Direct zero-sum problems for certain groups of rank three

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DIRECT ZERO-SUM PROBLEMS FOR CERTAIN GROUPS OF RANK THREE

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Abstract. We determine the exact value of the η-constant and the multiwise Davenport constants for finite abelian groups of rank three having the form $G \simeq C_2 \oplus C_{n_2} \oplus C_{n_3}$ with $2 \mid n_2 \mid n_3$. Moreover, we determine the Erdős–Ginzburg–Ziv constant of these groups under the assumption that $n_2/2$ has Property D or $n_2 = n_3$.

1. Introduction

A well-known direct zero-sum problem is to determine the Davenport constant of finite abelian groups. For such a group $(G, +, 0)$, this constant, denoted by $\mathsf{D}(G)$, is defined as the smallest positive integer t such that every sequence of t elements from G contains a non-empty subsequence whose terms sum to 0.

A closely related problem is to determine the Erdős–Ginzburg–Ziv constant, denoted by $\mathsf{s}(G)$, which is defined in the same way except that one requires the existence of a subsequence whose sum is 0 and whose length is equal to the exponent of the group. A variant of this constant is $\eta(G)$, where one seeks a non-empty subsequence with sum 0 whose length is at most the exponent of the group.

The investigation of these zero-sum constants has been a topic of active research for more than fifty years. We refer to [10, 11, 12, 14] for detailed expositions. Some results are also recalled in the next section. The exact values of these three constants are known for every finite abelian group of rank at most two, and only for fairly special types of groups of higher rank. Even for groups of rank three, that is, $G \simeq C_{n_1} \oplus C_{n_2} \oplus C_{n_3}$ with $1 < n_1 \mid n_2 \mid n_3$, the problem of determining these constants is wide open. For example, when $n_1 = 2$, the Davenport constant is known, but the exact values of the other two constants defined above is not.

In the present paper, we obtain these values for the η-constant and, assuming a now well-supported conjecture, for the Erdős–Ginzburg–Ziv constant as well. Our results confirm Gao's conjecture (Conjecture 2.2) for this type of groups, and generalize previous results obtained in the case $n_1 = n_2 = 2$ (see [5, Theorem 1.2(1)] and [4, Theorem 1.3]). Moreover, they show that recent results of Luo [16] are essentially optimal. In addition, we determine the multiwise Davenport constants for this type of groups (see the subsequent section for the definition). For a more detailed overview of our results and how they relate to the existing literature on the subject, we refer to Section 3.

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2. Preliminaries

We recall some notation and results; for more detailed information we refer again to [10, 11, 12, 14]. All intervals in this paper are intervals of integers, specifically $[a, b] = \{z \in \mathbb{Z} \colon a \le z \le b\}$.

Let G be a finite abelian group, written additively. For each g in G, we denote by $\mathrm{ord}(g)$ its order in G. For a subset $A \subset G$ we denote by $\langle A \rangle$ the subgroup it generates; we say that A is a *generating* set if $\langle A \rangle = G$. We say that elements $g_1, \ldots, g_k$ are *independent* if $\sum_{i=1}^k a_i g_i = 0$, with integers a_i, implies that $a_i g_i = 0$ for each i; we say that a set is independent when its elements are independent.

By $\exp(G)$ we denote the exponent of G, that is the least common multiple of the orders of elements of G. By $\mathsf{r}(G)$ we denote the rank of G, that is the minimum cardinality of a generating subset of G. For n a positive integer we denote by C_n the cyclic group of order n.

For a finite abelian group G there exist uniquely determined integers $1 < n_1 \mid \cdots \mid n_r$ such that $G \simeq C_{n_1} \oplus \cdots \oplus C_{n_r}$. For $|G| > 1$ we have $\mathsf{r}(G) = r$ and $\exp(G) = n_r$; the rank of a group of cardinality 1 is 0 and its exponent is 1.

By a *sequence* over G we mean an element of the free abelian monoid over G. In other words, this is a finite sequence of ℓ elements from G, where repetitions are allowed and the order of elements is disregarded. We use multiplicative notation for sequences. We denote its neutral element, that is the sequence of length zero, simply by 1. Let

$$S = g_1 \cdots g_\ell = \prod_{g \in G} g^{\mathsf{v}_g(S)}$$

be a sequence over G, where, for all $g \in G$, $\mathsf{v}_g(S)$ is a non-negative integer called the *multiplicity* of g in S. Moreover ℓ is the *length* of S.

A sequence T over G is said to be a *subsequence* of S if it is a divisor of S in the free abelian monoid over G, that is if $\mathsf{v}_g(T) \le \mathsf{v}_g(S)$ for all $g \in G$; in this case we write $T \mid S$. For a subsequence T of S we set $ST^{-1} = \prod_{g \in G} g^{(\mathsf{v}_g(S) - \mathsf{v}_g(T))}$, that is, it is the subsequence of S such that $T(ST^{-1}) = S$.

For subsequences $T_1, \ldots, T_k$ of S we say that they are *disjoint* subsequences if $T_1 \ldots T_k$ is also a subsequence of S. Moreover, for sequences $S_1, \ldots, S_k$ over G we denote by

$$\gcd(S_1, \ldots, S_k) = \prod_{g \in G} g^{\min\{\mathsf{v}_g(S_i) \colon 1 \le i \le k\}}$$

the greatest common divisor of the sequences $S_1, \ldots, S_k$ in the free abelian monoid. To avoid confusion we stress that it is not necessary for disjoint subsequences to have a trivial greatest common divisor.

We call the set $\mathrm{supp}(S) = \{g \in G \mid \mathsf{v}_g(S) > 0\}$ the *support* of S, and $\sigma(S) = \sum_{i=1}^\ell g_i = \sum_{g \in G} \mathsf{v}_g(S) g$ the *sum* of S. In addition, we say that $s \in G$ is a *subsum* of S if

$$s = \sum_{i \in I} g_i \text{ for some } \emptyset \subsetneq I \subseteq [1, \ell].$$

If 0 is not a subsum of S, we say that S is a *zero-sumfree* sequence. If $\sigma(S) = 0$, then S is said to be a *zero-sum sequence*. If, moreover, one has $\sigma(T) \neq 0$ for all proper and non-empty subsequences $T \mid S$, then S is called a *minimal zero-sum sequence*.

We set

$$\Sigma(S) = \{\sigma(T)\colon 1 \neq T \mid S\}.$$

For every integer k, we also set

$$\Sigma_k(S) = \{\sigma(T)\colon T \mid S, |T| = k\}$$

as well as

$$\Sigma_{\leq k}(S) = \bigcup_{i=1}^{k} \Sigma_i(S) = \{\sigma(T)\colon 1 \neq T \mid S, |T| \leq k\}.$$

We now recall in more detail the definitions and results alluded to in the introduction.

By $\mathsf{D}(G)$ we denote the smallest positive integer t such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence. This number $\mathsf{D}(G)$ is called the *Davenport constant* of the group G. More generally, given an integer $k \geq 1$, we denote by $\mathsf{D}_k(G)$ the smallest positive integer t such that every sequence S over G of length $|S| \geq t$ contains at least k non-empty disjoint zero-sum subsequences.

Note that, by definition, $\mathsf{D}_1(G) = \mathsf{D}(G)$ for every finite abelian group G. It is known that for every finite abelian group the sequence $(\mathsf{D}_k(G))_{k\geq 1}$ is eventually an arithmetic progression. More precisely, one has the following result (see [6, Lemma 5.1]).

Theorem 2.1. *Let G be a finite abelian group. There exist $\mathsf{D}_0(G) \in \mathbb{N}$ and an integer $k_0 \geq 1$ such that*

$$\mathsf{D}_k(G) = \mathsf{D}_0(G) + k\exp(G), \text{ for each } k \geq k_0.$$

Let $k_{\mathsf{D}}(G)$ denote the smallest possible value of k_0 in the above theorem.

