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A bound of the β -mixing coefficient for point processes in terms of their intensity functions

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Abstract

We prove a general inequality on β -mixing coefficients of point processes depending uniquely on their n -th order intensity functions. We apply this inequality in the case of determinantal point processes and show that the rate of decay of the β -mixing coefficients of a wide class of DPPs is optimal.

Keywords: lower sum transform, determinantal point processes

1. Introduction

In asymptotic inference for dependent random variables, it is necessary to quantify the dependence between σ -algebras. Some of the first measures of dependence that have been introduced are the alpha-mixing coefficients [28] and the beta-mixing coefficients [29]. They have been used to establish moment inequalities, exponential inequalities and central limit theorems for stochastic processes (see [7, 24, 27] for more details about mixing) with various applications in statistics, see for instance [6, 10]. In this paper, we focus on spatial point processes. As detailed below, for these models, alpha-mixing has been widely studied and exploited in the literature, but not beta-mixing in spite of its stronger properties. In a lesser extent, some alternative measures of dependence have also been used for spatial point processes, namely Brillinger mixing [4, 16] (which only applies to stationary point processes but has been established in [16] under suitable conditions on the β -mixing coefficients) and association [20, 25].

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The main models used in spatial point processes are Gibbs point processes, Cox processes and determinantal point processes, see [23] for a recent review. An α -mixing inequality is established for Gibbs point processes in the Dobrushin uniqueness region in [11]. It has been used to show asymptotic normality of maximum likelihood and pseudo-likelihood estimates [18]. Similarly, some inhomogeneous Cox processes like the Neyman-Scott process have also been showed to satisfy α -mixing inequalities in [31]. These inequalities are at the core of asymptotic inference results in [8, 26, 31]. Finally, an α -mixing inequality has also been showed for determinantal point processes in [25] and used to get the asymptotic normality of a wide class of estimators of these models.

On the other hand, β -mixing is a stronger property than α -mixing. It implies stronger covariance inequalities [27] as well as a coupling theorem known as Berbee's Lemma [3] used in various limit theorems (for example in [2, 30]). Nevertheless, it rarely appears in the literature in comparison to α -mixing. This is especially true for point processes where there has been no β -mixing property established for any of the above examples. Nevertheless, β -mixing coefficients have still been used several times in random geometry and point process statistics [12, 13, 14, 15]. In particular, it is argued in [14] that the β -mixing coefficient cannot be replaced by the α -mixing coefficient when used to obtain bounds for point process characteristics related with the Palm distribution. Our goal is to establish a general inequality for the β -mixing coefficients of a point process in terms of its intensity functions.

We begin in Section 2 by recalling the basic definitions and properties of the α -mixing and β -mixing coefficients and we introduce the lower sum transform which is the main technical tool that we use throughout the paper. Then, a general inequality for the β -mixing coefficients of a point process that depends only on its n -th order intensity functions is proved in Section 3. As an example, we deduce a β -mixing inequality in the special case of determinantal point processes (DPPs) in Section 4 whose rate of decay is optimal for a wide class of DPPs.

2. Preliminaries

2.1. Intensities of point processes

In this paper, we consider simple point processes on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d), \mu)$ equipped with the euclidean norm $\|\cdot\|$ where d is a fixed integer, $\mathcal{B}(\mathbb{R}^d)$ the Borel- σ -algebra and μ the Lebesgue measure (more information on spatial point processes can be found in [9, 22]). We denote by Ω (resp. Ω_F) the set of locally finite (resp. finite) point configurations in \mathbb{R}^d . For all functions $f : \Omega_F \rightarrow \mathbb{R}$, $n \in \mathbb{N}$ and $x = (x_1, \dots, x_n) \in (\mathbb{R}^d)^n$, we write $f(x)$ for $f(\{x_1, \dots, x_n\})$ by an abuse of notation. Finally, we write $|A|$ for the cardinal of a finite set A and $\|f\|_\infty$ for the uniform norm of a function f .

We begin by recalling that the n -th order intensity functions (also called n -th order product density) are defined the following way (see [22]).

