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# GENERALIZED COMPRESSIBLE FLUID FLOWS AND SOLUTIONS OF THE CAMASSA-HOLM VARIATIONAL MODEL

THOMAS GALLOUËT, ANDREA NATALE AND FRANÇOIS-XAVIER VIALARD

ABSTRACT. The Camassa-Holm equation on a domain  $M \subset \mathbb{R}^d$ , in one of its possible multi-dimensional generalizations, describes geodesics on the group of diffeomorphisms with respect to the  $H(\text{div})$  metric. It has been recently reformulated as a geodesic equation for the  $L^2$  metric on a subgroup of the diffeomorphism group of the cone over  $M$ . Using such an interpretation, we are able to develop a theoretical framework to solve the relative boundary value problem. This represents a development of Brenier's approach to solve the same problem but for the incompressible Euler equation. It involves describing the fluid motion using probability measures on the space of paths on the cone, which implies a probabilistic representation of the Jacobian of the flow map and allows one to capture minimizers characterized by both non-injective and non-surjective flows. We prove several fundamental results on our relaxed formulation: existence of solutions; existence of a unique pressure field associated with them; that, for short times, smooth solutions of the Camassa-Holm equations are the unique solutions of our model; that particular non-deterministic solutions emerge naturally as limits of deterministic flows. We also propose a numerical scheme to construct generalized solutions on the cone and present some numerical results illustrating the relation between the generalized Camassa-Holm and incompressible Euler solutions.

## 1. INTRODUCTION

The Camassa-Holm (CH) equation is the geodesic equation for the  $H^1$  metric on the group of diffeomorphisms of the circle or the real line [9]. This can be derived as an approximation for ideal fluid flow with a free boundary in the shallow water regime. In this context, [23] showed that its natural generalization to a higher dimensional domain  $M \subset \mathbb{R}^d$  consists in replacing the  $H^1$  norm with the  $H(\text{div})$  norm. In other words, the CH equation is the Euler-Lagrange equation for the Lagrangian

$$(1.1) \quad l(u) = a \int_M \|u\|^2 d\rho_0 + b \int_M |\text{div}(u)|^2 d\rho_0,$$

where  $u$  is the Eulerian velocity field,  $a, b > 0$  are constants and  $\rho_0$  is the Lebesgue measure on  $M$ . This is a particular instance of a class of right-invariant Lagrangians on the diffeomorphism group of  $M$  considered in [22], which for  $d = 3$  can be written as

$$(1.2) \quad l(u) = a \int_M \|u\|^2 d\rho_0 + b \int_M |\text{div}(u)|^2 d\rho_0 + c \int_M \|\text{curl}(u)\|^2 d\rho_0.$$

Such Lagrangians give rise to several important fluid dynamics models, including the EPDiff equation for the  $H^1$  Sobolev norm of vector fields and the Euler- $\alpha$  model [17, 18], both of which have also been regarded as possible multi-dimensional versions of the CH equation, but also the Hunter-Saxton equation [20] (see also [22], for a discussion on the geometric properties of this equation). Michor and Mumford proved in [28] that the  $H(\text{div})$  Lagrangian (1.1) defines a non-vanishing distance on the diffeomorphism group, in contrast to the  $L^2$  case (i.e. when  $b = c = 0$ ) for which the metric is degenerate. In addition, by a recent result of Jerrard and Maor [21], in dimension  $d \geq 2$  the distance induced by the  $H^s$  metric vanishes if and only if  $s < 1$ . This places the  $H(\text{div})$  case at the boundary of such vanishing distance phenomena and raises the natural question of finding the correct definition for the solutions to the associated boundary value problem, which is the main contribution of this paper.

In one dimension, the CH equation is bi-Hamiltonian and completely integrable. It also possesses soliton solutions named peakons, i.e. non-smooth traveling wave solutions which interact

and collide without changing their shapes. On the real line, these (weak) solutions have the following expression

$$(1.3) \quad u(x, t) = p(t)e^{-|x-q(t)|/\alpha},$$

where  $p(t)$  and  $q(t)$  determine the height and speed of the wave, respectively, and  $\alpha > 0$  is an independent constant determining its width [16]. Peakons always emerge from appropriate smooth initial data satisfying a certain decay property on the real line, yielding therefore a model for wave breaking [12, 29]. In other words, since the emergence of peakons corresponds to blow up in an appropriate norm, strong solutions may have finite existence time. Furthermore, even weak solutions cannot be defined globally [29]; the collision of peakons, for instance, gives an explicit example of finite time breakdown (blow up) of solutions. In this case, at the collision time, the Lagrangian map ceases to be injective and after this, weak solutions are not uniquely defined. The existence of these solutions has deep implications on the CH boundary value problem and requires the introduction of a novel concept of generalized flow which develops the original idea used by Brenier to treat the incompressible Euler case.

**1.1. Generalized incompressible Euler.** The boundary value problem associated with the incompressible Euler equation stems out of Arnold's interpretation of this model's solutions as geodesics on the group of volume-preserving diffeomorphisms [4]. More precisely, the configuration space of the system is given by a subgroup of the diffeomorphism group which consists of all diffeomorphisms  $\varphi$  preserving the Lebesgue measure  $\rho_0$ , i.e. satisfying

$$(1.4) \quad \varphi\#\rho_0 = \rho_0.$$

Incompressible Euler flows are minimizers of the action

$$(1.5) \quad \int_0^T \int_M \frac{1}{2} \|\dot{\varphi}_t\|^2 d\rho_0 dt,$$

subject to (1.4) at all times and with the additional constraint that  $\varphi_0 = \text{Id}$ , the identity map on  $M$ , and  $\varphi_T = h$ , a given diffeomorphism on  $M$ , which prescribes the final position of each particle in  $M$  at the final time  $T$ . Note that the configuration space defined by equation (1.4) can be seen as an isotropy subgroup once we interpret the push-forward as an action of the diffeomorphism group on the space of densities on  $M$ . This point of view establishes a remarkable connection with optimal transport theory [7] (see also [31] for a description of the geometrical connection between the diffeomorphism group and the space of densities).

Shnirelman proved that the infimum of this problem is not generally attained when  $d \geq 3$  and that even when  $d = 2$  there exist final configurations  $h$  which cannot be connected to the identity map with finite action [33]. This motivated Brenier to introduce a relaxation whose solutions are not diffeomorphisms, but rather describe the flow in a probabilistic fashion. More precisely, Brenier defined generalized incompressible flows as probability measures  $\mu$  on  $\Omega(M)$ , the space of continuous curves on the domain  $x : t \in [0, T] \rightarrow x_t \in M$ , satisfying

$$(1.6) \quad (e_t)\#\mu = \rho_0,$$

where  $e_t : \Omega(M) \rightarrow M$  is the evaluation map at time  $t$  defined by  $e_t(x) = x_t$ . In this interpretation, the marginals  $(e_0, e_t)\#\mu$  are probability measures on the product  $M \times M$  and describe how particles move and spread their mass across the domain. Of course, classical deterministic solutions, i.e. curves of volume preserving diffeomorphisms  $t \mapsto \varphi_t$ , also fit in this definition and correspond to the case where the marginals  $(e_0, e_t)\#\mu$  are concentrated on the graph of  $\varphi_t$ . Then, equation (1.6) is the equivalent of the incompressibility constraint in the generalized setting; in fact, when  $\mu$  is deterministic it coincides with (1.4). The minimization problem in terms of generalized flows consists in minimizing the action

$$(1.7) \quad \int_{\Omega(M)} \int_0^T \frac{1}{2} \|\dot{x}_t\|^2 dt d\mu(x)$$

among generalized incompressible flows, with the constraint  $(e_0, e_T)\#\mu = (\text{Id}, h)\#\rho_0$ . Brenier proved that this model is consistent with classical solutions of the incompressible Euler equations [7]. In particular, smooth solutions correspond to the unique minimizers of the generalized problem if the pressure has bounded Hessian and for sufficiently small times. On the other hand, for any coupling there exists a unique pressure, defined as a distribution, associated with

generalized solutions. This result was later improved by Ambrosio and Figalli [1] who showed that the pressure can be actually defined as a function and defined optimality conditions for generalized flows based on this result.

From the numerical side, the connection between Brenier's approach and optimal transport has paved the way for the development of a number of algorithms to simulate generalized flows. This is also due to the emergence of efficient methods for optimal transport problems, in particular those based on entropic regularization [13, 5] and semi-discrete optimal transport [26]. In [6], the authors used entropic regularization to solve the multi-marginal optimal transport problem arising from time discretization of Brenier's relaxation. Semi-discrete schemes for the incompressible Euler problem have been developed in [27] for the boundary value problem and in [15] for the initial value problem.

**1.2. A novel formulation for CH.** In [14], it was proven that the CH equation can be reformulated as a geodesic equation for the  $L^2$  cone metric on a certain isotropy subgroup of the diffeomorphism group of  $M \times \mathbb{R}_{>0}$ . More precisely, CH flows are represented by time dependent maps in the form

$$(1.8) \quad (\varphi, \lambda) : (x, r) \in M \times \mathbb{R}_{>0} \rightarrow (\varphi(x), \lambda(x)r) \in M \times \mathbb{R}_{>0},$$

where  $\varphi : M \rightarrow M$  and  $\lambda : M \rightarrow \mathbb{R}_{>0}$ . This set of maps is a group under composition and is known as the automorphism group of  $M \times \mathbb{R}_{>0}$ . The isotropy subgroup is given by

$$(1.9) \quad \varphi_{\#}(\lambda^2 \rho_0) = \rho_0.$$

Differently from (1.4), this condition does not enforce incompressibility but it relates  $\varphi$  and  $\lambda$  by requiring  $\lambda = \sqrt{\text{Jac}(\varphi)}$ . Therefore, automorphisms satisfying (1.9) provide us with an alternative way to represent diffeomorphisms of  $M$ . Importantly, in this picture we cannot capture the blow up of solutions as induced by peakon collisions, as in this case the Jacobian would locally vanish. In addition, the metric space  $M \times \mathbb{R}_{>0}$  equipped with the cone metric is not complete. We are then led to work with the cone  $\mathcal{C} = (M \times \mathbb{R}_{\geq 0}) / (M \times \{0\})$ , which allows us to represent solutions with vanishing Jacobian by paths on the cone reaching the apex.

The decoupling between the Lagrangian flow map and its Jacobian has also been used in [24] to construct global weak solutions of the CH equation. However, in their case, one continues solutions after the blowup by allowing the square root of the Jacobian to become negative, which does not occur in the formulation described above. It should also be noted that the cone construction has been developed and used extensively in [25] in order to characterize the metric side of the Wasserstein-Fisher-Rao (WFR) distance (which is also called Hellinger-Kantorovich distance) on the space of positive Radon measures. In fact, as noted in [14] this has the same relation to the CH equation as the Wasserstein  $L^2$  distance does to the incompressible Euler equations. In the geodesic problem associated the WFR distance the isotropy subgroup relation in (1.9) is used to prescribe the initial and final density. The resulting problem can then be expressed without recurring to the cone construction, yielding an optimal entropy-transport problem, a widespread form of unbalanced optimal transport based on the Kullback-Leibler divergence [11, 10, 25].

By analogy with the incompressible Euler case, we can try to solve the boundary value problem associated with the CH equation using generalized flows interpreted as probability measures  $\mu$  on the space  $\Omega(\mathcal{C})$  of continuous paths on the cone  $z : t \in [0, T] \rightarrow z_t = [x_t, r_t] \in \mathcal{C}$ . Such a problem consists in minimizing the action

$$(1.10) \quad \int_{\Omega(\mathcal{C})} \int_0^T \|\dot{z}_t\|_{g_c}^2 dt d\mu(z)$$

among generalized flows satisfying appropriate constraints enforcing a generalized version of (1.9) and the coupling between initial and final times, i.e. the boundary conditions. Choosing the correct form for such constraints is not trivial. In particular, it will be evident that enforcing the coupling of points on the cone in the same way Brenier did for incompressible Euler is not appropriate for our case. This will be the point of departure for our investigation and it will lead to a number of results on the CH boundary value problem which provide a comprehensive characterization of its solutions.

**1.3. Contributions and structure of the paper.** In section 2, we introduce the notations and the needed background. Then, in section 3, we provide a detailed description of the  $L^2$  variational formulation of the CH geodesic problem.

The first contribution of this paper is the definition of a generalized version of the CH boundary value problem, for which we prove existence of solutions as generalized compressible flows, intended as probability measures on  $\Omega(\mathcal{C})$ . This can be found in section 4. The main difficulty for this lies on the necessity to work on an unbounded cone domain and on the impossibility to “cut it” without limiting the class of functions that can be represented by the model. This issue is directly linked to the choice of the correct coupling constraint. Guided by the homogeneity of the problem, we will introduce a sufficiently weak coupling constraint in order to gain compactness by representing a sufficiently large class of generalized flows on the cone and consequently prove existence of solutions. Another key result of this section is the decomposition of solutions into two parts: a part which satisfies the boundary conditions in a classical sense and another part which involves paths which start and end at the cone singularity. When this last part is not trivial, it implies the appearance and disappearance of mass in the domain; we refer to such minimizers as singular solutions.

