{R}_K^+$  of positive roots.

In the next lemma we describe the critical set  $Z_\theta$  in the flag manifold  $\mathcal{F} = K/T \times K/T$  of  $G$ .

**Lemma 4.1.** *We have  $Z_\theta = \bigcup_{w \in W_K} Z_w$  with  $Z_w = K \cdot (wT, T)$ . In other words, the set  $H \backslash Z_\theta / W$  is a singleton.*

*Proof.* The element  $x = (aT, bT) \in \mathcal{F}$  belongs to  $Z_\theta$  if and only if  $(a^{-1}b, b^{-1}a) \in W \times W$ . If  $b^{-1}a = w \in W$  then  $(aT, bT) \in Z_w$ . □

We take  $x = (T, T) \in Z_\theta$ . For each  $w \in W_K$ , we write  $xw = (wT, T)$ . We take  $\lambda = (a, b) \in \Lambda_K^+ \times \Lambda_K^+ = \widehat{G}$ .

Our data are as follows:

- the group  $G_x$  is the maximal torus  $T \times T \subset K$ ,
- the group  $H_x$  is the maximal torus  $T \subset K$ ,
- $\mathbb{C}_{(w \bullet \lambda)_{x+\delta(x)}} = \mathbb{C}_{w(a+\rho)+b+\rho}$  as a character of  $T$ ,
- $n_x = |\mathfrak{R}_K^+|$ ,
- $k_{x,w}$  is equal to  $|w\mathfrak{R}_K^+ \cap \mathfrak{R}_K^+| + |\mathfrak{R}_K^+|$ , so  $(-1)^{k_{x,w}} = (-1)^w$ ,
- the vector spaces  $\mathbb{E}_x^{\text{ci}}, \mathbb{E}_x^{\text{nci}}$  are reduced to  $\{0\}$ .

In this context we obtain the following relation

$$(4.8) \quad V_a^K \otimes V_b^K = (-1)^{\dim(K/T)/2} \sum_{w \in W_K} (-1)^w \text{Ind}_T^K (\mathbb{C}_{w(a+\rho)+b+\rho}).$$

This type of generalized Clebsch-Gordan formula what first noticed by Steinberg [15] (see Section 5).

**Example 4.2.** *The irreducible representation  $SU(2)$  are parametrized by  $\mathbb{N}$ . If  $n \geq 0$ , the irreducible representation  $V_n$  of  $SU(2)$  satisfies*

$$V_n = \text{Ind}_{U(1)}^{SU(2)} ((\mathbb{C}_0 - \mathbb{C}_2) \otimes \mathbb{C}_n).$$

If we take  $m \geq n \geq 0$ , then (4.8) gives

$$\begin{aligned} V_n \otimes V_m &= \text{Ind}_{U(1)}^{SU(2)} (\mathbb{C}_{m-n}) - \text{Ind}_{U(1)}^{SU(2)} (\mathbb{C}_{m+n+2}) \\ &= \sum_{k=0}^n \text{Ind}_{U(1)}^{SU(2)} ((\mathbb{C}_0 - \mathbb{C}_2) \otimes \mathbb{C}_{m+n-2k}) \\ &= \sum_{k=0}^n V_{m+n-2k}. \end{aligned}$$

We recognize here the classical Clebsch-Gordan relations.

4.2.  $U(p) \times U(q) \subset U(p+q)$ . Let  $p \geq q \geq 1$  and  $n = p+q$ . We take  $G = U(n)$  with maximal torus  $T \simeq U(1)^n$  the subgroup formed by the diagonal matrices. We use the canonical map  $\tau$  from the symmetric group  $\mathfrak{S}_n$  into  $G$ . It induces an isomorphism between  $\mathfrak{S}_n$  and the Weyl group  $W$  of  $G$ .

We work with the involution  $\theta(g) = \Delta g \Delta^{-1}$  where  $\Delta := \text{diag}(I_p, -I_q)$ : the subgroup fixed by  $\theta$  is  $H = U(p) \times U(q)$ .

In the next section we describe the critical set  $Z_\theta \subset \mathcal{F}$ . For another type of parametrization of  $H_{\mathbb{C}} \backslash \mathcal{F}$ , see Section 5 of [13].

4.2.1. *The critical set.* We consider the following elements of  $O(2)$ :

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The element  $R$  is of order 8,  $R^2 = -J$  and  $R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R = S$ .

To any  $j \in \{0, \dots, q\}$  we associate :

- $g_j := \text{diag}(\underbrace{1, \dots, 1}_{p-j \text{ times}}, \underbrace{R, \dots, R}_j, \underbrace{1, \dots, 1}_{q-j \text{ times}}) \in G$ ,
- the permutation  $w_j \in \mathfrak{S}_n$  that fixes the elements of  $[1, \dots, p-j] \cup [p+j+1, \dots, n]$  and such that
 
$$w_j(p-j+2k-1) = p-j+k, \quad w_j(p-j+2k) = p+k, \quad \text{for } 1 \leq k \leq j,$$
- $k_j = \tau_j g_j \in G$ , where  $\tau_j = \tau(w_j) \in N(T)$ ,
- $x_j = k_j T \in \mathcal{F}$ .

The adjoint map  $Ad(\tau_j) : G \rightarrow G$  sends the matrix  $\text{diag}(a_1, \dots, a_{p-j}, b_1, \dots, b_{2j}, c_1, \dots, c_{q-j})$  to the matrix  $\text{diag}(a_1, \dots, a_{p-j}, b_1, b_3, \dots, b_{2j-1}, b_2, b_4, \dots, b_{2j}, c_1, \dots, c_{q-j})$ .

We see then that

$$\sigma_j := k_j^{-1} \Delta k_j = \text{diag}(\underbrace{1, \dots, 1}_{p-j \text{ times}}, \underbrace{S, \dots, S}_j, \underbrace{-1, \dots, -1}_{q-j \text{ times}})$$

and  $k_j^{-1} \theta(k_j) = \sigma_j \Delta$  belong to  $N(T)$ . Thus the elements  $x_0, \dots, x_q$  belongs to  $Z_\theta$ .

**Lemma 4.3.** *In the flag manifold  $\mathcal{F}$  the set  $Z_\theta$  has the following description:*

$$Z_\theta = \bigcup_{0 \leq j \leq q} \bigcup_{\bar{w} \in W_{x_j} \setminus W} Hx_j w$$

So we have  $H \setminus Z_\theta / W = \{\bar{x}_0, \dots, \bar{x}_q\}$ .

*Proof.* If  $1 \leq a < b \leq n$ , we denote by  $\tau_{a,b} \in N(T)$  the permutation matrix associated to the transposition  $(a, b)$ .