By $\eta(G)$ we denote the smallest positiver integer t such that every sequence S over G of length $|S| \geq t$ contains a non-empty zero-sum subsequence $S' \mid S$ of length $|S'| \leq \exp(G)$. Such a subsequence is called a *short zero-sum subsequence*.

By $\mathsf{s}(G)$ we denote the smallest positive integer t such that every sequence S over G of length $|S| \geq t$ contains a zero-sum subsequence $S' \mid S$ of length $|S'| = \exp(G)$. The number $\mathsf{s}(G)$ is called the *Erdős–Ginzburg–Ziv constant* of the group G.

It is not hard to see that $\mathsf{s}(G) \geq \eta(G) + \exp(G) - 1$ holds for each finite abelian group G. It was conjectured by Gao that in fact equality holds (see [10, Conjecture 6.5]).

Conjecture 2.2 (Gao)**.** *For every finite abelian group G, one has*

$$\mathsf{s}(G) = \eta(G) + \exp(G) - 1.$$

We now recall the values of $\eta(G)$ and $\mathsf{s}(G)$ as well as the ones of the multiwise Davenport constants for groups of rank at most two, see [12, Theorem 5.8.3] and [12, Theorem 6.1.5]. We parametrize these groups as $C_m \oplus C_{mn}$ with $m, n \geq 1$ integers rather than $C_{n_1} \oplus C_{n_2}$ with integers $n_1 \mid n_2$ since later on this will be more convenient.

Theorem 2.3. *Let $m, n \geq 1$ be two integers. Then*

$$\eta(C_m \oplus C_{mn}) = 2m + mn - 2 \quad \text{and} \quad \mathsf{s}(C_m \oplus C_{mn}) = 2m + 2mn - 3.$$

In addition, for every integer $k \ge 1$,

$$\mathsf{D}_k(C_m \oplus C_{mn}) = m + k(mn) - 1.$$

In particular, choosing $m = 1$, *we have* $\eta(C_n) = n$ *and* $\mathsf{s}(C_n) = 2n - 1$ *as well as* $\mathsf{D}_k(C_n) = kn$ *for all* $k \ge 1$.

Theorem 2.3 shows that Conjecture 2.2 is true for all finite abelian groups of rank at most two.

In the case of groups of rank at most two even the structure of extremal examples is well-understood. For cyclic groups in fact more is known, see, e.g., [18], yet we only recall what is needed in this paper.

A sequence S over C_n of length $n - 1 = \eta(C_n) - 1$ has no short zero-sum subsequence (and thus no non-empty zero-sum subsequence) if and only if $S = b^{n-1}$ for some generating element b of C_n. A sequence S over C_n of length $2n - 2 = \mathsf{s}(C_n) - 1$ has no zero-sum subsequence of length n if and only if $S = c^{n-1}(c+b)^{n-1}$ for some $c \in C_n$ and some generating element b of C_n.

For the η-constant one has the following result. It was obtained in [19]; a result of Reiher [17] was crucial in the proof.

Theorem 2.4. *Let* $H \simeq C_m \oplus C_{mn}$ *with integers* $m \ge 2$ *and* $n \ge 1$. *Every sequence* S *over* H *of length* $|S| = \eta(H) - 1$ *not containing any short zero-sum subsequence has the following form:*

$$S = b_1^{m-1} b_2^{sm-1} \left(-xb_1 + b_2\right)^{(n+1-s)m-1}$$

where $\{b_1, b_2\}$ *is a generating set of* H *with* $\mathrm{ord}(b_2) = mn$, $s \in [1, n]$, $x \in [1, m]$ *with* $\gcd(x, m) = 1$ *and either*

(1) $\{b_1, b_2\}$ *is an independent generating set of* H, *or*
(2) $s = n$ *and* $x = 1$.

For the Erdős–Ginzburg–Ziv constant a similar result is expected to hold true, yet it is so far only known conditionally or in special cases.

A positive integer m is said to have *Property D* if every sequence S over C_m^2 of length $|S| = \mathsf{s}(C_m^2) - 1 = 4m - 4$ and containing no zero-sum subsequence of length $\exp(G) = m$ has the form $S = T^{m-1}$ for some sequence T over C_m^2. We include the trivial case $m = 1$ in our definition as it simplifies the statement of certain results. This property was introduced by Gao who made the following conjecture [8, Conjecture 2].

Conjecture 2.5 (Gao)**.** *Every positive integer has Property D.*

For the time being, Property D has been proved to be multiplicative [8, Theorem 1.4] in the sense that whenever m, n have this property, then so does mn. Also, Property D is known to hold for $p \in \{2, 3, 5, 7\}$, hence for any m of the form $m = 2^\alpha 3^\beta 5^\gamma 7^\delta$, where $\alpha, \beta, \gamma, \delta \ge 0$ are non-negative integers (see [8, Theorem 1.5] and [21, Theorem 3.1]).

Whenever an integer m satisfies Property D, the sequences over $H \simeq C_m \oplus C_{mn}$ of length $\mathsf{s}(H) - 1$ and not containing any zero-sum subsequence of length $\exp(H)$ can be fully characterized for every integer $n \ge 1$ (see [19, Theorem 3.1(2)]).

Theorem 2.6. *Let $H \simeq C_m \oplus C_{mn}$, where $m \geq 2$ satisfies Property D and $n \geq 1$. Every sequence S over H of length $|S| = \mathsf{s}(H) - 1$ not containing any zero-sum subsequence of length $\exp(H)$ has the following form:*

$$S = c^{tm-1}(b_1 + c)^{(n+1-t)m-1}(b_2 + c)^{sm-1}\left(-xb_1 + b_2 + c\right)^{(n+1-s)m-1}$$

where $c \in H$, $\{b_1, b_2\}$ is a generating set of H with $\mathrm{ord}(b_2) = mn$, $s, t \in [1, n]$, $x \in [1, m]$ with $\gcd(x, m) = 1$ and either

1. *$\{b_1, b_2\}$ is an independent generating set of H, or*
2. *$s = t = n$ and $x = 1$.*

For definiteness we briefly recap some properties of the generating sets in the result above. Since $\{b_1, b_2\}$ is a generating set of H and $\mathrm{ord}(b_2) = mn$, the equalities

$$|H| = |\langle b_1\rangle + \langle b_2\rangle| = \frac{|\langle b_1\rangle||\langle b_2\rangle|}{|\langle b_1\rangle \cap \langle b_2\rangle|}$$

imply that $\mathrm{ord}(b_1) = md$ where $d = |\langle b_1\rangle \cap \langle b_2\rangle|$ is a positive divisor of n. More precisely, we have

$$\langle mb_1\rangle = \langle b_1\rangle \cap \langle b_2\rangle = \left\langle m\frac{n}{d}b_2\right\rangle.$$

In particular, every element $h \in H$ can be written $h = a_1b_1 + a_2b_2$ with $a_1 \in [0, m-1]$ and $a_2 \in [0, mn-1]$. In addition, it is easily seen that $\{b_1, b_2\}$ is an independent generating set of H if and only if $\langle b_1\rangle \cap \langle b_2\rangle = \{0\}$, that is to say if and only if $d = 1$. Finally, whenever $d > 1$, there is a unique integer $\ell \in [1, d-1]$ relatively prime to d such that $mb_1 = \ell m(n/d)b_2$.

We end by recalling the result on the Davenport constant for groups of the form $C_2 \oplus C_{2m} \oplus C_{2mn}$, which we mentioned in the introduction and that we need in the proof of our result on the multiwise Davenport constant.