In section 5 we prove that for any boundary conditions, there always exists a unique pressure field defined as a distribution on  $(0, T) \times M$  associated with any given generalized solution, be it singular or not.

In section 6 we prove that smooth solutions of the CH equation are the unique minimizers of our generalized model for sufficiently short times. The result hinges on the fact that if the pressure satisfies a certain pointwise bound, only dependent on the problem time scale, then the singular part of the solution vanishes and we are left with a minimizer satisfying the coupling in a classical sense. Effectively, such a result validates our choice of constraints and proves that our generalized formulation is a relaxation of the CH geodesic problem.

In section 7 we show that, at least in two dimensions, singular solutions emerge naturally from the continuous formulation whenever the displacement induced by the boundary conditions is sufficiently large. In fact, we will provide explicit examples for this type of solutions on the torus in one and two dimensions. We construct approximations for such minimizers using deterministic flows which describe dense formation of shocks, i.e. regions with vanishing Jacobian, and simultaneous formation of fractures, i.e. regions with unbounded Jacobian. The specific construction uses a particular form of peakon collision which arises from the Hunter-Saxton equation and describes the optimal way to compress finite volume to a point at small scales in the CH model. This result gives an interpretation for singular solutions and therefore it justifies our definition of generalized compressible flows.

Finally, in section 8 we construct a numerical scheme based on entropic regularization and Sinkhorn algorithm to compute generalized CH flows. Even if our scheme is not able to represent singular solution, we provide some numerical results in one dimension which illustrate the emergence of peakon-like solutions and the connection with generalized solutions of the incompressible Euler equations.

## 2. NOTATION AND PRELIMINARIES

In this section, we describe the notation and some basic results used throughout the paper. Because of the similarities between our setting and the one of [25], we will adopt a similar notation for the cone construction and the measure theory objects we will employ.

**2.1. Function spaces.** Given two metric spaces  $X$  and  $Y$ , we denote by  $C^0(X; Y)$  the space of continuous functions  $f : X \rightarrow Y$ , and with  $C^0(X)$  the space of real-valued continuous functions  $f : X \rightarrow \mathbb{R}$ . If  $X$  is compact  $C^0(X)$  is a Banach space with respect to the sup norm  $\|\cdot\|_{C^0}$ . The set of Lipschitz continuous function on  $X$  is denoted by  $C^{0,1}(X)$  and the associated norm is given by

$$(2.1) \quad \|f\|_{C^{0,1}} := \sup_{x \in X} |f(x)| + \sup_{x, y \in X, x \neq y} \frac{|f(x) - f(y)|}{d_X(x, y)},$$

where  $d_X$  denotes the distance function on  $X$ . If  $X$  is a manifold, we will denote by  $\text{Diff}(X)$  the group of smooth diffeomorphisms of  $X$ .

**2.2. The cone and metric structures.** Let  $M \subset \mathbb{R}^d$  be a compact domain. Occasionally, we will also consider the case  $M = S_R^1 := \mathbb{R}/2\pi R\mathbb{Z}$  the circle of radius  $R$ , or  $M = T_{R_1, R_2}^1 := S_{R_1}^1 \times S_{R_2}^1$  the torus with radii  $R_1, R_2 > 0$ . We will denote by  $g$  the Euclidean metric tensor on  $M$ , with  $d_M : M \times M \rightarrow \mathbb{R}_{\geq 0}$  the Euclidean distance on  $M$  and with  $\|\cdot\|_g$  the Euclidean norm. We denote by  $\mathcal{C} := (M \times \mathbb{R}_{\geq 0}) / (M \times \{0\})$  the cone over  $M$ . A point on the cone is an equivalence class  $p = [x, r]$ , with equivalence relation given by

$$(2.2) \quad (x_1, r_1) \sim (x_2, r_2) \Leftrightarrow (x_1, r_1) = (x_2, r_2) \text{ or } r_1 = r_2 = 0.$$

The distinguished point of the cone  $[x, 0]$  is the apex of  $\mathcal{C}$  and it is denoted by  $o$ . Every point on the cone different from the apex can be identified with a couple  $(x, r)$  where  $x \in M$  and  $r \in \mathbb{R}_{> 0}$ . Moreover, we fix a point  $\bar{x} \in M$  and we introduce the projections  $\pi_x : \mathcal{C} \rightarrow M$  and  $\pi_r : \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  defined by

$$(2.3) \quad \pi_x([x, r]) = \begin{cases} x & \text{if } r > 0, \\ \bar{x} & \text{if } r = 0, \end{cases} \quad \pi_r([x, r]) = r.$$

We endow the cone with the metric tensor  $g_{\mathcal{C}} = r^2 g + dr^2$  defined on  $M \times \mathbb{R}_{> 0}$ . We denote the associated norm by  $\|\cdot\|_{g_{\mathcal{C}}}$ . We use the superscripts  $g$  and  $g_{\mathcal{C}}$  for differential operators, e.g.,  $\nabla^g$ ,  $\text{div}^g$  and so on, to indicate that they are computed with respect to either one of these metrics. The distance on the cone  $d_{\mathcal{C}} : \mathcal{C} \times \mathcal{C} \rightarrow \mathbb{R}_{\geq 0}$  is given by

$$(2.4) \quad d_{\mathcal{C}}([x_1, r_1], [x_2, r_2])^2 = r_1^2 + r_2^2 - 2r_1 r_2 \cos(d_M(x_1, x_2) \wedge \pi).$$

The closed subset of the cone composed of points below a given radius  $R > 0$  is denoted by  $\mathcal{C}_R$ , or more precisely

$$(2.5) \quad \mathcal{C}_R := \{[x, r] \in \mathcal{C} ; r \leq R\}.$$

Given an interval  $I \subset \mathbb{R}$ , we denote by  $C^0(I; \mathcal{C})$  and  $AC(I; \mathcal{C})$  the spaces of, respectively, continuous and absolutely continuous curves  $z : t \in I \rightarrow z_t \in \mathcal{C}$ . We will generally use the notation

$$(2.6) \quad x : t \in I \rightarrow x_t = \pi_x(z_t) \in M, \quad r : t \in I \rightarrow r_t = \pi_r(z_t) \in [0, +\infty),$$

so that  $z = [x, r]$  and  $z_t = [x_t, r_t]$ . Note that if  $z$  is continuous (resp. absolutely continuous), then so is the path  $r$  but not  $x$ . However,  $x$  is continuous (resp. locally absolutely continuous) when restricted to the open set  $\{t \in I ; r_t > 0\}$ . Then, if we define  $\dot{z} : t \in I \rightarrow \dot{z}_t \in \mathbb{R}^{d+1}$  by

$$(2.7) \quad \dot{z}_t = \begin{cases} (\dot{x}_t, \dot{r}_t) & \text{if } r_t > 0 \text{ and the derivatives exist,} \\ (0, 0) & \text{otherwise,} \end{cases}$$

we have that  $\|\dot{z}_t\|_{g_{\mathcal{C}}}$  coincides for a.e.  $t \in I$  with the metric derivative of  $z$  with respect to the distance  $d_{\mathcal{C}}$  [25]. We denote by  $AC^p(I; \mathcal{C})$  the space of absolutely continuous curves such that  $\|\dot{z}\|_{g_{\mathcal{C}}} \in L^p(I)$ . Then, the following variational formula for the distance function holds

$$(2.8) \quad d_{\mathcal{C}}(p, q)^2 = \inf \left\{ \int_0^1 \|\dot{z}_t\|_{g_{\mathcal{C}}}^2 dt ; z \in AC^2([0, 1]; \mathcal{C}), z_0 = p, z_1 = q \right\}.$$

We will extensively use the class of homogeneous functions on the cone defined as follows. A function  $f : \mathcal{C}^n \rightarrow \mathbb{R}$  is  $p$ -homogeneous (in the radial direction) if for any constant  $\lambda > 0$  and for all  $n$ -tuples  $([x_1, r_1], \dots, [x_n, r_n]) \in \mathcal{C}^n$ ,

$$(2.9) \quad f([x_1, \lambda r_1], \dots, [x_n, \lambda r_n]) = \lambda^p f([x_1, r_1], \dots, [x_n, r_n]).$$

In particular, a  $p$ -homogeneous function  $f : \mathcal{C} \rightarrow \mathbb{R}$  satisfies  $f([x, \lambda r]) = \lambda^p f([x, r])$ . Similarly, a functional  $\sigma : C^0(I; \mathcal{C}) \rightarrow \mathbb{R}$  is  $p$ -homogeneous if for any constant  $\lambda > 0$  and for any path  $z \in C^0(I; \mathcal{C})$ ,

$$(2.10) \quad \sigma(t \mapsto [x_t, \lambda r_t]) = \lambda^p \sigma(z),$$

where  $z : t \in I \rightarrow [x_t, r_t] \in \mathcal{C}$ .

**2.3. Measure theoretic background.** Let  $X$  be a Polish space, i.e. a complete and separable metric space. We denote by  $\mathcal{M}(X)$  the set of non-negative and finite Borel measures on  $X$ . The set of probability measures on  $X$  is denoted by  $\mathcal{P}(X)$ . Let  $Y$  be another Polish space and  $F : X \rightarrow Y$  a Borel map. Given a measure  $\mu \in \mathcal{M}(X)$  we denote by  $F_{\#}\mu \in \mathcal{M}(Y)$  the push-forward measure defined by  $(F_{\#}\mu)(A) := \mu(F^{-1}(A))$  for any Borel set  $A \subset Y$ . Given a Borel set  $B \subset X$  we let  $\mu \llcorner B$  the restriction of  $\mu$  to  $B$  defined by  $\mu \llcorner B(C) := \mu(B \cap C)$  for any Borel set  $C \subseteq X$ . Note that we will generally use bold symbols to denote measures on product spaces, e.g.,  $\boldsymbol{\mu} \in \mathcal{M}(X \times \dots \times X)$ .

We endow  $\mathcal{P}(X)$  with the topology induced by narrow convergence, which is the convergence in duality with the space of real-valued continuous bounded functions  $C_b^0(X)$ . In other words, a sequence  $\mu_n \in \mathcal{P}(X)$ ,  $n \in \mathbb{N}$ , is said to converge narrowly to  $\mu \in \mathcal{P}(X)$  if for any  $f \in C_b^0(X)$

$$(2.11) \quad \lim_{n \rightarrow +\infty} \int_X f \, d\mu_n = \int_X f \, d\mu.$$

In practice, however, to check for narrow convergence it is sufficient to verify equation (2.11) for all bounded Lipschitz continuous functions. With such a topology,  $\mathcal{P}(X)$  can be identified with a subset of  $[C_b^0(X)]^*$  with the weak-\* topology (see Remark 5.1.2 in [3]). In addition, given a lower semi-continuous function  $f : X \rightarrow \mathbb{R} \cup \{+\infty\}$ , the functional  $\mathcal{F} : \mathcal{P}(X) \rightarrow \mathbb{R} \cup \{+\infty\}$  defined by

$$(2.12) \quad \mathcal{F}(\mu) := \int_X f \, d\mu$$

is also lower-semicontinuous (see Lemma 1.6 in [32]).

As usual in this setting, we will use Prokhorov's theorem for a characterization of compact subsets of  $\mathcal{P}(X)$  endowed with the narrow topology.

**Theorem 2.1** (Prokhorov's theorem). *A set  $\mathcal{K} \subset \mathcal{P}(X)$  is relatively sequentially compact in  $\mathcal{P}(X)$  if and only if it is tight, i.e. for any  $\epsilon > 0$  there exists a compact set  $K_\epsilon \subset X$  such that  $\mu(X \setminus K_\epsilon) < \epsilon$  for any  $\mu \in \mathcal{K}$ .*

We also need a criterion to pass to the limit when computing integrals of unbounded functions: for this will use the concept of uniform integrability. Given a set  $\mathcal{K} \subset \mathcal{P}(X)$ , we say that a Borel function  $f : X \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  is uniformly integrable with respect to  $\mathcal{K}$  if for any  $\epsilon > 0$  there exists a  $k > 0$  such that, for any  $\mu \in \mathcal{K}$ ,

$$(2.13) \quad \int_{f(x) > k} f(x) \, d\mu(x) < \epsilon.$$

**Lemma 2.2** (Lemma 5.1.17 in [3]). *Let  $\{\mu_n\}_{n \in \mathbb{N}}$  be a sequence in  $\mathcal{P}(X)$  narrowly convergent to  $\mu \in \mathcal{P}(X)$  and let  $f \in C^0(X)$ . If  $f$  is uniformly integrable with respect to the set  $\{\mu_n\}_{n \in \mathbb{N}}$  then*

$$(2.14) \quad \lim_{n \rightarrow +\infty} \int_X f \, d\mu_n = \int_X f \, d\mu.$$

For a fixed  $T > 0$ , we will denote by  $\Omega(X) := C^0([0, T]; X)$  the space of continuous paths on  $X$ . This is a Polish space so that we can use the tools introduced in this section also for probability measures  $\boldsymbol{\mu} \in \mathcal{P}(\Omega(X))$ . We call such probability measures *generalized flows* or also *dynamic plans*. When  $X = \mathcal{C}$ , where  $\mathcal{C}$  is the cone over the compact domain  $M \subset \mathbb{R}^d$ , we will often use  $\Omega$  to denote  $\Omega(\mathcal{C})$ .