Definition 2.1. *Let X be a simple point process on \mathbb{R}^d and $n \geq 1$ be an integer. If there exists a non negative function $\rho_n : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ such that*

$$\mathbb{E} \left[\sum_{\substack{\neq \\ x_1, \dots, x_n \in X}} f(x_1, \dots, x_n) \right] = \int_{(\mathbb{R}^d)^n} f(x) \rho_n(x) d\mu^n(x). \quad (1)$$

for all locally integrable functions $f : (\mathbb{R}^d)^n \rightarrow \mathbb{R}$ then ρ_n is called the n th order intensity function of X .

In the rest of the paper, all point processes will be considered to admit bounded n -th order intensity function for all $n \geq 1$.

2.2. Mixing

Consider a probability space $(\mathcal{X}, \mathcal{F}, \mathbb{P})$ and \mathcal{A}, \mathcal{B} two sub σ -algebras of \mathcal{F} . Let $\mathbb{P}_{\mathcal{A}}$ and $\mathbb{P}_{\mathcal{B}}$ be the respective restrictions of \mathbb{P} to \mathcal{A} and \mathcal{B} and define the probability $\mathbb{P}_{\mathcal{A} \otimes \mathcal{B}}$ on the product σ -algebra by $\mathbb{P}_{\mathcal{A} \otimes \mathcal{B}}(A \times B) = \mathbb{P}(A \cap B)$ for all $A \in \mathcal{A}$ and $B \in \mathcal{B}$. The α -mixing and β -mixing coefficients (also called strong-mixing and absolute regularity coefficients) are defined as the following measures of dependence between \mathcal{A} and \mathcal{B} [24, 27]:

$$\alpha(\mathcal{A}, \mathcal{B}) := \sup\{|\mathbb{P}(A \cap B) - \mathbb{P}(A)\mathbb{P}(B)| : A \in \mathcal{A}, B \in \mathcal{B}\}, \quad (2)$$

$$\beta(\mathcal{A}, \mathcal{B}) := \|\mathbb{P}_{\mathcal{A} \otimes \mathcal{B}} - \mathbb{P}_{\mathcal{A}} \otimes \mathbb{P}_{\mathcal{B}}\|_{TV}, \quad (3)$$

where $\|\cdot\|_{TV}$ is the total variation of a signed norm.

For a given point process X and a bounded set $A \subset \mathbb{R}^d$, we denote by $\mu(A) := \int_A d\mu(x)$ the volume of A and $\mathcal{E}(A)$ the σ -algebra generated by $X \cap A$. Finally, for all $A, B \subset \mathbb{R}^d$, we write $\text{dist}(A, B)$ for the infimum of $\|y - x\|$ where $(x, y) \in A \times B$. The β -mixing coefficients of the point process X are then defined by

$$\beta_{p,q}(r) := \sup\{\beta(\mathcal{E}(A), \mathcal{E}(B)) : \mu(A) \leq p, \mu(B) \leq q, \text{dist}(A, B) > r\},$$

and we say that the point process X is beta-mixing if $\beta_{p,q}(r)$ vanishes when $r \rightarrow +\infty$ for all $p, q > 0$. The α -mixing coefficients can be defined in a similar way.

Our goal is to prove that under appropriate assumptions over the intensity functions ρ_n of X we have a β -mixing property.

2.3. Lower sum transform

The main tool we use throughout this paper is the so-called lower sum operator (see [1]). Notice that when f is a symmetric function the term in the expectation in (1) can be written as $n! \sum_{Z \subset X} f(Z) \mathbb{1}_{|Z|=n}$. This motivates the following definition:

Definition 2.2. *Let f be a real function defined over Ω_F . The lower sum of f is the linear operator defined by*

$$\hat{f} : X \mapsto \sum_{Z \subset X} f(Z). \quad (4)$$

As shown in Example 4.19 in [1], this operator admits the following inverse transform.

