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SYMMETRIC PAIRS AND BRANCHING LAWS

PAUL-ÉMILE PARADAN

ABSTRACT. Let G be a compact connected Lie group and let H be a subgroup fixed by an involution. A classical result assures that the $H_{\mathbb{C}}$ -action on the flag variety \mathcal{F} of G admits a finite number of orbits. In this article we propose a formula for the branching coefficients of the symmetric pair (G, H) that is parametrized by $H_{\mathbb{C}} \backslash \mathcal{F}$.

CONTENTS

1. Introduction	1
2. Non abelian localization	4
3. Proof of the main theorem	9
4. Examples	16
5. Kostant multiplicity formula	23
References	28

1. INTRODUCTION

Let G be a compact connected Lie group equipped with an involution θ . Let $G^\theta := \{g \in G, \theta(g) = g\}$ be the subgroup fixed by the involution. We consider a subgroup $H \subset G$ such that $(G^\theta)_0 \subset H \subset G^\theta$. The purpose of this paper is the study of the branching laws between G and H .

Let T be a maximal torus of G that we choose θ -invariant. Let \mathfrak{t} be the Lie algebra of T . Let $\Lambda \subset \mathfrak{t}^*$ be the lattice of weights, and let \mathfrak{t}_+^* be a Weyl chamber. The irreducible representations of G are parametrized by the semi-group $\Lambda_+ := \Lambda \cap \mathfrak{t}_+^*$ of dominant weights.

Let $\lambda \in \Lambda_+$. In order to study the restriction $V_\lambda^G|_H$ of the irreducible G -representation V_λ^G , we consider the H -action on the flag variety $\mathcal{F} = G/T$ of G . An important object is the H -invariant subset

$$Z_\theta \subset \mathcal{F}$$

formed of the elements $x \in \mathcal{F}$ for which the stabilizer subgroup $G_x := \{g \in G, gx = x\}$ is stable under θ . In other words, $gT \in Z_\theta$ if and only if $g^{-1}\theta(g)$ belongs to the normalizer subgroup $N(T)$. A well-known result tells us that the group H has finitely many orbits in Z_θ , and that the map $\mathcal{O} \in H_{\mathbb{C}} \backslash \mathcal{F} \mapsto \mathcal{O} \cap Z_\theta \in H \backslash Z_\theta$ is bijective [8, 14, 12, 9].

Let $x \in Z_\theta$. The stabilizer subgroup G_x is a maximal torus in G , stable under θ , with Lie algebra \mathfrak{g}_x . We will also consider the abelian subgroup $H_x := G_x \cap H$ (that is

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not necessarily connected). Any weight $\mu \in \Lambda$ determines a character \mathbb{C}_{μ_x} of the torus G_x by taking $\mu_x = g \cdot \mu$ if $x = gT \in \mathcal{F}$.

We denote by $\mathfrak{R}_x \subset \mathfrak{g}_x^*$ the set of roots relative to the action of the Cartan subalgebra \mathfrak{g}_x on $\mathfrak{g} \otimes \mathbb{C}$. The map $\mu \in \mathfrak{R} \mapsto \mu_x \in \mathfrak{R}_x$ is an isomorphism, and we take $\mathfrak{R}_x^+ \subset \mathfrak{R}_x$ as the image of $\mathfrak{R}^+ \subset \mathfrak{R}$ through this isomorphism.

The involution θ leaves the set \mathfrak{R}_x invariant, and $\alpha \in \mathfrak{R}_x$ is an *imaginary root* if $\theta(\alpha) = \alpha$. If α is imaginary, the subspace $(\mathfrak{g} \otimes \mathbb{C})_\alpha$ is θ -stable. There are two cases. If the action of θ on $(\mathfrak{g} \otimes \mathbb{C})_\alpha$ is trivial then α is *compact imaginary*. If the action of $-\theta$ on $(\mathfrak{g} \otimes \mathbb{C})_\alpha$ is trivial, then α is *non-compact imaginary*. We denote respectively by $\mathfrak{R}_x^{\text{ci}}$ and by $\mathfrak{R}_x^{\text{nci}}$ the subsets of compact imaginary and non-compact imaginary roots, and we introduce the following G_x -modules

$$\mathbb{E}_x^{\text{ci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} (\mathfrak{g} \otimes \mathbb{C})_\alpha, \quad \mathbb{E}_x^{\text{nci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} (\mathfrak{g} \otimes \mathbb{C})_\alpha.$$

The weight

$$\delta(x) := \frac{1}{2} \sum_{\substack{\alpha \in \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \\ \theta(\alpha) \neq \alpha}} \alpha$$

defines a character $\mathbb{C}_{\delta(x)}$ of the abelian group H_x .

We denote by $R(H)$ and by $R(H_x)$ the representations rings of the compact Lie groups H and H_x . An element $E \in R(H)$ can be represented as a *finite* sum $E = \sum_{V \in \widehat{H}} m_V V$, with $m_V \in \mathbb{Z}$. We denote by $\widehat{R}(H)$ (resp. $\widehat{R}(H_x)$) the space of \mathbb{Z} -valued functions on \widehat{H} (resp. \widehat{H}_x). An element $E \in \widehat{R}(H)$ can be represented as an *infinite* sum $\sum_{V \in \widehat{H}} m_V V$, with $m_V \in \mathbb{Z}$. The induction map $\text{Ind}_{H_x}^H : \widehat{R}(H_x) \rightarrow \widehat{R}(H)$ is the dual of the restriction morphism $R(H) \rightarrow R(H_x)$.

Let $m_x = \frac{1}{2} |\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + \dim \mathbb{E}_x^{\text{nci}} \in \mathbb{N}$.

The main result of this paper is the following theorem.

Theorem 1.1. *Let $\lambda \in \Lambda_+$. We have the decomposition*

$$(1.1) \quad V_\lambda^G|_H = \sum_{Hx \in H \backslash Z_\theta} Q_{Hx}(\lambda)$$

where the terms $Q_{Hx}(\lambda) \in \widehat{R}(H)$ are defined by the following relation :

$$Q_{Hx}(\lambda) = (-1)^{m_x} \text{Ind}_{H_x}^H \left(\mathbb{C}_{\lambda_x + \delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \text{Sym}(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge \mathbb{E}_x^{\text{ci}} \right).$$

Here $\text{Sym}(\mathbb{E}_x^{\text{nci}})$, which is the symmetric algebra of $\mathbb{E}_x^{\text{nci}}$, is an admissible representation of H_x and $\bigwedge \mathbb{E}_x^{\text{ci}} = \bigwedge^+ \mathbb{E}_x^{\text{ci}} \ominus \bigwedge^- \mathbb{E}_x^{\text{ci}}$ is a virtual representation of H_x .

We give now another formulation for decomposition (1.1) using the (right) action of the Weyl group $W = N(T)/T$ on the flag variety \mathcal{F} . If $x = gT \in \mathcal{F}$ and $w \in W$ we take $xw := gwT$. We notice that Z_θ is stable under the action of W and that the quotient Z_θ/W parametrizes the set of maximal tori of G stable under θ .

We associate to an element $x = gT \in Z_\theta$ the subgroup $W_x^H \subset W$ defined by the relation $w \in W_x^H \iff Hxw = Hx$. We denote by $H \backslash Z_\theta / W$ the quotient of Z_θ by the

action of $H \times W$, and by $\bar{x} \in H \backslash Z_\theta / W$ the image of $x \in Z_\theta$ through the quotient map. We associate to $\bar{x} \in H \backslash Z_\theta / W$ the element $Q_{\bar{x}}(\lambda) \in \widehat{R}(H)$ defined as follows

$$Q_{\bar{x}}(\lambda) = \sum_{\bar{w} \in W_x^H \backslash W} Q_{Hxw}(\lambda).$$

The previous theorem says then that $V_\lambda^G|_H = \sum_{\bar{x} \in H \backslash Z_\theta / W} Q_{\bar{x}}(\lambda)$. Here is a new formulation of Theorem 1.1.

Theorem 1.2. *We have $V_\lambda^G|_H = \sum_{\bar{x} \in H \backslash Z_\theta / W} Q_{\bar{x}}(\lambda)$ where $Q_{\bar{x}}(\lambda) \in \widehat{R}(H)$ has the following description*

$$Q_{\bar{x}}(\lambda) = \text{Ind}_{H_x}^H \left(\mathbb{M}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^{\text{ci}} \right),$$

for some¹ $\mathbb{M}_x(\lambda) \in \widehat{R}(H_x)$.

We finish this section by giving two basic examples associated to the group $SU(2)$. Here the flag variety of $SU(2)$ is the 2-dimensional sphere \mathbb{S}^2 . For $n \geq 0$, we denote by V_n the irreducible representation of $SU(2)$ of dimension $n + 1$.

Example 1. $G = SU(2)$ and the involution θ is the conjugation by the matrix $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$. The subgroup fixed by θ is the torus $T \simeq U(1)$ and the critical set $Z_\theta \subset \mathbb{S}^2$ is composed of the poles S, N and the equator E , so that $T \backslash Z_\theta$ has three terms. We take $\lambda = n$ in $\widehat{SU(2)} \simeq \mathbb{N}$.

For $Hx = E$, we have $\mathbb{E}_x^{\text{nci}} = \mathbb{E}_x^{\text{ci}} = \{0\}$, $H_x \simeq \mathbb{Z}_2$, and $\mathbb{C}_{\lambda_x + \delta(x)} = \mathbb{C}_n|_{\mathbb{Z}_2}$. The contribution of E is then $\text{Ind}_{\mathbb{Z}_2}^{U(1)}(\mathbb{C}_n|_{\mathbb{Z}_2}) = \mathbb{C}_n \otimes \sum_{k \in \mathbb{Z}} \mathbb{C}_{2k}$.

For $Hx = N$, we have $H_x = T$, $\mathbb{E}_x^{\text{nci}} = \mathbb{C}_2$, $\mathbb{E}_x^{\text{ci}} = \{0\}$, and $\mathbb{C}_{\lambda_x + \delta(x)} = \mathbb{C}_n$. The contribution of N is then $-\mathbb{C}_{n+2} \otimes \text{Sym}(\mathbb{C}_2)$.

For $Hx = S$, we have $H_x = T$, $\mathbb{E}_x^{\text{nci}} = \mathbb{C}_{-2}$, $\mathbb{E}_x^{\text{ci}} = \{0\}$, and $\mathbb{C}_{\lambda_x + \delta(x)} = \mathbb{C}_{-n}$. The contribution of S is then $-\mathbb{C}_{-n-2} \otimes \text{Sym}(\mathbb{C}_{-2})$.

Finally, Relations (1.1) become

$$\begin{aligned} V_n|_T &= \mathbb{C}_n \otimes \sum_{k \in \mathbb{Z}} \mathbb{C}_{2k} - \mathbb{C}_{-n-2} \otimes \text{Sym}(\mathbb{C}_{-2}) - \mathbb{C}_{n+2} \otimes \text{Sym}(\mathbb{C}_2) \\ &= \sum_{k=-n}^0 \mathbb{C}_{2k+n}. \end{aligned}$$

Example 2. $G = SU(2) \times SU(2)$ and the involution θ is the map $(a, b) \mapsto (b, a)$. The subgroup fixed by θ is $SU(2)$ embedded diagonally and the critical set $Z_\theta \subset \mathbb{S}^2 \times \mathbb{S}^2$ is equal to the union of the orbits $SU(2) \cdot (N, N)$ and $SU(2) \cdot (S, N)$. Let $\lambda = (n, m) \in \widehat{G}$.

For $x = (N, N)$ or $x = (S, N)$ we have $\mathbb{E}_x^{\text{nci}} = \mathbb{E}_x^{\text{ci}} = \{0\}$ and $H_x \simeq T$. For $x = (N, N)$ we have $\lambda_x + \delta(x) = m + n + 2$, and for $x = (S, N)$ we have $\lambda_x + \delta(x) = m - n$. Relations (1.1) give then

$$V_n \otimes V_m = \text{Ind}_T^{SU(2)}(\mathbb{C}_{m-n}) - \text{Ind}_T^{SU(2)}(\mathbb{C}_{m+n+2}).$$

¹The precise expression of $\mathbb{M}_x(\lambda)$ is given in Proposition 3.8.

It is not difficult to see that the previous identities correspond to the classical Clebsch-Gordan relations (see Example 4.2).

Here is a brief overview of the article. Sections 2 and 3 are devoted to the proof of our main result. In Section 4, we detail the case of $U(p) \times U(q) \subset U(n)$: in particular, we explain the branching formula we obtain for the restriction of $U(n)$ to $U(n-1)$. In the last section, we recall Kostant's branching formula and explain the formula it gives in the case of the restriction of $U(n)$ to $U(n-1)$, in order to compare it with our own formula.

Notations

Throughout the paper :

- G denotes a compact connected Lie group with Lie algebra \mathfrak{g} .
- T is a maximal torus in G with Lie algebra \mathfrak{t} .
- $\Lambda \subset \mathfrak{t}^*$ is the weight lattice of T : every $\mu \in \Lambda$ defines a 1-dimensional T -representation, denoted by \mathbb{C}_μ , where $t = \exp(X)$ acts by $t^\mu := e^{i\langle \mu, X \rangle}$.
- The coadjoint action of $g \in G$ on $\xi \in \mathfrak{g}^*$ is denoted by $g \cdot \xi$.
- When a Lie group K acts on set X , the stabilizer subgroup of $x \in X$ is denoted by $K_x := \{k \in K \mid k \cdot x = x\}$ and the Lie algebra of K_x is denoted by \mathfrak{k}_x .
- When a Lie group K acts on a manifold M , we denote by $X \cdot m := \frac{d}{dt} e^{tX} \cdot m|_{t=0}$, $m \in M$, the infinitesimal action of $X \in \mathfrak{k}$ on M .

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2. NON ABELIAN LOCALIZATION

Our main result is obtained by means of a non-abelian localization of the Riemann-Roch character on the flag variety \mathcal{F} of G . For that purpose we will use the family $(\Omega_r)_r$ of symplectic structure parametrized by the interior of the Weyl chamber \mathfrak{t}_+^* . The symplectic structure Ω_r comes from the identification $gT \rightarrow g \cdot r$ of \mathcal{F} with the coadjoint orbit Gr . The moment map $\Phi_r : \mathcal{F} \rightarrow \mathfrak{g}^*$ associated to the action of G on (\mathcal{F}, Ω_r) is the map $gT \mapsto g \cdot r$.

At the level of Lie algebras we have $\mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$ where $\mathfrak{h} = \mathfrak{g}^\theta$ and $\mathfrak{q} = \mathfrak{g}^{-\theta}$. For any $\xi \in \mathfrak{g} = \mathfrak{h} \oplus \mathfrak{q}$, we denote by ξ^+ his \mathfrak{h} -part and by ξ^- his \mathfrak{q} -part. We use a G -invariant scalar product $(-, -)$ on \mathfrak{g} such that the involution θ is an orthogonal map. It induces identifications $\mathfrak{g}^* \simeq \mathfrak{g}$, $\mathfrak{h}^* \simeq \mathfrak{h}$ and $\mathfrak{q}^* \simeq \mathfrak{q}$.

The moment map $\Phi_r^H : \mathcal{F} \rightarrow \mathfrak{h}^*$ associated to the action of H on (\mathcal{F}, Ω_r) is the map $gT \mapsto (g \cdot r)^+$.