Theorem 2.7. *Let $m, n \geq 1$ be two integers. Then*

$$\mathsf{D}(C_2 \oplus C_{2m} \oplus C_{2mn}) = 2m + 2mn.$$

The proof of the above result involved two parts. First, the claim was established conditionally on a result on the structure of the set of subsums of zero-sum free sequences of maximal length over a group of rank two; this motivated the definition of the ν-invariant (see, e.g., [10, Definition 2.1]). Then, this property was established. The first part dates back to the very beginning of investigations of the Davenport constant (see [3]). The second part was only completed much later when Property B (and thus Property C) was established by Reiher [17] (see in particular Section 11). For further context, see for instance [7] or [20, Section 4.1].

3. New results

As mentioned in the introduction we investigate zero-sum constants for groups of rank three of the form $C_2 \oplus C_{n_2} \oplus C_{n_3}$ where $2 \mid n_2 \mid n_3$. For ease of notation we will use a different parametrization, namely $C_2 \oplus C_{2m} \oplus C_{2mn}$ with $m, n \geq 1$.

We determine $\eta(C_2 \oplus C_{2m} \oplus C_{2mn})$ for all $m, n \geq 1$, and $\mathsf{s}\left(C_2 \oplus C_{2m} \oplus C_{2mn}\right)$ under the condition that $n = 1$ or m has Property D. We recall that the constants were known for $m = 1$, see [5, Theorem 1.2(1)] and [4, Theorem 1.3]; in this case even the inverse problem is solved [13]. Moreover, we determine $\mathsf{D}_k\left(C_2 \oplus C_{2m} \oplus C_{2mn}\right)$ for all $k, m, n \geq 1$; as recalled the case $k = 1$ and the case $m = n = 1$ were known.

We will see that there is a quite significant difference between the two cases $n = 1$ and $n \neq 1$.

Our approach to determining $\eta(C_2 \oplus C_{2m} \oplus C_{2mn})$ is similar to the one used the Davenport constant, which we recalled above. In particular, the property of the set of restricted subsums established in Lemma 4.3 resembles the property underlying the definition of the ν-invariant. The subsequent result on $\mathsf{s}(C_2 \oplus C_{2m} \oplus C_{2mn})$ is obtained by establishing Gao's conjecture for this group using a generalization of a well-known technique (see Lemma 4.4). The proof of the result for $\mathsf{D}_k(C_2 \oplus C_{2m} \oplus C_{2mn})$ also uses the result on $\eta(C_2 \oplus C_{2m} \oplus C_{2mn})$.

Theorem 3.1. *Let $m \geq 1$ and $n \geq 2$ be two integers. Then*

$$\eta(C_2 \oplus C_{2m} \oplus C_{2m}) = 6m + 2$$

and

$$\eta(C_2 \oplus C_{2m} \oplus C_{2mn}) = 4m + 2mn.$$

When $G \simeq C_2 \oplus C_{2m} \oplus C_{2m}$ and m is a power of 2, G is a finite abelian 2-group such that

$$\mathsf{D}(G) = 2\exp(G) \text{ and } 2\mathsf{D}(G) - \exp(G) < \eta(G),$$

thus showing that [16, Theorem 1.6] is optimal in the sense that for $2\mathsf{D}(G) - \exp(G) = \eta(G)$ to hold, the condition $\mathsf{D}(G) \leq 2\exp(G) - 1$ in the statement of the theorem cannot be replaced by $\mathsf{D}(G) \leq 2\exp(G)$.

Theorem 3.2. *Let $m \geq 1$ and $n \geq 2$ be two integers. Then*

$$\mathsf{s}(C_2 \oplus C_{2m} \oplus C_{2m}) = 8m + 1.$$

Moreover, if m has Property D, then

$$\mathsf{s}(C_2 \oplus C_{2m} \oplus C_{2mn}) = 4m + 4mn - 1.$$

In combination the two results imply that Gao's conjecture holds true for these types of groups.

Corollary 3.3. *Let $m, n \geq 1$ be two integers. Then Conjecture 2.2 holds true for $C_2 \oplus C_{2m} \oplus C_{2m}$. Moreover, if m has Property D, then Conjecture 2.2 holds true for $C_2 \oplus C_{2m} \oplus C_{2mn}$.*

We end this section with our result on the multiwise Davenport constants.

Theorem 3.4. *Let $G \simeq C_2 \oplus C_{2m} \oplus C_{2mn}$, where $m, n \geq 1$ are integers. If $n \geq 2$, then $\mathsf{D}_0(G) = 2m$ and $k_{\mathsf{D}}(G) = 1$. If $n = 1$, then $\mathsf{D}_0(G) = 2m + 1$ and $k_{\mathsf{D}}(G) = 2$.*

Note that the case $n = 1$ extends to all $m \geq 1$ the result [1, Lemma 3.7] stating that $\mathsf{D}_0(C_2^3) = 3$ and $k_{\mathsf{D}}(C_2^3) = 2$.

When $G \simeq C_2 \oplus C_{2m} \oplus C_{2m}$, where $m \geq 1$, we have

$$\mathsf{D}(G) = \mathsf{D}(C_2 \oplus C_{2m}) + (2m - 1) \quad \text{and} \quad \eta(G) > \mathsf{D}(G) + 2m,$$

however $k_{\mathsf{D}}(G) \geq 2$, thus showing that [6, Remark 5.3.2] is nearly optimal in the sense that for $k_{\mathsf{D}}(G) = 1$ to hold, the condition $\eta(G) \leq \mathsf{D}(G) + \exp(G)$ stated in this remark cannot be replaced by a much weaker inequality.

4. Auxiliary results

In this section we establish several auxiliary results. In some cases, we prove results which are slightly more general than what is needed for our immediate applications, but we mostly focus on the needs of this paper.

Our first lemma shows that extremal sequences with respect to the Erdős–Ginzburg–Ziv constant for groups of rank at most two are stable in the sense that changing a unique element cannot yield another extremal sequence. It could be interesting to consider this problem for more general groups, and to determine for specifc groups the exact number of elements one has to change to get another extremal example.

Lemma 4.1. *Let $H \simeq C_m \oplus C_{mn}$, where $m, n \geq 1$ are integers and m satisfies Property D. Let S_1, S_2 be sequences over H of length $\mathsf{s}(H) - 1$ not containing any zero-sum subsequence of length $\exp(H)$. If $|\gcd(S_1, S_2)| \geq \mathsf{s}(H) - 2$, then $S_1 = S_2$.*

Proof. Let $T = \gcd(S_1, S_2)$. By assumption, we have $S_1 = g_1 T$ and $S_2 = g_2 T$ with $g_1, g_2 \in G$.

First, assume that $m \geq 3$. We know by Theorem 2.6 that $\mathsf{v}_g(S_i) \equiv m - 1 \pmod m$ for each $g \in \mathrm{supp}(S_i)$. It thus follows that $\mathsf{v}_{g_1}(T) \equiv m - 2 \pmod m$. Since this is non-zero it follows that $g_1 \in \mathrm{supp}(S_2)$, and thus $\mathsf{v}_{g_1}(S_2) \equiv m - 1 \pmod m$. Yet, $\mathsf{v}_{g_1}(S_2) = \mathsf{v}_{g_1}(g_2) + \mathsf{v}_{g_1}(T)$. Since $\mathsf{v}_{g_1}(T) \equiv m - 2 \pmod m$ we must have $\mathsf{v}_{g_1}(g_2) \neq 0$, that is, $g_1 = g_2$. This proves the claim in the case $m \geq 3$.