Proposition 2.3 ([1, Theorem 4.18]). *The operator (4) admits an inverse transform \check{f} , called the lower difference of f , defined by*

$$\check{f} : X \mapsto \sum_{Z \subset X} (-1)^{|X \setminus Z|} f(Z). \quad (5)$$

These definitions extend to functions over Ω_F^2 by defining

$$\hat{f} : (X_1, X_2) \mapsto \sum_{Z_1 \subset X_1, Z_2 \subset X_2} f(Z_1, Z_2) \quad \text{and} \quad \check{f} : (X_1, X_2) \mapsto \sum_{Z_1 \subset X_1, Z_2 \subset X_2} (-1)^{|X_1 \setminus Z_1| + |X_2 \setminus Z_2|} f(Z_1, Z_2).$$

In a similar way, we could also extend these definitions to Ω_F^n for any n but we will only need the case $n \leq 2$ for the remaining of the paper. These operators allow us to give an explicit expression for the expectation of a functional of a point process with respect to its intensity functions.

Proposition 2.4. *If X is an almost surely finite point process such that $\mathbb{E}[4^{|X|}] < +\infty$, then*

$$\mathbb{E}[f(X)] = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \check{f}(x) \rho_n(x) d\mu^n(y) \quad (6)$$

for all bounded functions $f : \Omega_F \rightarrow \mathbb{R}$. Moreover, if X' is a point process independent from X satisfying the same assumptions than X and with n -th order intensity functions ρ'_n , then

$$\mathbb{E}[f(X, X')] = \sum_{m,n=0}^{+\infty} \frac{1}{m!n!} \int_{(\mathbb{R}^d)^{m+n}} \check{f}(x, y) \rho_m(x) \rho'_n(y) d\mu^m(x) d\mu^n(x) \quad (7)$$

for all bounded functions $f : \Omega_F^2 \rightarrow \mathbb{R}$.

Proof. Using the bound $|\check{f}(x)| \leq \|f\|_\infty \text{card}\{Z, Z \subset X\} = \|f\|_\infty 2^{|x|}$ we get

$$\sum_{n \geq 0} \mathbb{E} \left[\left| \sum_{\substack{Z \subset X \\ |Z|=n}} \check{f}(Z) \right| \right] \leq \sum_{n \geq 0} \mathbb{E} \left[2^{|X|} \binom{|X|}{n} \right] \|f\|_\infty = \|f\|_\infty \mathbb{E} [4^{|X|}] < +\infty. \quad (8)$$

Since we can write

$$f(X) = \hat{f}(X) = \sum_{Z \subset X} \check{f}(Z) = \sum_{n \geq 0} \sum_{\substack{Z \subset X \\ |Z|=n}} \check{f}(Z) \quad \text{a.s.},$$

then

$$\mathbb{E}[f(X)] = \sum_{n \geq 0} \mathbb{E} \left[\sum_{\substack{Z \subset X \\ |Z|=n}} \check{f}(Z) \right] = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{(\mathbb{R}^d)^n} \check{f}(x) \rho_n(x) d\mu^n(x)$$

where the inversion of the first sum and the expectation is a consequence of (8). Similarly, for all functions $f : \Omega_F^2 \rightarrow \mathbb{R}$ we have

$$\begin{aligned} \mathbb{E}[f(X, X')] &= \mathbb{E}[\mathbb{E}[f(X, X')|X']] \\ &= \mathbb{E} \left[\sum_{m=0}^{+\infty} \frac{1}{m!} \int_{(\mathbb{R}^d)^m} \left(\sum_{z \subset x} (-1)^{m-|z|} f(z, X') \right) \rho_m(x) d\mu^m(x) \right] \\ &= \sum_{m,n=0}^{+\infty} \frac{1}{m!n!} \int_{(\mathbb{R}^d)^{m+n}} \check{f}(x, y) \rho_m(x) \rho'_n(y) d\mu^m(x) d\mu^n(y), \end{aligned}$$

where all inversions of expectation with sum and integrals can be justified in a similar way than (8). \square

3. β -mixing of point processes with known intensity functions

Our main result is the following inequality showing that if all $\rho_m(x)\rho_n(y) - \rho_{m+n}(x, y)$ vanish fast enough when $\|y - x\| \rightarrow +\infty$ for all $m, n \in \mathbb{N}$, then the underlying point process is β -mixing.