2.1. Matsuki duality. Consider the complex reductive groups $G_{\mathbb{C}}$ and $H_{\mathbb{C}}$ associated to the compact Lie groups G and H . Let $L \subset G_{\mathbb{C}}$ be the real form such that $H \subset L$ is a maximal compact subgroup of L .

Matsuki duality is the statement that a one-to-one correspondence exists between the $H_{\mathbb{C}}$ -orbits and the L -orbits in \mathcal{F} ; two orbits are in duality when their intersection is a single orbit of H .

Uzawa, and Mirkovic-Uzawa-Vilonen [16, 9] proved the Matsuki correspondence by showing that both $H_{\mathbb{C}}$ -orbits and L -orbits in \mathcal{F} are parametrized by the H -orbits in the set of critical points of the function $\|\Phi_r^H\|^2 : \mathcal{F} \rightarrow \mathbb{R}$.

First we recall the elementary but fundamental fact that the subset Z_θ is equal to the set of critical points of the function $\|\Phi_r^H\|^2$ [9, 3].

Lemma 2.1. *Let $x = gT \in \mathcal{F}$ and $r \in \text{Interior}(\mathfrak{t}_+^*)$. The following statements are equivalent:*

- i) the subalgebra \mathfrak{g}_x is invariant under θ (i.e. $x \in Z_\theta$),
- ii) $g^{-1}\theta(g) \in N(T)$,
- iii) x is a critical point of the function $\|\Phi_r^H\|^2$,
- iv) $(g \cdot r)^+$ and $(g \cdot r)^-$ commute.

Proof. Let $n_g = g^{-1}\theta(g)$ and let r be a regular element of $\mathfrak{t}^* \simeq \mathfrak{t}$. Since $\mathfrak{g}_x = \text{Ad}(g)\mathfrak{t}$ we see that

$$\begin{aligned} \theta(\mathfrak{g}_x) = \mathfrak{g}_x &\iff n_g \in N_G(T) \\ &\iff [n_g \cdot \theta(r), r] = 0 \\ &\iff [\theta(g \cdot r), g \cdot r] = 0 \\ &\iff [(g \cdot r)^+, (g \cdot r)^-] = 0. \end{aligned}$$

A small computation shows that for any $X \in \mathfrak{g}$ the derivative of the function $t \mapsto \|\Phi_r^H(e^{tX}x)\|^2$ at $t = 0$ is equal to $(X, [g \cdot r, \theta(g \cdot r)])$. Hence $x = gT$ is a critical point of the function $\|\Phi_r^H\|^2$ if and only if $[g \cdot r, \theta(g \cdot r)] = 0$. Finally we have proved that the statements *i), ii), iii)* and *iv)* are equivalent. \square

Let us check the other easy fact.

Lemma 2.2. *The set $H \setminus Z_\theta$ is finite.*

Proof. Let $x = gT \in Z_\theta$. A neighborhood of x is defined by elements of the form $e^X e^Y x$ where $X \in \mathfrak{h}$ and $Y \in \mathfrak{q}$. Now we see that $e^X e^Y gT \in Z_\theta$ if and only if $e^{-2g^{-1}Y} \in N(T)$. If Y is sufficiently small the former relation is equivalent to $g^{-1}Y \in \mathfrak{t}$, and in this case $e^X e^Y x = e^X x$. We have proved that any element in $H \setminus Z_\theta$ is isolated. As $H \setminus Z_\theta$ is compact, we can conclude that $H \setminus Z_\theta$ is finite. \square

2.2. Borel-Weil-Bott theorem. We first recall the Borel-Weil-Bott theorem. The flag manifold \mathcal{F} is equipped with the G -invariant complex structure such that

$$\mathbf{T}_{eT}\mathcal{F} \simeq \sum_{\alpha \in \mathfrak{R}^+} (\mathfrak{g} \otimes \mathbb{C})_\alpha$$

is an identity of T -modules. Let us consider the tangent bundle $\mathbf{T}\mathcal{F}$ as a complex vector bundle on \mathcal{F} with the invariant Hermitian structure $h_{\mathcal{F}}$ induced by the invariant scalar product on \mathfrak{g} .

Any weight $\lambda \in \Lambda$ defines a line bundle $\mathcal{L}_\lambda \simeq G \times_T \mathbb{C}_\lambda$ on \mathcal{F} .

Definition 2.3. *We associate to a weight $\lambda \in \Lambda$*

- the spin-c bundle on \mathcal{F}

$$\mathcal{S}_\lambda := \bigwedge_{\mathbb{C}} \mathbf{T}\mathcal{F} \otimes \mathcal{L}_\lambda,$$

• the Riemann-Roch character $\mathrm{RR}_G(\mathcal{F}, \mathcal{L}_\lambda) \in R(G)$ which is the equivariant index of the Dirac operator D_λ associated to the spin-c structure \mathcal{S}_λ .

The Borel-Weil-Bott theorem asserts that $V_\lambda^G = \mathrm{RR}_G(\mathcal{F}, \mathcal{L}_\lambda)$ when λ is dominant. Now we consider the restriction $V_\lambda^G|_H = \mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda)$. In the next section, we will explain how we can localize the H -equivariant Riemann-Roch character $\mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda)$ on the critical set of the function $\|\Phi_r^H\|^2$ [10].

2.3. Localization of the Riemann-Roch character. In this section, we recall how we perform the “Witten non-abelian localization” of the Riemann-Roch character with the help of the moment map $\Phi_r^H : \mathcal{F} \rightarrow \mathfrak{h}^*$ attached to a regular element r of the Weyl chamber [10, 6, 11].

Let us denote by $X \mapsto [X]_{\mathfrak{g}/\mathfrak{t}}$ the projection $\mathfrak{g} \rightarrow \mathfrak{g}/\mathfrak{t}$. The Kirwan vector field κ_r on \mathcal{F} is defined as follows:

$$\kappa_r(x) = -\Phi_r^H(x) \cdot x \in \mathbf{T}_x\mathcal{F}.$$

Through the identification $\mathfrak{g}/\mathfrak{t} \simeq \mathbf{T}_x\mathcal{F}$, $X \mapsto \frac{d}{dt}|_{t=0} g e^{tX} T$, the vector $\kappa_r(x) \in \mathbf{T}_x\mathcal{F}$ is equal to $[g^{-1}\theta(g) \cdot r]_{\mathfrak{g}/\mathfrak{t}}$. Hence the set $Z_\theta \subset \mathcal{F}$ is exactly the set where κ_r vanishes.

Let D_0 be the Dirac operator associated to the spin-c structure $\mathcal{S}_0 = \bigwedge_{\mathbb{C}} \mathbf{T}\mathcal{F}$. The principal symbol of the elliptic operator D_0 is the bundle map $\sigma(\mathcal{F}) \in \Gamma(\mathbf{T}^*\mathcal{F}, \mathrm{hom}(\bigwedge_{\mathbb{C}}^+ \mathbf{T}\mathcal{F}, \bigwedge_{\mathbb{C}}^- \mathbf{T}\mathcal{F}))$ defined by the Clifford action

$$\sigma(\mathcal{F})(x, \nu) = \mathbf{c}_x(\tilde{\nu}) : \bigwedge_{\mathbb{C}}^+ \mathbf{T}_x\mathcal{F} \rightarrow \bigwedge_{\mathbb{C}}^- \mathbf{T}_x\mathcal{F}.$$

Here $\nu \in \mathbf{T}_x^*\mathcal{O} \simeq \tilde{\nu} \in \mathbf{T}_x\mathcal{O}$ is the one to one map associated to the identification $\mathfrak{g}^* \simeq \mathfrak{g}$ (see [2]).

Now we deform the elliptic symbol $\sigma(\mathcal{F})$ by means of the vector field κ_r [10, 11].

Definition 2.4. *The symbol $\sigma(\mathcal{F})$ shifted by the vector field κ_r is the symbol on \mathcal{F} defined by*

$$\sigma_r(\mathcal{F})(x, \nu) = \mathbf{c}_x(\tilde{\nu} - \kappa_r(x))$$

for any $(x, \nu) \in \mathbf{T}^*\mathcal{F}$.

Consider an H -invariant open subset $\mathcal{U} \subset \mathcal{F}$ such that $\mathcal{U} \cap Z_\theta$ is compact in \mathcal{F} . Then the restriction $\sigma_r(\mathcal{F})|_{\mathcal{U}}$ is a H -transversally elliptic symbol on \mathcal{U} , and so its equivariant index is a well defined element in $\widehat{R}(H)$ (see [1, 10, 11]).

Thus we can define the following localized equivariant indices.

Definition 2.5. *Let $Hx \subset Z_\theta$. We denote by*

$$\mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx) \in \widehat{R}(H)$$

the equivariant index of $\sigma_r(\mathcal{F}) \otimes \mathcal{L}_\lambda|_{\mathcal{U}}$ where \mathcal{U} is an invariant neighbourhood of Hx so that $\mathcal{U} \cap Z_\theta = Hx$.

We proved in [10] that the following decomposition holds in $\widehat{R}(H)$:

$$\mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda) = \sum_{Hx \in H \setminus Z_\theta} \mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx).$$

The computation of the characters $\mathrm{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx)$ will be handle in Section 3.1. To undertake these calculations we need to describe geometrically a neighborhood of Hx in \mathcal{F} . This is the goal of the next section.

2.4. **Local model near $Hx \subset Z_\theta$.** Let $x = gT \in Z_\theta$. We need to compute a symplectic model of a neighborhood of Hx in (\mathcal{F}, Ω_r) . Here we use the identification $\mathfrak{g} \simeq \mathfrak{g}^*$ given by the choice of an invariant scalar product. Let $\mu = g \cdot r$ that we write $\mu = \mu^+ + \mu^-$ where $\mu^+ \in \mathfrak{h}$ and $\mu^- \in \mathfrak{q}$.

The tangent space $\mathbf{T}_x \mathcal{F}$ is equipped with the symplectic two form $\Omega_r|_x$:

$$\Omega_r|_x(X \cdot x, Y \cdot x) = (\mu, [X, Y]), \quad X, Y \in \mathfrak{g}.$$

We need to understand the structure of the symplectic vector space $(\mathbf{T}_x \mathcal{F}, \Omega_r|_x)$. If $\mathfrak{a} \subset \mathfrak{g}$ is a vector subspace we denote by $\mathfrak{a} \cdot x := \{X \cdot x, X \in \mathfrak{a}\}$ the corresponding subspace of $\mathbf{T}_x \mathcal{F}$. The symplectic orthogonal of $\mathfrak{a} \cdot x$ is denoted by $(\mathfrak{a} \cdot x)^{\perp, \Omega}$.

If $\mathfrak{a}, \mathfrak{b}$ are two subspaces, a small computation gives that

$$(2.2) \quad (\mathfrak{a} \cdot x)^{\perp, \Omega} \cap \mathfrak{b} \cdot x \simeq \mathfrak{a}^\perp \cap [\mathfrak{b}, \mu],$$

where $\mathfrak{a}^\perp \subset \mathfrak{g}$ is the orthogonal of \mathfrak{a} relatively to the scalar product.

We denote by $\mathfrak{g}_{\mu^+} = \mathfrak{h}_{\mu^+} \oplus \mathfrak{q}_{\mu^+}$ the subspaces fixed by $ad(\mu^+)$. Notice that $\mathfrak{g}_\mu = \mathfrak{g}_x$ is an abelian subalgebra containing μ^+ since $[\mu^+, \mu^-] = 0$. It follows that $\mathfrak{g}_x \subset \mathfrak{g}_{\mu^+}$.

Lemma 2.6. $[\mathfrak{g}, \mu^+] \cdot x, \mathfrak{g}_{\mu^+} \cdot x$ and $[\mathfrak{h}, \mu^+] \cdot x$ are symplectic subspaces of $\mathbf{T}_x \mathcal{F}$.

Proof. It is a direct consequence of (2.2). \square

We consider now the symplectic subspace $V_x \subset \mathbf{T}_x \mathcal{F}$ defined by the relation

$$(2.3) \quad V_x = ([\mathfrak{h}, \mu^+] \cdot x)^{\perp, \Omega} \cap [\mathfrak{g}, \mu^+] \cdot x.$$

A small computation shows that $X \cdot x \in V_x$ if and only if $[X, \mu] \in [\mathfrak{q}, \mu^+]$.

We have the following important Lemma.

Lemma 2.7. • We have the following decomposition

$$(2.4) \quad \mathbf{T}_x \mathcal{F} = \mathfrak{g}_{\mu^+} \cdot x \overset{\perp}{\oplus} [\mathfrak{h}, \mu^+] \cdot x \overset{\perp}{\oplus} V_x$$

where \perp stands for the orthogonal relative to $\Omega_r|_x$.

- $\mathfrak{g}_{\mu^+} \cdot x$ is symplectomorphic to $\mathfrak{h}_{\mu^+}/\mathfrak{h}_x \oplus (\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^*$.
- $[\mathfrak{h}, \mu^+] \cdot x$ is symplectomorphic to $\mathfrak{h}/\mathfrak{h}_{\mu^+}$ equipped with the symplectic structure $\Omega_{\mu^+}(\bar{u}, \bar{v}) = (\mu^+, [u, v])$.
- V_x is symplectomorphic to $(\mathfrak{h} \cdot x)^{\perp, \Omega} / [(\mathfrak{h} \cdot x)^{\perp, \Omega} \cap \mathfrak{h} \cdot x]$.

Proof. If we use the decomposition $\mathfrak{g} = \mathfrak{g}_{\mu^+} \oplus [\mathfrak{g}, \mu^+]$ and the fact that the abelian subalgebra \mathfrak{g}_x is contained in \mathfrak{g}_{μ^+} we obtain

$$\mathbf{T}_x \mathcal{F} = \mathfrak{g}_{\mu^+} \cdot x \oplus [\mathfrak{g}, \mu^+] \cdot x.$$

It is obvious to check that the subspaces $[\mathfrak{g}, \mu^+] \cdot x$ and $\mathfrak{g}_{\mu^+} \cdot x$ are orthogonal relatively to the symplectic form $\Omega_r|_x$. Since $[\mathfrak{h}, \mu^+] \cdot x$ is a symplectic subspace we have $[\mathfrak{g}, \mu^+] \cdot x = [\mathfrak{h}, \mu^+] \cdot x \overset{\perp}{\oplus} V_x$ where V_x is defined by (2.3). The first point is proved.

The identities $\mathfrak{g}_x = \theta(\mathfrak{g}_x) = \mathfrak{g}_{\theta(x)}$ imply the decompositions $\mathfrak{g}_x = \mathfrak{h}_x \oplus \mathfrak{q}_x$ and $\mathfrak{g}_{\mu^+} \cdot x = \mathfrak{q}_{\mu^+} \cdot x \oplus \mathfrak{h}_{\mu^+} \cdot x$. The vector subspace $\mathfrak{h}_{\mu^+} \cdot x$ is isomorphic to $\mathfrak{h}_{\mu^+}/\mathfrak{h}_x$, and the map $v \mapsto \Omega_r|_x(v, -)$ defines an isomorphism between $\mathfrak{q}_{\mu^+} \cdot x$ and the dual of $\mathfrak{h}_{\mu^+} \cdot x$. The second point is proved.