Let  $gT \in Z_\theta$ . Then  $k := g^{-1} \theta(g) \Delta = g^{-1} \Delta g$  is an element of order two in  $N(T)$ . The Weyl group element  $\bar{k} \in W$  is of order two, then there exists  $0 \leq l \leq n/2$ , and a family  $(a_1 < b_1), \dots, (a_l < b_l)$  of disjoint couples in  $\{1, \dots, n\}$  such that  $kT = \tau_{a_1, b_1} \dots \tau_{a_l, b_l} T$ .

Now, if we use the fact that the characteristic polynomial of  $k \in G$  is equal to  $(X-1)^p (X+1)^q$  with  $p \geq q \geq 1$ , we see that

- $l \leq q$ ,
- there exists  $n \in N(T)$  such that  $nk n^{-1} = \sigma_l = k_l^{-1} \Delta k_l$ .

If we take  $w = \bar{n} \in W$ , the previous identity says that  $g \in Hk_l w T$ .  $\square$

**4.2.2. Localized indices.** We work with the groups  $T \subset H = U(p) \times U(q) \subset G = U(n)$  and the corresponding Lie algebras  $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}$ . Let  $\mathfrak{R} = \{\varepsilon_r - \varepsilon_s\}$  be the set of non-zero roots for the action of  $T$  on  $\mathfrak{g} \otimes \mathbb{C}$ . We choose the Weyl chamber so that  $\mathfrak{R}^+ := \{\varepsilon_r - \varepsilon_s, 1 \leq r < s \leq n\}$ .

Let  $j \in \{0, \dots, q\}$ . The aim of this section is to compute the localized index  $Q_{\bar{x}_j}(\lambda) \in \widehat{R}(H)$ . In order to have a fairly simple expression we will rewrite the terms of the form  $\text{Ind}_{H_{x_j}}^H(\mathbb{C}_\beta \otimes \bigwedge \mathbb{E}_{x_j}^{\text{ci}})$ .

Let  $\{1, \dots, n\} = I_j^1 \cup I_j^2 \cup I_j^3 \cup I_j^4$  where  $I_j^1 = \{1 \leq k \leq p-j\}$ ,  $I_j^2 = \{p-j+1 \leq k \leq p\}$ ,  $I_j^3 = \{p+1 \leq k \leq p+j\}$ , and  $I_j^4 = \{p+j+1 \leq k \leq n\}$ .

For the maximal torus  $T \subset G$  we have a decomposition

$$T \simeq T_j^1 \times T_j^2 \times T_j^3 \times T_j^4$$

where  $T_j^p = \{(t_k)_{k=1}^n, t_k \in U(1), t_k = 1 \text{ unless } k \in I_j^p\}$ . Let  $T_j \subset T_j^2 \times T_j^3$  be the subtorus defined by the relations: an element  $((t_k)_{k=1}^n, (s_k)_{k=1}^n) \in T_j^2 \times T_j^3$  belongs to  $T_j$  if and only if  $t_{p-j+k} = s_{p+k}$  for all  $1 \leq k \leq j$ .

The elements of order two  $\sigma_j \in G$  induce involutions on  $G$  (by conjugation) that we still denote by  $\sigma_j$ . We start with a basic lemma whose proof is left to the reader.

**Lemma 4.4.** *Let  $x_j = k_j T \in \mathcal{F}$ .*

- *The adjoint map  $Ad(k_j) : \mathfrak{g} \rightarrow \mathfrak{g}$  realizes an isomorphism between the vector space  $\mathfrak{t}$  equipped with the involution induced by  $\sigma_j$  and the vector space  $\mathfrak{g}_{x_j}$  equipped with the involution  $\theta$ .*

- *The group  $N(T)^{\sigma_j}/T^{\sigma_j}$  is isomorphic with  $\mathfrak{S}_{p-j} \times \mathfrak{S}_{q-j} \times \mathfrak{S}_j \times \{\pm\}^j$ .*
- *The adjoint map  $Ad(k_j) : G \rightarrow G$  induces an isomorphism  $N(T)^{\sigma_j}/T^{\sigma_j} \simeq W_{x_j}$ .*
- *The stabilizer subgroup  $H_{x_j}$  is equal to  $T_j^1 \times T_j \times T_j^4 \subset T$ .*
- *If  $\mathbb{C}_\alpha$  is a character of  $T$ , then  $\mathbb{C}_{k_j\alpha}$  is a character of  $G_{x_j}$  and  $\mathbb{C}_{\tau_j\alpha}$  is a character of  $T$ . We have the relation*

$$\mathbb{C}_{k_j\alpha}|_{H_{x_j}} = \mathbb{C}_{\tau_j\alpha}|_{H_{x_j}}.$$

- *The set of roots  $\mathfrak{R}_{x_j}^{\text{ci}}$  is equal to*

$$k_j \cdot \{\varepsilon_r - \varepsilon_s, 1 \leq r < s \leq p-j\} \cup k_j \cdot \{\varepsilon_r - \varepsilon_s, p+j+1 \leq r < s \leq n\}$$

and  $\mathfrak{R}_{x_j}^{\text{nci}} = k_j \cdot \{\varepsilon_r - \varepsilon_s, 1 \leq r \leq p-j \ \& \ p+j+1 \leq s \leq n\}$ .

We denote by  $\mathbb{M}_j$  the  $T$ -module  $\mathbb{C}^{p-j} \otimes (\mathbb{C}^{q-j})^*$  where the subgroup  $T_j^2 \times T_j^3$  acts trivially and the  $T_j^1 \times T_j^4$ -action is the canonical one. Thanks to Lemma 4.4, we have the following isomorphisms of  $H_{x_j}$ -modules:  $\mathbb{E}_{x_j}^{\text{nci}} \simeq \mathbb{M}_j$ . Following Lemma 3.7, one can associate the modules  $(\mathbb{M}_j)_w^\pm$  and  $|\mathbb{M}_j|_w$  to each  $w \in W$ .

We consider the Lie group

$$K_j := U(p-j) \times U(q-j)$$

that we view as a subgroup of  $H$  in such a way that  $T_j^1 \times T_j^4$  is a maximal torus of  $K_j$ . A set of positive roots for  $(K_j, T_j^1 \times T_j^4)$  is  $\varepsilon_r - \varepsilon_s$  for  $1 \leq r < s \leq p-j$  and  $p+j+1 \leq r < s \leq n$ . We equip  $\mathfrak{k}_j / [\mathfrak{t}_j^1 \times \mathfrak{t}_j^4]$  with a complex structure such that

$$\mathbb{E}_{x_j}^{\text{ci}} \simeq \mathfrak{k}_j / [\mathfrak{t}_j^1 \times \mathfrak{t}_j^4]$$

is an isomorphism of  $T_j^1 \times T_j^4$ -modules.