Since we will work with homogeneous functions on the cone, we also introduce the space of probability measures  $\mathcal{P}_p(X)$ , for  $p > 0$ , defined by

$$(2.15) \quad \mathcal{P}_p(X) := \left\{ \mu \in \mathcal{P}(X); \int_X d_X(x, \bar{x})^p \, d\mu(x) < +\infty \text{ for some } \bar{x} \in X \right\}.$$

Then, if  $\boldsymbol{\mu} \in \mathcal{P}_p(\mathcal{C}^n)$  it is easy to verify that any locally-bounded  $p$ -homogeneous function on  $\mathcal{C}^n$  is  $\boldsymbol{\mu}$ -integrable.

Finally, we will denote by  $\rho_0$  the Lebesgue measure on  $M$  normalized so that  $\rho_0(M) = 1$ .



## 3. THE VARIATIONAL FORMULATION ON THE CONE

In this section we describe the geometric structure of the CH equation using the group of automorphisms of the cone. Such a formulation was introduced in [14] and it was used to interpret the CH equation as an incompressible Euler equations on the cone. In fact, in itself it is similar to that of the incompressible Euler equations originally considered by Arnold [4]. In this section we will only focus on smooth solutions, but we will later use the variational interpretation presented here to guide the construction of generalized solutions of the CH equation. We will keep the discussion formal at this stage and we will use some standard geometric tools and notation commonly adopted in similar contexts.

Consider a compact smooth domain  $M \subset \mathbb{R}^d$ . For any  $\varphi \in \text{Diff}(M)$  and  $\lambda \in C^\infty(M; \mathbb{R}_{>0})$ , we let  $(\varphi, \lambda) : \mathcal{C} \rightarrow \mathcal{C}$  be the map defined by  $(\varphi, \lambda)([x, r]) = [\varphi(x), \lambda(x)r]$ . The automorphism group  $\text{Aut}(\mathcal{C})$  is the collection of such maps, i.e.

$$(3.1) \quad \text{Aut}(\mathcal{C}) = \{(\varphi, \lambda) : \mathcal{C} \rightarrow \mathcal{C}; \varphi \in \text{Diff}(M), \lambda \in C^\infty(M; \mathbb{R}_{>0})\}.$$

The group composition law is given by

$$(3.2) \quad (\varphi, \lambda) \cdot (\psi, \mu) = (\varphi \circ \psi, (\lambda \circ \psi)\mu),$$

the identity element is  $(\text{Id}, 1)$ , where  $\text{Id}$  is the identity map on  $M$ , and the inverse is given by

$$(3.3) \quad (\varphi, \lambda)^{-1} = (\varphi^{-1}, \lambda^{-1} \circ \varphi^{-1}).$$

The tangent space of  $\text{Aut}(\mathcal{C})$  at  $(\varphi, \lambda)$  is denoted by  $T_{(\varphi, \lambda)}\text{Aut}(\mathcal{C})$ . This is the set of tangent vectors

$$(3.4) \quad (\dot{\varphi}, \dot{\lambda}) = \left. \frac{d}{dt} \right|_{t=0} (\varphi_t, \lambda_t),$$

where  $t \mapsto (\varphi_t, \lambda_t)$  is a curve on  $\text{Aut}(\mathcal{C})$  defined on an open interval around 0 and satisfying  $(\varphi_0, \lambda_0) = (\varphi, \lambda)$ . The tangent space  $T_{(\varphi, \lambda)}\text{Aut}(\mathcal{C})$  can be identified with the space of vector fields  $C^\infty(M, \mathbb{R}^{d+1})$ . The collection all the tangent spaces is the tangent bundle  $T\text{Aut}(\mathcal{C})$ .

We endow  $T\text{Aut}(\mathcal{C})$  with the  $L^2(M; \mathcal{C})$  metric inherited from  $g_{\mathcal{C}}$ . This is defined as follows: given  $(\dot{\varphi}, \dot{\lambda}) \in T_{(\varphi, \lambda)}\text{Aut}(\mathcal{C})$ ,

$$(3.5) \quad \|(\dot{\varphi}, \dot{\lambda})\|_{L^2(M; \mathcal{C})}^2 := \int_M (\lambda^2 \|\dot{\varphi}\|_g^2 + \dot{\lambda}^2) d\rho_0,$$

where  $\|\cdot\|_g$  is the norm on  $M$  associated with  $g$  and  $\rho_0$  is the Lebesgue measure on  $M$ .

In [14] the authors found that the CH equation on  $M$  coincides with the geodesic equation on the subgroup  $\text{Aut}_{\rho_0}(\mathcal{C}) \subset \text{Aut}(\mathcal{C})$  defined as follows:

$$(3.6) \quad \text{Aut}_{\rho_0}(\mathcal{C}) := \{(\varphi, \lambda) \in \text{Aut}(\mathcal{C}); \varphi_{\#}(\lambda^2 \rho_0) = \rho_0\}.$$

In other words, the group  $\text{Aut}_{\rho_0}(\mathcal{C})$  can be regarded as the configuration space for the CH equation in the same way as the  $\text{Diff}_{\rho_0}(M)$  is the configuration space for the incompressible Euler equations, with

$$(3.7) \quad \text{Diff}_{\rho_0}(M) := \{\varphi \in \text{Diff}(M); \varphi_{\#} \rho_0 = \rho_0\}.$$

In order to see this, we first observe that the  $L^2(M; \mathcal{C})$  metric is right invariant when restricted to  $\text{Aut}_{\rho_0}(\mathcal{C})$ , meaning that it does not change when moving on this subgroup by right translations. In particular, for any  $(\psi, \vartheta) \in \text{Aut}_{\rho_0}(\mathcal{C})$ , consider the right translation map  $R_{(\psi, \vartheta)} : \text{Aut}_{\rho_0}(\mathcal{C}) \rightarrow \text{Aut}_{\rho_0}(\mathcal{C})$  defined by  $R_{(\psi, \vartheta)}(\varphi, \lambda) = (\varphi, \lambda) \cdot (\psi, \vartheta)$ . Its tangent map at  $(\varphi, \lambda)$  is given by

$$(3.8) \quad TR_{(\psi, \vartheta)}(\dot{\varphi}, \dot{\lambda}) = (\dot{\varphi} \circ \psi, (\dot{\lambda} \circ \psi)\vartheta).$$

Then,

$$(3.9) \quad \begin{aligned} \|TR_{(\psi, \vartheta)}(\dot{\varphi}, \dot{\lambda})\|_{L^2(M; \mathcal{C})}^2 &= \int_M (\lambda^2 \circ \psi \vartheta^2 \|\dot{\varphi}\|_g^2 \circ \psi + \dot{\lambda}^2 \circ \psi \vartheta^2) d\rho_0 \\ &= \int_M (\lambda^2 \|\dot{\varphi}\|_g^2 + \dot{\lambda}^2) \circ \psi \vartheta^2 d\rho_0 \\ &= \int_M (\lambda^2 \|\dot{\varphi}\|_g^2 + \dot{\lambda}^2) d\psi_{\#}(\vartheta^2 \rho_0) \\ &= \|(\dot{\varphi}, \dot{\lambda})\|_{L^2(M; \mathcal{C})}^2. \end{aligned}$$



Geodesics on  $\text{Aut}_{\rho_0}(\mathcal{C})$  correspond to stationary paths on  $T\text{Aut}_{\rho_0}(\mathcal{C})$  for the action functional

$$(3.10) \quad \int_0^T L((\varphi, \lambda), (\dot{\varphi}, \dot{\lambda})) dt$$

for a given  $T > 0$ , where the Lagrangian  $L((\varphi, \lambda), (\dot{\varphi}, \dot{\lambda})) = \|(\dot{\varphi}, \dot{\lambda})\|_{L^2(M; \mathcal{C})}^2$ . The invariance of the metric implies that the geodesic equation can be expressed in terms of right trivialized (Eulerian) velocities only, or in other words in terms of the variables

$$(3.11) \quad (u, \alpha) = TR_{(\varphi, \lambda)^{-1}}(\dot{\varphi}, \dot{\lambda}) = (\dot{\varphi} \circ \varphi^{-1}, (\dot{\lambda} \lambda^{-1}) \circ \varphi^{-1}).$$

Now, the constraint  $\varphi_{\#}(\lambda^2 \rho) = \rho$  can be rewritten as  $\lambda = \sqrt{\text{Jac}(\varphi)}$ . Moreover, we have that for any  $f \in C^\infty(M)$ ,

$$(3.12) \quad \begin{aligned} \frac{d}{dt} \int_M f d\varphi_{\#}(\lambda^2 \rho_0) &= \int_M g(\nabla^g f \circ \varphi, \dot{\varphi}) \lambda^2 d\rho_0 + \int_M 2\lambda \dot{\lambda} f \circ \varphi d\rho_0 \\ &= \int_M (g(\nabla^g f, u) + 2\alpha f) \circ \varphi \lambda^2 d\rho_0 \\ &= \int_M (-\text{div}^g u + 2\alpha) f d\rho_0. \end{aligned}$$

Hence the constraint becomes  $2\alpha = \text{div}^g u$  in terms of Eulerian variables. Moreover, by right invariance,

$$(3.13) \quad L((\varphi, \lambda), (\dot{\varphi}, \dot{\lambda})) = L((\text{Id}, 1), (u, \alpha)) = \int_M \|u\|_g^2 + \frac{1}{4}(\text{div}^g u)^2 d\rho_0,$$

which is the Lagrangian for the CH equation. Note that the coefficient  $1/4$  is directly related to the choice of  $g_{\mathcal{C}}$  as cone metric. Using different coefficients in  $g_{\mathcal{C}}$  we can obtain the general form of the Lagrangian in equation (1.1).

In order to compute the geodesic equation we consider the following augmented Lagrangian

$$(3.14) \quad L((\varphi, \lambda), (\dot{\varphi}, \dot{\lambda})) = \int_M (\lambda^2 \|\dot{\varphi}\|_g^2 + \dot{\lambda}^2) d\rho_0 - \int_M P d(\varphi_{\#}(\lambda^2 \rho_0) - \rho_0),$$

where  $P : M \rightarrow \mathbb{R}$  is the Lagrange multiplier enforcing the constraint. Taking variations we obtain

$$(3.15) \quad \delta L = \int_M (2\lambda \delta \lambda \|\dot{\varphi}\|_g^2 + 2\lambda^2 g(\dot{\varphi}, \delta \dot{\varphi}) + 2\dot{\lambda} \delta \dot{\lambda}) d\rho_0 - \int_M (g(\nabla^g P \circ \varphi, \delta \varphi) \lambda^2 + 2P \circ \varphi \lambda \delta \lambda) d\rho_0.$$

Hence the Euler-Lagrange equations associated with  $L$  read as follows

$$(3.16) \quad \begin{cases} \lambda \ddot{\varphi} + 2\dot{\lambda} \dot{\varphi} + \frac{1}{2} \lambda \nabla^g P \circ \varphi = 0, \\ \ddot{\lambda} - \lambda \|\dot{\varphi}\|_g^2 + \lambda P \circ \varphi = 0, \end{cases}$$

which can be expressed in terms of  $(u, \alpha)$  via right trivialization, yielding

$$(3.17) \quad \begin{cases} \dot{u} + \nabla_u^g u + 2u\alpha = -\frac{1}{2} \nabla^g P, \\ \dot{\alpha} + u \cdot \nabla \alpha + \alpha^2 - \|v\|_g^2 = -P. \end{cases}$$

Using the relation  $\alpha = \text{div}^g(u)/2$ , finally gives us the CH equation for  $u$ .

**Remark 3.1.** Note that in the literature for the CH equation the ‘‘pressure field’’ is sometimes defined in a different way so that, when  $M$  is one-dimensional, the first equation in (3.17) can be written as

$$(3.18) \quad \partial_t u + u \partial_x u = -\partial_x p,$$

for an appropriate function  $p$  (see, e.g., [19]). Throughout the paper we will instead intend by pressure the Lagrange multiplier  $P$  considered above.

## 4. THE GENERALIZED CH FORMULATION

In view of the interpretation of the CH equation as geodesic flow on  $\text{Aut}_{\rho_0}(\mathcal{C})$ , we now turn our attention to the following minimization problem:

**Problem 4.1** (Deterministic CH flow problem). *Given a diffeomorphism  $h \in \text{Diff}(M)$ , find a curve  $t \in [0, T] \mapsto (\varphi_t, \lambda_t) \in \text{Aut}_{\rho_0}(\mathcal{C})$  satisfying*

$$(4.1) \quad (\varphi_0, \lambda_0) = (\text{Id}, 1), \quad (\varphi_T, \lambda_T) = (h, \sqrt{\text{Jac}(h)}),$$

and minimizing the action in equation (3.10).