Now, assume that $m = 2$. By Theorem 2.6, every sequence S over H containing no zero-sum subsequence of length $2n$ can be decomposed as

$$S = c^{2t-1}(b_1 + c)^{2(n+1-t)-1}(b_2 + c)^{2s-1}\left(-b_1 + b_2 + c\right)^{2(n+1-s)-1},$$

where $c \in H$, $\{b_1, b_2\}$ is a generating set of H with $\mathrm{ord}(b_2) = 2n$ and $s, t \in [1, n]$, such that either $\{b_1, b_2\}$ is an independent generating set of H or $s = t = n$. In particular, one has

$$\sigma\left(c^{2t-1}(b_1 + c)^{2(n+1-t)-1}\right) = -(2t-1)b_1,$$

and

$$\sigma\left((b_2 + c)^{2s-1}\left(-b_1 + b_2 + c\right)^{2(n+1-s)-1}\right) = (2s-1)b_1.$$

Therefore, either $\{b_1, b_2\}$ is an independent generating set of H, in which case $\mathrm{ord}(b_1) = 2$ (see the remarks at the end of Section 2) so that $\sigma(S) = b_1 - b_1 = 0$, or $s = t = n$, in which case $\sigma(S) = b_1 - b_1 = 0$ also. It thus follows in both cases that $g_1 + \sigma(T) = \sigma(S_1) = 0 = \sigma(S_2) = g_2 + \sigma(T)$, which yields $g_1 = g_2$ indeed.

Finally, let us consider the case $m = 1$. Then, we know by the results recalled before Theorem 2.4 that $\mathsf{v}_g(S_i) \equiv n - 1 \pmod n$ for each $g \in \mathrm{supp}(S_i)$. For $n \geq 3$ we can argue as in the case $m \geq 3$. For $n \leq 2$, that is for C_1 and C_2, the claim is trivial as there is only one sequence of length $\mathsf{s}(H) - 1$ not containing any zero-sum subsequence of length $\exp(H)$. □

The analogous result for the η-constant holds true as well; we record it for its own sake.

Lemma 4.2. *Let $H \simeq C_m \oplus C_{mn}$ where $m, n \geq 1$ are integers. Let S_1, S_2 be sequences over H of length $\eta(H) - 1$ not containing a short zero-sum subsequence. If $|\gcd(S_1, S_2)| \geq \eta(H) - 2$, then $S_1 = S_2$.*

Proof. For $m \geq 3$, the same argument as in Lemma 4.1 works. For $m = 2$, we assume for a contradiction that there are two distinct sequences S_1, S_2 of length $\eta(H)-1$ not containing a short zero-sum subsequence with $|\gcd(S_1, S_2)| \geq \eta(H)-2$. Then $S_1' = S_1 0^{\exp(H)-1}$ and $S_2' = S_2 0^{\exp(H)-1}$ are two distinct sequences of length $\mathsf{s}(H)-1$ not containing a zero-sum subsequence of length $\exp(H)$ with $|\gcd(S_1, S_2)| \geq \mathsf{s}(H)-2$, contradicting Lemma 4.1. Finally, for $m = 1$ and $n \geq 3$ the claim follows from the fact that a sequence over H of length $\eta(H)-1$ not containing a short zero-sum subsequence is of the form h^{n-1} for some generating element of $h \in H$ (recall the results before Theorem 2.4). The case $m = 1$ and $n \leq 2$ is trivial. □

Next we obtain results on the set of restricted subsums of sequences that are extremal examples with respect to the η-constant and the Erdős–Ginzburg–Ziv constant. The result we obtain is reminiscent of the condition in the definition of the ν-invariant (see, e.g., [7]).

Lemma 4.3. *Let $H \simeq C_m \oplus C_{mn}$ where m, n are positive integers and $n \geq 2$.*

1. *Let S be a sequence over H of length $|S| = \eta(H)-1$ not containing any short zero-sum subsequence. Then $\Sigma_{\leq mn-2}(S) \supseteq H \setminus ((-k'+K) \cup \{0\})$ for a proper subgroup K and some $k' \notin K$.*
2. *Suppose that m has Property D. Let S be a sequence over H of length $|S| = \mathsf{s}(H)-1$ not containing any zero-sum subsequence of length $\exp(H)$. Then $\Sigma_{mn-2}(S) \supseteq H \setminus (-k'+K)$ for a proper subgroup K and some $k' \notin K$.*

The condition $n \geq 2$ is necessary. Indeed, the claim is not true for groups of the form C_m^2. To see this it suffices to note that for $\{b_1, b_2\}$ an independent generating set and $S = b_1^{m-1} b_2^{m-1} (b_1 + b_2)^{m-1}$ the set $\Sigma_{\leq m-2}(S)$ contains no element of $-b_1 + \langle b_2 \rangle$ and $-b_2 + \langle b_1 \rangle$.

Proof. We first deal with the main case $m \geq 2$.

(1). Let S be a sequence over H of length $|S| = \eta(H)-1$ not containing any short zero-sum subsequence. By Theorem 2.4 we know that

$$S = b_1^{m-1} b_2^{sm-1} \left(-xb_1 + b_2\right)^{(n+1-s)m-1}$$

where $\{b_1, b_2\}$ is a generating set of H with $\mathrm{ord}(b_2) = mn$, $s \in [1, n]$, $x \in [1, m]$ with $\gcd(x, m) = 1$ and either $\{b_1, b_2\}$ is an independent generating set of H, or $s = n$ and $x = 1$.

Let $d \in [1, n]$ such that $\mathrm{ord}(b_1) = md$; as recalled after Theorem 2.6 this always exists.

We now distinguish the following three cases.

Case 1. $d = 1$, that is to say $\{b_1, b_2\}$ is an independent generating set of H. In particular, $mb_1 = 0$. Let $h = a_1 b_1 + a_2 b_2$ with $a_1 \in [0, m-1]$ and $a_2 \in [0, mn-1]$. If $a_1 = 0$ then $h \notin \Sigma_{\leq mn-2}(S)$ only if $a_2 = 0$ or $a_2 = mn-1$, that is to say only if $h = 0$ or $h = -b_2$. If $a_2 \leq m-1$, we observe that $h \in \Sigma_{a_1+a_2}(S)$ and $a_1 + a_2 \leq 2m-2 \leq mn-2$. Now, assume that $a_1 \neq 0$ and $a_2 \geq m$. Let $v \in [1, m-1]$ be the unique integer such that $a_1 \equiv -vx \pmod m$. In particular, one has $v \leq m-1 < a_2$ so that $1 \leq a_2 - v \leq mn-2 < mn-1 = ms-1+m(n-s)$. Therefore, there exists $q \in [0, n-s]$ such that $0 \leq a_2 - v - qm \leq ms-1$. Such an

integer q readily satisfies $1 \le v+qm \le m-1+m(n-s) = (n+1-s)m-1$. As a consequence,

$$S' = (-xb_1+b_2)^{v+qm}b_2^{a_2-v-qm}$$

is a subsequence of S of length $|S'| = a_2$ such that

$$\sigma(S') = -vxb_1 + vb_2 - vb_2 - qx(mb_1) + qmb_2 - qmb_2 + a_2b_2 = a_1b_1 + a_2b_2 = h,$$

so that $h \in \Sigma_{\le a_2}(S)$ and the claim follows with $k' = b_2$ and $K = \langle b_1\rangle$.