The holomorphic induction map  $\text{Hol}_{T_j^1 \times T_j^4}^{K_j} : \widehat{R}(T_j^1 \times T_j^4) \rightarrow \widehat{R}(K_j)$  is defined as follows:

$$\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(V) := \text{Ind}_{T_j^1 \times T_j^4}^{K_j}(V \otimes \bigwedge \mathfrak{k}_j / [\mathfrak{t}_j^1 \times \mathfrak{t}_j^4]).$$

If  $a = (a_1 \geq \dots \geq a_{p-j}) \in \mathbb{Z}^{p-j}$  and  $b = (b_1 \geq \dots \geq b_{q-j}) \in \mathbb{Z}^{q-j}$ , then  $\mathbb{C}_{(a,b)}$  defines a character of  $T_j^1 \times T_j^4$  and

$$\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_{(a,b)}) = V_a^{U(p-j)} \otimes V_b^{U(q-j)}$$

is the irreducible representation of  $K_j$  with highest weight  $(a, b)$ .

A character  $\mathbb{C}_\beta$  of the torus  $T$  can be written  $\mathbb{C}_\beta = \mathbb{C}_{\beta^{14}} \otimes \mathbb{C}_{\beta^{23}}$  where  $\mathbb{C}_{\beta^{14}}$  is a character of  $T_j^1 \times T_j^4$  and  $\mathbb{C}_{\beta^{23}}$  is a character of  $T_j^2 \times T_j^3$ . Note that  $\mathbb{C}_{\tau_j \beta}|_{H_{x_j}} = \mathbb{C}_{\beta^{14}} \otimes \mathbb{C}_{\beta'}$  where  $\beta' = \tau_j \beta^{23}$  defines a character of  $T_j \subset T_j^2 \times T_j^3$ .

**Lemma 4.5.** *Let  $\mathbb{C}_\beta$  be a character of  $T$ . Then  $\text{Ind}_{H_{x_j}}^H(\mathbb{C}_\beta|_{H_{x_j}} \otimes \wedge \mathbb{E}_{x_j}^{\text{ci}})$  is equal to*

$$\text{Ind}_{K_j \times T_j^2 \times T_j^3}^H \left( \text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_{\beta^{14}}) \otimes \mathbb{C}_{\beta^{23}} \otimes L^2([T_j^2 \times T_j^3]/T_j) \right),$$

where  $L^2([T_j^2 \times T_j^3]/T_j) = \text{Ind}_{T_j}^{T_j^2 \times T_j^3}(1) \in \widehat{R}(T_j^2 \times T_j^3)$ .

**Remark 4.6.** *To gain some space in our formulas, we will write  $\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_\beta)$  instead of  $\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_{\beta^{14}}) \otimes \mathbb{C}_{\beta^{23}}$*

We need to fix some notations.

**Definition 4.7.** • *Let  $\chi : H \rightarrow \mathbb{C}$  be the character  $(A, B) \mapsto \det(A) \det(B)^{-1}$ .*

• *Let  $\psi_j$  be the character<sup>2</sup> of  $T$  associated to the weight*

$$\sum_{1 \leq k \leq j} (q - p + 2 + 2j - 4k) \varepsilon_{p-j+k}.$$

• *For any  $(j, w) \in [0, q] \times W$ , we define the integer  $d_{j,w}$  by the relation*

$$d_{j,w} = \dim(\mathbb{M}_j)_w^+ + |\{1 \leq k \leq j, w^{-1}(p - j + 2k - 1) < w^{-1}(p - j + 2k)\}|.$$

A small computation gives the following lemma.

**Lemma 4.8.** • *The  $H_{x_j}$ -character  $\mathbb{C}_{\delta(x_j)}$  is equal to  $\chi^{\otimes j} \otimes \psi_j|_{H_{x_j}}$ .*

• *For any  $(j, w) \in [0, q] \times W$ , we have  $(-1)^{n_{x_j} + k_{x_j, w}} = (-1)^{j(n+1)} (-1)^w (-1)^{d_{j,w}}$ .*

The main result of this section is the following proposition.

**Proposition 4.9.**

$$V_\lambda^{U(n)}|_{U(p) \times U(q)} = \sum_{j=0}^q Q_{\bar{x}_j}(\lambda)$$

where  $Q_{\bar{x}_j}(\lambda) \in \widehat{R}(U(p) \times U(q))$  is determined by the relation

$$Q_{\bar{x}_j}(\lambda) = \frac{(-1)^{j(n+1)}}{|W_{x_j}|} \chi^{\otimes j} \otimes \sum_{w \in W} (-1)^w (-1)^{d_{j,w}} \text{Ind}_{K_j \times T_j^2 \times T_j^3}^{U(p) \times U(q)} (\mathbb{A}_j^w(\lambda) \otimes \psi_j).$$

Here the elements  $\mathbb{A}_j^w(\lambda) \in \widehat{R}(K_j \times T_j^2 \times T_j^3)$  are defined as follows:

$$\mathbb{A}_j^w(\lambda) = \text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_{\tau_j(w \bullet \lambda)} \otimes \det((\mathbb{M}_j)_w^+) \otimes \text{Sym}(|\mathbb{M}_j|_w)) \otimes L^2([T_j^2 \times T_j^3]/T_j).$$

We finish this section by considering particular situations.

<sup>2</sup>Remark that  $\psi_j$  is trivial  $T_j^1 \times T_j^3 \times T_j^4$ .

4.2.3. *The extreme cases :  $j = 0$  or  $j = q$ .* When  $j = 0$ , the torus  $T_0^2$  and  $T_0^3$  are trivial and  $K_0 = U(p) \times U(q) = H$ . Moreover  $\mathbb{M}_0 = \mathbb{C}^p \otimes (\mathbb{C}^q)^*$  and  $d_{0,w} = \dim(\mathbb{M}_0)_w^+$ . Thanks to Lemma 4.4, we know also that  $W_{x_0} \simeq \mathfrak{S}_p \times \mathfrak{S}_q$ .

So we get the formula

$$Q_{\bar{x}_0}(\lambda) = \frac{1}{p!q!} \sum_{w \in W} (\pm)_w \text{Hol}_T^H(\mathbb{C}_{w \bullet \lambda} \otimes \det((\mathbb{M}_0)_w^+) \otimes \text{Sym}(|\mathbb{M}_0|_w))$$

where  $(\pm)_w = (-1)^w (-1)^{\dim(\mathbb{M}_0)_w^+}$ .

**Remark 4.10.** *An useful exercise is to consider the term*

$$\mathbb{A}_w := (\pm)_w \text{Hol}_T^H(\mathbb{C}_{w \bullet \lambda} \otimes \det((\mathbb{M}_0)_w^+) \otimes \text{Sym}(|\mathbb{M}_0|_w))$$

and verify that  $\mathbb{A}_{w'w} = \mathbb{A}_w$  when  $w' \in W_{x_0}$ .