For the third point we use the isomorphism $j : [\mathfrak{h}, \mu^+] \rightarrow \mathfrak{h}/\mathfrak{h}_{\mu^+}$ induced by the projection $\mathfrak{h} \rightarrow \mathfrak{h}/\mathfrak{h}_{\mu^+}$. Then the map $\bar{u} \mapsto j(\bar{u}) \cdot x$ defines a symplectomorphism between $(\mathfrak{h}/\mathfrak{h}_{\mu^+}, \Omega_{\mu^+})$ and $[\mathfrak{h}, \mu^+] \cdot x$.

Now we see that (2.4) together with the decomposition $\mathfrak{h} \cdot x = \mathfrak{h}_{\mu^+} \cdot x \oplus [\mathfrak{h}, \mu^+] \cdot x$ leads to

$$\begin{aligned} (\mathfrak{h} \cdot x)^{\perp, \Omega} &= ([\mathfrak{h}, \mu^+] \cdot x)^{\perp, \Omega} \cap (\mathfrak{h}_{\mu^+} \cdot x)^{\perp, \Omega} \\ &= \mathfrak{h}_{\mu^+} \cdot x \oplus V_x \\ &= [(\mathfrak{h} \cdot x)^{\perp, \Omega} \cap \mathfrak{h} \cdot x] \oplus V_x. \end{aligned}$$

The last point follows. \square

We denote by Ω_{V_x} the restriction of $\Omega_r|_x$ on the symplectic vector subspace V_x . The action of H_x on (V_x, Ω_{V_x}) is Hamiltonian, with moment map $\Phi_{V_x} : V_x \rightarrow \mathfrak{h}_x^*$ defined by the relation

$$\langle \Phi_{V_x}(v), A \rangle = \frac{1}{2} \Omega_{V_x}(v, Av), \quad v \in V_x, \quad A \in \mathfrak{h}_x.$$

Thanks to Lemma 2.7, we know that the H_x -symplectic vector space $(\mathbf{T}_x \mathcal{F}, \Omega_r|_x)$ admits the following decomposition

$$\mathbf{T}_x \mathcal{F} \simeq \mathfrak{h}_{\mu^+}/\mathfrak{h}_x \oplus (\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \oplus \mathfrak{h}/\mathfrak{h}_{\mu^+} \oplus V_x.$$

Thanks to the normal form Theorem of Marle [7] and Guillemin-Sternberg [5], we get the following result.

Corollary 2.8. *An H -equivariant symplectic model of a neighborhood of Hx in \mathcal{F} is $\mathcal{F}_x := H \times_{H_{\mu^+}} Y_x$ where*

$$Y_x = H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x).$$

The corresponding moment map on \mathcal{F}_x is

$$\Phi_{\mathcal{F}_x}([h; \eta, v]) = h(\eta + \mu^+ + \Phi_{V_x}(v))$$

for $[h; \eta, v] \in H \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x)$.

We finish this section by computing a compatible complex structure on V_x .

By definition, the map that sends $X \cdot x$ to $[X, \mu]$ defines an isomorphism $i : V_x \rightarrow [\mathfrak{q}, \mu^+]$. The adjoint map $ad(\mu)$ defines also an automorphism of $[\mathfrak{g}, \mu^+]$: for any $X \in [\mathfrak{g}, \mu^+]$ we denote by $\tilde{X} \in [\mathfrak{g}, \mu^+]$ the unique element such that $ad(\mu)\tilde{X} = X$.

The symplectic structure $\Omega_{\mu} := (i^{-1})^* \Omega_{V_x}$ satisfies the relations

$$\Omega_{\mu}(X, Y) = (\mu, [\tilde{X}, \tilde{Y}]) = (X, \tilde{Y}) = -(\tilde{X}, Y), \quad \forall X, Y \in [\mathfrak{q}, \mu^+].$$

We work with the following H_x -equivariant maps

- the one to one map $T_{\mu} := -ad(\mu)ad(\theta(\mu)) : [\mathfrak{g}, \mu^+] \rightarrow [\mathfrak{g}, \mu^+]$,
- the complex structure $J_{\mu^+} = ad(\mu^+)(-ad(\mu^+)^2)^{-1/2}$ on $[\mathfrak{g}, \mu^+]$.

The map T_μ restricts to a one to one map $T_x : [\mathfrak{q}, \mu^+] \rightarrow [\mathfrak{q}, \mu^+]$ and J_{μ^+} defines a complex structure on $[\mathfrak{q}, \mu^+]$ (still denoted by J_{μ^+}).

Let $S_x := (T_x^2)^{-1/2} T_x$. The map $J_{V_x} := J_{\mu^+} \circ S_x$ defines a H_x -invariant complex structure on $[\mathfrak{q}, \mu^+]$.

Lemma 2.9. *The H_x -symplectic space (V_x, Ω_{V_x}) is isomorphic to $[\mathfrak{q}, \mu^+]$ equipped with the symplectic form $\Omega_\mu^1(v, w) = (J_{V_x} v, w)$.*

Proof. We know already that $(V_x, \Omega_{V_x}) \simeq ([\mathfrak{q}, \mu^+], \Omega_\mu)$. If one takes $L = T_x \circ (-ad(\mu^+)^2)^{-1/4} \circ (T_x^2)^{-1/4}$, we check easily that $\Omega_\mu(L(v), L(w)) = (J_{V_x} v, w)$. \square

3. PROOF OF THE MAIN THEOREM

We start with the following lemma.

Lemma 3.1. *The quantity $\text{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx)$ does not depend on the choice of the regular element r in the Weyl chamber. In the following we will denote it by $Q_{Hx}(\lambda) \in \widehat{R}(H)$.*

Proof. Let r_0, r_1 be two regular elements of the Weyl chamber. For $t \in [0, 1]$, we consider the regular element $r(t) = tr_1 + (1-t)r_0$: the Kirwan vector field $\kappa_{r(t)}$ vanishes exactly on Z_θ for any $t \in [0, 1]$. If \mathcal{U} is an invariant neighbourhood of Hx so that $\mathcal{U} \cap Z_\theta = Hx$, then $t \in [0, 1] \mapsto \sigma_{r(t)}(\mathcal{F}) \otimes \mathcal{L}_\lambda|_{\mathcal{U}}$ defines an homotopy of transversally elliptic symbols. Accordingly, the equivariant index of $\sigma_{r_0}(\mathcal{F}) \otimes \mathcal{L}_\lambda|_{\mathcal{U}}$ and $\sigma_{r_1}(\mathcal{F}) \otimes \mathcal{L}_\lambda|_{\mathcal{U}}$ are equal. \square

3.1. Computation of $Q_{Hx}(\lambda)$. The computation of $Q_{Hx}(\lambda)$ is done in three steps.

3.1.1. Step 1: holomorphic induction. Let $H_{\mu^+} \subset H$ be the stabilizer subgroup of $\mu^+ := \Phi_r^H(x)$. By Corollary 2.8, a symplectic H -equivariant model of a neighborhood of Hx in \mathcal{F} is the manifold $H \times_{H_{\mu^+}} Y_x$ where

$$Y_x = H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x).$$

The symplectic two form on Y_x is built from the canonical symplectic structure on $H_{\mu^+} \times_{H_x} (\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \simeq \mathbf{T}^*(H_{\mu^+}/H_x)$ and the symplectic structure on V_x . The moment map relative to the action of H_{μ^+} on Y_x is

$$\Phi_{Y_x}([h; \eta, v]) = h(\eta + \mu^+ + \Phi_{V_x}(v)) \in \mathfrak{h}_{\mu^+}^*,$$

for $[h; \eta, v] \in H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x)$.

Let κ_{Y_x} the Kirwan vector field on Y_x . It is immediate to check that $[h; \eta, v] \in \{\kappa_{Y_x} = 0\}$ if and only if $\eta = 0$ and $(\mu^+ + \Phi_{V_x}(v)) \cdot v = 0$. The map $v \in V_x \mapsto \mu^+ \cdot v \in V_x$ is bijective and $v \mapsto \Phi_{V_x}(v) \cdot v$ is homogeneous of degree equal to 3. Then there exists $\epsilon > 0$ such that

$$(\mu^+ + \Phi_{V_x}(v)) \cdot v = 0 \quad \text{and} \quad \|v\| \leq \epsilon \implies v = 0.$$

In Y_x , we still denote by x the point $[e, 0, 0]$. We equip Y_x with an invariant almost complex structure that is compatible with the symplectic structure, and we denote by $\text{RR}_{H_{\mu^+}}(Y_x, \mathcal{L}_\lambda|_{Y_x}, \Phi_{Y_x}, H_{\mu^+}x)$ the Riemann-Roch character on Y_x localized on the component $H_{\mu^+}x \subset \{\kappa_{Y_x} = 0\}$.

The quotient $\mathfrak{h}/\mathfrak{h}_{\mu^+}$, which is equipped with the invariant complex structure $J_{\mu^+} := ad(\mu^+)(-ad(\mu^+)^2)^{-1/2}$, is a complex H_{μ^+} -module.

In [10][Theorem 7.5], we proved that $Q_{H_x}(\lambda) = \text{RR}_H(\mathcal{F}, \mathcal{L}_\lambda, \Phi_r^H, Hx)$ is equal to

$$(3.5) \quad \text{Ind}_{H_{\mu^+}}^H \left(\text{RR}_{H_{\mu^+}}(Y_x, \mathcal{L}_\lambda|_{Y_x}, \Phi_{Y_x}, H_{\mu^+}x) \otimes \bigwedge \mathfrak{h}/\mathfrak{h}_{\mu^+} \right).$$

3.1.2. *Step 2: cotangent induction.* The map $\Phi_x(v) := \mu^+ + \Phi_{V_x}(v)$ is a moment map for the Hamiltonian action of H_x on V_x . The moment map on the H_{μ^+} -manifold

$$Y_x = H_{\mu^+} \times_{H_x} ((\mathfrak{h}_{\mu^+}/\mathfrak{h}_x)^* \times V_x)$$

is $\Phi_{Y_x}([h; \eta, v]) = h(\eta + \Phi_x(v)) \in \mathfrak{h}_\mu^*$.

Let $\kappa_{V_x}(v) = -\Phi_x(v) \cdot v$ be the Kirwan vector field on V_x . We are interested in the connected component $\{0\}$ of $\{\kappa_{V_x} = 0\}$. We choose a compatible almost complex structure on the symplectic vector space V_x and we denote by $\text{RR}_{H_x}(V_x, \Phi_x, \{0\}) \in \widehat{R}(H_x)$ the Riemann-Roch character localized on $\{0\} \subset \{\kappa_{V_x} = 0\}$.

In Section 3.3 of [11] we have proved that

$$(3.6) \quad \text{RR}_{H_{\mu^+}}(Y_x, \mathcal{L}_\lambda|_{Y_x}, \Phi_{Y_x}, H_{\mu^+}x) = \text{Ind}_{H_x}^{H_{\mu^+}} (\text{RR}_{H_x}(V_x, \Phi_x, \{0\}) \otimes \mathcal{L}_\lambda|_x).$$

3.1.3. *Step 3: linear case.* We write $\mathfrak{q}/\mathfrak{q}_{\mu^+}$ for the vector space $[\mathfrak{q}, \mu^+]$ equipped with the complex structure J_{μ^+} . So $\mathfrak{q}/\mathfrak{q}_{\mu^+}$ is a H_{μ^+} -module and we denote by $\text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+})$ the corresponding symmetric algebra.

We need to compare the virtual H_x -modules $\bigwedge_{J_{V_x}} V_x$ and $\bigwedge_{-J_{\mu^+}} V_x$. The weight

$$\delta(x) := \frac{1}{2} \sum_{\substack{\alpha \in \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \\ \theta(\alpha) \neq \alpha}} \alpha$$

defines a character $\mathbb{C}_{\delta(x)}$ of the abelian group H_x . Recall that $m_x \in \mathbb{N}$ corresponds to the quantity $\frac{1}{2} |\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + \dim \mathbb{E}_x^{\text{nci}}$.

The following lemma will be proved in Section 3.2.

Lemma 3.2. *The following identity holds :*

$$\bigwedge_{J_{V_x}} V_x \simeq (-1)^{m_x} \mathbb{C}_{\delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge_{-J_{\mu^+}} V_x.$$

On the vector space V_x , we can work with two localized Riemann-Roch characters:

- $\text{RR}_{H_x}(V_x, \Phi_x, \{0\})$ is defined with the complex structure J_{V_x} ,
- $\widetilde{\text{RR}}_{H_x}(V_x, \Phi_x, \{0\})$ is defined with the complex structure $-J_{\mu^+}$.

Thanks to the previous Lemma we know that $\text{RR}_{H_x}(V_x, \Phi_x, \{0\})$ is equal to $(-1)^{m_x} \mathbb{C}_{\delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \widetilde{\text{RR}}_{H_x}(V_x, \Phi_x, \{0\})$.

Proposition 3.3. *We have*

$$(3.7) \quad \text{RR}_{H_x}(V_x, \Phi_x, \{0\}) = (-1)^{m_x} \mathbb{C}_{\delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+}).$$

Proof. For $s \in [0, 1]$, we consider the H_x -equivariant map $\Phi^s : V_x \rightarrow \mathfrak{h}_x^*$ defined by the relation $\Phi^s(v) = \mu^+ + s\Phi_{V_x}(v)$. The corresponding Kirwan vector field on V_x is $\kappa^s(v) = -\Phi^s(v) \cdot v$. It is not difficult to see that there exists $\epsilon > 0$ such that

$\{\kappa^s = 0\} \cap \{\|v\| \leq \epsilon\} = \{0\}$ for any $s \in [0, 1]$. Then a simple deformation argument gives that $\widetilde{\text{RR}}_{H_x}(V_x, \Phi^s, \{0\})$ does not depend on $s \in [0, 1]$. We have proved that

$$\widetilde{\text{RR}}_{H_x}(V_x, \Phi_x, \{0\}) = \widetilde{\text{RR}}_{H_x}(V_x, \mu^+, \{0\})$$

where μ^+ denotes the constant map Φ^0 . Standard computations give $\widetilde{\text{RR}}_{H_x}(V_x, \mu^+, \{0\}) = \text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+})$ (see [10][Proposition 5.4]). Our proof is completed. \square

3.1.4. *Conclusion.* If we use the formulas (3.5), (3.6) and (3.7) we obtain the following expression

$$Q_{H_x}(\lambda) = (-1)^{m_x} \text{Ind}_{H_x}^H \left(\mathbb{C}_{\lambda_x + \delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+}) \otimes \bigwedge_{\mathbb{C}} \mathfrak{h}/\mathfrak{h}_{\mu^+} \right)$$

in $\widehat{R}(H)$. Here \mathbb{C}_{λ_x} is the character of G_x associated to the weight $\lambda_x = g\lambda$.

The previous formula depends on a choice of a regular element r in the Weyl chamber. In the next section we will propose another expression for $Q_{H_x}(\lambda)$ that does not depend on this choice.

3.2. **Another expression for $Q_{H_x}(\lambda)$.** Let $\mathfrak{R}_x \subset \mathfrak{g}_x^*$ be the roots for the action of the torus G_x on $\mathfrak{g} \otimes \mathbb{C}$. The involution $\theta : \mathfrak{t}^* \rightarrow \mathfrak{t}^*$ leaves the set \mathfrak{R}_x invariant and a root $\alpha \in \mathfrak{R}_x$ is called *imaginary* if $\theta(\alpha) = \alpha$. We denote respectively by $\mathfrak{R}_x^{\text{ci}}$ and by $\mathfrak{R}_x^{\text{nci}}$ the subsets of *compact imaginary* and *non-compact imaginary* roots.