There is a remarkable analogy between this problem and its equivalent version for the incompressible Euler equations. This raises the question of whether we can define minimizers using the concept of generalized flow which Brenier used for the Euler case. In this section we address this question by formulating the generalized CH flow problem, proving existence of solutions and discussing their nature. In the following the Lebesgue measure on the base space  $M$  is renormalized in such a way that  $\rho_0(M) = 1$ .

By generalized flow or dynamic plan we mean a probability measure on the space of continuous paths of the cone  $\mu \in \mathcal{P}(\Omega)$ . This is a generalization for curves on the automorphism group since for any smooth curve  $(\varphi, \lambda) : t \in [0, T] \rightarrow (\varphi_t, \lambda_t) \in \text{Aut}_{\rho_0}(\mathcal{C})$ , we can associate the generalized flow  $\mu$  defined by

$$(4.2) \quad \mu = (\varphi, \lambda)_{\#} \rho_0.$$

More explicitly, for any Borel functional  $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ ,

$$(4.3) \quad \int_{\Omega} \mathcal{F}(z) d\mu(z) = \int_M \mathcal{F}([\varphi(x), \lambda(x)]) d\rho_0(x),$$

where  $[\varphi(x), \lambda(x)] : t \in [0, T] \rightarrow [\varphi_t(x), \lambda_t(x)] \in \mathcal{C}$ .

The condition  $(\varphi_t)_{\#} \lambda_t^2 \rho_0 = \rho_0$  is equivalent to requiring  $\lambda : t \in [0, T] \rightarrow \lambda_t := \sqrt{\text{Jac}(\varphi_t)} \in C^\infty(M; \mathbb{R}_{>0})$ . We want to generalize this condition for arbitrary  $\mu \in \mathcal{P}(\Omega)$ . Let  $e_t : \Omega \rightarrow \mathcal{C}$  be the evaluation map at time  $t \in [0, T]$ . Then, if  $\mu$  is defined as in (4.2), we have

$$(4.4) \quad \mathfrak{h}_t^2(\mu) := (\pi_x)_{\#} [r^2(e_t)_{\#} \mu] = \rho_0.$$

In fact, for any  $f \in C^0(M)$ ,

$$(4.5) \quad \begin{aligned} \int_M f d\mathfrak{h}_t^2(\mu) &= \int_{\Omega} f(x_t) r_t^2 d\mu(z) \\ &= \int_{\Omega} f(x_t) r_t^2 d(\varphi, \lambda)_{\#} \rho_0 \\ &= \int_M f \circ \varphi_t \lambda_t^2 d\rho_0 \\ &= \int_M f d(\varphi_t)_{\#} \lambda_t^2 \rho_0 \\ &= \int_M f d\rho_0, \end{aligned}$$

where for any path  $z$  and any time  $t$ ,  $x_t := \pi_x(z_t)$  and  $r_t := \pi_r(z_t)$ . By similar calculations, we also obtain

$$(4.6) \quad (e_0, e_T)_{\#} \mu = \gamma := [(\varphi_0, \lambda_0), (\varphi_T, \lambda_T)]_{\#} \rho_0.$$

In other words, enforcing the boundary conditions in the generalized setting boils down to constraining a certain marginal of  $\mu$  to coincide with a given *coupling plan*  $\gamma$  on the cone, i.e. a probability measure in  $\mathcal{P}(\mathcal{C} \times \mathcal{C})$ .

Consider now the energy functional  $\mathcal{A} : \Omega \rightarrow \mathbb{R}_{\geq 0} \cup \{+\infty\}$  defined by

$$(4.7) \quad \mathcal{A}(z) := \begin{cases} \int_0^T \|\dot{z}_t\|_{g_C}^2 dt & \text{if } z \in AC^2([0, T]; \mathcal{C}), \\ +\infty & \text{otherwise.} \end{cases}$$

Setting  $\mathcal{F}(z) = \mathcal{A}(z)$  in (4.3) we obtain the action for the CH equation expressed in Lagrangian coordinates. This motivates the following definition for the generalized CH flow problem.

**Problem 4.2** (Generalized CH flow problem). *Given a coupling plan on the cone  $\gamma \in \mathcal{P}_2(\mathcal{C}^2)$ , find the dynamic plan  $\mu \in \mathcal{P}(\Omega)$  satisfying: the homogeneous coupling constraint*

$$(4.8) \quad \int_{\Omega} f(z_0, z_T) d\mu(z) = \int_{\mathcal{C}^2} f d\gamma,$$

for all 2-homogeneous continuous functions  $f : \mathcal{C}^2 \rightarrow \mathbb{R}$ ; the homogeneous marginal constraint

$$(4.9) \quad \int_{\Omega} \int_0^T f(t, x_t) r_t^2 dt d\mu(z) = \int_M \int_0^T f(t, x) dt d\rho_0(x) \quad \forall f \in C^0([0, T] \times M);$$

and minimizing the action

$$(4.10) \quad \mathcal{A}(\mu) := \int_{\Omega} \mathcal{A}(z) d\mu(z).$$

We remark three basic facts on this formulation:

- we substituted the constraint in (4.4) by its integral version in equation (4.9) as this form will be easier to manipulate in the following. However, the two formulations are equivalent when restricting to generalized flows with finite action (see lemma 4.3);
- we replaced the strong coupling constraint (4.6) by a weaker version, which is always implied by the former as long as  $\gamma \in \mathcal{P}_2(\mathcal{C}^2)$  and in particular when  $\gamma$  is deterministic, i.e. when it is induced by a diffeomorphism as in equation (4.6);
- we allow for general coupling plans in  $\mathcal{P}_2(\mathcal{C}^2)$  so that the integral on the right-hand side of equation (4.8) is finite. However, we will mostly be interested in the case where the coupling is deterministic.

The first of the points above is made explicit in the following lemma, whose proof is postponed to the appendix.

**Lemma 4.3.** *For any generalized flow  $\mu$  with  $\mathcal{A}(\mu) < +\infty$  and satisfying the homogeneous coupling constraint in equation (4.8), the homogeneous marginal constraint in equation (4.9) is equivalent to the constraint*

$$(4.11) \quad \mathfrak{h}_t^2(\mu) = \rho_0$$

for all  $t \in [0, T]$ .

The main result of this section is contained in the following proposition, which states that generalized CH flows are well-defined as solutions of problem (4.2).

**Proposition 4.4** (Existence of minimizers). *Provided that there exists a dynamic plan  $\mu^*$  such that  $\mathcal{A}(\mu^*) < +\infty$ , the minimum of the action in problem 4.2 is attained.*

Before providing the proof of proposition 4.4, we introduce a useful rescaling operation which will allow us to preserve the homogenous constraint when passing to the limit using sequences of narrowly convergent dynamic plans. Such an operation was introduced in [25] in order to deal with the analogous problem arising from the formulation of optimal entropy-transport (i.e. unbalanced transport) on the cone. Adapting the notation in [25] to our setting, we define for a functional  $\theta : \Omega \rightarrow \mathbb{R}$ ,

$$(4.12) \quad \text{prod}_{\theta}(z) := (t \in [0, T] \mapsto [x_t, r_t/\theta(z)]).$$

Then, given a dynamic plan  $\mu$ , if  $\theta(z) > 0$  for  $\mu$ -almost any path  $z$ , we can define the dilation map

$$(4.13) \quad \text{dil}_{\theta,2}(\mu) := \text{prod}_{\theta\#}(\theta^2\mu).$$

Since the constraint in equation (4.9) is 2-homogeneous in the radial coordinate  $r$ , it is invariant under the dilation map, meaning that if  $\mu$  satisfies (4.9), also  $\text{dil}_{\theta,2}(\mu)$  does. For the same reason, we also have

$$(4.14) \quad \mathcal{A}(\text{dil}_{\theta,2}(\mu)) = \mathcal{A}(\mu).$$

The map  $\text{dil}_{\theta,2}$  performs a *rescaling* on the measure  $\mu$  in the sense specified by the following lemma.

**Lemma 4.5.** *Given a measure  $\mu \in \mathcal{M}(\Omega)$  and a 1-homogeneous functional  $\sigma : \Omega \rightarrow \mathbb{R}$  such that  $\sigma(z) > 0$  for  $\mu$ -almost every path  $z$ , suppose that*

$$(4.15) \quad C := \left( \int_{\Omega} (\sigma(z))^2 d\mu(z) \right)^{1/2} < +\infty;$$

if  $\tilde{\mu} = \text{dil}_{\sigma/C, 2}(\mu)$  then  $\tilde{\mu}(\Omega) = 1$  and

$$(4.16) \quad \tilde{\mu}(\{z \in \Omega; \sigma(z) = C\}) = 1.$$

*Proof.* We prove this by direct calculation. Let  $\theta := \sigma/C$ . By 1-homogeneity of  $\sigma$ , for  $\mu$ -almost every path  $z$

$$(4.17) \quad \sigma(\text{prod}_{\theta}(z)) = \frac{\sigma(z)}{|\theta(z)|} = C.$$

Then,

$$(4.18) \quad \begin{aligned} \int_{\{z \in \Omega; \sigma(z)=C\}} d\tilde{\mu}(z) &= \int_{\{z \in \Omega; \sigma(z)=C\}} d\text{prod}_{\theta\#}(\theta^2\mu)(z) \\ &= \int_{\{z \in \Omega; \sigma(\text{prod}_{\theta}(z))=C\}} \theta^2 d\mu(z) \\ &= \frac{1}{C^2} \int_{\Omega} (\sigma(z))^2 d\mu(z) = 1. \end{aligned}$$

By similar calculations we also have  $\tilde{\mu}(\Omega) = 1$ .  $\square$

Besides the rescaling operator and lemma 4.5, we will also need the following result which will allow us to construct suitable minimizers of the action in problem 4.2.

**Lemma 4.6.** *The set of measures with uniformly bounded action  $\mathcal{A}(\mu) \leq C$  and satisfying the homogeneous constraint in equation (4.9) is relatively sequentially compact for the narrow topology.*

*Proof.* Due to Theorem 2.1, it is sufficient to prove that sequences of admissible measures are tight. For a given path  $z$  with  $\mathcal{A}(z) \leq K$ , for all  $0 \leq s \leq t \leq T$ ,

$$(4.19) \quad d_{\mathcal{C}}(z_s, z_t) \leq \int_s^t \|\dot{z}_{t^*}\|_{gc} dt^* \leq K^{1/2}|t-s|^{1/2},$$

which implies that level sets of  $\mathcal{A}(z)$  are equicontinuous. Consider now the set

$$(4.20) \quad \Omega_R := \Omega(\mathcal{C}_R) = \{z \in \Omega; \forall t \in [0, T], r_t \leq R\};$$

For any  $K > 0$ , the set  $\{z \in \Omega_R; \mathcal{A}(z) \leq K\}$  is also equicontinuous; moreover, since paths in this set are bounded at any time, it is contained in a compact subset of  $\Omega$ , by the Ascoli-Arzelà theorem.

In order to use such sets to prove tightness we need to be able to control the measure of  $\Omega \setminus \Omega_R$ . In particular, we now show that there exists a constant  $C' > 0$  such that

$$(4.21) \quad \mu(\Omega \setminus \Omega_R) \leq \frac{C'}{R^2}.$$

Let us fix a  $t^* \in (0, T)$ ,  $\epsilon > 0$  and an interval  $I_{\epsilon} = (t^* - \epsilon/2, t^* + \epsilon/2) \subset [0, T]$ . Moreover, consider the following set of paths

$$(4.22) \quad \{z \in \Omega; \forall t \in I_{\epsilon}, r_t > R\}.$$

Then, integrating the constraint in equation (4.9) over such a set with  $f$  being any continuous function such that  $f(t, \cdot) = 1$  for  $t \in I_{\epsilon}$ ,  $f(t, \cdot) = 0$  for  $t \in [0, T] \setminus I_{2\epsilon}$  and  $0 \leq f \leq 1$ , we obtain

$$(4.23) \quad \mu(\{z \in \Omega; \forall t \in I_{\epsilon}, r_t > R\}) \leq \frac{2}{R^2}.$$

Since the estimate is uniform in  $\epsilon$  this means that

$$(4.24) \quad \mu(\{z \in \Omega; r_{t^*} > R\}) \leq \frac{2}{R^2}.$$

Now, consider the set  $\mathcal{A}(z) < Q$ ; because of equation (4.19) we have

$$(4.25) \quad |r_t - r_s| \leq Q^{1/2}|t-s|^{1/2}.$$

This implies that if  $Q$  is sufficiently small

$$(4.26) \quad \{z \in \Omega \setminus \Omega_R; \mathcal{A}(z) < Q\} \subseteq \{z \in \Omega; r_{T/2} > R/2\}.$$

More precisely this holds for

$$(4.27) \quad Q^{1/2} \left| \frac{T}{2} \right|^{1/2} < \frac{R}{2},$$

and hence for  $Q < R^2/(2T)$ . Therefore, if  $Q < R^2/(2T)$ ,

$$(4.28) \quad \begin{aligned} \mu(\Omega \setminus \Omega_R) &\leq \mu((\Omega \setminus \Omega_R) \cap \{z; \mathcal{A}(z) < Q\}) + \mu(\{z; \mathcal{A}(z) \geq Q\}) \\ &\leq \mu(\{z \in \Omega; r_{T/2} > R/2\}) + \frac{C}{Q} \\ &\leq \frac{8}{R^2} + \frac{C}{Q}. \end{aligned}$$

Taking  $Q = R^2/(4T)$ , we deduce that

$$(4.29) \quad \mu(\Omega \setminus \Omega_R) \leq \frac{4(CT + 2)}{R^2},$$

which proves equation (4.21).