When  $j = q$ , the torus  $T_q^4$  is trivial,  $K_q = U(p - q)$  and  $\mathbb{M}_q = \{0\}$ . Moreover  $W_{x_q} \simeq \mathfrak{S}_{p-q} \times \mathfrak{S}_q \times \{\pm\}^q$ . In this case we obtain

$$Q_{\bar{x}_q}(\lambda) = \frac{(-1)^{q(n+1)}}{(p-q)!q!2^q} \chi^{\otimes q} \otimes \sum_{w \in W} (-1)^w (-1)^{d_{q,w}} Q_q^w(\lambda)$$

with

$$Q_q^w(\lambda) = \text{Ind}_{U(p-q) \times T_q^2 \times T_q^3}^{U(p) \times U(q)} \left( \text{Hol}_{T_q^1}^{U(p-q)}(\mathbb{C}_{\tau_q(w \bullet \lambda)}) \otimes \psi_q \otimes L^2([T_q^2 \times T_q^3] / T_q) \right).$$

4.2.4.  $U(n-1) \times U(1) \subset U(n)$ . Here we are in the case where  $q = 1$ , and so

$$V_\lambda^{U(n)}|_{U(n-1) \times U(1)} = Q_{\bar{x}_0}(\lambda) + Q_{\bar{x}_1}(\lambda).$$

To simplify the expression of  $Q_{\bar{x}_0}(\lambda)$  we use the fact that the quotient  $W_{x_0} \backslash W$  is represented by the class of the elements  $\tau_{k,n} \in G$  associated to the transposition  $(k, n)$  for  $1 \leq k \leq n$ . We write  $T = T' \times U(1)$  where  $T'$  is a maximal torus of  $U(n-1)$ . The  $T'$ -module  $\mathbb{C}^{n-1}$  can be decomposed as  $\mathbb{V}_k \oplus \mathbb{V}'_k$  where  $\mathbb{V}_k = \sum_{j=1}^{k-1} \mathbb{C}_{\varepsilon_j}$  and  $\mathbb{V}'_k = \sum_{j=k}^{n-1} \mathbb{C}_{\varepsilon_j}$ .

The  $T$ -module  $\mathbb{M}_0$  is equal to  $\mathbb{C}^{n-1} \otimes \mathbb{C}^* = \mathbb{V}_k \otimes \mathbb{C}_{-\varepsilon_n} \oplus \mathbb{V}'_k \otimes \mathbb{C}_{-\varepsilon_n}$  and the polarized  $T$ -module  $|\mathbb{M}_0|_{\tau_{k,n}}$  is equal to  $\mathbb{V}_k \otimes \mathbb{C}_{-\varepsilon_n} \oplus \overline{\mathbb{V}'_k} \otimes \mathbb{C}_{\varepsilon_n}$ . We have  $\dim(\mathbb{M}_0)_{\tau_{k,n}}^+ = k - 1$  and  $\det(\mathbb{M}_0)_{\tau_{k,n}}^+ = \mathbb{C}_{\mu_k} \otimes \mathbb{C}_{\varepsilon_n}^{\otimes 1-k}$  with  $\mu_k = \sum_{j=1}^{k-1} \varepsilon_j$ .

So we obtain

$$Q_{\bar{x}_0}(\lambda) = \sum_{\substack{a, b \geq 0 \\ 1 \leq k \leq n}} (\pm)_k \text{Hol}_{T'}^{U(n-1)}(\mathbb{C}_{\tau_{k,n} \bullet \lambda + \mu_k} \otimes \text{Sym}^b(\mathbb{V}_k) \otimes \text{Sym}^a(\overline{\mathbb{V}'_k})) \otimes \mathbb{C}_{\varepsilon_n}^{\otimes 1+a-b-k},$$

where  $(\pm)_k = (-1)^k$  if  $k < n$  and  $(\pm)_n = (-1)^{n-1}$ .

We consider now the term  $Q_{\bar{x}_1}(\lambda)$ . When  $j = q = 1$ , the torus  $T_1^4$  is trivial,  $K_1 = U(n-2)$  and  $\mathbb{M}_1 = \{0\}$ . Moreover  $W_{x_1} \simeq \mathfrak{S}_{n-2} \times \{\pm\}$ ,  $\tau_1 = Id$  and  $\psi_1 = (2-n)\varepsilon_{n-1}$ . Here the quotient  $W_{x_1} \backslash W$  is represented by the class of the elements  $\tau_{l,n} \tau_{k,n-1}$  for  $1 \leq k < l \leq n$ . We denote by  $\lambda_{kl}$  the term  $\tau_{l,n} \tau_{k,n-1} \bullet \lambda$ .

In this case we obtain

$$Q_{\bar{x}_1}(\lambda) = (-1)^n \chi \otimes \left( Q_{\bar{x}_1}^{n-1,n}(\lambda) - \sum_{1 \leq k < n-1} Q_{\bar{x}_1}^{k,n}(\lambda) + \sum_{1 \leq k < l \leq n-1} Q_{\bar{x}_1}^{k,l}(\lambda) \right)$$

with

$$\begin{aligned} Q_{\bar{x}_1}^{k,l}(\lambda) &= \text{Ind}_{U(n-2) \times T_1^2 \times T_1^3}^{U(n-1) \times T_1^3} \left( \text{Hol}_{T_1^1}^{U(n-2)}(\mathbb{C}_{\lambda_{kl}}) \otimes \psi_1 \otimes L^2([T_1^2 \times T_1^3] / T_1) \right) \\ &= \sum_{a \in \mathbb{Z}} \text{Ind}_{U(n-2) \times T_1^2}^{U(n-1)} \left( \text{Hol}_{T_1^1}^{U(n-2)}(\mathbb{C}_{\lambda_{kl}}) \otimes \mathbb{C}_{\varepsilon_{n-1}}^{\otimes a} \right) \otimes \mathbb{C}_{\varepsilon_n}^{\otimes 2-n-a}. \end{aligned}$$

Let us finish this section by considering the simplest example:  $U(1) \times U(1) \subset U(2)$ . Take  $\lambda = (\lambda_1 \geq \lambda_2) \in \widehat{U(2)}$ . We have  $V_\lambda^{U(2)}|_{U(1) \times U(1)} = Q_{\bar{x}_0}(\lambda) + Q_{\bar{x}_1}(\lambda)$  where

$$Q_{\bar{x}_0}(\lambda) = -\mathbb{C}_\lambda \otimes \sum_{-\infty}^{\lambda_2 - \lambda_1 - 1} \mathbb{C}_{\varepsilon_1 - \varepsilon_2}^{\otimes k} - \mathbb{C}_\lambda \otimes \sum_{k \geq 1} \mathbb{C}_{\varepsilon_1 - \varepsilon_2}^{\otimes k}$$

and  $Q_{\bar{x}_1}(\lambda) = \mathbb{C}_\lambda \otimes \sum_{k \in \mathbb{Z}} \mathbb{C}_{\varepsilon_1 - \varepsilon_2}^{\otimes k}$ . We recover the basic relation

$$V_\lambda^{U(2)}|_{U(1) \times U(1)} = \mathbb{C}_\lambda \otimes \sum_{k=\lambda_2 - \lambda_1}^0 \mathbb{C}_{\varepsilon_1 - \varepsilon_2}^{\otimes k}.$$