We choose a generic element $r \in \mathfrak{t}_+^*$ such that $\mu^+ = (g \cdot r)^+$ satisfies the following relation : for any $\alpha \in \mathfrak{R}_x$, we have

$$(\alpha, \mu^+) = 0 \iff \theta(\alpha) = -\alpha.$$

Notice that an imaginary roots α is positive if and only if $(\alpha, \mu^+) > 0$.

Definition 3.4. We consider the subset $\mathfrak{A}_x \subset \mathfrak{R}_x$ defined by the following relations:

$$\alpha \in \mathfrak{A}_x \iff \alpha(\mu^+) > 0 \quad \text{and} \quad \theta(\alpha) \neq \alpha.$$

The involution θ defines a free action of \mathbb{Z}_2 on the set \mathfrak{A}_x . We denote by $\mathfrak{A}_x/\mathbb{Z}_2$ its quotient. For any $\alpha \in \mathfrak{R}_x$, we denote by \mathbb{C}_α the corresponding 1-dimensional representation of G_x , and $\mathbb{C}_\alpha|_{H_x}$ its restriction to the subgroup H_x . We have a natural map $[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2 \mapsto \mathbb{C}_\alpha|_{H_x} \in R(H_x)$.

For any $\alpha \in \mathfrak{R}_x$ we define

$$\tilde{\alpha} = \pm \alpha$$

where \pm is the sign of $\alpha(\mu)\alpha(\theta(\mu))$.

We consider the H_x -modules $\mathfrak{h}/\mathfrak{h}_{\mu^+} := ([\mathfrak{h}, \mu^+], J_{\mu^+})$, $\mathfrak{q}/\mathfrak{q}_{\mu^+} := (V_x, J_{\mu^+})$ and (V_x, J_{V_x}) .

Lemma 3.5. We have the following isomorphisms of H_x -modules

$$\begin{aligned} \mathfrak{h}/\mathfrak{h}_{\mu^+} &\simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha|_{H_x} & [A], \\ \mathfrak{q}/\mathfrak{q}_{\mu^+} &\simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha|_{H_x} & [B], \\ (V_x, J_{V_x}) &\simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_{\tilde{\alpha}}|_{H_x} \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_{\tilde{\alpha}}|_{H_x} & [C]. \end{aligned}$$

Proof. Thanks to Lemma 2.9, we know that the H_x -module (V_x, J_{V_x}) is isomorphic to the vector space $[\mathfrak{q}, \mu^+]$ equipped with the complex structure $J_{V_x} := J_{\mu^+} \circ S_x$. We consider the vector spaces $[\mathfrak{q}, \mu^+]$ and $[\mathfrak{g}, \mu^+]$ equipped with the complex structure J_{μ^+} . The projection (taking the real part) $\mathbf{r} : \mathfrak{g} \otimes \mathbb{C} \rightarrow \mathfrak{g}$ induces an isomorphism of G_x -modules

$$\mathbf{r} : \bigoplus_{\alpha(\mu^+) > 0} (\mathfrak{g} \otimes \mathbb{C})_\alpha \longrightarrow [\mathfrak{g}, \mu^+].$$

The orthogonal projections $\mathbf{p}_1 : [\mathfrak{g}, \mu^+] \rightarrow [\mathfrak{q}, \mu^+]$ and $\mathbf{p}_2 : [\mathfrak{g}, \mu^+] \rightarrow [\mathfrak{h}, \mu^+]$ commute with the H_x -action, so the maps

$$\begin{aligned} \mathbf{p}_1 \circ \mathbf{r} &: \bigoplus_{\alpha(\mu^+) > 0} (\mathfrak{g} \otimes \mathbb{C})_\alpha \longrightarrow [\mathfrak{q}, \mu^+], \\ \mathbf{p}_2 \circ \mathbf{r} &: \bigoplus_{\alpha(\mu^+) > 0} (\mathfrak{g} \otimes \mathbb{C})_\alpha \longrightarrow [\mathfrak{h}, \mu^+] \end{aligned}$$

are surjective morphisms of H_x -modules.

Let $V_x^1(\alpha) = \mathbf{p}_1 \circ \mathbf{r}((\mathfrak{g} \otimes \mathbb{C})_\alpha)$. We notice that $\dim_{\mathbb{C}} V_x^1(\alpha) \in \{0, 1\}$: $V_x^1(\alpha) = \{0\}$ only if α is a non-compact imaginary root and $V_x^1(\alpha) \simeq \mathbb{C}_\alpha|_{H_x}$ when $V_x^1(\alpha) \neq \{0\}$. We notice also that $V_x^1(\alpha) = V_x^1(\theta(\alpha))$, hence

$$\mathfrak{q}/\mathfrak{q}_{\mu^+} = ([\mathfrak{q}, \mu^+], J_{\mu^+}) \simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} V_x^1(\alpha) \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} V_x^1(\alpha).$$

The identity [B] is proved.

Similarly we consider $V_x^2(\alpha) = \mathbf{p}_2 \circ \mathbf{r}((\mathfrak{g} \otimes \mathbb{C})_\alpha)$. We notice that $\dim_{\mathbb{C}} V_x^2(\alpha) \in \{0, 1\}$: $V_x^2(\alpha) = \{0\}$ only if α is a compact imaginary root and $V_x^2(\alpha) \simeq \mathbb{C}_\alpha|_{H_x}$ when $V_x^2(\alpha) \neq \{0\}$. We notice also that $V_x^2(\alpha) = V_x^2(\theta(\alpha))$, hence

$$\mathfrak{h}/\mathfrak{h}_{\mu^+} = ([\mathfrak{h}, \mu^+], J_{\mu^+}) \simeq \bigoplus_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} V_x^2(\alpha) \oplus \bigoplus_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} V_x^2(\alpha).$$

The identity [A] is proved.

Finally we check that the complex structures J_{μ^+} and J_{V_x} preserve each $V_x^1(\alpha)$ and that $(V_x(\alpha), J_{V_x}) \simeq \mathbb{C}_{\tilde{\alpha}}|_{H_x}$ when $(\alpha, \mu^+) > 0$. The identity [C] follows. \square

We consider the H_x -module $\mathbb{V}_x := \sum_{[\alpha] \in \mathfrak{A}_x/\mathbb{Z}_2} \mathbb{C}_\alpha|_{H_x}$, and the G_x -modules $\mathbb{E}_x^{\text{nci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha$ and $\mathbb{E}_x^{\text{ci}} := \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \mathbb{C}_\alpha$. In the previous lemma we have proved that H_x -modules $\mathfrak{h}/\mathfrak{h}_{\mu^+}$ and $\mathfrak{q}/\mathfrak{q}_{\mu^+}$ are respectively isomorphic to $\mathbb{V}_x \oplus \mathbb{E}_x^{\text{ci}}$ and $\mathbb{V}_x \oplus \mathbb{E}_x^{\text{nci}}$. If we use the fact that $\text{Sym}(\mathbb{V}_x) \otimes \bigwedge \mathbb{V}_x = 1$, we get the following corollary.

Corollary 3.6. *We have the following identity of virtual H_x -modules:*

$$\text{Sym}(\mathfrak{q}/\mathfrak{q}_{\mu^+}) \otimes \bigwedge \mathfrak{h}/\mathfrak{h}_{\mu^+} \simeq \text{Sym}(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge \mathbb{E}_x^{\text{ci}}.$$

Proof of Lemma 3.2. Let $\mathcal{B} := \mathfrak{A}_x/\mathbb{Z}_2 \cup (\mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+)$. We see that $\bigwedge_{J_{V_x}} V_x = \prod_{\alpha \in \mathcal{B}} (1 - t^{\tilde{\alpha}})$ whereas $\bigwedge_{-J_{\mu^+}} V_x = \prod_{\alpha \in \mathcal{B}} (1 - t^{-\alpha})$. Accordingly we get $\bigwedge_{J_{V_x}} V_x \simeq (-1)^{|\mathcal{B}'|} \mathbb{C}_\eta \otimes \bigwedge_{-J_{\mu^+}} V_x$ where $\mathcal{B}' = \{\alpha \in \mathcal{B}, \tilde{\alpha} = \alpha\}$ and $\eta = \sum_{\alpha \in \mathcal{B}'} \alpha$. Now it is easy

to check that an element $\alpha \in \mathcal{B}$ belongs to \mathcal{B}' if and only if α and $\theta(\alpha)$ both belong to \mathfrak{R}_x^+ . In other words

$$\mathcal{B}' = \{\alpha \in \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+), \theta(\alpha) \neq \alpha\} / \mathbb{Z}_2 \bigcup \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+.$$

We have proved that

$$\bigwedge_{J_{V_x}} V_x \simeq (-1)^{m_x} \mathbb{C}_{\delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge_{-J_{\mu^+}} V_x.$$

□

Finally, thanks to Lemma 3.2 and Corollary 3.6, we obtain the final formula for $Q_{H_x}(\lambda)$ (that does not depend on the choice of r):

$$Q_{H_x}(\lambda) = (-1)^{m_x} \text{Ind}_{H_x}^H \left(\mathbb{C}_{\lambda_x + \delta(x)} \otimes \det(\mathbb{E}_x^{\text{nci}}) \otimes \text{Sym}(\mathbb{E}_x^{\text{nci}}) \otimes \bigwedge \mathbb{E}_x^{\text{ci}} \right).$$

3.3. Computation of the virtual module $\mathbb{M}_x(\lambda)$. According to Theorem 1.1, we have the decomposition $V_\lambda^G|_H = \sum_{\bar{x}} Q_{\bar{x}}(\lambda)$ where $Q_{\bar{x}}(\lambda) = \text{Ind}_{H_x}^H (\mathbb{A}_x(\lambda))$, and $\mathbb{A}_x(\lambda) \in \widehat{R}(H_x)$ has the following description

$$\mathbb{A}_x(\lambda) = \frac{1}{|W_x^H|} \sum_{w \in W} (-1)^{m_{xw}} \mathbb{C}_{\lambda_{xw} + \delta(xw)} \otimes \det(\mathbb{E}_{xw}^{\text{nci}}) \otimes \text{Sym}(\mathbb{E}_{xw}^{\text{nci}}) \otimes \bigwedge \mathbb{E}_{xw}^{\text{ci}}.$$

The aim of this section is to simplify the expression of the virtual H_x -module $\mathbb{A}_x(\lambda)$. We start by comparing the G_x -modules $\mathbb{E}_{xw}^{\text{nci}}$ and $\mathbb{E}_x^{\text{nci}}$. We use the decomposition $\mathbb{E}_x^{\text{nci}} = (\mathbb{E}_x^{\text{nci}})_w^+ \oplus (\mathbb{E}_x^{\text{nci}})_w^-$ where

$$(\mathbb{E}_x^{\text{nci}})_w^+ := \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+} \mathbb{C}_\alpha, \quad \text{and} \quad (\mathbb{E}_x^{\text{nci}})_w^- = \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+ \cap -\mathfrak{R}_{xw}^+} \mathbb{C}_\alpha.$$

We have the following basic lemma (see Lemma 3.10).

Lemma 3.7. *The G_x -module $|\mathbb{E}_x^{\text{nci}}|_w := (\mathbb{E}_x^{\text{nci}})_w^+ \oplus \overline{(\mathbb{E}_x^{\text{nci}})_w^-}$ is isomorphic to $\mathbb{E}_{xw}^{\text{nci}}$.*

Let $\rho = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}^+} \alpha$. We denote by $w \bullet \lambda = w(\lambda + \rho) - \rho$ the affine action of the Weyl group on the lattice Λ .

The main result of this section is the following proposition.

Proposition 3.8. *Let $x \in Z_\theta$. We have*

$$\mathbb{A}_x(\lambda) = \mathbb{M}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^{\text{ci}}$$

where $\mathbb{M}_x(\lambda) \in \widehat{R}(H_x)$ is defined by the following expression

$$\mathbb{M}_x(\lambda) = \frac{(-1)^{n_x}}{|W_x^H|} \sum_{w \in W} (-1)^{k_{x,w}} \mathbb{C}_{(w \bullet \lambda)_x} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+) \otimes \text{Sym}(|\mathbb{E}_x^{\text{nci}}|_w),$$

and

- $k_{x,w} = |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq \pm\alpha\}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{ci}}|,$
- $n_x := |\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_x^+| - \frac{1}{2} |\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_x^+ \cap \{\theta(\alpha) \neq \alpha\}|.$

Remark 3.9. We can describe $Q_{\bar{x}}(\lambda)$ differently by taking $\{w_1, \dots, w_p\} \subset W$ such that $W_x^H \setminus W \simeq \{\bar{w}_1, \dots, \bar{w}_p\}$. We have $Q_{\bar{x}}(\lambda) = \text{Ind}_{H_x}^H \left(\tilde{\mathbb{A}}_x(\lambda) \right)$ with

$$\tilde{\mathbb{A}}_x(\lambda) = \tilde{\mathbb{M}}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^{\text{ci}}$$

and where $\tilde{\mathbb{M}}_x(\lambda) \in \widehat{R}(H_x)$ is defined by the following expression

$$\tilde{\mathbb{M}}_x(\lambda) = (-1)^{n_x} \sum_{k=1}^p (-1)^{k_{x,w_k}} \mathbb{C}_{(w_k \bullet \lambda)_x} \otimes \det((\mathbb{E}_x^{\text{nci}})_{w_k}^+) \otimes \text{Sym}(|\mathbb{E}_x^{\text{nci}}|_{w_k}).$$

We need to introduce some notations. To $x \in Z_\theta$, we associate :

- The polarized roots : to $\alpha \in \mathfrak{R}_x$ and $w \in W$, we associate $|\alpha|_w \in \mathfrak{R}_x$ defined as follows

$$|\alpha|_w = \begin{cases} \alpha & \text{if } \alpha \in \mathfrak{R}_{xw}^+, \\ -\alpha & \text{if } \alpha \notin \mathfrak{R}_{xw}^+. \end{cases}$$

- The following G_x -weights :

$$\gamma_{x,w}^{\text{ci}} := \sum_{\substack{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+ \\ |\alpha|_w \neq \alpha}} \alpha, \quad \gamma_{x,w}^{\text{nci}} := \sum_{\substack{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+ \\ |\alpha|_w \neq \alpha}} \alpha, \quad \gamma_{x,w} := \sum_{\substack{\alpha \in \mathfrak{R}_x^+ \\ |\alpha|_w \neq \alpha}} \alpha.$$

The proof of Proposition 3.8 is based on the following Lemma.

Lemma 3.10. Let $x \in Z_\theta$ and $w \in W$. Let $d_{x,w}^{\text{ci}}$ be the cardinal of the set $\{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+, |\alpha|_w \neq \alpha\}$. We have the following relations

- (1) $\mathbb{E}_{xw}^{\text{nci}} \simeq \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \mathbb{C}_{|\alpha|_w}$ and $\mathbb{E}_{xw}^{\text{ci}} \simeq \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \mathbb{C}_{|\alpha|_w}$,
- (2) $\det(\mathbb{E}_{xw}^{\text{nci}}) = \mathbb{C}_{-\gamma_{x,w}^{\text{nci}}} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+)$,
- (3) $\bigwedge \mathbb{E}_{xw}^{\text{ci}} = (-1)^{d_{x,w}^{\text{ci}}} \mathbb{C}_{-\gamma_{x,w}^{\text{ci}}} \otimes \bigwedge \mathbb{E}_x^{\text{ci}}$.
- (4) The H_x -weight $\delta(xw) - \delta(x)$ is equal to the restriction of G_x -weight $\gamma_{x,w}^{\text{nci}} + \gamma_{x,w}^{\text{ci}} - \gamma_{x,w}$ to H_x .