Recall that  $\{z \in \Omega_R; \mathcal{A}(z) \leq K\}$  is contained in a compact set for any  $K > 0$  and  $R > 0$ . For any  $\epsilon > 0$ , set  $R = (8(CT + 1)/\epsilon)^{1/2}$ . For any admissible  $\mu$ , we have

$$(4.30) \quad \begin{aligned} \mu(\Omega \setminus \{z \in \Omega_R; \mathcal{A}(z) \leq 2C\epsilon^{-1}\}) &\leq \mu(\Omega \setminus \{z; \mathcal{A}(z) \leq 2C\epsilon^{-1}\}) + \mu(\Omega \setminus \Omega_R) \\ &\leq \frac{\epsilon}{2C} \int_{\Omega} \mathcal{A}(z) d\mu(z) + \frac{\epsilon}{2} \leq \epsilon, \end{aligned}$$

which proves tightness.  $\square$

We are now ready to prove existence of optimal solutions for the generalized CH problem.

*Proof of proposition 4.4.* The functional  $\mathcal{A}(z)$  is lower semi-continuous; hence so is  $\mathcal{A}(\mu)$ . Consider a minimizing sequence  $\mu_n$  with  $n \in \mathbb{N}$ . By assumption we can take  $\mathcal{A}(\mu_n) \leq C$  for all  $n \in \mathbb{N}$ . Let  $o : t \in [0, T] \rightarrow o \in \mathcal{C}$  the path on the cone assigning to every time the apex of the cone  $o$ . Let  $\mu_n^\circ := \mu_n \llcorner \Omega^\circ \in \mathcal{M}(\Omega)$  the restriction of  $\mu_n$  to  $\Omega^\circ := \Omega \setminus \{o\}$ . Such an operation preserves both the action and the constraints.

Let  $\sigma : \Omega \rightarrow \mathbb{R}$  be the 1-homogeneous functional defined by

$$(4.31) \quad \sigma(z) := \left( r_0^2 + r_T^2 + \int_0^T r_t^2 dt \right)^{1/2}.$$

For any  $\mu_n^\circ$  in the sequence, we obviously have that  $\sigma(z) > 0$  for  $\mu_n^\circ$ -almost every path. Moreover, since  $\mu_n^\circ$  satisfies both the homogeneous marginal and coupling constraint, for all  $n \in \mathbb{N}$ ,

$$(4.32) \quad \int_{\Omega} \sigma(z)^2 d\mu_n(z) = T + 2.$$

Hence we can apply lemma 4.5 and define a sequence  $\tilde{\mu}_n \in \mathcal{P}(\Omega)$  by  $\tilde{\mu}_n := \text{dil}_{\sigma/\sqrt{T+2}, 2} \mu_n^\circ$ . In particular, for all  $n \in \mathbb{N}$ ,  $\tilde{\mu}_n$  is concentrated on the set of paths such that  $\sigma(z) = \sqrt{T+2}$ , i.e.

$$(4.33) \quad \tilde{\mu}_n \left( \left\{ z \in \Omega; r_0^2 + r_T^2 + \int_0^T r_t^2 dt = T + 2 \right\} \right) = 1.$$

Moreover,  $\tilde{\mu}_n$  satisfies the homogeneous constraint and the coupling constraint, since these are both 2-homogeneous in the radial direction, and for the same reason  $\mathcal{A}(\tilde{\mu}_n) = \mathcal{A}(\mu_n) \leq C$ . This is enough to apply lemma 4.6; thus, we can extract a subsequence  $(\tilde{\mu}_n)_n \rightarrow \tilde{\mu}_\infty \in \mathcal{P}(\Omega)$ .

We now show that for any  $f \in C^0([0, T] \times M)$  the functional

$$(4.34) \quad \mathcal{F}(z) := \int_0^T f(t, x_t) r_t^2 dt$$

is uniformly integrable with respect to the sequence  $(\tilde{\mu}_n)_n$ , that is, for any  $\epsilon > 0$  there exists a constant  $K > 0$  such that for all  $n \in \mathbb{N}$

$$(4.35) \quad \int_{\Omega, \mathcal{F}(z) > K} \mathcal{F}(z) d(\tilde{\mu}_n)_n(z) < \epsilon.$$

It is sufficient to consider the case  $\|f\|_{C^0} = 1$ , because the case  $\|f\|_{C^0} = 0$  is trivial and otherwise we can always rescale the functional by dividing it by  $\|f\|_{C^0}$ . Recall the definition of the functional  $\sigma$  in equation (4.31); we have

$$(4.36) \quad \int_{\Omega, \mathcal{F}(z) > K} \mathcal{F}(z) d(\tilde{\mu}_n)_n(z) \leq \int_{\Omega, \sigma(z)^2 > K} \sigma(z)^2 d(\tilde{\mu}_n)_n(z).$$

However, by equation (4.33) the right-hand side is zero if  $K > T + 2$ , which proves uniform integrability. Hence, using lemma 2.2, we deduce that  $\tilde{\mu}_\infty$  satisfies the homogeneous marginal constraint. Similarly, we can deduce that  $\tilde{\mu}_\infty$  also satisfies the homogeneous coupling constraint since  $(e_0, e_T)_\#(\tilde{\mu}_n)_n$  is concentrated on  $\mathcal{C}_R^2$  with  $R = \sqrt{T + 2}$ ; hence it is an optimal solution of problem 4.2.  $\square$

**Corollary 4.7.** *Suppose that  $h \in \text{Diff}(M)$  is in the connected component containing Id. Then, if  $\gamma = [(\text{Id}, 1), (h, \sqrt{\text{Jac}(h)})]_\# \rho_0$ , the minimum of the action in problem 4.2 is attained.*

In general, we cannot ensure that there exists a minimizer  $\mu$  of problem 4.2 satisfying the strong coupling constraint:

$$(4.37) \quad (e_0, e_T)_\# \mu = \gamma.$$

However, we can easily obtain a characterization for the existence of such minimizers when  $\gamma$  is deterministic. This relies on the following crucial result which allows us to isolate the part of the solution involving the cone singularity.

**Proposition 4.8.** *Suppose that  $\gamma = [(\text{Id}, 1), (h, \sqrt{\text{Jac}(h)})]_\# \rho_0$ . Any measure  $\mu \in \mathcal{M}(\Omega)$  satisfying the homogeneous coupling constraint admits the decomposition*

$$(4.38) \quad \mu = \tilde{\mu} + \tilde{\mu}^0,$$

where  $\tilde{\mu} = \mu \llcorner \{z \in \Omega; r_0 \neq 0, r_T \neq 0\}$  and  $\tilde{\mu}^0 = \mu \llcorner \{z \in \Omega; r_0 = r_T = 0\}$ . Moreover  $\tilde{\mu}^1 := \text{dil}_{r_0, 2} \tilde{\mu}$  satisfies the strong coupling constraint, i.e.  $(e_0, e_T)_\# \tilde{\mu}^1 = \gamma$ .

*Proof.* Let  $\mu \in \mathcal{M}(\Omega)$  be any dynamic plan satisfying the homogeneous coupling constraint. We decompose  $\mu = \tilde{\mu} + \tilde{\mu}^0$  where

$$(4.39) \quad \tilde{\mu} := \mu \llcorner \{z \in \Omega; r_0 \neq 0\}, \quad \tilde{\mu}^0 := \mu \llcorner \{z \in \Omega; r_0 = 0\}.$$

Consider the 1-homogeneous functional  $\tilde{\sigma}(z) : \Omega \rightarrow \mathbb{R}$  defined by  $\tilde{\sigma}(z) = r_0$ . Clearly  $\tilde{\sigma}(z) > 0$  for  $\tilde{\mu}$ -almost every path  $z$ . Moreover, we have

$$(4.40) \quad \int_{\Omega} (\tilde{\sigma}(z))^2 d\tilde{\mu}(z) = \int_{\Omega} r_0^2 d\tilde{\mu}(z) = 1.$$

Hence, by lemma 4.5, the measure  $\tilde{\mu}^1 := \text{dil}_{r_0, 2} \tilde{\mu} \in \mathcal{P}(\Omega)$  is concentrated on paths such that  $r_0 = 1$ . Moreover,  $\tilde{\mu}^0 + \tilde{\mu}^1$  still satisfies the homogeneous coupling constraint and in particular, for any  $\alpha \in [0, 2)$ ,

$$(4.41) \quad \begin{aligned} \int_{\Omega} r_T^\alpha d\tilde{\mu}^1(z) &= \int_{\Omega} r_0^{2-\alpha} r_T^\alpha d\tilde{\mu}^1(z) \\ &= \int_{\Omega} r_0^{2-\alpha} r_T^\alpha d(\tilde{\mu}^0 + \tilde{\mu}^1)(z) \\ &= \int_M \zeta^\alpha d\rho_0. \end{aligned}$$

Taking the limit for  $\alpha \rightarrow 2$ , by the dominated convergence theorem,

$$(4.42) \quad \int_{\Omega} r_T^2 d\tilde{\mu}^1(z) = \int_M \zeta^2 d\rho_0 = 1.$$

In turn, this implies that

$$(4.43) \quad \int_{\Omega} r_T^2 d\tilde{\mu}^0(z) = 0,$$

which means that  $\tilde{\mu}^0$ -almost every path  $z$  has  $r_T = 0$ . This proves that  $\tilde{\mu}^0 = \mu \llcorner \{z \in \Omega; r_0 = r_T = 0\}$  and that  $\tilde{\mu}$  satisfies the homogeneous coupling constraint.

Next, we prove that  $(e_0, e_T)_{\#} \tilde{\mu}^1 = \gamma$ . For any  $g \in C^0(M^2)$  we can take  $f = gr_0^2$  in equation (4.8) yielding

$$(4.44) \quad \int_{\Omega} g(x_0, x_T) d\tilde{\mu}^1(z) = \int_M g(x, h(x)) d\rho_0(x).$$

Similarly, letting  $\zeta := \sqrt{\text{Jac}(h)}$ ,

$$(4.45) \quad \begin{aligned} \int_{\Omega} (r_T - \zeta(x_0))^2 d\tilde{\mu}^1(z) &= \int_{\Omega} (r_T^2 + \zeta(x_0)^2 - 2\zeta(x_0)r_T) d\tilde{\mu}^1(z) \\ &= \int_{\Omega} (r_T^2 + r_0^2\zeta(x_0)^2 - 2\zeta(x_0)r_0r_T) d\tilde{\mu}^1(z) \\ &= 2 \int_M \zeta(x)^2 d\rho_0(x) - 2 \int_M \zeta(x)^2 d\rho_0(x) = 0, \end{aligned}$$

which means that for  $\tilde{\mu}^1$ -almost every path  $r_T = \zeta(x_0)$ . Then, for any continuous bounded function  $f : \mathcal{C}^2 \rightarrow \mathbb{R}$ , we have

$$(4.46) \quad \begin{aligned} \int_{\Omega} f(z_0, z_T) d\tilde{\mu}^1(z) &= \int_{\Omega} f([x_0, 1], [x_T, \zeta(x_0)]) d\tilde{\mu}^1(z) \\ &= \int_M f([x, 1], [\varphi(x), \zeta(x)]) d\rho_0(x), \end{aligned}$$

which proves the second part of the proposition. Finally, we must also have  $\tilde{\mu} = \mu \llcorner \{z \in \Omega; r_0 \neq 0, r_T \neq 0\}$ , since by definition of the dilation map

$$(4.47) \quad \int_{\{z \in \Omega; r_T=0\}} r_0^2 d\tilde{\mu} = \int_{\{z \in \Omega; r_T=0\}} r_0^2 d\tilde{\mu}^1 = \tilde{\mu}^1(\{z \in \Omega; r_T = 0\}) = 0.$$

□

**Remark 4.9.** *It should be noted that proposition 4.8 can be proved also if the coupling constraint in equation (4.8) is enforced only for homogeneous functions  $f \in C^0(\mathcal{C}^2)$  in the form  $f(z_0, z_1) = g(x_0, x_1)r_0^{2-\alpha}r_1^\alpha$  and  $\alpha \in [0, 2]$ , for example. Nonetheless, if we defined the constraint in this way, given the fact that  $\tilde{\mu}^1$  satisfies the strong coupling constraint, we would still retrieve that (when the coupling is deterministic)  $\mu$  satisfies the coupling constraint with respect to any homogeneous function.*

**Corollary 4.10** (Existence of minimizers satisfying the strong coupling constraint). *Suppose that  $\gamma = [(\text{Id}, 1), (h, \sqrt{\text{Jac}(h)})]_{\#} \rho_0$ . Then, problem 4.2 admits a minimizer satisfying the strong coupling constraint if and only if there exists a minimizer  $\mu \in \mathcal{M}(\Omega)$  (hence not necessarily a probability measure) such that*

$$(4.48) \quad \mu(\{z \in \Omega; r_0 = r_T = 0\}) = 0.$$

The proofs of proposition 4.4 and 4.8 give us several insights on the nature of the generalized solutions of the CH variational problem. First of all, it is evident that such solutions can only be unique up to rescaling. In fact, since all constraints are homogeneous and preserved by rescaling, given one minimizer one can generate others using the dilation map as in lemma 4.5. In addition, if the coupling is deterministic, even using rescaling, in principle one might not be able to find a minimizer satisfying the coupling constraint in the classical sense. By proposition 4.8, this happens if all minimizers charge paths which start and end at the apex of the cone and are not trivial. In this case the optimal solutions use the reservoir of mass in the apex to enforce the homogeneous marginal constraint on some time interval contained in  $(0, T)$ . We will refer to such minimizers as *singular solutions* since they involve the cone singularity. More precisely:

**Definition 4.11** (Singular generalized CH flows). A singular solution of the generalized CH problem is a minimizer  $\mu \in \mathcal{P}(\Omega)$  such that

$$(4.49) \quad \mu(\{z \in \Omega \setminus \{o\}; r_0 = r_T = 0\}) > 0,$$

where  $o : t \in [0, T] \rightarrow o \in \mathcal{C}$ .