Case 2. $1 < d < n$, or $d = n$ and $\ell \ge 2$. Let $h = a_1b_1 + a_2b_2$ with $a_1 \in [0, m-1]$ and $a_2 \in [0, mn-1]$. If $a_1 = 0$ then $h \notin \Sigma_{\le mn-2}(S)$ only if $a_2 = 0$ or $a_2 = mn-1$, that is to say only if $h = 0$ or $h = -b_2$. Now, assume that $a_1 \ne 0$. Then, either $a_1 + a_2 \le mn-2$ in which case $h \in \Sigma_{\le mn-2}(S)$ or $a_2 \ge mn-1-a_1 \ge mn-1-(m-1) = m(n-1)$ so that

$$2mn-1-m \ge mn-1+mn\frac{d-1}{d} \ge a_2 + \ell m\frac{n}{d} \ge m(n-1)+2m = mn+m.$$

Therefore, setting $a_2' = a_2 + \ell m(n/d) - mn$, we have $a_2' \in [m, mn-2]$ and since $m-a_1 \in [1, m-1]$ we obtain that

$$S' = (-b_1+b_2)^{m-a_1}b_2^{a_2'-(m-a_1)}$$

is a subsequence of S of length $|S'| = a_2' \le mn-2$ verifying

$$\sigma(S') = a_1b_1 - mb_1 + a_2'b_2 = a_1b_1 - mb_1 + \ell m\frac{n}{d}b_2 + a_2b_2 = h,$$

so that $h \in \Sigma_{\le mn-2}(S)$ and the claim follows with $k' = b_2$ and $K = \{0\}$.

Case 3. $d = n$ and $\ell = 1$. In this case, $mb_1 = mb_2$. Let $h = a_1b_1 + a_2b_2$ with $a_1 \in [0, m-1]$ and $a_2 \in [0, mn-1]$. If $a_1 = 0$ then $h \notin \Sigma_{\le mn-2}(S)$ only if $a_2 = 0$ or $a_2 = mn-1$, that is to say only if $h = 0$ or $h = -b_2$. Now, assume that $a_1 \ne 0$. Then, either $a_1 + a_2 \le mn-2$ in which case $h \in \Sigma_{\le mn-2}(S)$ or $a_1 + a_2 = mn-1$ or $a_1 + a_2 \ge mn$ so that

$$2mn-2 \ge mn-1+m-1 \ge a_1+a_2 \ge mn$$

Therefore, setting $a_2' = a_1 + a_2 - mn$, we have $a_2' \in [0, mn-2]$ and since $m-a_1 \in [1, m-1]$ we obtain that

$$S' = (-b_1+b_2)^{m-a_1}b_2^{a_2'}$$

is a subsequence of S of length $|S'| = m-a_1+a_2' = m+a_2-mn \le m+mn-1-mn \le m-1$ verifying

$$\sigma(S') = a_1b_1 - mb_1 + mb_2 - a_1b_2 + a_1b_2 + a_2b_2 = h,$$

so that $h \in \Sigma_{\le mn-2}(S)$ and the claim follows with $k' = b_2$ and $K = \langle b_1 - b_2\rangle$.

(2). Suppose that m has Property D. Let S be a sequence over H of length $|S| = \mathsf{s}(H) - 1$ not containing any zero-sum subsequence of length $\exp(H)$. By Theorem 2.6 we know that

$$S = c^{tm-1}(b_1+c)^{(n+1-t)m-1}b_2^{sm-1}\,(-xb_1+b_2)^{(n+1-s)m-1}$$

where $c \in H$, $\{b_1, b_2\}$ is a generating set of H with $\mathrm{ord}(b_2) = mn$, $s, t \in [1, n]$, $x \in [1, m]$ with $\gcd(x, m) = 1$ and either $\{b_1, b_2\}$ is an independent generating set of H, or $s = t = n$ and $x = 1$.

Since $\Sigma_{mn-2}(-c+S) = 2c+\Sigma_{mn-2}(S)$, we can assume without loss of generality that $c = 0$. We now distinguish the following two cases.

Case 1. If $\{b_1, b_2\}$ is an independent generating set of H, we see that $S = 0^{tm-1}b_1^{(n-t)m}T$, where T is a sequence of length $\eta(H)-1$ that has no short zero-sum subsequence. By Lemma 4.3, it follows that then $\Sigma_{\le mn-2}(T) \supseteq H\setminus((-k'+K)\cup\{0\})$ for a proper subgroup K and some $k' \notin K$. Therefore, it suffices to assert that $\{0\} \cup \Sigma_{\le mn-2}(T) \subseteq \Sigma_{mn-2}(0^{tm-1}b_1^{(n-t)m}T)$. Indeed, note that for each subsequence $T' \mid T$ of length at most $mn-2 = mt-1+m(n-t)-1$, and any integer $t' \in [0, n-t]$ such that

$$mn-2 \ge |T'| + mt' \ge m(n-t)-1,$$

we obtain

$$0 \le mn-2-|T'|-mt' \le mt-1$$

so that, since $mb_1 = 0$ in this case, the sequence $0^{mn-2-|T'|-mt'}b_1^{mt'}T'$ is a subsequence of S of length $mn-2$ with the same sum. The fact that we get 0 in addition to $\Sigma_{\le mn-2}(T)$ is due to the fact that T' can be chosen to be the empty sequence.

Case 2. If $s = t = n$ and $x = 1$, we see that $S = 0^{mn-1}T$, where T is a sequence of length $\eta(H)-1$ that has no short zero-sum subsequence. By Lemma 4.3, it follows that then $\Sigma_{\le mn-2}(T) \supseteq H \setminus ((-k'+K)\cup\{0\})$ for a proper subgroup K and some $k' \notin K$. We assert that $\{0\} \cup \Sigma_{\le mn-2}(T) \subseteq \Sigma_{mn-2}(0^{mn-1}T)$, then the claim is proved. As above, it suffices to note that for each subsequence T' of T, the sequence $0^{mn-2-|T'|}|T'$ is a subsequence of S of length $mn-2$ with the same sum. he fact that we get 0 in addition to $\Sigma_{\le mn-2}(T)$ is due to the fact that T' can be chosen to be the empty sequence.

To finish the argument we consider the case $m = 1$. For assertion (1), we have a sequence S over H of length $|S| = \eta(H)-1 = n-1$ not containing any short zero-sum subsequence. By the results recalled before Theorem 2.4 we know that $S = b^{n-1}$ for some generating element b of H. It follows that $\Sigma_{\le n-2}(S) = \{b, 2b, \dots, (n-2)b\}$. Thus the claim is established with $K = \{0\}$. For assertion (2), we have a sequence S over H of length $|S| = \mathsf{s}(H)-1 = 2n-2$ not containing any zero-sum subsequence of length n. By the results recalled above we know that $S = c^{n-1}(c+b)^{n-1}$ for some generating element b of H. Without loss of generality we can assume that $c = 0$. It follows that $\Sigma_{=n-2}(S) = \{0, b, 2b, \dots, (n-2)b\}$. Thus the claim is established with $K = \{0\}$. □

The following lemma slightly develops a well-known technique useful to establish Conjecture 2.2; see, e.g., [9, Proposition 2.7] or [4, Theorem 1.3] for earlier versions. We do not need the second part in this paper, but include it as it might be useful elsewhere. We note that the condition in the lemma is trivial for $\exp(G) \le 4$.

Lemma 4.4. *Let G be a finite abelian group. The following two statements hold.*

1. *Let S be a sequence over G of length $\eta(G)+\exp(G)-1$. Let $C \mid S$ be a subsequence such that there exists some $h \in G$ with $jh \in \Sigma_j(C)$ for each $j \le |C|$. If $|C| \ge \lfloor(\exp(G)-1)/2\rfloor$, then S has a zero-sum subsequence of length $\exp(G)$.*
2. *Let S be a sequence over G of length $(\eta(G)-1)+\exp(G)-1$ that does not contain any zero-sum subsequence of length $\exp(G)$. Let $C \mid S$ be a subsequence such that there exists some $h \in G$ with $jh \in \Sigma_j(C)$ for each*

$j \le |C|$. *If* $|C| \ge \lfloor (\exp(G)-1)/2 \rfloor$, *then* $-h+S$ *has a subsequence of length* $\eta(G)-1$ *without any short zero-sum subsequence.*

Proof. Without loss of generality suppose $h = 0$.