4.2.5.  $U(n-1) \subset U(n)$ . If we restrict the representation  $V_\lambda^{U(n)}$  to the subgroup  $U(n-1)$ , we get

$$(4.9) \quad V_\lambda^{U(n)}|_{U(n-1)} = Q_0(\lambda) + Q_1(\lambda),$$

where the characters  $Q_0(\lambda), Q_1(\lambda) \in \widehat{R}(U(n-1))$  are given by the relations

$$Q_0(\lambda) = \sum_{k=1}^n (\pm)_k \text{Hol}_{T'}^{U(n-1)}(\mathbb{C}_{\tau_{k,n} \bullet \lambda + \mu_k} \otimes \text{Sym}(\mathbb{V}_k) \otimes \text{Sym}(\overline{\mathbb{V}'_k})),$$

and

$$Q_1(\lambda) = (-1)^n \det \otimes \left( Q_1^{n-1,n}(\lambda) - \sum_{1 \leq k < n-1} Q_1^{k,n}(\lambda) + \sum_{1 \leq k < l \leq n-1} Q_1^{k,l}(\lambda) \right),$$

with  $Q_1^{k,l}(\lambda) = \text{Ind}_{U(n-2)}^{U(n-1)} \left( \text{Hol}_{T_1^1}^{U(n-2)}(\mathbb{C}_{\lambda_{kl}}) \right)$ .

Let's detail expression (4.9) when  $n = 3$ .

Small calculations give  $Q_0(\lambda) = \mathbb{B}_1(\lambda) + \mathbb{B}_2(\lambda) + \mathbb{B}_3(\lambda)$  with

$$\mathbb{B}_1(\lambda) = \text{Hol}_T^{U(2)}(\mathbb{C}_{\tau_{1,3} \bullet \lambda + \mu_1}) \otimes \text{Sym}(\overline{\mathbb{C}^2}) = \sum_{\lambda_2 - 1 \geq a \geq \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)},$$

$$\mathbb{B}_2(\lambda) = \text{Hol}_T^{U(2)}(\mathbb{C}_{\tau_{2,3} \bullet \lambda + \mu_2} \otimes \text{Sym}(\mathbb{C}_{\varepsilon_1}) \otimes \text{Sym}(\overline{\mathbb{C}_{\varepsilon_2}})) = \sum_{a \geq \lambda_1 + 1, \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)},$$

$$\mathbb{B}_3(\lambda) = \text{Hol}_T^{U(2)}(\mathbb{C}_{\tau_{3,3} \bullet \lambda + \mu_3}) \otimes \text{Sym}(\mathbb{C}^2) = \sum_{a \geq \lambda_1 + 1 \geq b \geq \lambda_2 + 1} V_{(a,b)}^{U(2)}.$$

For the other term, we obtain  $Q_1(\lambda) = \mathbb{A}_1(\lambda) - \mathbb{A}_2(\lambda) - \mathbb{A}_3(\lambda)$  with

$$\begin{aligned}\mathbb{A}_1(\lambda) &= \det \otimes Q_1^{1,3}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(\mathbb{C}_{\lambda_2-1}) = \sum_{a \geq \lambda_2 \geq b} V_{(a,b)}^{U(2)}, \\ \mathbb{A}_2(\lambda) &= \det \otimes Q_1^{2,3}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(\mathbb{C}_{\lambda_1}) = \sum_{a \geq \lambda_1+1 \geq b} V_{(a,b)}^{U(2)}, \\ \mathbb{A}_3(\lambda) &= \det \otimes Q_1^{1,2}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(\mathbb{C}_{\lambda_3-2}) = \sum_{a \geq \lambda_3-1 \geq b} V_{(a,b)}^{U(2)}.\end{aligned}$$

Finally one checks that the decomposition

$$V_\lambda^{U(3)}|_{U(2)} = \mathbb{A}_1(\lambda) - \mathbb{A}_2(\lambda) - \mathbb{A}_3(\lambda) + \mathbb{B}_1(\lambda) + \mathbb{B}_2(\lambda) + \mathbb{B}_3(\lambda)$$

permits to recover the classical relation  $V_\lambda^{U(3)}|_{U(2)} = \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3} V_{(a,b)}^{U(2)}$  (see [4]).

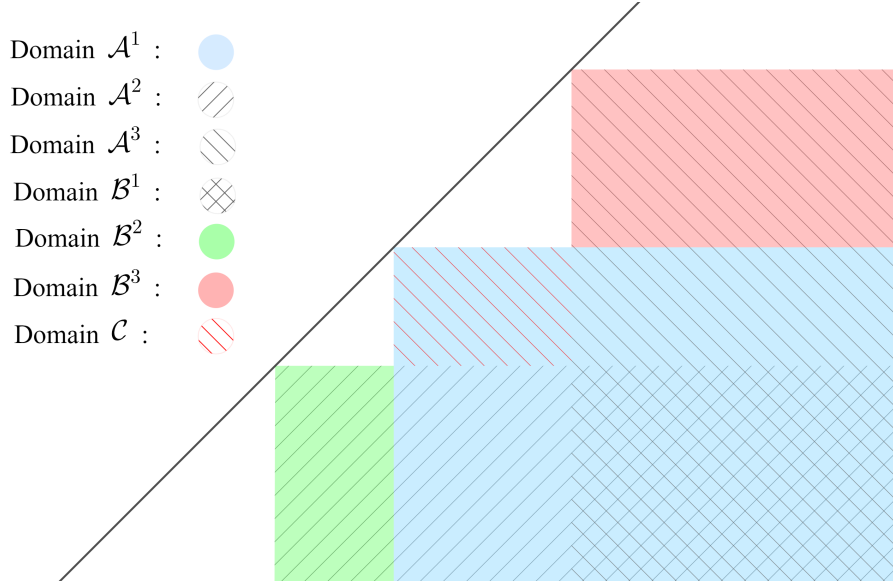


FIGURE 1. Restriction from  $U(3)$  to  $U(2)$

In Figure 1, we can visualise the supports of the different characters: we have

$$\begin{aligned}\mathbb{A}_1(\lambda) &= \sum_{\mu \in \mathcal{A}_1} V_\mu^{U(2)}, & \mathbb{A}_2(\lambda) &= \sum_{\mu \in \mathcal{A}_2 \cup \mathcal{B}_1} V_\mu^{U(2)}, & \mathbb{A}_3(\lambda) &= \sum_{\mu \in \mathcal{A}_3 \cup \mathcal{B}_1} V_\mu^{U(2)}, \\ \mathbb{B}_1(\lambda) &= \sum_{\mu \in \mathcal{B}_1} V_\mu^{U(2)}, & \mathbb{B}_2(\lambda) &= \sum_{\mu \in \mathcal{B}_2} V_\mu^{U(2)}, & \mathbb{B}_3(\lambda) &= \sum_{\mu \in \mathcal{B}_3} V_\mu^{U(2)},\end{aligned}$$

so that  $V_\lambda^{U(3)}|_{U(2)} = \sum_{\mu \in \mathcal{C}} V_\mu^{U(2)}$ .