Proposition 4.8 can also help us visualize such solutions. In fact, for deterministic boundary conditions, to any singular minimizer  $\boldsymbol{\mu}$  we can still associate a measure  $\tilde{\boldsymbol{\mu}}^1 = \text{dil}_{r_0,2}\tilde{\boldsymbol{\mu}}$  which satisfies the strong coupling constraint but not necessarily the homogeneous marginal constraint. The lack of mass in the homogeneous marginals indicates a mismatch between the Lagrangian flow map and its Jacobian so that particles do not fill the whole volume. On the other hand  $\tilde{\boldsymbol{\mu}}^0$  charges paths reaching the apex and it is therefore associated with the occurrence Lagrangian maps with vanishing Jacobian, which in turn corresponds to the breakdown of classical (weak) solutions. In section 7 we will construct some specific examples of singular minimizers, which will provide further intuition on their meaning.

## 5. EXISTENCE AND UNIQUENESS OF THE PRESSURE

In the previous section, we proved existence of minimizers of the generalized CH problem. In general, given that all constraints are homogeneous, such minimizers are only defined up to rescaling. However, even using rescaling, it might not always be possible to find a minimizer that satisfies the strong coupling constraint. Here, we show that independently of this, the pressure field  $P$  in the CH equation (3.17) is uniquely defined as a distribution for any given deterministic coupling constraint. This reproduces a similar result proved by Brenier for the incompressible Euler case [8].

The idea is to extend the set of admissible generalized flows in order to define appropriate variations of the action. By analogy to the Euler case, we consider dynamic plans whose homogeneous marginals are not the Lebesgue measure  $\rho_0$ , but are sufficiently close to it. Given a dynamic plan  $\boldsymbol{\nu} \in \mathcal{P}(\Omega)$  we denote by  $\rho^\boldsymbol{\nu} : [0, T] \times M \rightarrow \mathbb{R}$  the function defined by

$$(5.1) \quad \rho^\boldsymbol{\nu}(t, \cdot) := \frac{d\mathfrak{h}_t^2 \boldsymbol{\nu}}{d\rho_0},$$

for any  $t \in [0, T]$ . For an admissible generalized flow  $\boldsymbol{\nu}$ ,  $\rho^\boldsymbol{\nu} = 1$ . Dynamic plans  $\boldsymbol{\nu}$  with  $\rho^\boldsymbol{\nu} \neq 1$  correspond to generalized automorphisms of the cone with a mismatch between the radial variable and the Jacobian of the flow map on the base space.

**Definition 5.1** (Almost diffeomorphisms). A generalized almost diffeomorphism is a probability measure  $\boldsymbol{\nu} \in \mathcal{P}(\Omega)$  such that  $\rho^\boldsymbol{\nu} \in C^{0,1}([0, T] \times M)$  and

$$(5.2) \quad \|\rho^\boldsymbol{\nu} - 1\|_{C^{0,1}([0, T] \times M)} \leq \frac{1}{2}.$$

For any  $\rho \in C^{0,1}([0, T] \times M)$  with  $\rho > 0$ , let  $\Phi^\rho : \Omega \rightarrow \Omega$  be the map defined by

$$(5.3) \quad \Phi^\rho(z) := (t \in [0, T] \mapsto [x_t, r_t \sqrt{\rho(t, x_t)}] \in \mathcal{C}).$$

We use this map in the following proposition, which is the equivalent of proposition 2.1 in [8] and justifies our choice for the space of densities in definition 5.1.

**Proposition 5.2.** Fix a  $\rho \in C^{0,1}([0, T] \times M)$  such that

$$(5.4) \quad \|\rho - 1\|_{C^{0,1}} \leq \frac{1}{2}, \quad \rho(0, \cdot) = \rho(1, \cdot) = 1.$$

Then, given any dynamic plan  $\boldsymbol{\mu} \in \mathcal{P}(\Omega)$  with finite action  $\mathcal{A}(\boldsymbol{\mu}) < +\infty$ , satisfying the homogeneous constraint in equation (4.9), i.e.  $\rho^\boldsymbol{\mu} = \rho_0$ , and the coupling constraint (4.8), the dynamic plan  $\boldsymbol{\nu} := \Phi^\rho_\# \boldsymbol{\mu} \in \mathcal{P}(\Omega)$  still satisfies the coupling constraint and we have  $\rho^\boldsymbol{\nu} = \rho$ ; moreover,

$$(5.5) \quad \mathcal{A}(\boldsymbol{\nu}) \leq \mathcal{A}(\boldsymbol{\mu}) + \|\rho - 1\|_{C^{0,1}} \mathcal{A}(\boldsymbol{\mu}) + \|\rho - 1\|_{C^{0,1}}^2 (T + \mathcal{A}(\boldsymbol{\mu})).$$

*Proof.* The fact that  $\rho^\boldsymbol{\nu} = \rho$  and that  $\boldsymbol{\nu}$  satisfies the coupling constraint follows from direct computation. As for equation (5.5), observe that

$$(5.6) \quad \begin{aligned} \mathcal{A}(\boldsymbol{\nu}) &= \int_\Omega \int_0^T \mathcal{A}(\Phi^\rho(z)) dt d\boldsymbol{\mu}(z) \\ &= \int_\Omega \int_0^T \rho(t, x_t) \|\dot{z}_t\|_{g_C}^2 + r_t \dot{r}_t \partial_t(\rho(t, x_t)) + r_t^2 (\partial_t \sqrt{\rho(t, x_t)})^2 dt d\boldsymbol{\mu}(z) \\ &\leq \|\rho\|_{C^0} \mathcal{A}(\boldsymbol{\mu}) + \int_\Omega \int_0^T r_t \dot{r}_t \partial_t(\rho(t, x_t)) + r_t^2 (\partial_t \sqrt{\rho(t, x_t)})^2 dt d\boldsymbol{\mu}(z). \end{aligned}$$

Moreover,

$$(5.7) \quad \int_{\Omega} \int_0^T r_t \dot{r}_t \partial_t(\rho(t, x_t)) dt d\boldsymbol{\mu}(z) \leq |\rho - 1|_{C^{0,1}} \int_{\Omega} \int_0^T r_t \dot{r}_t (1 + \|\dot{x}_t\|_g) dt d\boldsymbol{\mu}(z) \\ \leq \frac{1}{2} |\rho - 1|_{C^{0,1}} \mathcal{A}(\boldsymbol{\mu}),$$

and similarly, since  $\rho \geq 1/2$ ,

$$(5.8) \quad \int_{\Omega} \int_0^T r_t^2 (\partial_t \sqrt{\rho(t, x_t)})^2 dt d\boldsymbol{\mu}(z) \leq \frac{1}{2} \int_{\Omega} \int_0^T r_t^2 (\partial_t(\rho(t, x_t)))^2 dt d\boldsymbol{\mu}(z) \\ \leq \frac{1}{2} |\rho - 1|_{C^{0,1}}^2 \int_{\Omega} \int_0^T r_t^2 (1 + \|\dot{x}_t\|_g)^2 dt d\boldsymbol{\mu}(z) \\ \leq |\rho - 1|_{C^{0,1}}^2 (T + \mathcal{A}(\boldsymbol{\mu})).$$

Reinserting these estimates into equation (5.6) we obtain (5.5).  $\square$

Consider now the following space

$$(5.9) \quad B_0 := \{\rho \in C^{0,1}([0, T] \times M); \rho(0, \cdot) = \rho(1, \cdot) = 0\},$$

which we regard as a Banach space with the  $C^{0,1}$  norm. The following theorem shows that we can define the pressure as an element  $p \in B_0^*$  and it is the analogue of Theorem 6.2 in [2].

**Theorem 5.3.** *Let  $\boldsymbol{\mu}^*$  be a minimizer for the generalized CH problem such that  $\mathcal{A}(\boldsymbol{\mu}^*) < +\infty$ . Then there exists  $p \in B_0^*$  such that*

$$(5.10) \quad \langle p, \rho^{\boldsymbol{\nu}} - 1 \rangle \leq \mathcal{A}(\boldsymbol{\nu}) - \mathcal{A}(\boldsymbol{\mu}^*),$$

for all generalized almost diffeomorphisms  $\boldsymbol{\nu}$  satisfying the coupling constraint (4.8).

*Proof.* First of all, observe that for any generalized almost diffeomorphism  $\boldsymbol{\nu}$  satisfying the coupling constraint,

$$(5.11) \quad \rho^{\boldsymbol{\nu}}(0, \cdot) = \rho^{\boldsymbol{\nu}}(1, \cdot) = 1;$$

hence  $\rho^{\boldsymbol{\nu}} - 1 \in B_0$  and the pairing in equation (5.10) is well defined. Now, consider the convex set  $C := \{\rho \in B_0; \|\rho\|_{C^{0,1}} \leq \frac{1}{2}\}$  and the functional  $\phi : B_0 \rightarrow \mathbb{R}^+ \cup \{+\infty\}$  defined by

$$(5.12) \quad \phi(\rho) := \begin{cases} \inf\{\mathcal{A}(\boldsymbol{\nu}); \rho^{\boldsymbol{\nu}} = \rho + 1 \text{ and (4.8) holds}\} & \text{if } \rho \in C, \\ +\infty & \text{otherwise.} \end{cases}$$

We observe that  $\phi(0) = \mathcal{A}(\boldsymbol{\mu}^*) < +\infty$  and so  $\phi$  is a proper convex function. We prove that it is bounded in a neighborhood of  $\rho = 0$ . By proposition 5.2, for any  $\rho \in C$  there exists a  $\boldsymbol{\nu} \in \mathcal{P}(\Omega)$  satisfying  $\rho^{\boldsymbol{\nu}} = \rho + 1$  and the coupling constraint, such that

$$(5.13) \quad \mathcal{A}(\boldsymbol{\nu}) \leq \mathcal{A}(\boldsymbol{\mu}^*) + \|\rho\|_{C^{0,1}} \mathcal{A}(\boldsymbol{\mu}^*) + |\rho|_{C^{0,1}}^2 (T + \mathcal{A}(\boldsymbol{\mu}^*)),$$

which implies

$$(5.14) \quad \phi(\rho) \leq \phi(1) + \|\rho\|_{C^{0,1}} \mathcal{A}(\boldsymbol{\mu}^*) + |\rho|_{C^{0,1}}^2 (T + \mathcal{A}(\boldsymbol{\mu}^*)).$$

Therefore,  $\phi$  is bounded in a neighborhood of  $\rho = 0$ . As a consequence, by standard convex analysis arguments,  $\phi$  is also locally Lipschitz on the same neighborhood and the subdifferential of  $\phi$  at 0 is not empty, i.e. there exists  $p \in B_0^*$  such that

$$(5.15) \quad \langle p, \rho \rangle \leq \phi(\rho) - \phi(0).$$

By the definition of  $\phi$ , this implies

$$(5.16) \quad \langle p, \rho \rangle \leq \mathcal{A}(\boldsymbol{\nu}) - \mathcal{A}(\boldsymbol{\mu}^*),$$

for all generalized almost diffeomorphisms  $\boldsymbol{\nu}$  satisfying  $\rho^{\boldsymbol{\nu}} = \rho + 1$  and the coupling constraint in (4.8).  $\square$

Theorem 5.3 tells us that  $\boldsymbol{\mu}^*$  is also a minimizer for the augmented action

$$(5.17) \quad \mathcal{A}^p(\boldsymbol{\nu}) := \mathcal{A}(\boldsymbol{\nu}) - \langle p, \rho^\nu - 1 \rangle,$$

defined on generalized almost diffeomorphisms. Then, for any  $\rho \in B_0$ ,  $\boldsymbol{\mu}_\epsilon^* := \Phi_{\#}^{1+\epsilon\rho} \boldsymbol{\mu}^*$  is a generalized almost diffeomorphism if  $\epsilon$  is sufficiently small. Moreover, we must have

$$(5.18) \quad \left. \frac{d}{d\epsilon} \mathcal{A}(\boldsymbol{\mu}_\epsilon^*) \right|_{\epsilon=0} = 0.$$

By the same calculation as in the proof of proposition 5.2, this implies

$$(5.19) \quad \langle p, \rho \rangle = \int_{\Omega} \int_0^T \rho(t, x_t) \|\dot{z}_t\|_{g_c}^2 + \partial_t(\rho(t, x_t)) r_t \dot{r}_t dt d\boldsymbol{\mu}^*(z),$$

for any  $\rho \in B_0$ , which defines  $p$  uniquely as a distribution. This also implies that the functional  $\phi$  is actually differentiable at 0 since its subdifferential reduces to a single element.