Proof. We remark that $\mathfrak{R}_x = \mathfrak{R}_{xw}$, $\mathfrak{R}_x^+ = g(\mathfrak{R}^+)$ and $\mathfrak{R}_{xw}^+ = g(w\mathfrak{R}^+)$. The first point follows and points (ii) and (iii) derive from the first.

Let us check the last point. The term $\rho_x := \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_x^+} \alpha$ is the image of $\rho := \frac{1}{2} \sum_{\alpha' \in \mathfrak{R}^+} \alpha'$ through the map $\mu \mapsto \mu_x$. We see that

$$\rho_x + \theta(\rho_x) = \sum_{\alpha \in \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+)} \alpha = 2\delta(x) + 2\rho_x^{\text{nci}} + 2\rho_x^{\text{ci}}$$

where $\rho_x^{\text{nci}} = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_x^{\text{nci}} \cap \mathfrak{R}_x^+} \alpha$ and $\rho_x^{\text{ci}} = \frac{1}{2} \sum_{\alpha \in \mathfrak{R}_x^{\text{ci}} \cap \mathfrak{R}_x^+} \alpha$. Similarly we have

$$\rho_{xw} + \theta(\rho_{xw}) = 2\delta(xw) + 2\rho_{xw}^{\text{nci}} + 2\rho_{xw}^{\text{ci}}.$$

Thus the H_x -weight $\delta(xw) - \delta(x)$ is equal to the restriction to H_x of the G_x -weight

$$\beta(x, w) := \rho_{xw} - \rho_x + (\rho_x^{\text{nci}} - \rho_{xw}^{\text{nci}}) + (\rho_x^{\text{ci}} - \rho_{xw}^{\text{ci}}).$$

We notice that $\rho_{xw} - \rho_x = (w\rho - \rho)_x = -\gamma_{x,w}$. Furthermore, small computations give that $\rho_x^{\text{nci}} - \rho_{xw}^{\text{nci}} = \gamma_{x,w}^{\text{nci}}$ and $\rho_x^{\text{ci}} - \rho_{xw}^{\text{ci}} = \gamma_{x,w}^{\text{ci}}$. We have proved that $\beta(x, w) = \gamma_{x,w}^{\text{nci}} + \gamma_{x,w}^{\text{ci}} - \gamma_{x,w}$. The last point follows. \square

Now, we can finish the proof of the Proposition 3.8. We must check that the virtual H_x -module

$$\mathbf{A} := (-1)^{m_{xw}} \mathbb{C}_{\lambda_{xw} + \delta(xw)} \otimes \det(\mathbb{E}_{xw}^{\text{nci}}) \otimes \bigwedge \mathbb{E}_{xw}^{\text{ci}}$$

is equal to the virtual H_x -module

$$\mathbf{B} := (-1)^{n_x + k_{x,w}} \mathbb{C}_{(w \bullet \lambda)_x + \delta(x)} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+) \otimes \bigwedge \mathbb{E}_x^{\text{ci}}.$$

If we use Lemma 3.10, we get

$$\mathbf{A} = (-1)^{m_{xw} + d_{x,w}^{\text{ci}}} \mathbb{C}_{(w(\lambda + \rho) - \rho)_x + \delta(x)} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+) \otimes \bigwedge \mathbb{E}_x^{\text{ci}}.$$

Thus the equality $\mathbf{A} = \mathbf{B}$ follows from the following lemma.

Lemma 3.11. *For any $x \in Z_\theta$ and $w \in W$, we have $n_x + k_{x,w} = m_{xw} + d_{x,w}^{\text{ci}} \pmod{2}$.*

Proof. In order to simplify our notations, we write $a \equiv b$ for $a = b \pmod{2}$.

We have $\dim \mathbb{E}_x^{\text{nci}} = \dim \mathbb{E}_{xw}^{\text{nci}}$ and $\dim \mathbb{E}_x^{\text{ci}} = \dim \mathbb{E}_{xw}^{\text{ci}}$, then

$$\begin{aligned} m_{xw} - m_x &= \frac{1}{2} (|\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) \cap \{\theta(\alpha) \neq \alpha\}| - |\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}|) \\ &= \frac{1}{2} (|\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)| - |\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+)|). \end{aligned}$$

We remark now that

$$\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+) = A_{++} \cup A_{--} \cup A_{+-} \cup A_{-+}$$

with $A_{++} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$, $A_{--} = -\mathfrak{R}_x^+ \cap -\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$, $A_{+-} = \mathfrak{R}_x^+ \cap \theta(-\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$ and $A_{-+} = -\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$.

Similarly we have

$$\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) = B_{++} \cup B_{--} \cup B_{+-} \cup B_{-+}$$

with $B_{++} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$, $B_{--} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap -\mathfrak{R}_{xw}^+ \cap \theta(-\mathfrak{R}_{xw}^+)$, $B_{+-} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_{xw}^+ \cap \theta(-\mathfrak{R}_{xw}^+)$ and $B_{-+} = \mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap -\mathfrak{R}_{xw}^+ \cap \theta(\mathfrak{R}_{xw}^+)$.

We have the obvious relations : $A_{++} = B_{++}$, $A_{--} = -B_{--}$, $\theta(A_{+-}) = A_{-+}$, $\theta(B_{+-}) = B_{-+}$ and $A_{++} = B_{++}$. So we get $m_{xw} - m_x \equiv |A_{+-}| + |B_{+-}|$.

Let consider $\mathcal{A} := \mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+$ and $\mathcal{B} := \mathfrak{R}_x^+ \cap -\mathfrak{R}_{xw}^+$. We have

$$\begin{aligned} m_{xw} - m_x &\equiv |\mathcal{A} \cap \theta(\mathcal{B})| + |\mathcal{A} \cap -\theta(\mathcal{B})| \\ &\equiv |\mathcal{A}| + |\mathcal{A} \cap \theta(\mathcal{A})| + |\mathcal{A} \cap -\theta(\mathcal{A})|. \end{aligned}$$

Now we remark that

$$\begin{aligned} |\mathcal{A} \cap \theta(\mathcal{A})| &\equiv |\mathcal{A} \cap \theta(\mathcal{A}) \cap \{\theta(\alpha) = \alpha\}| \\ &\equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}|. \end{aligned}$$

Similarly

$$\begin{aligned} |\mathcal{A} \cap -\theta(\mathcal{A})| &\equiv |\mathcal{A} \cap -\theta(\mathcal{A}) \cap \{\theta(\alpha) = -\alpha\}| \\ &\equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = -\alpha\}|. \end{aligned}$$

At this stage we have proved that

$$\begin{aligned} m_{xw} - m_x &\equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = -\alpha\}| \\ &\equiv |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq -\alpha\}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}|. \end{aligned}$$

As $d_{x,w}^{\text{ci}} = |\mathfrak{R}_x^+ \cap -\mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{ci}}|$, we have $|\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) = \alpha\}| + d_{x,w}^{\text{ci}}$ is equal to $\dim \mathbb{E}_x^{\text{ci}} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{nci}}|$. This implies that $m_{xw} + d_{x,w}^{\text{ci}}$ is equal, modulo 2, to

$$\begin{aligned} &m_x + \dim \mathbb{E}_x^{\text{ci}} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{nci}}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq -\alpha\}| \\ &\equiv m_x + \dim \mathbb{E}_x^{\text{ci}} + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{ci}}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq \pm\alpha\}|. \end{aligned}$$

By definition $m_x = \frac{1}{2}|\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + \dim \mathbb{E}_x^{\text{nci}}$ and then

$$\begin{aligned} m_x + \dim \mathbb{E}_x^{\text{ci}} &\equiv \frac{1}{2}|\mathfrak{R}_x^+ \cap \theta(\mathfrak{R}_x^+) \cap \{\theta(\alpha) \neq \alpha\}| + |\mathfrak{R}_x^+ \cap \{\theta(\alpha) = \alpha\}| \\ &\equiv n_x. \end{aligned}$$

Finally we have proved that $m_{xw} + d_{x,w}^{\text{ci}}$ is equal, modulo 2, to $n_x + k_{x,w}$. □

4. EXAMPLES

In this section we will study in details some examples of our formula

$$V_\lambda^G|_H = \sum_{\bar{x} \in H \backslash Z_\theta / W} Q_{\bar{x}}(\lambda)$$

where $Q_{\bar{x}}(\lambda) = \text{Ind}_{H_x}^H (\mathbb{M}_x(\lambda) \otimes \mathbb{C}_{\delta(x)} \otimes \bigwedge \mathbb{E}_x^{\text{ci}})$ and

$$\mathbb{M}_x(\lambda) = \frac{(-1)^{n_x}}{|W_x^H|} \sum_{w \in W} (-1)^{k_{x,w}} \mathbb{C}_{(w \bullet \lambda)_x} \otimes \det((\mathbb{E}_x^{\text{nci}})_w^+) \otimes \text{Sym}(|\mathbb{E}_x^{\text{nci}}|_w).$$

Here the integers $k_{x,w}$ and n_x are defined as follows:

- $k_{x,w} = |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \{\theta(\alpha) \neq \pm\alpha\}| + |\mathfrak{R}_x^+ \cap \mathfrak{R}_{xw}^+ \cap \mathfrak{R}_x^{\text{ci}}|$,
- $n_x = |\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_x^+| - \frac{1}{2}|\theta(\mathfrak{R}_x^+) \cap \mathfrak{R}_x^+ \cap \{\theta(\alpha) \neq \alpha\}|$.

4.1. $K \subset K \times K$. Let K be a connected compact Lie group. Here we work with the Lie group $G = K \times K$ and the involution $\theta(k_1, k_2) = (k_2, k_1)$. The subgroup $H = G^\theta$ is the group K embedded diagonally in G .

Let T be a maximal torus of K and let $W_K = N_K(T)/T$ be the Weyl group of K . We denote by \mathfrak{R}_K the set of roots for (K, T) , and we make the choice of a set \mathfrak{R}_K^+ of positive roots.

In the next lemma we describe the critical set Z_θ in the flag manifold $\mathcal{F} = K/T \times K/T$ of G .

Lemma 4.1. *We have $Z_\theta = \bigcup_{w \in W_K} Z_w$ with $Z_w = K \cdot (wT, T)$. In other words, the set $H \backslash Z_\theta / W$ is a singleton.*

Proof. The element $x = (aT, bT) \in \mathcal{F}$ belongs to Z_θ if and only if $(a^{-1}b, b^{-1}a) \in W \times W$. If $b^{-1}a = w \in W$ then $(aT, bT) \in Z_w$. □

We take $x = (T, T) \in Z_\theta$. For each $w \in W_K$, we write $xw = (wT, T)$. We take $\lambda = (a, b) \in \Lambda_K^+ \times \Lambda_K^+ = \widehat{G}$.

Our data are as follows:

- the group G_x is the maximal torus $T \times T \subset K$,
- the group H_x is the maximal torus $T \subset K$,
- $\mathbb{C}_{(w \bullet \lambda)_{x+\delta(x)}} = \mathbb{C}_{w(a+\rho)+b+\rho}$ as a character of T ,
- $n_x = |\mathfrak{R}_K^+|$,
- $k_{x,w}$ is equal to $|w\mathfrak{R}_K^+ \cap \mathfrak{R}_K^+| + |\mathfrak{R}_K^+|$, so $(-1)^{k_{x,w}} = (-1)^w$,
- the vector spaces $\mathbb{E}_x^{\text{ci}}, \mathbb{E}_x^{\text{nci}}$ are reduced to $\{0\}$.

In this context we obtain the following relation

$$(4.8) \quad V_a^K \otimes V_b^K = (-1)^{\dim(K/T)/2} \sum_{w \in W_K} (-1)^w \text{Ind}_T^K (\mathbb{C}_{w(a+\rho)+b+\rho}).$$

This type of generalized Clebsch-Gordan formula what first noticed by Steinberg [15] (see Section 5).

Example 4.2. *The irreducible representation $SU(2)$ are parametrized by \mathbb{N} . If $n \geq 0$, the irreducible representation V_n of $SU(2)$ satisfies*

$$V_n = \text{Ind}_{U(1)}^{SU(2)} ((\mathbb{C}_0 - \mathbb{C}_2) \otimes \mathbb{C}_n).$$

If we take $m \geq n \geq 0$, then (4.8) gives

$$\begin{aligned} V_n \otimes V_m &= \text{Ind}_{U(1)}^{SU(2)} (\mathbb{C}_{m-n}) - \text{Ind}_{U(1)}^{SU(2)} (\mathbb{C}_{m+n+2}) \\ &= \sum_{k=0}^n \text{Ind}_{U(1)}^{SU(2)} ((\mathbb{C}_0 - \mathbb{C}_2) \otimes \mathbb{C}_{m+n-2k}) \\ &= \sum_{k=0}^n V_{m+n-2k}. \end{aligned}$$

We recognize here the classical Clebsch-Gordan relations.

4.2. $U(p) \times U(q) \subset U(p+q)$. Let $p \geq q \geq 1$ and $n = p+q$. We take $G = U(n)$ with maximal torus $T \simeq U(1)^n$ the subgroup formed by the diagonal matrices. We use the canonical map τ from the symmetric group \mathfrak{S}_n into G . It induces an isomorphism between \mathfrak{S}_n and the Weyl group W of G .

We work with the involution $\theta(g) = \Delta g \Delta^{-1}$ where $\Delta := \text{diag}(I_p, -I_q)$: the subgroup fixed by θ is $H = U(p) \times U(q)$.

In the next section we describe the critical set $Z_\theta \subset \mathcal{F}$. For another type of parametrization of $H_{\mathbb{C}} \backslash \mathcal{F}$, see Section 5 of [13].

4.2.1. *The critical set.* We consider the following elements of $O(2)$:

$$R = \begin{pmatrix} \frac{1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \\ \frac{-1}{\sqrt{2}} & \frac{1}{\sqrt{2}} \end{pmatrix}, \quad S = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \quad J = \begin{pmatrix} 0 & -1 \\ 1 & 0 \end{pmatrix}.$$

The element R is of order 8, $R^2 = -J$ and $R^{-1} \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix} R = S$.

To any $j \in \{0, \dots, q\}$ we associate :

- $g_j := \text{diag}(\underbrace{1, \dots, 1}_{p-j \text{ times}}, \underbrace{R, \dots, R}_j, \underbrace{1, \dots, 1}_{q-j \text{ times}}) \in G$,
- the permutation $w_j \in \mathfrak{S}_n$ that fixes the elements of $[1, \dots, p-j] \cup [p+j+1, \dots, n]$ and such that

$$w_j(p-j+2k-1) = p-j+k, \quad w_j(p-j+2k) = p+k, \quad \text{for } 1 \leq k \leq j,$$
- $k_j = \tau_j g_j \in G$, where $\tau_j = \tau(w_j) \in N(T)$,
- $x_j = k_j T \in \mathcal{F}$.

The adjoint map $Ad(\tau_j) : G \rightarrow G$ sends the matrix $\text{diag}(a_1, \dots, a_{p-j}, b_1, \dots, b_{2j}, c_1, \dots, c_{q-j})$ to the matrix $\text{diag}(a_1, \dots, a_{p-j}, b_1, b_3, \dots, b_{2j-1}, b_2, b_4, \dots, b_{2j}, c_1, \dots, c_{q-j})$.