## 5. KOSTANT MULTIPLICITY FORMULA

The aim of this section is first to recall the Kostant multiplicity formula : we follow the line of [4], Section 8.2. Then, we rewrite it in a form similar to the one we use in this article (see Proposition 5.4). Finally, we detail Kostant's multiplicity formula for



the restriction of  $U(n)$  to  $U(n-1)$ , in order to compare it with the calculations done in Section 4.2.5.

Let  $G' \subset G$  be two connected compact Lie groups with maximal tori  $T' \subset T$ . The corresponding Lie algebras are  $\mathfrak{t} \subset \mathfrak{g}$  and  $\mathfrak{t}' \subset \mathfrak{g}'$ . In this section, we make the following regularity assumption:

(R) The centralizer  $Z_{\mathfrak{g}}(\mathfrak{t}')$  of  $\mathfrak{t}'$  in  $\mathfrak{g}$  is abelian.

We recall the following well-known fact.

**Lemma 5.1.** *The assumption (R) is valid when  $G'$  is the connected component of a fixed-point subgroup of an involution.*

*Proof :* Suppose that  $G' = (G^\tau)_0$  for some involution  $\tau$ . Then, we have a decomposition  $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{q}$  where  $\mathfrak{q} = \{X \in \mathfrak{g}, \tau(X) = -X\}$ . The centralizer  $Z_{\mathfrak{g}}(\mathfrak{t}')$  is stable under the involution  $\tau$  and under the adjoint action of  $T$ . Thus  $\mathfrak{t} \subset Z_{\mathfrak{g}}(\mathfrak{t}') = \mathfrak{t}' \oplus Z_{\mathfrak{q}}(\mathfrak{t}')$ : in particular the torus  $T$  is invariant under  $\tau$ .

If  $Z_{\mathfrak{g}}(\mathfrak{t}')$  is not abelian, there exists a roots  $\alpha \in \mathfrak{R}$  such that  $(\mathfrak{g}_{\mathbb{C}})_{\alpha} \subset Z_{\mathfrak{g}}(\mathfrak{t}')_{\mathbb{C}} = \mathfrak{t}'_{\mathbb{C}} \oplus Z_{\mathfrak{q}}(\mathfrak{t}')_{\mathbb{C}}$ . Then we obtain a contradiction: on one hand  $(\mathfrak{g}_{\mathbb{C}})_{\alpha} \subset Z_{\mathfrak{g}}(\mathfrak{t}')_{\mathbb{C}}$  implies that  $\alpha|_{\mathfrak{t}'} = 0$  and on the other hand since  $(\mathfrak{g}_{\mathbb{C}})_{\alpha} \subset \mathfrak{q}_{\mathbb{C}}$ , we must have  $\sigma(\alpha) = \alpha$ . The two conditions  $\alpha|_{\mathfrak{t}'} = 0$  and  $\sigma(\alpha) = \alpha$  implies that  $\alpha = 0$ .  $\square$ .

Let  $\mathfrak{R}$  and  $\mathfrak{R}'$  and be the set of roots for the pairs  $T \subset G$  and  $T' \subset G'$ . Note that assumption (R) is equivalent to :

(R') There exists  $X_o \in \mathfrak{t}'$  such that  $\langle \alpha, X_o \rangle \neq 0$  for all  $\alpha \in \mathfrak{R}$ .

If  $\xi \in \mathfrak{t}'^*$  we write  $\bar{\xi}$  for the restriction of  $\xi$  to  $(\mathfrak{t}')^*$ . Because of our assumption,  $\bar{\alpha} \neq 0$  for all  $\alpha \in \mathfrak{R}$ .

The positive roots are  $\mathfrak{R}_+ := \{\alpha \in \mathfrak{R}, \langle \alpha, X_o \rangle > 0\}$  and  $\mathfrak{R}'_+ := \{\beta \in \mathfrak{R}', \langle \beta, X_o \rangle > 0\}$ . We write  $\overline{\mathfrak{R}_+} := \{\bar{\alpha}, \alpha \in \mathfrak{R}_+\}$  for the set of positive restricted roots: we keep track of the multiplicity  $n_a = \#\{\alpha \in \mathfrak{R}_+, \bar{\alpha} = a\}$  of each element  $a \in \overline{\mathfrak{R}_+}$ .

Since  $\mathfrak{R}'_+$  is contained in  $\overline{\mathfrak{R}_+}$ , we may consider the set of roots  $\Sigma := \overline{\mathfrak{R}_+} - \mathfrak{R}'_+$ : the multiplicity of  $\beta \in \overline{\mathfrak{R}_+}$  in  $\Sigma$  is equal to

$$m_{\beta} := \begin{cases} n_{\beta} & \text{if } \beta \notin \mathfrak{R}'_+, \\ n_{\beta} - 1 & \text{if } \beta \in \mathfrak{R}'_+. \end{cases}$$

Let  $\Lambda' \subset (\mathfrak{t}')^*$  be the lattice of weights for the torus  $T'$ . Let  $(\mathfrak{t}')_+^*$  be the Weyl chamber associated to the system  $\mathfrak{R}'_+$ . The irreducible representations of  $G'$  are parameterized by  $\Lambda'_+ = \Lambda' \cap (\mathfrak{t}')_+^*$ .

**Definition 5.2.** *We denote by  $\mathcal{P}_{\Sigma} : \Lambda' \rightarrow \mathbb{N}$  the partition function associated to the set  $\Sigma$ . For all  $\xi' \in \Lambda'$ ,  $\mathcal{P}_{\Sigma}(\xi')$  is the number of way of writing  $\xi' = \sum_{\beta \in \Sigma} x_{\beta} \beta$ , where  $x_{\beta} \in \mathbb{N}$  and each  $\beta$  that occurs is counted with multiplicity  $m_{\beta}$ .*

For dominant weights  $\lambda \in \Lambda_+$  and  $\mu \in \Lambda'_+$ , we denote by  $m_{\lambda}(\mu)$  the multiplicity of the irreducible  $G'$ -representation  $V_{\mu}^{G'}$  with highest weight  $\mu$  in the irreducible  $G$ -representation  $V_{\lambda}^G$  with highest weight  $\lambda$ .