The existence of a unique pressure for generalized CH flows is a natural extension of a similar surprising result discovered by Brenier for incompressible Euler. In fact, it can be regarded as the second instance of the appearance of a recurring behavior in the solutions of a variational fluid model. It should also be noted that in our case, we explicitly used the cone structure to construct appropriate variations of the Lagrangian which simplified the proof if compared to the incompressible Euler case.

## 6. CORRESPONDENCE WITH DETERMINISTIC SOLUTIONS

In this section we study the correspondence between generalized and classical solutions of the CH equation. In particular, we show that for sufficiently short times classical solutions generate dynamic plans which are the unique minimizers of problem 4.2. There are two main complications that arise in this context. One is due to the singularity of the cone geometry and the other to the homogeneity of the coupling constraint. The first will imply an additional bound on the time  $T$  for which the correspondence holds. The second will intervene in the proof of uniqueness and it will be addressed by using the characterization of minimizers in proposition 4.8. Note that when  $M$  is the circle of unit radius  $S_1^1$ , we do not have to deal with the cone singularity since the cone can be identified with the plane  $\mathbb{R}^2$  with the Euclidean metric. This will result in a stronger result for this specific case.

We start by proving a modified version of a result presented in [14] stating that smooth solutions of the CH equations are length minimizing for short times in an  $L^\infty$  neighborhood on  $\text{Aut}_{\rho_0}(\mathcal{C})$ . Let  $(\varphi, \lambda)$  be a smooth solution of the system (3.16) on the interval  $[0, T]$ . Let  $P$  be the associated pressure and  $\Psi_p(t, x, r) := P(t, x)r^2$ . Following [7] we introduce the following functional on  $\Omega$ ,

$$(6.1) \quad \mathcal{B}(z) := \begin{cases} \int_0^T \|\dot{z}_t\|_{g_c}^2 - \Psi_p(t, x_t, r_t) dt & \text{if } z \in AC^2([0, T]; \mathcal{C}), \\ +\infty & \text{otherwise.} \end{cases}$$

Moreover, we consider the function  $b : \mathcal{C}^2 \rightarrow \mathbb{R}$  defined by

$$(6.2) \quad b(p, q) := \inf\{\mathcal{B}(z) ; z_0 = p, z_T = q\}.$$

**Lemma 6.1.** *Let  $M \subset \mathbb{R}^d$  be a convex domain and let  $(\varphi, \lambda)$  be a smooth solution of (3.16) on  $[0, T] \times M$ , with  $P$  being the associated pressure and  $\Psi_p(t, x, r) := P(t, x)r^2$ . For any fixed  $x \in M$ , let  $z^* = [x^*, r^*] \in \Omega$  be the curve defined by  $x^* : t \rightarrow x_t^* := \varphi_t(x)$  and  $r^* : t \rightarrow r_t^* := \lambda_t(x)$ . Let  $r_{\min} := \min_{(t,x) \in [0,T] \times M} \lambda_t(x)$ ,  $r_{\max} := \max_{(t,x) \in [0,T] \times M} \lambda_t(x)$  and  $\varrho := 2r_{\max}/r_{\min}$ . There exists a constant  $C_0 > 0$  such that, if*

- for all  $t \in [0, T]$  and for all  $w \in T_{z_t^*} \mathcal{C}$ ,

$$(6.3) \quad |\text{Hess}^{g_c} \Psi_p(w, w)| \leq \frac{C_0 \pi^2}{T^2} \|w\|_{g_c}^2 ;$$

- for all  $t_0, t_1 \in [0, T]$ ,

$$(6.4) \quad d_{\mathcal{C}}(z_{t_0}, z_{t_1}) \leq \frac{r_{\min}}{4} ;$$

- the following inequality holds:

$$(6.5) \quad \left[ \varrho^2 + (\varrho + 1)^2 \right] \|P\|_{C^0} \leq \frac{3}{2T^2} ;$$

then,  $\mathcal{B}(z^*) = b(z_0^*, z_T^*)$ ; moreover, for any other  $z \in AC^2([0, T]; \mathcal{C})$  such that  $z_0 = z_0^*$  and  $z_T = z_T^*$ ,  $\mathcal{B}(z) = \mathcal{B}(z^*)$  if and only if  $z = z^*$ . When  $M$  is the circle of unit radius  $S_1^1 := \mathbb{R}/2\pi\mathbb{Z}$  the same holds with  $C_0 = 2$  but without the conditions in equations (6.4) and (6.5).

**Remark 6.2.** The assumption in (6.3) amounts to requiring that the spectral norm of the matrix

$$(6.6) \quad g_C^{-1/2}(\text{Hess}^{g_C} \Psi_p)g_C^{-1/2} = \begin{pmatrix} 2P + (\nabla^g)^2 P & \nabla^g P \\ (\nabla^g P)^T & 2P \end{pmatrix}$$

be bounded by  $C_0\pi^2/T^2$ . This is verified for sufficiently small  $T$  if, e.g.,  $P \in L^\infty([0, T]; C^2(M))$ . Similarly, the assumption in (6.5) is verified for sufficiently small  $T$  if  $P \in C^0([0, T] \times M)$ , since for a given smooth solution  $\varphi$  with  $\varphi_0 = \text{Id}$ ,  $\varrho = 2r_{\max}/r_{\min} \rightarrow 2$  as  $T \rightarrow 0$ . In addition, note that when  $M = S_1^1$  the cone  $\mathcal{C}$  can be identified with  $\mathbb{R}^2$  and we do not have to deal with the singularity introduced by the apex. This is the reason why the assumptions in (6.4) and (6.5) are not necessary in this case.

The proof of lemma 6.1 is postponed to the appendix. Lemma 6.1 is the equivalent of lemma 5.2 in [7] on the cone. As in [7], we can use it to prove the optimality of the plan concentrated on the continuous solution. For this, however, we also need the following additional characterization of the function  $b$ .

**Lemma 6.3.** Suppose  $P \in C^0([0, T] \times M)$  and  $P_{\max} := \max_{(t,x) \in [0,T] \times M} P(t,x) \leq (\pi/T)^2$ . Then  $b(o, o) = \mathcal{B}(o) = 0$  where  $o : t \in [0, T] \rightarrow o \in \mathcal{C}$ . If the inequality is strict then for any other  $z \in AC^2([0, T]; \mathcal{C})$  such that  $z_0 = o$  and  $z_T = o$ ,  $\mathcal{B}(z) = \mathcal{B}(o)$  if and only if  $z = o$ .

*Proof.* For the first part, observe that for any  $z \in AC^2([0, T]; \mathcal{C})$  such that  $r_0 = r_T = 0$ , using Poincaré inequality

$$(6.7) \quad \begin{aligned} \mathcal{B}(z) &\geq \int_0^T \|\dot{z}_t\|_{g_C}^2 - r_t^2 P_{\max} dt \\ &\geq \int_0^T r_t^2 \|\dot{x}_t\|_g^2 + \frac{\pi^2}{T^2} r_t^2 - r_t^2 P_{\max} dt \\ &\geq \left( \frac{\pi^2}{T^2} - P_{\max} \right) \mathcal{A}(z). \end{aligned}$$

This implies that  $b(o, o) \geq 0$ . Clearly,  $b(o, o) \leq \mathcal{B}(o) = 0$  and therefore  $b(o, o) = 0$ . For the second part, if the inequality is strict,  $C := \frac{\pi^2}{T^2} - P_{\max} > 0$ . Then, for any other  $z \in AC^2([0, T]; \mathcal{C})$  such that  $z_0 = o$  and  $z_T = o$ , and satisfying  $\mathcal{B}(z) = \mathcal{B}(o)$ , we have

$$(6.8) \quad 0 = \mathcal{B}(z) \geq C\mathcal{A}(z),$$

which implies  $z = o$ . □

**Theorem 6.4.** Under the assumptions of lemma 6.1, the dynamic plan  $\mu^* = (\varphi, \lambda)_{\#} \rho_0$  is an optimal solution of problem 4.2 with  $\gamma = [(\varphi_0, \lambda_0), (\varphi_T, \lambda_T)]_{\#} \rho_0$ . If the inequalities (6.3) and (6.5) are strict, it is unique up to rescaling (in the sense of lemma 4.5).

*Proof.* Let  $\mu$  be any dynamic plan with finite action, i.e.  $\mathcal{A}(\mu) < +\infty$ , and satisfying the constraints in (4.8) and (4.9). Consider the functional

$$(6.9) \quad \mathcal{P}(z) = \int_0^T \Psi_p(t, x_t, r_t) dt.$$

Then,

$$(6.10) \quad \begin{aligned} \int_{\Omega} \mathcal{P}(z) d\mu(z) &= \int_{\Omega} \int_0^T \Psi_p(t, x_t, r_t) dt d\mu(z) \\ &= \int_{\Omega} \int_0^T P(t, x_t) r_t^2 dt d\mu(z) \\ &= \int_0^T \int_M P d\rho_0 dt. \end{aligned}$$

Hence,

$$(6.11) \quad \mathcal{B}(\boldsymbol{\mu}) = \mathcal{A}(\boldsymbol{\mu}) - \int_0^T \int_M P \, d\rho_0 dt,$$

and since equation (6.11) also holds replacing  $\boldsymbol{\mu}$  with  $\boldsymbol{\mu}^*$ ,

$$(6.12) \quad \mathcal{B}(\boldsymbol{\mu}) - \mathcal{B}(\boldsymbol{\mu}^*) = \mathcal{A}(\boldsymbol{\mu}) - \mathcal{A}(\boldsymbol{\mu}^*).$$

Now, by proposition 4.8 we have the decomposition  $\boldsymbol{\mu} = \tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}}^0$  where  $\tilde{\boldsymbol{\mu}} = \boldsymbol{\mu} \llcorner \{z \in \Omega; r_0 \neq 0\}$  and  $\tilde{\boldsymbol{\mu}}^0 = \boldsymbol{\mu} \llcorner \{z \in \Omega; r_0 = r_T = 0\}$ . Therefore, integrating the function  $b$  defined in (6.2) with respect to  $\boldsymbol{\mu}$  we obtain

$$(6.13) \quad \begin{aligned} \int_{\Omega} b(z_0, z_T) \, d\boldsymbol{\mu}(z) &= \int_{\Omega} b(z_0, z_T) \, d\tilde{\boldsymbol{\mu}}(z) + \int_{\Omega} b(o, o) \, d\tilde{\boldsymbol{\mu}}^0(z) \\ &= \int_{\Omega} b(z_0, z_T) \, d\tilde{\boldsymbol{\mu}}(z), \end{aligned}$$

where we used the fact that  $b(o, o) = 0$  by lemma 6.3. By proposition 4.8,  $\tilde{\boldsymbol{\mu}}^1 := \text{dil}_{r_0, 2} \tilde{\boldsymbol{\mu}}$  satisfies the strong coupling constraint  $(e_0, e_T)_{\#} \tilde{\boldsymbol{\mu}}^1 = \gamma$ . Moreover,  $b$  is 2-homogeneous (because  $\mathcal{B}$  is) and therefore

$$(6.14) \quad \int_{\Omega} b(z_0, z_T) \, d\tilde{\boldsymbol{\mu}}(z) = \int_{\Omega} b(z_0, z_T) \, d\tilde{\boldsymbol{\mu}}^1(z) = \int_{\mathcal{C}^2} b(p, q) \, d\gamma(p, q).$$

We get the same result integrating  $b$  with respect to  $\boldsymbol{\mu}^*$ . In particular, by lemma 6.1,

$$(6.15) \quad \int_{\Omega} b(z_0, z_T) \, d\boldsymbol{\mu}(z) = \mathcal{B}(\boldsymbol{\mu}^*).$$

By definition of  $b$  in (6.2), for any path  $z \in \Omega$ ,  $\mathcal{B}(z) \geq b(z_0, z_T)$  and therefore

$$(6.16) \quad \mathcal{B}(\boldsymbol{\mu}) \geq \int_{\Omega} b(z_0, z_T) \, d\boldsymbol{\mu}(z) = \mathcal{B}(\boldsymbol{\mu}^*),$$

which implies the same inequality for  $\mathcal{A}$  due to equation (6.12). This proves that  $\boldsymbol{\mu}^*$  is an optimal solution.