We see then that

$$\sigma_j := k_j^{-1} \Delta k_j = \text{diag}(\underbrace{1, \dots, 1}_{p-j \text{ times}}, \underbrace{S, \dots, S}_j, \underbrace{-1, \dots, -1}_{q-j \text{ times}})$$

and $k_j^{-1} \theta(k_j) = \sigma_j \Delta$ belong to $N(T)$. Thus the elements x_0, \dots, x_q belongs to Z_θ .

Lemma 4.3. *In the flag manifold \mathcal{F} the set Z_θ has the following description:*

$$Z_\theta = \bigcup_{0 \leq j \leq q} \bigcup_{\bar{w} \in W_{x_j} \setminus W} Hx_j w$$

So we have $H \setminus Z_\theta / W = \{\bar{x}_0, \dots, \bar{x}_q\}$.

Proof. If $1 \leq a < b \leq n$, we denote by $\tau_{a,b} \in N(T)$ the permutation matrix associated to the transposition (a, b) .

Let $gT \in Z_\theta$. Then $k := g^{-1} \theta(g) \Delta = g^{-1} \Delta g$ is an element of order two in $N(T)$. The Weyl group element $\bar{k} \in W$ is of order two, then there exists $0 \leq l \leq n/2$, and a family $(a_1 < b_1), \dots, (a_l < b_l)$ of disjoint couples in $\{1, \dots, n\}$ such that $kT = \tau_{a_1, b_1} \dots \tau_{a_l, b_l} T$.

Now, if we use the fact that the characteristic polynomial of $k \in G$ is equal to $(X-1)^p (X+1)^q$ with $p \geq q \geq 1$, we see that

- $l \leq q$,
- there exists $n \in N(T)$ such that $nk n^{-1} = \sigma_l = k_l^{-1} \Delta k_l$.

If we take $w = \bar{n} \in W$, the previous identity says that $g \in Hk_l w T$. \square

4.2.2. Localized indices. We work with the groups $T \subset H = U(p) \times U(q) \subset G = U(n)$ and the corresponding Lie algebras $\mathfrak{t} \subset \mathfrak{h} \subset \mathfrak{g}$. Let $\mathfrak{R} = \{\varepsilon_r - \varepsilon_s\}$ be the set of non-zero roots for the action of T on $\mathfrak{g} \otimes \mathbb{C}$. We choose the Weyl chamber so that $\mathfrak{R}^+ := \{\varepsilon_r - \varepsilon_s, 1 \leq r < s \leq n\}$.

Let $j \in \{0, \dots, q\}$. The aim of this section is to compute the localized index $Q_{\bar{x}_j}(\lambda) \in \widehat{R}(H)$. In order to have a fairly simple expression we will rewrite the terms of the form $\text{Ind}_{H_{x_j}}^H(\mathbb{C}_\beta \otimes \bigwedge \mathbb{E}_{x_j}^{\text{ci}})$.

Let $\{1, \dots, n\} = I_j^1 \cup I_j^2 \cup I_j^3 \cup I_j^4$ where $I_j^1 = \{1 \leq k \leq p-j\}$, $I_j^2 = \{p-j+1 \leq k \leq p\}$, $I_j^3 = \{p+1 \leq k \leq p+j\}$, and $I_j^4 = \{p+j+1 \leq k \leq n\}$.

For the maximal torus $T \subset G$ we have a decomposition

$$T \simeq T_j^1 \times T_j^2 \times T_j^3 \times T_j^4$$

where $T_j^p = \{(t_k)_{k=1}^n, t_k \in U(1), t_k = 1 \text{ unless } k \in I_j^p\}$. Let $T_j \subset T_j^2 \times T_j^3$ be the subtorus defined by the relations: an element $((t_k)_{k=1}^n, (s_k)_{k=1}^n) \in T_j^2 \times T_j^3$ belongs to T_j if and only if $t_{p-j+k} = s_{p+k}$ for all $1 \leq k \leq j$.

The elements of order two $\sigma_j \in G$ induce involutions on G (by conjugation) that we still denote by σ_j . We start with a basic lemma whose proof is left to the reader.

Lemma 4.4. *Let $x_j = k_j T \in \mathcal{F}$.*

- *The adjoint map $Ad(k_j) : \mathfrak{g} \rightarrow \mathfrak{g}$ realizes an isomorphism between the vector space \mathfrak{t} equipped with the involution induced by σ_j and the vector space \mathfrak{g}_{x_j} equipped with the involution θ .*

- *The group $N(T)^{\sigma_j}/T^{\sigma_j}$ is isomorphic with $\mathfrak{S}_{p-j} \times \mathfrak{S}_{q-j} \times \mathfrak{S}_j \times \{\pm\}^j$.*
- *The adjoint map $Ad(k_j) : G \rightarrow G$ induces an isomorphism $N(T)^{\sigma_j}/T^{\sigma_j} \simeq W_{x_j}$.*
- *The stabilizer subgroup H_{x_j} is equal to $T_j^1 \times T_j \times T_j^4 \subset T$.*
- *If \mathbb{C}_α is a character of T , then $\mathbb{C}_{k_j\alpha}$ is a character of G_{x_j} and $\mathbb{C}_{\tau_j\alpha}$ is a character of T . We have the relation*

$$\mathbb{C}_{k_j\alpha}|_{H_{x_j}} = \mathbb{C}_{\tau_j\alpha}|_{H_{x_j}}.$$

- *The set of roots $\mathfrak{R}_{x_j}^{\text{ci}}$ is equal to*

$$k_j \cdot \{\varepsilon_r - \varepsilon_s, 1 \leq r < s \leq p-j\} \cup k_j \cdot \{\varepsilon_r - \varepsilon_s, p+j+1 \leq r < s \leq n\}$$

and $\mathfrak{R}_{x_j}^{\text{nci}} = k_j \cdot \{\varepsilon_r - \varepsilon_s, 1 \leq r \leq p-j \ \& \ p+j+1 \leq s \leq n\}$.

We denote by \mathbb{M}_j the T -module $\mathbb{C}^{p-j} \otimes (\mathbb{C}^{q-j})^*$ where the subgroup $T_j^2 \times T_j^3$ acts trivially and the $T_j^1 \times T_j^4$ -action is the canonical one. Thanks to Lemma 4.4, we have the following isomorphisms of H_{x_j} -modules: $\mathbb{E}_{x_j}^{\text{nci}} \simeq \mathbb{M}_j$. Following Lemma 3.7, one can associate the modules $(\mathbb{M}_j)_w^\pm$ and $|\mathbb{M}_j|_w$ to each $w \in W$.

We consider the Lie group

$$K_j := U(p-j) \times U(q-j)$$

that we view as a subgroup of H in such a way that $T_j^1 \times T_j^4$ is a maximal torus of K_j . A set of positive roots for $(K_j, T_j^1 \times T_j^4)$ is $\varepsilon_r - \varepsilon_s$ for $1 \leq r < s \leq p-j$ and $p+j+1 \leq r < s \leq n$. We equip $\mathfrak{k}_j / [\mathfrak{t}_j^1 \times \mathfrak{t}_j^4]$ with a complex structure such that

$$\mathbb{E}_{x_j}^{\text{ci}} \simeq \mathfrak{k}_j / [\mathfrak{t}_j^1 \times \mathfrak{t}_j^4]$$

is an isomorphism of $T_j^1 \times T_j^4$ -modules.

The holomorphic induction map $\text{Hol}_{T_j^1 \times T_j^4}^{K_j} : \widehat{R}(T_j^1 \times T_j^4) \rightarrow \widehat{R}(K_j)$ is defined as follows:

$$\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(V) := \text{Ind}_{T_j^1 \times T_j^4}^{K_j}(V \otimes \bigwedge \mathfrak{k}_j / [\mathfrak{t}_j^1 \times \mathfrak{t}_j^4]).$$

If $a = (a_1 \geq \dots \geq a_{p-j}) \in \mathbb{Z}^{p-j}$ and $b = (b_1 \geq \dots \geq b_{q-j}) \in \mathbb{Z}^{q-j}$, then $\mathbb{C}_{(a,b)}$ defines a character of $T_j^1 \times T_j^4$ and

$$\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_{(a,b)}) = V_a^{U(p-j)} \otimes V_b^{U(q-j)}$$

is the irreducible representation of K_j with highest weight (a, b) .

A character \mathbb{C}_β of the torus T can be written $\mathbb{C}_\beta = \mathbb{C}_{\beta^{14}} \otimes \mathbb{C}_{\beta^{23}}$ where $\mathbb{C}_{\beta^{14}}$ is a character of $T_j^1 \times T_j^4$ and $\mathbb{C}_{\beta^{23}}$ is a character of $T_j^2 \times T_j^3$. Note that $\mathbb{C}_{\tau_j \beta}|_{H_{x_j}} = \mathbb{C}_{\beta^{14}} \otimes \mathbb{C}_{\beta'}$ where $\beta' = \tau_j \beta^{23}$ defines a character of $T_j \subset T_j^2 \times T_j^3$.

Lemma 4.5. *Let \mathbb{C}_β be a character of T . Then $\text{Ind}_{H_{x_j}}^H(\mathbb{C}_\beta|_{H_{x_j}} \otimes \wedge \mathbb{E}_{x_j}^{\text{ci}})$ is equal to*

$$\text{Ind}_{K_j \times T_j^2 \times T_j^3}^H \left(\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_{\beta^{14}}) \otimes \mathbb{C}_{\beta^{23}} \otimes L^2([T_j^2 \times T_j^3]/T_j) \right),$$

where $L^2([T_j^2 \times T_j^3]/T_j) = \text{Ind}_{T_j}^{T_j^2 \times T_j^3}(1) \in \widehat{R}(T_j^2 \times T_j^3)$.

Remark 4.6. *To gain some space in our formulas, we will write $\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_\beta)$ instead of $\text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_{\beta^{14}}) \otimes \mathbb{C}_{\beta^{23}}$*

We need to fix some notations.

Definition 4.7. • *Let $\chi : H \rightarrow \mathbb{C}$ be the character $(A, B) \mapsto \det(A) \det(B)^{-1}$.*

• *Let ψ_j be the character² of T associated to the weight*

$$\sum_{1 \leq k \leq j} (q - p + 2 + 2j - 4k) \varepsilon_{p-j+k}.$$

• *For any $(j, w) \in [0, q] \times W$, we define the integer $d_{j,w}$ by the relation*

$$d_{j,w} = \dim(\mathbb{M}_j)_w^+ + |\{1 \leq k \leq j, w^{-1}(p - j + 2k - 1) < w^{-1}(p - j + 2k)\}|.$$

A small computation gives the following lemma.

Lemma 4.8. • *The H_{x_j} -character $\mathbb{C}_{\delta(x_j)}$ is equal to $\chi^{\otimes j} \otimes \psi_j|_{H_{x_j}}$.*

• *For any $(j, w) \in [0, q] \times W$, we have $(-1)^{n_{x_j} + k_{x_j, w}} = (-1)^{j(n+1)} (-1)^w (-1)^{d_{j,w}}$.*

The main result of this section is the following proposition.

Proposition 4.9.

$$V_\lambda^{U(n)}|_{U(p) \times U(q)} = \sum_{j=0}^q Q_{\bar{x}_j}(\lambda)$$

where $Q_{\bar{x}_j}(\lambda) \in \widehat{R}(U(p) \times U(q))$ is determined by the relation

$$Q_{\bar{x}_j}(\lambda) = \frac{(-1)^{j(n+1)}}{|W_{x_j}|} \chi^{\otimes j} \otimes \sum_{w \in W} (-1)^w (-1)^{d_{j,w}} \text{Ind}_{K_j \times T_j^2 \times T_j^3}^{U(p) \times U(q)} (\mathbb{A}_j^w(\lambda) \otimes \psi_j).$$

Here the elements $\mathbb{A}_j^w(\lambda) \in \widehat{R}(K_j \times T_j^2 \times T_j^3)$ are defined as follows:

$$\mathbb{A}_j^w(\lambda) = \text{Hol}_{T_j^1 \times T_j^4}^{K_j}(\mathbb{C}_{\tau_j(w \bullet \lambda)} \otimes \det((\mathbb{M}_j)_w^+) \otimes \text{Sym}(|\mathbb{M}_j|_w)) \otimes L^2([T_j^2 \times T_j^3]/T_j).$$

We finish this section by considering particular situations.

²Remark that ψ_j is trivial $T_j^1 \times T_j^3 \times T_j^4$.

4.2.3. *The extreme cases* : $j = 0$ or $j = q$. When $j = 0$, the torus T_0^2 and T_0^3 are trivial and $K_0 = U(p) \times U(q) = H$. Moreover $\mathbb{M}_0 = \mathbb{C}^p \otimes (\mathbb{C}^q)^*$ and $d_{0,w} = \dim(\mathbb{M}_0)_w^+$. Thanks to Lemma 4.4, we know also that $W_{x_0} \simeq \mathfrak{S}_p \times \mathfrak{S}_q$.

So we get the formula

$$Q_{\bar{x}_0}(\lambda) = \frac{1}{p!q!} \sum_{w \in W} (\pm)_w \text{Hol}_T^H (\mathbb{C}_{w \bullet \lambda} \otimes \det((\mathbb{M}_0)_w^+) \otimes \text{Sym}(|\mathbb{M}_0|_w))$$

where $(\pm)_w = (-1)^w (-1)^{\dim(\mathbb{M}_0)_w^+}$.

Remark 4.10. *An useful exercise is to consider the term*

$$\mathbb{A}_w := (\pm)_w \text{Hol}_T^H (\mathbb{C}_{w \bullet \lambda} \otimes \det((\mathbb{M}_0)_w^+) \otimes \text{Sym}(|\mathbb{M}_0|_w))$$

and verify that $\mathbb{A}_{w'w} = \mathbb{A}_w$ when $w' \in W_{x_0}$.

When $j = q$, the torus T_q^4 is trivial, $K_q = U(p - q)$ and $\mathbb{M}_q = \{0\}$. Moreover $W_{x_q} \simeq \mathfrak{S}_{p-q} \times \mathfrak{S}_q \times \{\pm\}^q$. In this case we obtain

$$Q_{\bar{x}_q}(\lambda) = \frac{(-1)^{q(n+1)}}{(p-q)!q!2^q} \chi^{\otimes q} \otimes \sum_{w \in W} (-1)^w (-1)^{d_{q,w}} Q_q^w(\lambda)$$

with

$$Q_q^w(\lambda) = \text{Ind}_{U(p-q) \times T_q^2 \times T_q^3}^{U(p) \times U(q)} \left(\text{Hol}_{T_q^1}^{U(p-q)} (\mathbb{C}_{\tau_q(w \bullet \lambda)}) \otimes \psi_q \otimes L^2([T_q^2 \times T_q^3] / T_q) \right).$$

4.2.4. $U(n-1) \times U(1) \subset U(n)$. Here we are in the case where $q = 1$, and so

$$V_\lambda^{U(n)}|_{U(n-1) \times U(1)} = Q_{\bar{x}_0}(\lambda) + Q_{\bar{x}_1}(\lambda).$$

To simplify the expression of $Q_{\bar{x}_0}(\lambda)$ we use the fact that the quotient $W_{x_0} \backslash W$ is represented by the class of the elements $\tau_{k,n} \in G$ associated to the transposition (k, n) for $1 \leq k \leq n$. We write $T = T' \times U(1)$ where T' is a maximal torus of $U(n-1)$. The T' -module \mathbb{C}^{n-1} can be decomposed as $\mathbb{V}_k \oplus \mathbb{V}'_k$ where $\mathbb{V}_k = \sum_{j=1}^{k-1} \mathbb{C}_{\varepsilon_j}$ and $\mathbb{V}'_k = \sum_{j=k}^{n-1} \mathbb{C}_{\varepsilon_j}$.