Theorem 3.1. *Let X be a simple point process on (\mathbb{R}^d, μ) such that $\mathbb{E}[4^{|X \cap A|}] < +\infty$ for all bounded subsets $A \subset \mathbb{R}^d$. Then, for all $p, q, r \in \mathbb{R}_+$,*

$$\beta_{p,q}(r) \leq \sup_{\substack{\mu(A) < p, \mu(B) < q \\ \text{dist}(A,B) > r}} \left(\sum_{m,n=0}^{+\infty} \frac{2^{n+m-1}}{m!n!} \int_{A^m \times B^n} |\rho_m(x)\rho_n(y) - \rho_{m+n}(x, y)| d\mu^m(x) d\mu^n(y) \right). \quad (9)$$

Before giving the proof of Theorem 3.1, we need the following lemmas showing the behaviour of $f(X \cap A, X \cap B)$ and $f(X \cap A, X' \cap B)$ under the lower difference operator.

Lemma 3.2. *Let $A \subset \mathbb{R}^d$, $f : \Omega_F \rightarrow \mathbb{R}$ and define $f_A : X \mapsto f(X \cap A)$. Then,*

$$\widetilde{f}_A(X) = \check{f}(X) \mathbb{1}_{X \subset A}.$$

Proof. If $X \subset A$ then the result is trivial. Otherwise, there exists $x \in X \setminus A$ and we can write

$$\begin{aligned} \widetilde{f}_A(X) &= \sum_{Z \subset X, Z \ni x} (-1)^{|X \setminus Z|} f(Z \cap A) + \sum_{Z \subset X, Z \not\ni x} (-1)^{|X \setminus Z|} f(Z \cap A) \\ &= \sum_{Z \subset X, Z \not\ni x} (-1)^{|X \setminus Z| - 1} f((Z \cup \{x\}) \cap A) + \sum_{Z \subset X, Z \not\ni x} (-1)^{|X \setminus Z|} f(Z \cap A) \\ &= \sum_{Z \subset X, Z \not\ni x} (-1)^{|X \setminus Z| - 1} f(Z \cap A) + \sum_{Z \subset X, Z \not\ni x} (-1)^{|X \setminus Z|} f(Z \cap A) = 0. \end{aligned}$$

\square

This result can be extended to multivariate functions: The lower difference of $(X_1, X_2) \rightarrow f(X_1 \cap A_1, X_2 \cap A_2)$ is $\check{f}(X_1, X_2) \mathbb{1}_{\{X_1 \subset A_1\}} \mathbb{1}_{\{X_2 \subset A_2\}}$.

Lemma 3.3. For all $f : \Omega_F^2 \rightarrow \mathbb{R}$ and A, B disjoint subsets of \mathbb{R}^d , let us define the function $g : X \mapsto f(X \cap A, X \cap B)$. The lower difference of g satisfies

$$\check{g}(X) = \check{f}(X \cap A, X \cap B) \mathbb{1}_{\{X \subset A \cup B\}}.$$

Proof. Using Lemma 3.2 we get that $\check{g}(X) = 0$ whenever X is not a subset of $A \cup B$. Otherwise, since A and B are disjoint sets,

$$\check{g}(X) = \sum_{Z \subset X} (-1)^{|X \setminus Z|} f(Z \cap A, Z \cap B) = \sum_{\substack{U \subset X \cap A \\ V \subset X \cap B}} (-1)^{|(X \cap A) \setminus U| + |(X \cap B) \setminus V|} f(U, V)$$

which, by definition, is equal to $\check{f}(X \cap A, X \cap B)$. \square

We now have the necessary tools required for the proof of Theorem 3.1.