If  $w \in W$ , we note  $w \bullet \lambda := w(\lambda + \rho) - \rho$  where  $\rho$  is the half sum of the positive roots.

**Theorem 5.3.** *The branching multiplicities are*

$$(5.10) \quad m_\lambda(\mu) = \sum_{w \in W} (-1)^w \mathcal{P}_\Sigma(\overline{w \bullet \lambda} - \mu).$$

We briefly recall how to obtain (5.10). Let  $\chi_\lambda^G$  be the character of  $V_\lambda^G$ . The Weyl relation gives

$$\chi_\lambda^G|_T \prod_{\alpha \in \mathfrak{R}_+} (1 - e^{-\alpha}) = \sum_{w \in W} (-1)^w e^{w \bullet \lambda}.$$

When we restrict this identity to  $T' \subset T$ , the relations

$$\prod_{\alpha \in \mathfrak{R}_+} (1 - e^{-\alpha})|_{T'} = \prod_{\beta \in \mathfrak{R}'_+} (1 - e^{-\beta}) \prod_{\gamma \in \Sigma} (1 - e^{-\gamma})$$

and

$$\prod_{\gamma \in \Sigma} (1 - e^{-\gamma}) \left( \sum_{\xi' \in \Lambda'} \mathcal{P}_\Sigma(\xi') e^{-\xi'} \right) = 1$$

permit to obtain

$$\begin{aligned} \chi_\lambda^G|_{T'} \prod_{\beta \in \mathfrak{R}'_+} (1 - e^{-\beta}) &= \left( \sum_{w \in W} (-1)^w e^{\overline{w \bullet \lambda}} \right) \left( \sum_{\xi' \in \Lambda'} \mathcal{P}_\Sigma(\xi') e^{-\xi'} \right) \\ &= \sum_{\xi' \in \Lambda'} N_\lambda(\xi') e^{\xi'}. \end{aligned}$$

with  $N_\lambda(\xi') := \sum_{w \in W} (-1)^w \mathcal{P}_\Sigma(\overline{w \bullet \lambda} - \xi')$ .

On the other hand, we have the decomposition  $\chi_\lambda^G|_H = \sum_{\mu \in \Lambda'_+} m_\lambda(\mu) \chi_\mu^{G'}$  and then<sup>3</sup>

$$\begin{aligned} \chi_\lambda^G|_{T'} \prod_{\beta \in \mathfrak{R}'_+} (1 - e^{-\beta}) &= \sum_{\mu \in \Lambda'_+} m_\lambda(\mu) \chi_\mu^{G'} \prod_{\beta \in \mathfrak{R}'_+} (1 - e^{-\beta}) \\ &= \sum_{\mu \in \Lambda'_+} \sum_{w' \in W'} (-1)^{w'} m_\lambda(\mu) e^{w' \bullet \mu}. \end{aligned}$$

Finally, we obtain the identity

$$\sum_{\xi' \in \Lambda'} N_\lambda(\xi') e^{\xi'} = \sum_{\mu \in \Lambda'_+} \sum_{w' \in W'} (-1)^{w'} m_\lambda(\mu) e^{w' \bullet \mu},$$

that shows two things:

- $N_\lambda(\mu) = m_\lambda(\mu)$  if  $\mu$  is dominant,
- $N_\lambda(w' \bullet \xi') = (-1)^{w'} N_\lambda(\xi')$ , for every  $(w', \xi') \in W' \times \Lambda'$ .

At this stage, we have proved Kostant's multiplicity formula. In the following we rewrite this formula in another form. Let's consider the following  $T'$ -module

$$\mathfrak{n}_\Sigma := \bigoplus_{\beta \in \Sigma} \mathbb{C}_{-\beta}.$$

<sup>3</sup>Here  $w' \bullet \xi' := w'(\lambda + \rho') - \rho'$  where  $\rho' = \frac{1}{2} \sum_{\beta \in \mathfrak{R}'_+} \beta$ .

**Proposition 5.4.** *For any  $\lambda \in \Lambda_+$ , we have the following restriction formula*

$$(5.11) \quad V_\lambda^G|_{G'} = \frac{1}{\#W'} \sum_{w \in W} (-1)^w \text{Hol}_{T'}^{G'}(\mathbb{C}_{w \bullet \lambda} \otimes \text{Sym}(\mathfrak{n}_\Sigma)).$$

*Proof:* Since  $\text{Sym}(\mathfrak{n}_\Sigma) = \sum_{\xi' \in \Lambda'} \mathcal{P}_\Sigma(\xi') \mathbb{C}_{-\xi'}$  we have

$$\begin{aligned} \sum_{w \in W} (-1)^w \mathbb{C}_{w \bullet \lambda} \otimes \text{Sym}(\mathfrak{n}_\Sigma) &= \sum_{w \in W} \sum_{\xi' \in \Lambda'} (-1)^w \mathcal{P}_\Sigma(\xi') \mathbb{C}_{w \bullet \lambda - \xi'} \\ &= \sum_{\xi' \in \Lambda'} N_\lambda(\xi') \mathbb{C}_{\xi'}. \end{aligned}$$

Hence the right hand side of (5.11) is equal to  $\frac{1}{\#W'} \sum_{\xi' \in \Lambda'} N_\lambda(\xi') \text{Hol}_{T'}^{G'}(\mathbb{C}_{\xi'})$ . We use now the following facts:

- $N_\lambda(w' \bullet \xi') \text{Hol}_{T'}^{G'}(\mathbb{C}_{w' \bullet \xi'}) = N_\Sigma(\xi') \text{Hol}_{T'}^{G'}(\mathbb{C}_{\xi'})$  for every  $(w', \xi') \in W' \times \Lambda'$ .
- $N_\lambda(\xi') \text{Hol}_{T'}^{G'}(\mathbb{C}_{\xi'}) = 0$  if  $\xi' \notin W' \bullet \Lambda'_+$ .
- $N_\lambda(\mu) \text{Hol}_{T'}^{G'}(\mathbb{C}_\mu) = m_\lambda(\mu) V_\mu^{G'}$  if  $\mu \in \Lambda'_+$ .

We have completed the proof of (5.11).  $\square$

We conclude this section with a few examples.