The T -module \mathbb{M}_0 is equal to $\mathbb{C}^{n-1} \otimes \mathbb{C}^* = \mathbb{V}_k \otimes \mathbb{C}_{-\varepsilon_n} \oplus \mathbb{V}'_k \otimes \mathbb{C}_{-\varepsilon_n}$ and the polarized T -module $|\mathbb{M}_0|_{\tau_{k,n}}$ is equal to $\mathbb{V}_k \otimes \mathbb{C}_{-\varepsilon_n} \oplus \overline{\mathbb{V}'_k} \otimes \mathbb{C}_{\varepsilon_n}$. We have $\dim(\mathbb{M}_0)_{\tau_{k,n}}^+ = k - 1$ and $\det(\mathbb{M}_0)_{\tau_{k,n}}^+ = \mathbb{C}_{\mu_k} \otimes \mathbb{C}_{\varepsilon_n}^{\otimes 1-k}$ with $\mu_k = \sum_{j=1}^{k-1} \varepsilon_j$.

So we obtain

$$Q_{\bar{x}_0}(\lambda) = \sum_{\substack{a,b \geq 0 \\ 1 \leq k \leq n}} (\pm)_k \text{Hol}_{T'}^{U(n-1)} (\mathbb{C}_{\tau_{k,n} \bullet \lambda + \mu_k} \otimes \text{Sym}^b(\mathbb{V}_k) \otimes \text{Sym}^a(\overline{\mathbb{V}'_k})) \otimes \mathbb{C}_{\varepsilon_n}^{\otimes 1+a-b-k},$$

where $(\pm)_k = (-1)^k$ if $k < n$ and $(\pm)_n = (-1)^{n-1}$.

We consider now the term $Q_{\bar{x}_1}(\lambda)$. When $j = q = 1$, the torus T_1^4 is trivial, $K_1 = U(n-2)$ and $\mathbb{M}_1 = \{0\}$. Moreover $W_{x_1} \simeq \mathfrak{S}_{n-2} \times \{\pm\}$, $\tau_1 = Id$ and $\psi_1 = (2-n)\varepsilon_{n-1}$. Here the quotient $W_{x_1} \backslash W$ is represented by the class of the elements $\tau_{l,n} \tau_{k,n-1}$ for $1 \leq k < l \leq n$. We denote by λ_{kl} the term $\tau_{l,n} \tau_{k,n-1} \bullet \lambda$.

In this case we obtain

$$Q_{\bar{x}_1}(\lambda) = (-1)^n \chi \otimes \left(Q_{\bar{x}_1}^{n-1,n}(\lambda) - \sum_{1 \leq k < n-1} Q_{\bar{x}_1}^{k,n}(\lambda) + \sum_{1 \leq k < l \leq n-1} Q_{\bar{x}_1}^{k,l}(\lambda) \right)$$

with

$$\begin{aligned} Q_{\bar{x}_1}^{k,l}(\lambda) &= \text{Ind}_{U(n-2) \times T_1^2 \times T_1^3}^{U(n-1) \times T_1^3} \left(\text{Hol}_{T_1^1}^{U(n-2)}(\mathbb{C}_{\lambda_{kl}}) \otimes \psi_1 \otimes L^2([T_1^2 \times T_1^3] / T_1) \right) \\ &= \sum_{a \in \mathbb{Z}} \text{Ind}_{U(n-2) \times T_1^2}^{U(n-1)} \left(\text{Hol}_{T_1^1}^{U(n-2)}(\mathbb{C}_{\lambda_{kl}}) \otimes \mathbb{C}_{\varepsilon_{n-1}}^{\otimes a} \right) \otimes \mathbb{C}_{\varepsilon_n}^{\otimes 2-n-a}. \end{aligned}$$

Let us finish this section by considering the simplest example: $U(1) \times U(1) \subset U(2)$. Take $\lambda = (\lambda_1 \geq \lambda_2) \in \widehat{U(2)}$. We have $V_\lambda^{U(2)}|_{U(1) \times U(1)} = Q_{\bar{x}_0}(\lambda) + Q_{\bar{x}_1}(\lambda)$ where

$$Q_{\bar{x}_0}(\lambda) = -\mathbb{C}_\lambda \otimes \sum_{-\infty}^{\lambda_2 - \lambda_1 - 1} \mathbb{C}_{\varepsilon_1 - \varepsilon_2}^{\otimes k} - \mathbb{C}_\lambda \otimes \sum_{k \geq 1} \mathbb{C}_{\varepsilon_1 - \varepsilon_2}^{\otimes k}$$

and $Q_{\bar{x}_1}(\lambda) = \mathbb{C}_\lambda \otimes \sum_{k \in \mathbb{Z}} \mathbb{C}_{\varepsilon_1 - \varepsilon_2}^{\otimes k}$. We recover the basic relation

$$V_\lambda^{U(2)}|_{U(1) \times U(1)} = \mathbb{C}_\lambda \otimes \sum_{k=\lambda_2 - \lambda_1}^0 \mathbb{C}_{\varepsilon_1 - \varepsilon_2}^{\otimes k}.$$

4.2.5. $U(n-1) \subset U(n)$. If we restrict the representation $V_\lambda^{U(n)}$ to the subgroup $U(n-1)$, we get

$$(4.9) \quad V_\lambda^{U(n)}|_{U(n-1)} = Q_0(\lambda) + Q_1(\lambda),$$

where the characters $Q_0(\lambda), Q_1(\lambda) \in \widehat{R}(U(n-1))$ are given by the relations

$$Q_0(\lambda) = \sum_{k=1}^n (\pm)_k \text{Hol}_{T'}^{U(n-1)}(\mathbb{C}_{\tau_{k,n} \bullet \lambda + \mu_k} \otimes \text{Sym}(\mathbb{V}_k) \otimes \text{Sym}(\overline{\mathbb{V}'_k})),$$

and

$$Q_1(\lambda) = (-1)^n \det \otimes \left(Q_1^{n-1,n}(\lambda) - \sum_{1 \leq k < n-1} Q_1^{k,n}(\lambda) + \sum_{1 \leq k < l \leq n-1} Q_1^{k,l}(\lambda) \right),$$

with $Q_1^{k,l}(\lambda) = \text{Ind}_{U(n-2)}^{U(n-1)} \left(\text{Hol}_{T_1^1}^{U(n-2)}(\mathbb{C}_{\lambda_{kl}}) \right)$.

Let's detail expression (4.9) when $n = 3$.

Small calculations give $Q_0(\lambda) = \mathbb{B}_1(\lambda) + \mathbb{B}_2(\lambda) + \mathbb{B}_3(\lambda)$ with

$$\mathbb{B}_1(\lambda) = \text{Hol}_T^{U(2)}(\mathbb{C}_{\tau_{1,3} \bullet \lambda + \mu_1}) \otimes \text{Sym}(\overline{\mathbb{C}^2}) = \sum_{\lambda_2 - 1 \geq a \geq \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)},$$

$$\mathbb{B}_2(\lambda) = \text{Hol}_T^{U(2)}(\mathbb{C}_{\tau_{2,3} \bullet \lambda + \mu_2} \otimes \text{Sym}(\mathbb{C}_{\varepsilon_1}) \otimes \text{Sym}(\overline{\mathbb{C}_{\varepsilon_2}})) = \sum_{a \geq \lambda_1 + 1, \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)},$$

$$\mathbb{B}_3(\lambda) = \text{Hol}_T^{U(2)}(\mathbb{C}_{\tau_{3,3} \bullet \lambda + \mu_3}) \otimes \text{Sym}(\mathbb{C}^2) = \sum_{a \geq \lambda_1 + 1 \geq b \geq \lambda_2 + 1} V_{(a,b)}^{U(2)}.$$

For the other term, we obtain $Q_1(\lambda) = \mathbb{A}_1(\lambda) - \mathbb{A}_2(\lambda) - \mathbb{A}_3(\lambda)$ with

$$\begin{aligned}\mathbb{A}_1(\lambda) &= \det \otimes Q_1^{1,3}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(\mathbb{C}_{\lambda_2-1}) = \sum_{a \geq \lambda_2 \geq b} V_{(a,b)}^{U(2)}, \\ \mathbb{A}_2(\lambda) &= \det \otimes Q_1^{2,3}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(\mathbb{C}_{\lambda_1}) = \sum_{a \geq \lambda_1+1 \geq b} V_{(a,b)}^{U(2)}, \\ \mathbb{A}_3(\lambda) &= \det \otimes Q_1^{1,2}(\lambda) = \det \otimes \text{Ind}_{U(1)}^{U(2)}(\mathbb{C}_{\lambda_3-2}) = \sum_{a \geq \lambda_3-1 \geq b} V_{(a,b)}^{U(2)}.\end{aligned}$$

Finally one checks that the decomposition

$$V_\lambda^{U(3)}|_{U(2)} = \mathbb{A}_1(\lambda) - \mathbb{A}_2(\lambda) - \mathbb{A}_3(\lambda) + \mathbb{B}_1(\lambda) + \mathbb{B}_2(\lambda) + \mathbb{B}_3(\lambda)$$

permits to recover the classical relation $V_\lambda^{U(3)}|_{U(2)} = \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3} V_{(a,b)}^{U(2)}$ (see [4]).

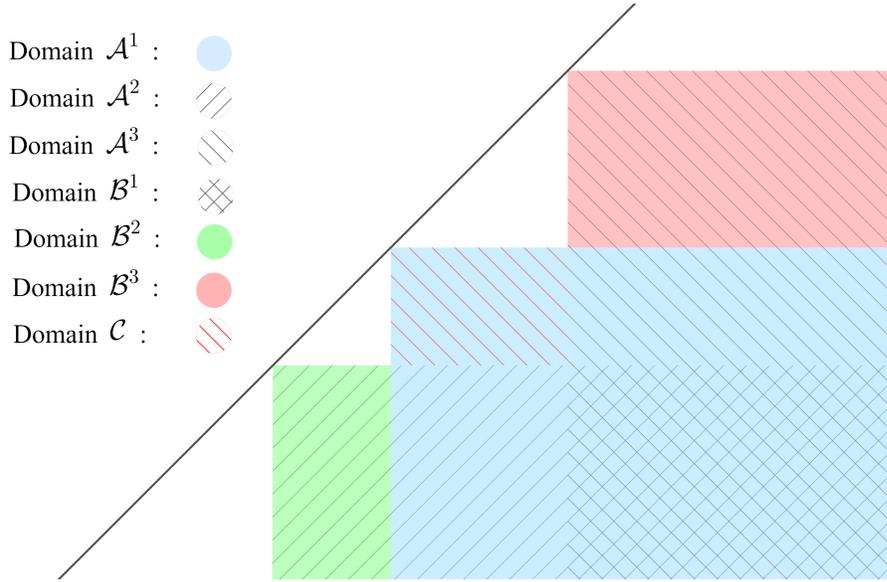


FIGURE 1. Restriction from $U(3)$ to $U(2)$

In Figure 1, we can visualise the supports of the different characters: we have

$$\begin{aligned}\mathbb{A}_1(\lambda) &= \sum_{\mu \in \mathcal{A}_1} V_\mu^{U(2)}, & \mathbb{A}_2(\lambda) &= \sum_{\mu \in \mathcal{A}_2 \cup \mathcal{B}_1} V_\mu^{U(2)}, & \mathbb{A}_3(\lambda) &= \sum_{\mu \in \mathcal{A}_3 \cup \mathcal{B}_1} V_\mu^{U(2)}, \\ \mathbb{B}_1(\lambda) &= \sum_{\mu \in \mathcal{B}_1} V_\mu^{U(2)}, & \mathbb{B}_2(\lambda) &= \sum_{\mu \in \mathcal{B}_2} V_\mu^{U(2)}, & \mathbb{B}_3(\lambda) &= \sum_{\mu \in \mathcal{B}_3} V_\mu^{U(2)},\end{aligned}$$

so that $V_\lambda^{U(3)}|_{U(2)} = \sum_{\mu \in \mathcal{C}} V_\mu^{U(2)}$.

5. KOSTANT MULTIPLICITY FORMULA

The aim of this section is first to recall the Kostant multiplicity formula : we follow the line of [4], Section 8.2. Then, we rewrite it in a form similar to the one we use in this article (see Proposition 5.4). Finally, we detail Kostant's multiplicity formula for

the restriction of $U(n)$ to $U(n-1)$, in order to compare it with the calculations done in Section 4.2.5.

Let $G' \subset G$ be two connected compact Lie groups with maximal tori $T' \subset T$. The corresponding Lie algebras are $\mathfrak{t} \subset \mathfrak{g}$ and $\mathfrak{t}' \subset \mathfrak{g}'$. In this section, we make the following regularity assumption:

(R) The centralizer $Z_{\mathfrak{g}}(\mathfrak{t}')$ of \mathfrak{t}' in \mathfrak{g} is abelian.

We recall the following well-known fact.

Lemma 5.1. *The assumption (R) is valid when G' is the connected component of a fixed-point subgroup of an involution.*

Proof : Suppose that $G' = (G^\tau)_0$ for some involution τ . Then, we have a decomposition $\mathfrak{g} = \mathfrak{g}' \oplus \mathfrak{q}$ where $\mathfrak{q} = \{X \in \mathfrak{g}, \tau(X) = -X\}$. The centralizer $Z_{\mathfrak{g}}(\mathfrak{t}')$ is stable under the involution τ and under the adjoint action of T . Thus $\mathfrak{t} \subset Z_{\mathfrak{g}}(\mathfrak{t}') = \mathfrak{t}' \oplus Z_{\mathfrak{q}}(\mathfrak{t}')$: in particular the torus T is invariant under τ .

If $Z_{\mathfrak{g}}(\mathfrak{t}')$ is not abelian, there exists a roots $\alpha \in \mathfrak{R}$ such that $(\mathfrak{g}_{\mathbb{C}})_{\alpha} \subset Z_{\mathfrak{g}}(\mathfrak{t}')_{\mathbb{C}} = \mathfrak{t}'_{\mathbb{C}} \oplus Z_{\mathfrak{q}}(\mathfrak{t}')_{\mathbb{C}}$. Then we obtain a contradiction: on one hand $(\mathfrak{g}_{\mathbb{C}})_{\alpha} \subset Z_{\mathfrak{g}}(\mathfrak{t}')_{\mathbb{C}}$ implies that $\alpha|_{\mathfrak{t}'} = 0$ and on the other hand since $(\mathfrak{g}_{\mathbb{C}})_{\alpha} \subset \mathfrak{q}_{\mathbb{C}}$, we must have $\sigma(\alpha) = \alpha$. The two conditions $\alpha|_{\mathfrak{t}'} = 0$ and $\sigma(\alpha) = \alpha$ implies that $\alpha = 0$. \square .

Let \mathfrak{R} and \mathfrak{R}' and be the set of roots for the pairs $T \subset G$ and $T' \subset G'$. Note that assumption (R) is equivalent to :

(R') There exists $X_o \in \mathfrak{t}'$ such that $\langle \alpha, X_o \rangle \neq 0$ for all $\alpha \in \mathfrak{R}$.