Proof of Theorem 3.1. Let $p, q > 0$ and A, B be two disjoint subsets of \mathbb{R}^d such that $\mu(A) \leq p$ and $\mu(B) \leq q$. Using one of the characterizations of the total variation distance, the β -mixing coefficient between $\mathcal{E}(A)$ and $\mathcal{E}(B)$ can be expressed as

$$\beta(\mathcal{E}(A), \mathcal{E}(B)) = \frac{1}{2} \sup_{\|f\|_\infty=1} |E[f(X \cap A, X \cap B)] - E[f(X \cap A, X' \cap B)]|$$

where X' is an independent copy of X . Since $X \cap A$, $X' \cap B$ and $X \cap B$ are finite a.s. we can apply (7) which, combined with Lemma 3.2, gives us

$$\mathbb{E}[f(X \cap A, X' \cap B)] = \sum_{m,n=0}^{+\infty} \frac{1}{m!n!} \int_{A^m \times B^n} \check{f}(x, y) \rho_m(x) \rho_n(y) d\mu^m(x) d\mu^n(y). \quad (10)$$

On the other hand, by combining (6) with Lemma 3.3, we get

$$\mathbb{E}[f(X \cap A, X \cap B)] = \sum_{n=0}^{+\infty} \frac{1}{n!} \int_{(A \cup B)^n} \check{f}(x \cap A, x \cap B) \rho_n(x) d\mu^n(x).$$

Since A and B are disjoint sets and by symmetry of $\check{f}(x \cap A, x \cap B) \rho_n(x)$, we can simplify the above expression into

$$\begin{aligned} \mathbb{E}[f(X \cap A, X \cap B)] &= \sum_{n=0}^{+\infty} \sum_{m=0}^n \frac{1}{n!} \binom{n}{m} \int_{A^m \times B^{n-m}} \check{f}(x, y) \rho_n(x, y) d\mu^m(x) d\mu^{n-m}(y) \\ &= \sum_{m,n=0}^{+\infty} \frac{1}{m!n!} \int_{A^m \times B^n} \check{f}(x, y) \rho_{m+n}(x, y) d\mu^m(x) d\mu^n(y). \end{aligned} \quad (11)$$

Combining (10) and (11) yields that $|\mathbb{E}[f(X \cap A, X \cap B)] - \mathbb{E}[f(X \cap A, X' \cap B)]|$ is equal to

$$\left| \sum_{m,n=0}^{+\infty} \frac{1}{m!n!} \int_{A^m \times B^n} \check{f}(x,y)(\rho_m(x)\rho_n(y) - \rho_{m+n}(x,y))d\mu^m(x)d\mu^n(y) \right|$$

which is bounded by

$$\sum_{m,n=0}^{+\infty} \frac{2^{n+m}}{m!n!} \int_{A^m \times B^n} |\rho_m(x)\rho_n(y) - \rho_{m+n}(x,y)|d\mu^m(x)d\mu^n(y)$$

when $\|f\|_\infty = 1$ and where we used the bound $|\check{f}(x,y)| \leq 2^{|x|+|y|}$. □

4. Application to determinantal point processes

We can directly apply Theorem 3.1 to determinantal point processes. First introduced in [21] under its current form to model fermion systems, DPPs are a broad class of repulsive point processes. We recall that a DPP X with kernel $K : (\mathbb{R}^d)^2 \rightarrow \mathbb{R}$ is defined by its intensity functions

$$\rho_n(x_1, \dots, x_n) = \det(K[x]) \quad \forall x \in (\mathbb{R}^d)^n, \forall n \in \mathbb{N}$$

where we denote by $K[x]$ the matrix $(K(x_i, x_j))_{1 \leq i, j \leq n}$. Existence and uniqueness conditions as well as general information on DPPs can be found in [17]. The application of Theorem 3.1 to DPPs gives us the following β -mixing condition:

Theorem 4.1. *Let X be a DPP with kernel K and define*

$$\omega(r) := \sup_{\|y-x\| \geq r} |K(x,y)|.$$

If K is bounded and $\omega(r) \xrightarrow{r \rightarrow +\infty} 0$ then X is β -mixing. In particular,

$$\beta_{p,q}(r) \leq 2pq(1 + 2p\|K\|_\infty)(1 + 2q\|K\|_\infty)e^{2\|K\|_\infty(p+q)}\omega(r)^2.$$

Unfortunately, this result does not give a bound for $\beta_{p,\infty}(r)$ which yet is necessary in almost all limit theorems based on beta-mixing.