(1). Consider SC^{-1}. Let $T \mid SC^{-1}$ be a short zero-sum subsequence (or the empty sequence) of maximal length. If $|T| > \exp(G)/2$, then $|C| \ge \exp(G) - |T|$. Thus, C has a subsequence C' of length $\exp(G) - |T|$ with sum 0. Since $|TC'| = \exp(G)$ and its sum is 0, the argument is complete in this case. Consequently, we can assume that $|T| \le \exp(G)/2$. It follows that $SC^{-1}T^{-1}$ has no short zero-sum subsequence. To see this it suffices to note that a short zero-sum subsequence T' would satisfy $|T'| \le |T| \le \exp(G)$ and thus TT' would also be a short zero-sum subsequence of SC^{-1} contradicting the maximality of T. If $|C| \ge \exp(G) - |T|$, then we get a zero-sum subsequence of length $\exp(G)$ as above. Thus, $|C| + |T| \le \exp(G) - 1$ and thus $|SC^{-1}T^{-1}| \ge \eta(G)$, contradicting the fact that $SC^{-1}T^{-1}$ has no short zero-sum subsequence.

(2). The proof is similar to the first part. Consider SC^{-1}. Let $T \mid SC^{-1}$ be a short zero-sum subsequence (or the empty sequence) of maximal length. If $|T| > \exp(G)/2$, then as above C has a subsequence C' of length $\exp(G) - |T|$ with sum 0, yielding again a zero-sum subsequence of S of length $\exp(G)$, which is a contradiction. Consequently, we have $|T| \le \exp(G)/2$. It follows as above that $SC^{-1}T^{-1}$ has no short zero-sum subsequence. If $|C| \ge \exp(G) - |T|$, then we get a contradiction as above. Thus, $|C| + |T| \le \exp(G) - 1$ and thus $|SC^{-1}T^{-1}| \ge \eta(G) - 1$, establishing our claim. □

5. Proofs of the main results

In this section we prove our Theorems 3.1, 3.2 and 3.4. The proofs of the latter two will rely on the first.

Proof of Theorem 3.1. We start by discussing the lower bounds. If $n = 1$, then [2, Proposition 3.1(3)] yields

$$\eta(C_2 \oplus C_{2m} \oplus C_{2m}) \ge 2 + (2^2 - 1)(2m - 1 + 2 - 1) = 6m + 2,$$

and if $n \ge 2$, then [2, Lemma 3.2] gives

$$\eta(C_2 \oplus C_{2m} \oplus C_{2mn}) \ge 2(\mathsf{D}(C_2 \oplus C_{2m}) - 1) + 2mn = 4m + 2mn.$$

We can now turn to the upper bounds. Let H be a subgroup of G isomorphic to $C_m \oplus C_{mn}$ such that G/H is isomorphic to C_2^3. We apply the inductive method with

$$H \hookrightarrow G \xrightarrow{\pi} G/H.$$

First, suppose $n = 1$, that is, let $G = C_2 \oplus C_{2m} \oplus C_{2m}$. By [12, Proposition 5.7.11] and as $\eta(C_2^3) = 8$ and $\eta(C_m^2) = 3m - 2$, it follows that

$$\eta(G) \le (\eta(H) - 1)\exp(G/H) + \eta(G/H) = (3m - 3)2 + 8 = 6m + 2.$$

Second, suppose $n \ge 2$. Let S be a sequence over G with $|S| = 4m + 2mn$. We have to show that S has a short zero-sum subsequence. Note that applying [12, Proposition 5.7.11] yields only an upper bound of $2mn + 4m + 2$ and a more refined analysis is needed.

Since $4m + 2mn = 2(2m + mn - 4) + \eta(C_2^3)$, it follows that there exist $(2m + mn - 3)$ non-empty and disjoint subsequences of S, say, $S_1 \ldots S_{2m+mn-3} \mid S$, with

$|S_i| \le 2$ and $\sigma(\pi(S_i)) = 0$ for each i. Let T be the subsequence of S such that $S_1 \ldots S_{2m+mn-3}T = S$. We note that $|T| \ge 6$.

We observe that $R = \sigma(S_1) \ldots \sigma(S_{2m+mn-3})$ is a sequence over H of length $\eta(H) - 1$. If R has a zero-sum subsequence of length at most mn, that is one that is short relative to H, we can complete the argument as follows. We note that if $\sum_{i\in I} \sigma(S_i) = 0$ for some $\emptyset \ne I \subseteq [1, 2m + mn - 3]$ with at most mn elements, then $\prod_{i\in I} S_i$ is a non-empty zero-sum subsequence of S of length at most $2|I| \le 2mn$. Thus we assume R does not have a short zero-sum subsequence. This means that R fulfills the conditions of Lemma 4.3. Thus, we get that there exist a proper subgroup K of H and some $k' \in H \setminus K$ such that the complement (in H) of $\Sigma_{\le mn-2}(R) \cup \{0\}$ is contained in $-k' + K$.

We continue by analyzing the sequence T. First, we note that we may assume that $\pi(T)$ does not have a non-empty zero-sum subsequence of length at most 2. Otherwise, let $S_0 \mid T$ with $1 \le |S_0| \le 2$ and we consider the sequence $\sigma(S_0)R$ over H that has length $\eta(H)$. Thus it has a short (relative to H) zero-sum subsequence. Using the same argument as above, this yields a short (relative to G) zero-sum subsequence of S.

Second, somewhat in the same vein, we note that for each $S_0 \mid T$ with $1 \le |S_0| \le 4$ such that $\pi(S_0)$ is a zero-sum sequence we may assume that $\sigma(S_0) \notin -(\Sigma_{\le mn-2}(R) \cup \{0\})$. To see this, just observe that otherwise we would get a non-empty zero-sum subsequence of $\sigma(S_0)R$ of length at most $mn - 1$, which contains $\sigma(S_0)$. This then establishes the existence of a zero-sum subsequence of S of length at most $|S_0| + 2(mn - 2) \le 2mn$, that is, it is short. Therefore, recalling what we know about $\Sigma_{\le mn-2}(R) \cup \{0\}$, we get a short zero-sum subsequence of S unless $\sigma(S_0) \in k' + K$ for each $S_0 \mid T$ with $1 \le |S_0| \le 4$ and $\sigma(\pi(S_0)) = 0$.

It remains to show that $\sigma(S_0) \in k' + K$ for each $S_0 \mid T$ with $1 \le |S_0| \le 4$ and $\sigma(\pi(S_0)) = 0$ is impossible. We know that $|T| \ge 6$ and that $\pi(T)$ consists of distinct non-zero elements, as otherwise we would get a zero-sum subsequence of length at most 2, which we excluded above. Fixing an appropriate independent generating set $\{e_1, e_2, e_3\}$ of G/H we may assume that $\mathrm{supp}(\pi(T))$ contains all non-zero elements except $e_1 + e_2 + e_3$. For $I \subseteq \{1, 2, 3\}$ with two elements, let $e_I = \sum_{i\in I} e_i$. We note that $\pi(T)$ has at least the following zero-sum subsequences of length at most 4: $V_k = e_i e_j e_{\{i,j\}}$, $V_0 = e_{\{1,2\}} e_{\{2,3\}} e_{\{1,3\}}$, and $V_i' = e_{\{i,j\}} e_{\{i,k\}} e_j e_k$ for $\{i, j, k\} = \{1, 2, 3\}$. Let $T_i^{(\prime)}$ denote the subsequence of T whose image under π is $V_i^{(\prime)}$. We want to show that at least one of these sequences $T_i^{(\prime)}$ has a sum that is not in $k' + K$. Assume to the contrary that the sum of each of these sequences is in $k' + K$. Now, note that $V_0V_1V_2V_3 = V_1'V_2'V_3'$ and thus $T_0T_1T_2T_3 = T_1'T_2'T_3'$. However, that yields $\sigma(T_0T_1T_2T_3) \in 4k' + K$ while $\sigma(T_1'T_2'T_3') \in 3k' + K$. Since $k' \notin K$, this is a contradiction, since $k' \notin K$. Thus, $\sigma(S_0) \in k' + K$ for each $S_0 \mid T$ with $1 \le |S_0| \le 4$ and $\sigma(\pi(S_0)) = 0$ is indeed impossible, and consequently S has a short zero-sum subsequence. □

We continue with the proof of our result on the Erdős–Ginzbirg–Ziv constant.