5.1.  $K \subset K \times K$ . Let  $K$  be a connected compact Lie group. Here we work with the Lie group  $G = K \times K$  containing  $K$  diagonally. Here  $\Sigma \subset \mathfrak{k}^*$  is equal to the set  $\mathfrak{A}_+$  of positive roots for  $K$ . We denote by  $\mathcal{P} : \Lambda \rightarrow \mathbb{N}$  the partition function associated to the set  $\mathfrak{A}_+$ .

If  $\lambda, \mu, \nu$  are three dominant weights, we denote by  $c'_{\lambda, \mu}$  the multiplicity of  $V_\nu^K$  in  $V_\lambda^K \otimes V_\mu^K$ . The branching formula (5.10) becomes

$$c'_{\lambda, \mu} = \sum_{w_1, w_2 \in W} (-1)^{w_1 w_2} \mathcal{P}(w_1 \bullet \lambda + w_2 \bullet \mu - \nu).$$

This formula was first observed by Steinberg [15].

Let's take a closer look at the branching formula (5.11). The  $T$ -module  $\mathfrak{n}_\Sigma$  is equal to  $\mathfrak{n} := \sum_{\alpha > 0} \mathbb{C}_{-\alpha}$ . By definition of the holomorphic induction map  $\text{Hol}_T^K$ , we have

$$\text{Hol}_T^K(\Theta \otimes \text{Sym}(\mathfrak{n})) = (-1)^d \text{Ind}_T^K(\Theta \otimes \mathbb{C}_{2\rho})$$

for any  $\Theta \in R(T)$ , with  $d = \frac{1}{2} \dim K/T$ . Finally, (5.11) becomes

$$\begin{aligned} V_\lambda^K \otimes V_\mu^K|_K &= \frac{(-1)^d}{\#W} \sum_{w_1, w_2 \in W} (-1)^{w_1 w_2} \text{Ind}_T^K(\mathbb{C}_{w_1(\lambda+\rho)+w_2(\mu+\rho)}) \\ &= (-1)^d \sum_{w \in W} (-1)^w \text{Ind}_T^K(\mathbb{C}_{w(\lambda+\rho)+\mu+\rho}). \end{aligned}$$

The latter formula is also obtained in (4.8).

5.2.  $U(p) \times U(q) \subset U(n)$ . In this example the torus  $T$  of diagonal matrices is the maximal torus for both  $U(n)$  and the subgroup  $U(p) \times U(q)$ . Here the  $T$ -module  $\mathfrak{n}_\Sigma$  is the  $T$ -restriction of the  $U(p) \times U(q)$ -module  $(\mathbb{C}^p)^* \otimes \mathbb{C}^q$ , and the quotient  $W' \backslash W$  is isomorphic to the subset  $\text{Shuffle}(p, q)$  formed by the elements  $w \in \mathfrak{S}_n$  satisfying  $w(1) < \dots < w(p)$  and  $w(p+1) < \dots < w(p+q)$ . Here, the branching formula (5.11) gives

$$(5.12) \quad V_\lambda^{U(n)}|_{U(p) \times U(q)} = \left( \sum_{w \in \text{Shuffle}(p, q)} (-1)^w \text{Hol}_T^{U(p) \times U(q)}(\mathbb{C}_{w \bullet \lambda}) \right) \otimes \text{Sym}((\mathbb{C}^p)^* \otimes \mathbb{C}^q).$$

Let's consider the case  $q = 1$ . From (5.12), we derive the following branching formula for the restriction to the subgroup  $U(n-1)$ :

$$V_\lambda^{U(n)}|_{U(n-1)} = \left( \sum_{k=1}^n (-1)^{n-k} V_{\lambda[k]}^{U(n-1)} \right) \otimes \text{Sym}((\mathbb{C}^{n-1})^*),$$

with  $\lambda[n] = (\lambda_1, \dots, \lambda_{n-1})$  and  $\lambda[k] = (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1} - 1, \dots, \lambda_n - 1)$  for  $1 \leq k \leq n-1$ .

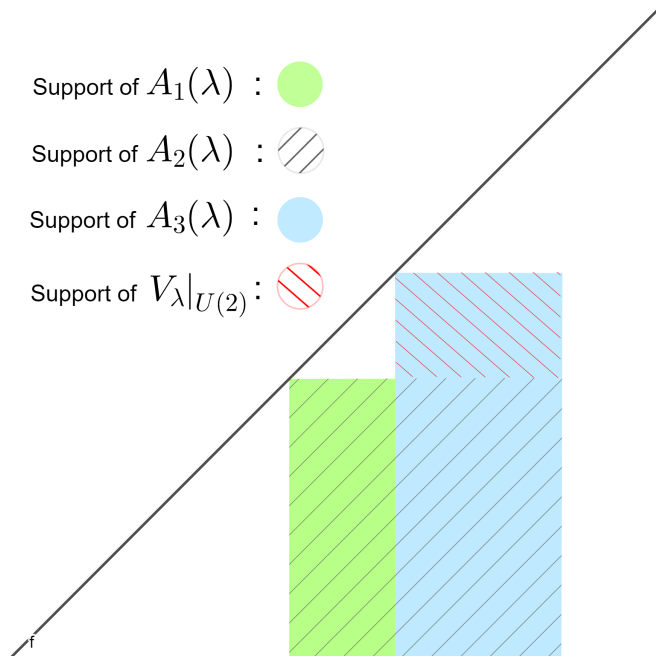


FIGURE 2. Kostant decomposition

Let's consider the case  $n = 3$ . For any  $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$  we obtain the following formula

$$V_\lambda^{U(3)}|_{U(2)} = \left( V_{\lambda[3]}^{U(2)} - V_{\lambda[2]}^{U(2)} + V_{\lambda[1]}^{U(2)} \right) \otimes \text{Sym}((\mathbb{C}^2)^*).$$

Hence  $V_\lambda^{U(3)}|_{U(2)} = A_1(\lambda) - A_2(\lambda) + A_3(\lambda)$  with

$$\begin{aligned} A_3(\lambda) &= V_{\lambda[3]}^{U(2)} \otimes \text{Sym}((\mathbb{C}^2)^*) = \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b} V_{(a,b)}^{U(2)}, \\ A_2(\lambda) &= V_{\lambda[2]}^{U(2)} \otimes \text{Sym}((\mathbb{C}^2)^*) = \sum_{\lambda_1 \geq a \geq \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)}, \\ A_1(\lambda) &= V_{\lambda[1]}^{U(2)} \otimes \text{Sym}((\mathbb{C}^2)^*) = \sum_{\lambda_2 - 1 \geq a \geq \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)}. \end{aligned}$$

We recover the classical branching formula  $V_\lambda^{U(3)}|_{U(2)} := \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3} V_{(a,b)}^{U(2)}$  (see [4], section 8.1). In Figure 2, one can visualize the support of each characters  $A_k(\lambda)$ .

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