Proof. Since $\mathbb{E}[4^{|X \cap A|}] < +\infty$ for all bounded sets A (see [25, Lemma B.5]) then the β -mixing coefficients of X satisfy (9) by Theorem 3.1. Let $x = (x_1, \dots, x_n)$ and $y = (y_1, \dots, y_m)$, we need to control $|\det(K[x]) \det(K[y]) - \det(K[x, y])|$ where $\|x - y\| \geq r$. By [25, Lemma B.4], we get the bound

$$0 \leq \det(K[x]) \det(K[y]) - \det(K[x, y]) \leq nm \|K\|_\infty^{n+m-2} \sum_{i=1}^n \sum_{j=1}^m K(x_i, y_j)^2.$$

Injecting this bound into (9) gives us

$$\begin{aligned} \beta_{p,q}(r) &\leq \sum_{n,m=0}^{+\infty} \frac{n^2 m^2 2^{n+m-1} p^{n-1} q^{m-1} \|K\|_\infty^{n+m-2}}{n! m!} \sup_{\substack{|A| < p, |B| < q \\ \text{dist}(A, B) > r}} \int_{A \times B} |K(x, y)|^2 d\mu(x) d\mu(y) \quad (12) \\ &\leq \sum_{n,m=0}^{+\infty} \frac{n^2 m^2 2^{n+m-1} p^n q^m \|K\|_\infty^{n+m-2}}{n! m!} \omega(r)^2 \\ &= 2pq(1 + 2p\|K\|_\infty)(1 + 2q\|K\|_\infty) e^{2(p+q)\|K\|_\infty} \omega(r)^2. \end{aligned}$$

In particular, if $\omega(r)$ vanishes when $r \rightarrow +\infty$ then X is β -mixing. \square

In conclusion, the β -mixing coefficients of DPPs decay at the same rate than $|K(x, y)|^2$ does when x and y deviate from each other. For example, kernels of the Ginibre ensemble or the Gaussian unitary ensemble have an exponential decay (see [17]). Moreover, among translation-invariant kernels used in spatial statistics (see [5, 19]), all kernels of the Laguerre-Gaussian family also have an exponential decay while kernels of the Whittle-Matérn and Cauchy family satisfy $\omega(r) = o(r^{-d})$ and kernels of the Bessel family satisfy $\omega(r) = o(r^{-(d+1)/2})$.

It is also worth noticing that Theorem 3.1 is optimal in the sense that for a wide class of DPPs, the β -mixing coefficients $\beta_{p,q}(r)$ do not decay faster, when r goes to infinity, than the supremum of $\int_{A \times B} |K(x, y)|^2 d\mu(x) d\mu(y)$ for all A, B such that $\mu(A) \leq p$, $\mu(B) \leq q$ and $\text{dist}(A, B) \geq r$ as stated in the following proposition.

Proposition 4.2. *Let X be a DPP with a non-negative bounded kernel K such that the eigenvalues of its associated integral operator are all in $[0, M]$ where $M < 1$. Then, for all*

$p, q, r > 0$,

$$\begin{aligned}
& 2(1 - M)^{\frac{(p+q)\|K\|_\infty}{M}} \sup_{\substack{\mu(A) < p, \mu(B) < q \\ \text{dist}(A, B) > r}} \int_{A \times B} |K(x, y)|^2 d\mu(x) d\mu(y) \leq \beta_{p, q}(r) \\
& \leq 2(1 + 2p\|K\|_\infty)(1 + 2q\|K\|_\infty) e^{2(p+q)\|K\|_\infty} \sup_{\substack{\mu(A) < p, \mu(B) < q \\ \text{dist}(A, B) > r}} \int_{A \times B} |K(x, y)|^2 d\mu(x) d\mu(y).
\end{aligned}$$

Proof. The first inequality is a consequence of the fact that $\beta_{p, q}(r) \geq 2\alpha_{p, q}(r)$ and [25, Proposition 4.3]. The second inequality is equivalent to (12) once the sum has been developed. \square

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