If $\xi \in \mathfrak{t}'^*$ we write $\bar{\xi}$ for the restriction of ξ to $(\mathfrak{t}')^*$. Because of our assumption, $\bar{\alpha} \neq 0$ for all $\alpha \in \mathfrak{R}$.

The positive roots are $\mathfrak{R}_+ := \{\alpha \in \mathfrak{R}, \langle \alpha, X_o \rangle > 0\}$ and $\mathfrak{R}'_+ := \{\beta \in \mathfrak{R}', \langle \beta, X_o \rangle > 0\}$. We write $\overline{\mathfrak{R}_+} := \{\bar{\alpha}, \alpha \in \mathfrak{R}_+\}$ for the set of positive restricted roots: we keep track of the multiplicity $n_a = \#\{\alpha \in \mathfrak{R}_+, \bar{\alpha} = a\}$ of each element $a \in \overline{\mathfrak{R}_+}$.

Since \mathfrak{R}'_+ is contained in $\overline{\mathfrak{R}_+}$, we may consider the set of roots $\Sigma := \overline{\mathfrak{R}_+} - \mathfrak{R}'_+$: the multiplicity of $\beta \in \overline{\mathfrak{R}_+}$ in Σ is equal to

$$m_{\beta} := \begin{cases} n_{\beta} & \text{if } \beta \notin \mathfrak{R}'_+, \\ n_{\beta} - 1 & \text{if } \beta \in \mathfrak{R}'_+. \end{cases}$$

Let $\Lambda' \subset (\mathfrak{t}')^*$ be the lattice of weights for the torus T' . Let $(\mathfrak{t}')_+^*$ be the Weyl chamber associated to the system \mathfrak{R}'_+ . The irreducible representations of G' are parameterized by $\Lambda'_+ = \Lambda' \cap (\mathfrak{t}')_+^*$.

Definition 5.2. *We denote by $\mathcal{P}_{\Sigma} : \Lambda' \rightarrow \mathbb{N}$ the partition function associated to the set Σ . For all $\xi' \in \Lambda'$, $\mathcal{P}_{\Sigma}(\xi')$ is the number of way of writing $\xi' = \sum_{\beta \in \Sigma} x_{\beta} \beta$, where $x_{\beta} \in \mathbb{N}$ and each β that occurs is counted with multiplicity m_{β} .*

For dominant weights $\lambda \in \Lambda_+$ and $\mu \in \Lambda'_+$, we denote by $m_{\lambda}(\mu)$ the multiplicity of the irreducible G' -representation $V_{\mu}^{G'}$ with highest weight μ in the irreducible G -representation V_{λ}^G with highest weight λ .

If $w \in W$, we note $w \bullet \lambda := w(\lambda + \rho) - \rho$ where ρ is the half sum of the positive roots.

Theorem 5.3. *The branching multiplicities are*

$$(5.10) \quad m_\lambda(\mu) = \sum_{w \in W} (-1)^w \mathcal{P}_\Sigma(\overline{w \bullet \lambda} - \mu).$$

We briefly recall how to obtain (5.10). Let χ_λ^G be the character of V_λ^G . The Weyl relation gives

$$\chi_\lambda^G|_T \prod_{\alpha \in \mathfrak{R}_+} (1 - e^{-\alpha}) = \sum_{w \in W} (-1)^w e^{w \bullet \lambda}.$$

When we restrict this identity to $T' \subset T$, the relations

$$\prod_{\alpha \in \mathfrak{R}_+} (1 - e^{-\alpha})|_{T'} = \prod_{\beta \in \mathfrak{R}'_+} (1 - e^{-\beta}) \prod_{\gamma \in \Sigma} (1 - e^{-\gamma})$$

and

$$\prod_{\gamma \in \Sigma} (1 - e^{-\gamma}) \left(\sum_{\xi' \in \Lambda'} \mathcal{P}_\Sigma(\xi') e^{-\xi'} \right) = 1$$

permit to obtain

$$\begin{aligned} \chi_\lambda^G|_{T'} \prod_{\beta \in \mathfrak{R}'_+} (1 - e^{-\beta}) &= \left(\sum_{w \in W} (-1)^w e^{w \bullet \lambda} \right) \left(\sum_{\xi' \in \Lambda'} \mathcal{P}_\Sigma(\xi') e^{-\xi'} \right) \\ &= \sum_{\xi' \in \Lambda'} N_\lambda(\xi') e^{\xi'}. \end{aligned}$$

with $N_\lambda(\xi') := \sum_{w \in W} (-1)^w \mathcal{P}_\Sigma(\overline{w \bullet \lambda} - \xi')$.

On the other hand, we have the decomposition $\chi_\lambda^G|_H = \sum_{\mu \in \Lambda'_+} m_\lambda(\mu) \chi_\mu^{G'}$ and then³

$$\begin{aligned} \chi_\lambda^G|_{T'} \prod_{\beta \in \mathfrak{R}'_+} (1 - e^{-\beta}) &= \sum_{\mu \in \Lambda'_+} m_\lambda(\mu) \chi_\mu^{G'} \prod_{\beta \in \mathfrak{R}'_+} (1 - e^{-\beta}) \\ &= \sum_{\mu \in \Lambda'_+} \sum_{w' \in W'} (-1)^{w'} m_\lambda(\mu) e^{w' \bullet \mu}. \end{aligned}$$

Finally, we obtain the identity

$$\sum_{\xi' \in \Lambda'} N_\lambda(\xi') e^{\xi'} = \sum_{\mu \in \Lambda'_+} \sum_{w' \in W'} (-1)^{w'} m_\lambda(\mu) e^{w' \bullet \mu},$$

that shows two things:

- $N_\lambda(\mu) = m_\lambda(\mu)$ if μ is dominant,
- $N_\lambda(w' \bullet \xi') = (-1)^{w'} N_\lambda(\xi')$, for every $(w', \xi') \in W' \times \Lambda'$.

At this stage, we have proved Kostant's multiplicity formula. In the following we rewrite this formula in another form. Let's consider the following T' -module

$$\mathfrak{n}_\Sigma := \bigoplus_{\beta \in \Sigma} \mathbb{C}_{-\beta}.$$

³Here $w' \bullet \xi' := w'(\lambda + \rho') - \rho'$ where $\rho' = \frac{1}{2} \sum_{\beta \in \mathfrak{R}'_+} \beta$.

Proposition 5.4. *For any $\lambda \in \Lambda_+$, we have the following restriction formula*

$$(5.11) \quad V_\lambda^G|_{G'} = \frac{1}{\#W'} \sum_{w \in W} (-1)^w \text{Hol}_{T'}^{G'}(\mathbb{C}_{w \bullet \lambda} \otimes \text{Sym}(\mathfrak{n}_\Sigma)).$$

Proof: Since $\text{Sym}(\mathfrak{n}_\Sigma) = \sum_{\xi' \in \Lambda'} \mathcal{P}_\Sigma(\xi') \mathbb{C}_{-\xi'}$ we have

$$\begin{aligned} \sum_{w \in W} (-1)^w \mathbb{C}_{w \bullet \lambda} \otimes \text{Sym}(\mathfrak{n}_\Sigma) &= \sum_{w \in W} \sum_{\xi' \in \Lambda'} (-1)^w \mathcal{P}_\Sigma(\xi') \mathbb{C}_{w \bullet \lambda - \xi'} \\ &= \sum_{\xi' \in \Lambda'} N_\lambda(\xi') \mathbb{C}_{\xi'}. \end{aligned}$$

Hence the right hand side of (5.11) is equal to $\frac{1}{\#W'} \sum_{\xi' \in \Lambda'} N_\lambda(\xi') \text{Hol}_{T'}^{G'}(\mathbb{C}_{\xi'})$. We use now the following facts:

- $N_\lambda(w' \bullet \xi') \text{Hol}_{T'}^{G'}(\mathbb{C}_{w' \bullet \xi'}) = N_\Sigma(\xi') \text{Hol}_{T'}^{G'}(\mathbb{C}_{\xi'})$ for every $(w', \xi') \in W' \times \Lambda'$.
- $N_\lambda(\xi') \text{Hol}_{T'}^{G'}(\mathbb{C}_{\xi'}) = 0$ if $\xi' \notin W' \bullet \Lambda'_+$.
- $N_\lambda(\mu) \text{Hol}_{T'}^{G'}(\mathbb{C}_\mu) = m_\lambda(\mu) V_\mu^{G'}$ if $\mu \in \Lambda'_+$.

We have completed the proof of (5.11). \square

We conclude this section with a few examples.

5.1. $K \subset K \times K$. Let K be a connected compact Lie group. Here we work with the Lie group $G = K \times K$ containing K diagonally. Here $\Sigma \subset \mathfrak{k}^*$ is equal to the set \mathfrak{A}_+ of positive roots for K . We denote by $\mathcal{P} : \Lambda \rightarrow \mathbb{N}$ the partition function associated to the set \mathfrak{A}_+ .

If λ, μ, ν are three dominant weights, we denote by $c'_{\lambda, \mu}$ the multiplicity of V_ν^K in $V_\lambda^K \otimes V_\mu^K$. The branching formula (5.10) becomes

$$c'_{\lambda, \mu} = \sum_{w_1, w_2 \in W} (-1)^{w_1 w_2} \mathcal{P}(w_1 \bullet \lambda + w_2 \bullet \mu - \nu).$$

This formula was first observed by Steinberg [15].

Let's take a closer look at the branching formula (5.11). The T -module \mathfrak{n}_Σ is equal to $\mathfrak{n} := \sum_{\alpha > 0} \mathbb{C}_{-\alpha}$. By definition of the holomorphic induction map Hol_T^K , we have

$$\text{Hol}_T^K(\Theta \otimes \text{Sym}(\mathfrak{n})) = (-1)^d \text{Ind}_T^K(\Theta \otimes \mathbb{C}_{2\rho})$$

for any $\Theta \in R(T)$, with $d = \frac{1}{2} \dim K/T$. Finally, (5.11) becomes

$$\begin{aligned} V_\lambda^K \otimes V_\mu^K|_K &= \frac{(-1)^d}{\#W} \sum_{w_1, w_2 \in W} (-1)^{w_1 w_2} \text{Ind}_T^K(\mathbb{C}_{w_1(\lambda+\rho)+w_2(\mu+\rho)}) \\ &= (-1)^d \sum_{w \in W} (-1)^w \text{Ind}_T^K(\mathbb{C}_{w(\lambda+\rho)+\mu+\rho}). \end{aligned}$$

The latter formula is also obtained in (4.8).

5.2. $U(p) \times U(q) \subset U(n)$. In this example the torus T of diagonal matrices is the maximal torus for both $U(n)$ and the subgroup $U(p) \times U(q)$. Here the T -module \mathfrak{n}_Σ is the T -restriction of the $U(p) \times U(q)$ -module $(\mathbb{C}^p)^* \otimes \mathbb{C}^q$, and the quotient $W' \backslash W$ is isomorphic to the subset $\text{Shuffle}(p, q)$ formed by the elements $w \in \mathfrak{S}_n$ satisfying $w(1) < \dots < w(p)$ and $w(p+1) < \dots < w(p+q)$. Here, the branching formula (5.11) gives

$$(5.12) \quad V_\lambda^{U(n)}|_{U(p) \times U(q)} = \left(\sum_{w \in \text{Shuffle}(p, q)} (-1)^w \text{Hol}_T^{U(p) \times U(q)}(\mathbb{C}_{w \bullet \lambda}) \right) \otimes \text{Sym}((\mathbb{C}^p)^* \otimes \mathbb{C}^q).$$

Let's consider the case $q = 1$. From (5.12), we derive the following branching formula for the restriction to the subgroup $U(n-1)$:

$$V_\lambda^{U(n)}|_{U(n-1)} = \left(\sum_{k=1}^n (-1)^{n-k} V_{\lambda[k]}^{U(n-1)} \right) \otimes \text{Sym}((\mathbb{C}^{n-1})^*),$$

with $\lambda[n] = (\lambda_1, \dots, \lambda_{n-1})$ and $\lambda[k] = (\lambda_1, \dots, \lambda_{k-1}, \lambda_{k+1} - 1, \dots, \lambda_n - 1)$ for $1 \leq k \leq n - 1$.

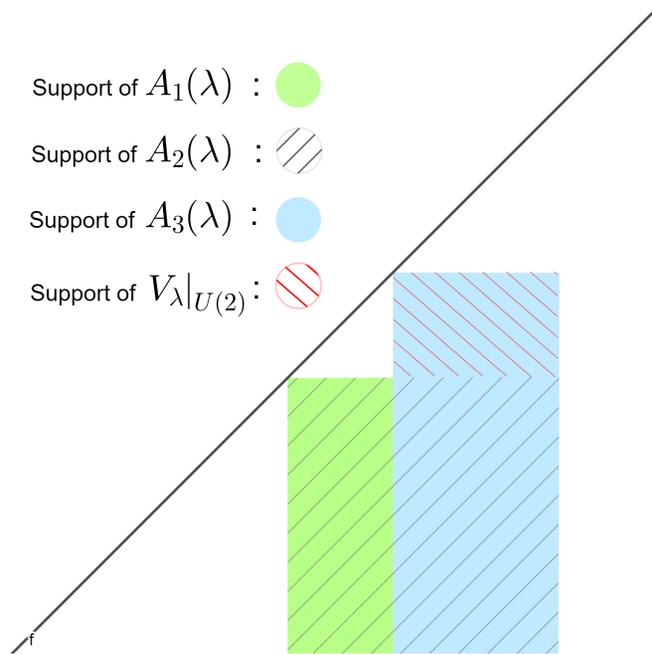


FIGURE 2. Kostant decomposition

Let's consider the case $n = 3$. For any $\lambda = (\lambda_1 \geq \lambda_2 \geq \lambda_3)$ we obtain the following formula

$$V_\lambda^{U(3)}|_{U(2)} = \left(V_{\lambda[3]}^{U(2)} - V_{\lambda[2]}^{U(2)} + V_{\lambda[1]}^{U(2)} \right) \otimes \text{Sym}((\mathbb{C}^2)^*).$$

Hence $V_\lambda^{U(3)}|_{U(2)} = A_1(\lambda) - A_2(\lambda) + A_3(\lambda)$ with

$$\begin{aligned} A_3(\lambda) &= V_{\lambda[3]}^{U(2)} \otimes \text{Sym}((\mathbb{C}^2)^*) = \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b} V_{(a,b)}^{U(2)}, \\ A_2(\lambda) &= V_{\lambda[2]}^{U(2)} \otimes \text{Sym}((\mathbb{C}^2)^*) = \sum_{\lambda_1 \geq a \geq \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)}, \\ A_1(\lambda) &= V_{\lambda[1]}^{U(2)} \otimes \text{Sym}((\mathbb{C}^2)^*) = \sum_{\lambda_2 - 1 \geq a \geq \lambda_3 - 1 \geq b} V_{(a,b)}^{U(2)}. \end{aligned}$$

We recover the classical branching formula $V_\lambda^{U(3)}|_{U(2)} := \sum_{\lambda_1 \geq a \geq \lambda_2 \geq b \geq \lambda_3} V_{(a,b)}^{U(2)}$ (see [4], section 8.1). In Figure 2, one can visualize the support of each characters $A_k(\lambda)$.

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