Proof of Theorem 3.2. Since $\mathsf{s}(G) \ge \eta(G) + \exp(G) - 1$ for every finite abelian group G (see the remark before Conjecture 2.2), Theorem 3.1 readily yields the desired lower bounds. We now show that these bounds are indeed optimal. Our strategy is

to obtain a situation in which we can invoke Lemma 4.4 and then apply Theorem 3.1. As already noted, we can always apply Lemma 4.4 if $\exp(G) \le 4$. Thus, we assume $mn > 2$.

Let H be a subgroup of $G \simeq C_2 \oplus C_{2m} \oplus C_{2mn}$ isomorphic to $C_m \oplus C_{mn}$ such that G/H is isomorphic to C_2^3. As before, we apply the inductive method with

$$H \hookrightarrow G \stackrel{\pi}{\to} G/H.$$

For $n = 1$, by [12, Proposition 5.7.11] and since $\mathsf{s}(C_2^3) = 9$ and $\mathsf{s}(C_m^2) = 4m - 3$, it follows that

$$\mathsf{s}(G) \le (\mathsf{s}(H) - 1)\exp(G/H) + \mathsf{s}(G/H) = (4m - 4)2 + 9 = 8m + 1.$$

Now, suppose $n \ge 2$ and m has Property D. Let S be a sequence over G with $|S| = 4m+4mn-1$. Assume for a contradiction that S has no zero-sum subsequence of length $\exp(G) = 2mn$.

Since $4m + 4mn - 1 = 2(2m + 2mn - 5) + \mathsf{s}(C_2^3)$, it follows that there exist $(2m + 2mn - 4)$ disjoint subsequences of S, say, $S_1 \ldots S_{2m+2mn-4} \mid S$ with $|S_i| = 2$ and $\sigma(\pi(S_i)) = 0$ for each i. Let T be the subsequence of S such that $S_1 \ldots S_{2m+2mn-4}T = S$. We note that $|T| = 7$.

We observe that $R = \sigma(S_1) \ldots \sigma(S_{2m+2mn-4})$ is a sequence over H of length $\mathsf{s}(H) - 1$. If R has a zero-sum subsequence of length mn we are done, since $\sum_{i\in I} \sigma(S_i) = 0$ for some $I \subseteq [1, 2m + mn - 4]$ with $|I| = mn$ implies that $\prod_{i\in I} S_i$ is a zero-sum subsequence of S of length $2|I| = 2mn$. Thus, the assumption that S has no zero-sum subsequence of length $2mn$, implies that R does not have a zero-sum subsequence of length mn. Hence R fulfills the conditions of Lemma 4.3.

In addition, we note that if $\pi(T)$ still has a zero-sum subsequence of length 2, then we also get a zero-sum subsequence of S of length $2mn$. Thus, we get that $\pi(T)$ has no zero-sum subsequence of length 2.

We continue by establishing an auxiliary fact.

F. If $g \mid T$ and $h \mid S$ such that $\pi(g) = \pi(h)$, then $h = g$ or S contains a zero-sum subsequence of length $\exp(G)$.

Assume there are distinct g, h with $g \mid T$ and $h \mid S$ such that $\pi(g) = \pi(h)$. Since $\pi(T)$ does not contain a zero-sum subsequence of length 2, it follows that $h \nmid T$ and thus $h \mid S_i$ for some i, say $i = 1$. Then $R' = \sigma(gS_1h^{-1})\sigma(S_2) \ldots \sigma(S_{2m+2mn-4})$ is a sequence over H of length $\mathsf{s}(H) - 1$. If this sequence contains a zero-sum subsequence of length mn, then as above S contains a zero-sum subsequence of length $2mn$. Thus, we know that it does not contain such a subsequence, and since we have $|\gcd(R, R')| \ge \mathsf{s}(H) - 2$, Lemma 4.1 gives that $R = R'$. This means that $\sigma(gS_1h^{-1}) = \sigma(S_1)$, contradicting the assumption that g, h are distinct. This establishes (F).

After these preparations, we proceed to show that the conditions of Lemma 4.4 are satisfied. For $r \in \mathrm{supp}(R)$, let $I_r \subset [1, 2m + mn - 4]$ denote the set of all i such that $\sigma(S_i) = r$ and let $Q_r = \prod_{i\in I_r} S_i$. If for some $i \in I_r$ we have that S_i does not contain two distinct elements, say $S_i = h_i^2$ for some $h_i \in G$, then it is not hard to see that $jh_i \in \Sigma_j(Q_r)$ for every $j \in [1, |Q_r|]$; just note that $\sigma(S_j) = r = 2h_i$ for every $j \in I_r$.

Let $v_r = \mathsf{v}_r(R)$. We have $|Q_r| = 2v_r$. If $v_r \geq (n+1)m/2 - 1$, then we have $|Q_r| \geq mn = \exp(G)/2$. Thus, if for such an r there is some $i \in I_r$ with $S_i = h_i^2$, then by Lemma 4.4 we would get a zero-sum subsequence of S.

Since $\pi(S_i)$ is a zero-sum sequence of length 2 over $G/H \simeq C_2^3$ we have $\pi(S_i) = e^2$ for some $e \in G/H$. If we have $\pi(S_i) = e^2$ for some $e \mid \pi(T)$, then by **F** we know that $S_i = h_i^2$; this is because both elements of S_i are equal to the one corresponding element in T.

Thus, the only situation in which we cannot establish by **F** that there is some $i \in I_r$ with $S_i = h_i^2$ is that $\pi(Q_r) = e_0^{|Q_r|}$ where e_0 is the *unique* element of G/H not in $\pi(T)$.

Let $r, r' \in \operatorname{supp}(R)$ be the two elements with the greatest multiplicity. We know that their respective multiplicities $v_r, v_{r'}$ are at least $(n+1)m/2 - 1$; and if $m = 1$ then $v_r = v_{r'} = n - 1$ (see the results recalled before Theorem 2.4). Suppose $\pi(Q_r) = e_0^{2v_r}$ where e_0 is the unique element of G/H not in $\pi(T)$, and likewise for r'. It suffices to show that there exists some $i \in I_r \cup I_{r'}$ such that $S_i = h_i^2$ for some $h_i \in G$.

We proceed to show this is always the case. Since $mn > 2$, we get that $v_r, v_{r'} \geq 2$. Let $i \in I_r$ and $i' \in I_{r'}$. Say $S_i = s_1 s_2$ and $S_{i'} = s_1' s_2'$. Assume for a contradiction that each of these two sequences consists of two distinct elements. We have $\sigma(S_i) = s_1 + s_2 = r$ as well as $\sigma(S_{i'}) = s_1' + s_2' = r'$.

We can consider instead of $S_i = s_1 s_2$ and $S_{i'} = s_1' s_2'$, the sequences $S_i' = s_1 s_1'$, $S_{i'}' = s_2 s_2'$. We set $S_j' = S_j$ for all $j \notin \{i, i'\}$. Then $R' = \sigma(S_1') \ldots \sigma(S_{2m+2mn-4}')$ is a sequence over H of length $\mathsf{s}(H) - 1$ and it has no zero-sum subsequence of length $\exp(H)$. Since $\operatorname{supp}(R) \subseteq \operatorname{supp}(R')$, the supports are in fact equal. If $m = 1$, it is immediate by the resumts recalled before Theorem 2.4 that $R = R'$. If $m > 1$, then since the multiplicity of each element in R is $m - 1 \pmod m$ and the same must be true for R', it follows again that in fact $R = R'$. This means that $\{\sigma(S_i'), \sigma(S_{i'}')\} = \{r, r'\}$.

Likewise, we can consider instead of $S_i = s_1 s_2$ and $S_{i'} = s_1' s_2'$, the sequences $S_i'' = s_1 s_2'$, $S_{i'}'' = s_2 s_1'$. And we set $S_j'' = S_j$ for all $j \notin \{i, i'\}$. Again, it follows that $R = R''$. This means that $\{\sigma(S_i''), \sigma(S_{i'}'')\} = \{r, r'\}$.

Since $\sigma(S_i') \neq \sigma(S_i'')$, recall that we assumed $s_1' \neq s_2'$, we get $\{\sigma(S_i'), \sigma(S_i'')\} = \{r, r'\}$. Without loss of generality we can assume $\sigma(S_i') = r$. Then, we get $\sigma(S_i'') = r'$ and thus $\sigma(S_{i'}'') = r$. Since also $\sigma(S_i) = r$, we get a contradiction. As $s_1 + s_2 = s_1 + s_1' = s_2 + s_1'$ implies that $s_2 = s_1'$ and $s_1 = s_1'$ and finally $s_1 = s_2$, which is a contradiction. □

We end with the proof of our result on the multiwise Davenport constant.

Proof of Theorem 3.4. When $n \geq 2$, it follows from Theorems 2.7, 2.3 and 3.1 that

$$\mathsf{D}(G) = \mathsf{D}(C_2 \oplus C_{2m}) + (2mn - 1) \quad \text{and} \quad \eta(G) \leq \mathsf{D}(G) + 2mn,$$

so that [12, Theorem 6.1.5(1)] (see also [15] and [1]) yields the desired result.

Now, assume that $n = 1$. If $k = 1$, Theorem 2.7 readily gives

$$\mathsf{D}_1(G) = \mathsf{D}(G) = 2m + 2mn.$$

If $k \geq 2$, let H be a subgroup of G isomorphic to C_m^2 such that G/H is isomorphic to C_2^3. On the one hand, [1, Lemma 3.7] gives $\mathsf{D}_0(C_2^3) = 3$, $k_{\mathsf{D}}(C_2^3) = 2$. By Theorem 2.3,

$$\mathsf{D}_k(C_m^2) = m + km - 1 \geq 2 = k_{\mathsf{D}}(C_2^3).$$

Therefore, we have (the first inequality by [1, Proposition 2.6], which is can be proved using the inductive method)

$$\begin{aligned}
\mathsf{D}_k(C_2 \oplus C_{2m} \oplus C_{2m}) &\leq \mathsf{D}_{\mathsf{D}_k(H)}(G/H) \\
&= \mathsf{D}_{\mathsf{D}_k(C_m^2)}(C_2^3) \\
&= \mathsf{D}_{(m(k+1)-1)}(C_2^3) \\
&= \mathsf{D}_0(C_2^3) + 2(m + km - 1) \\
&= 3 + 2(m + km - 1) \\
&= (2m + 1) + k(2m).
\end{aligned}$$

On the other hand, let $\{e_1, e_2, e_3\}$ be an independent generating set of G such that $\mathrm{ord}(e_1) = 2$ and $\mathrm{ord}(e_2) = \mathrm{ord}(e_3) = 2m$. For $k \geq 2$, we consider the sequence

$$S_k = (e_2 + e_3)^{2(k-1)m-1} e_2^{2m-1} e_3^{2(m-1)} e_1(e_1 + e_2 + e_3)(e_1 + e_2)(e_1 + e_3).$$

We show that S_k does not contain k disjoint non-empty zero-sum subsequences. This yields $\mathsf{D}_k(C_2 \oplus C_{2m} \oplus C_{2m}) > |S_k| = 2m + k(2m)$. We set

$$T = e_2^{2m-1} e_3^{2(m-1)} e_1(e_1 + e_2 + e_3)(e_1 + e_2)(e_1 + e_3) \mid S_k,$$

and

$$U = e_1(e_1 + e_2 + e_3)(e_1 + e_2)(e_1 + e_3) \mid T.$$

Now, suppose we have $A_1 \ldots A_k \mid S_k$ where $A_1, \ldots, A_k$ are k zero-sum subsequences. Since every non-empty zero-sum sequence is a product of minimal zero-sum subsequences, we can assume that A_i is minimal for every $i \in [1, k]$.

For every $i \in [1, k]$, we set $k_i = \mathsf{v}_{e_2+e_3}(A_i) \in [0, 2m]$ and $A_i' = A_i((e_2+e_3)^{k_i})^{-1} \mid T$. If A_i' does not contain any element of U then $A_i' = e_2^{2m-k_i} e_3^{2m-k_i}$ so that $|A_i| = 4m - k_i$. If A_i' contains at least one of the elements of U then it contains exactly two or four of them. Therefore, we obtain the following seven cases.

- $A_i' = e_1(e_1 + e_2) e_2^{2m-(k_i+1)} e_3^{2m-k_i}$, so that $|A_i| = 4m - k_i + 1$
- $A_i' = e_1(e_1 + e_3) e_2^{2m-k_i} e_3^{2m-(k_i+1)}$, so that $|A_i| = 4m - k_i + 1$
- $A_i' = e_1(e_1 + e_2 + e_3) e_2^{2m-(k_i+1)} e_3^{2m-(k_i+1)}$, so that $|A_i| = 4m - k_i$
- $A_i' = (e_1 + e_2)(e_1 + e_3) e_2^{2m-(k_i+1)} e_3^{2m-(k_i+1)}$, so that $|A_i| = 4m - k_i$
- $A_i' = e_1(e_1 + e_2 + e_3)(e_1 + e_2)(e_1 + e_3) e_2^{2m-(k_i+2)} e_3^{2m-(k_i+2)}$, so that $|A_i| = 4m - k_i$.
- $A_i' = (e_1 + e_2)(e_1 + e_2 + e_3) e_2^{2m-(k_i+2)} e_3^{2m-(k_i+1)}$, so that $|A_i| = 4m - k_i - 1$
- $A_i' = (e_1 + e_3)(e_1 + e_2 + e_3) e_2^{2m-(k_i+2)} e_3^{2m-(k_i+1)}$, so that $|A_i| = 4m - k_i - 1$

Note that since $\mathsf{v}_{e_1+e_2+e_3}(S_k) = 1$ and $A_1, \ldots, A_k$ are disjoint, there is at most one $i \in [1, k]$ such that $|A_i| = 4m - k_i - 1$. This yields

$$\begin{aligned}\sum_{i=1}^{k} |A_i| &\geq \sum_{i=1}^{k} (4m - k_i) - 1 \\ &= 2k(2m) - \sum_{i=1}^{k} k_i - 1 \\ &\geq 2k(2m) - \mathsf{v}_{e_2+e_3}(S_k) - 1 \\ &= 2k(2m) - (2(k-1)m - 1) - 1 \\ &= 2m + k(2m) \\ &= |S_k|.\end{aligned}$$

Therefore, we obtain $A_1 \ldots A_k = S_k$. Yet, then $0 = \sigma(A_1 \ldots A_k) = \sigma(S_k) = -e_3$, a contradiction. □

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