In order to prove uniqueness, let  $\boldsymbol{\mu}$  be a solution of problem 4.2. Then, equations (6.12) and (6.14) imply

$$(6.17) \quad \int_{\Omega} \mathcal{B}(z) - b(z_0, z_T) \, d\boldsymbol{\mu}(z) = \mathcal{B}(\boldsymbol{\mu}) - \mathcal{B}(\boldsymbol{\mu}^*) = \mathcal{A}(\boldsymbol{\mu}) - \mathcal{A}(\boldsymbol{\mu}^*) = 0.$$

Since for any  $z \in \Omega$  we have  $\mathcal{B}(z) \geq b(z_0, z_T)$ , then for  $\boldsymbol{\mu}$ -almost every path  $z$ ,  $\mathcal{B}(z) = b(z_0, z_T)$ . Clearly, also for  $\boldsymbol{\mu}^*$ -almost every path  $z$ ,  $\mathcal{B}(z) = b(z_0, z_T)$ . Now, if  $\boldsymbol{\mu}$  satisfies the strong coupling constraint, for  $\boldsymbol{\mu}$ -almost every path  $z$  and for  $\boldsymbol{\mu}^*$ -almost every path  $z^*$ , we have  $\mathcal{B}(z) = \mathcal{B}(z^*)$  but also  $z_0 = z_0^*$  and  $z_T = z_T^*$ . This implies  $z = z^*$  by lemma 6.1. In other words,  $\boldsymbol{\mu}$  and  $\boldsymbol{\mu}^*$  are concentrated on the same paths so they must coincide. On the other hand, if  $\boldsymbol{\mu}$  does not satisfy the strong coupling constraint, we need to prove that

$$(6.18) \quad \boldsymbol{\mu}^{\circ}(\{z \in \Omega; r_0 = r_T = 0\}) = 0,$$

where  $\boldsymbol{\mu}^{\circ} := \boldsymbol{\mu} \llcorner \Omega^{\circ}$ ,  $\Omega^{\circ} := \Omega \setminus \{o\}$  and  $o : t \in [0, T] \rightarrow o \in \mathcal{C}$ . In fact, in this case by corollary 4.10 we know that  $\boldsymbol{\mu}^{\circ}$  can be rescaled to a minimizer satisfying the strong coupling constraint.

Recall that for  $\boldsymbol{\mu}$ -almost every path  $z$  we have  $\mathcal{B}(z) = b(z_0, z_T)$ . Then, if we define  $\tilde{\Omega} := \{z \in \Omega; r_0 = r_T = 0\}$ , we have

$$(6.19) \quad \int_{\tilde{\Omega}} \mathcal{B}(z) \, d\boldsymbol{\mu}(z) = \int_{\tilde{\Omega}} b(z_0, z_T) \, d\boldsymbol{\mu}(z) = b(o, o) \boldsymbol{\mu}(\tilde{\Omega}) = 0.$$

For any  $z \in \tilde{\Omega}$  we also have  $\mathcal{B}(z) \geq b(o, o) = 0$ . Hence, we find that for  $\boldsymbol{\mu}$ -almost every path  $z$  such that  $z_0 = z_T = o$ , we have  $\mathcal{B}(z) = 0$  which by lemma 6.3 is equivalent to  $z = o$ . This implies equation (6.18) and we are done.  $\square$

The assumptions on the pressure in lemma 6.1 are less strict for the case of the circle. This leads to the following result.

**Corollary 6.5.** *Let  $M = S_1^1$  and let  $(\varphi, \lambda)$  be a smooth solution of (3.16) on  $[0, T] \times M$ , with  $P$  being the associated pressure and  $\Psi_p(t, x, r) := P(t, x)r^2$ . If for all  $t \in [0, T]$  and for all  $w \in T_{z_t^*}\mathcal{C}$ ,*

$$(6.20) \quad |\text{Hess}^{gc} \Psi_p(w, w)| \leq \frac{2\pi^2}{T^2} \|w\|_{gc}^2,$$

*then the dynamic plan  $\mu^* = (\varphi, \lambda)_{\#}\rho_0$  is an optimal solution of problem 4.2 for the coupling  $\gamma = [(\varphi_0, \lambda_0), (\varphi_T, \lambda_T)]_{\#}\rho_0$ . If the inequality in equation (6.20) is strict, it is unique up to rescaling (in the sense of lemma 4.5).*

The proof of theorem 6.4 finally validates our formulation for the generalized CH problem 4.2. In particular, it clearly shows that the choice of a homogeneous coupling constraint is appropriate for the problem. In fact, it allowed us to prove well-posedness on an unbounded cone domain in section 4 and crucially, it also allowed us to produce a simple characterization for minimizers satisfying the strong coupling constraint (see proposition 4.8 and corollary 4.10), which led here to the correspondence with deterministic solutions. It should also be noted that in this section we used a certain bound on a given pressure field (namely that in lemma 6.3) to infer that minimizers are deterministic and not singular (see definition 4.11). In fact, from section 5, we know that a pressure field always exists as a distribution. Improving such a result in terms of regularity could help to provide a better characterization of generalized flow solutions also in cases not covered in this section.

## 7. SOME EXAMPLES OF GENERALIZED CH FLOWS

In this section we construct some instructive examples of generalized CH solutions which shed some light on the need of the relaxation and its tightness. In particular, we will focus on deterministic boundary conditions and construct singular solutions, i.e. minimizers that charge (non-trivial) paths starting and ending at the apex of the cone. The occurrence of such solutions was anticipated in section 4. The discussion in this section will provide an intuition on their meaning. Specifically, we will see that they correspond to the limiting behavior of diffeomorphisms that tend to create voids in the domain or, in other words, that stretch arbitrarily small portions of the domain to occupy finite volume.

We start by considering an important generalized flow which provides an upper bound on the action on any domain and for any deterministic coupling.

**Lemma 7.1.** *Consider the generalized CH problem on a compact domain  $M \subset \mathbb{R}^d$  with coupling given by  $\gamma = (h, \sqrt{\text{Jac}(h)})_{\#}\rho_0$  where  $h \in \text{Diff}(M)$ . Denote by  $\rho_0$  the normalized Lebesgue measure on  $M$ . Then the measure*

$$(7.1) \quad \mu^* = \frac{1}{2}(\text{Id}, \zeta^0)_{\#}\rho_0 + \frac{1}{2}(\psi^1, \zeta^1)_{\#}\rho_0,$$

with

$$(7.2) \quad \zeta_t^0(x) = \sqrt{2} \sin(\sqrt{P^*}t), \quad \zeta_t^1(x) = \begin{cases} \sqrt{2} \cos(\sqrt{P^*}t) & t \leq T/2, \\ -\sqrt{2} \text{Jac}(h(x)) \cos(\sqrt{P^*}t) & t > T/2, \end{cases}$$

$$(7.3) \quad \psi_t^1(x) = \begin{cases} x & t \leq T/2, \\ h(x) & t > T/2, \end{cases}$$

where  $P^* = \pi^2/T^2$ , is an admissible generalized flow and the action of the minimizer is bounded from above by  $\mathcal{A}(\mu^*) = \pi^2/T$ .

*Proof.* We need to check that  $\mu^*$  is a probability measure and that it satisfies the homogeneous marginal and coupling constraints. The fact that  $\mu^*(\Omega) = 1$  is immediate from the definition.

As for the marginal constraint, observe that for any  $f \in C^0([0, T] \times M)$ ,

$$\begin{aligned}
\int_{\Omega} \int_0^T f(t, x_t) r_t^2 dt d\boldsymbol{\mu}(z) &= \frac{1}{2} \int_M \int_0^T f(t, x) 2 \sin^2(\sqrt{P^*}t) dt d\rho_0(x) \\
&+ \frac{1}{2} \int_M \int_0^{T/2} f(t, x) 2 \cos^2(\sqrt{P^*}t) dt d\rho_0(x) \\
(7.4) \qquad \qquad \qquad &+ \frac{1}{2} \int_M \int_{T/2}^T f(t, h(x)) 2 \text{Jac}(h(x)) \cos^2(\sqrt{P^*}t) dt d\rho_0(x) \\
&= \int_M \int_0^T f(t, x) dt d\rho_0(x).
\end{aligned}$$

By similar calculations also the homogeneous coupling constraint holds and therefore  $\boldsymbol{\mu}^*$  is admissible. Moreover, the action associated with  $\boldsymbol{\mu}^*$  is given by

$$\begin{aligned}
\mathcal{A}(\boldsymbol{\mu}^*) &= \frac{1}{2} \int_M \int_0^T \|\dot{\zeta}_t^0(x)\|^2 + \|\dot{\zeta}_t^1(x)\|^2 dt d\rho_0(x) \\
(7.5) \qquad \qquad \qquad &= \int_M \int_0^T P^* dt d\rho_0(x) = \frac{\pi^2}{T}.
\end{aligned}$$

□

The dynamic plan in lemma 7.1 shows that in our generalized formulation we can reach any final configuration only by changes in the Jacobian, although in a non-deterministic sense. In the following we will consider several instances of this flow for different domains and couplings and we will prove that in some cases it also minimizes the generalized CH action. In fact, the idea behind the construction of the generalized flow  $\boldsymbol{\mu}^*$  is that as for geodesics on the cone, we expect that for a sufficiently large displacement optimal solutions would concentrate on straight lines in the radial direction passing by the apex of the cone. If there is no motion on the base space  $M$ , the geodesic equation (3.16) in the radial direction reduces to

$$(7.6) \qquad \qquad \qquad \ddot{\lambda} + \lambda P = 0$$

The dynamic plan  $\boldsymbol{\mu}^*$  concentrates precisely on solutions to this equation with constant pressure  $P = P^*$ .

It should also be noted that  $\boldsymbol{\mu}^*$  is exactly in the form discussed in proposition 4.8, i.e. it is decomposed in the sum of two measures,  $\boldsymbol{\mu}^* = \tilde{\boldsymbol{\mu}} + \tilde{\boldsymbol{\mu}}^0$ , where

$$(7.7) \qquad \qquad \qquad \tilde{\boldsymbol{\mu}}^0 = \frac{1}{2}(\text{Id}, \zeta^0)_{\#} \rho_0, \quad \tilde{\boldsymbol{\mu}} = \frac{1}{2}(\psi^1, \zeta^1)_{\#} \rho_0.$$

In particular,  $\tilde{\boldsymbol{\mu}}$  does not charge paths starting and ending at the apex, so it can be rescaled to a probability measure satisfying the strong coupling constraint but not the homogeneous marginal constraint. This is given by

$$(7.8) \qquad \qquad \qquad \tilde{\boldsymbol{\mu}}^1 = \text{dil}_{r_0, 2} \tilde{\boldsymbol{\mu}} = (\psi^1, \zeta^1 / \sqrt{2})_{\#} \rho_0.$$

The dynamic plan  $\tilde{\boldsymbol{\mu}}^1$  describes an exotic solution in which particles gradually disappear up to time  $T/2$ , when the whole domain vanishes, and then gradually reappear in the final configuration. Heuristically, such a solution represents in a generalized sense the simultaneous formation of *shocks* and *fractures* in the domain, which correspond respectively to non-injective and non-surjective Lagrangian maps. The occurrence of this phenomenon can be seen as the consequence of the fact that there exist solutions of the CH equation which concentrate finite volume of particles to a single point at finite cost. This is the case for the collision of a peakon and an anti-peakon in one dimension. Reversing in time such solutions one discovers that arbitrarily small portions of the domain can be stretched to occupy finite area at finite cost. The generalized solution we constructed replicates this behavior in an averaged sense across the domain.

**7.1. Construction of a generalized solution on the circle.** We now consider the generalized CH problem on  $S_R^1$ , the circle of radius  $R$ . For specific boundary conditions given by uniform rotation and when  $R = 1$ , we show that the generalized flow in lemma 7.1 is a minimizer although not unique, having the same cost as the deterministic solution. When  $R > 1$ , the constant speed



rotation is not a minimizer since its action is strictly larger than  $\pi^2/T$ . This is made precise in the following theorem.

**Theorem 7.2.** *Consider the generalized CH problem on  $S_R^1$  with coupling constraint given by uniform rotation by half of the circle length, i.e. in polar coordinates  $h : \theta \in \mathbb{R}/2\pi\mathbb{Z} \rightarrow \theta + \pi$  so that  $\text{Jac}(h) = 1$ . Denote by  $\rho_0 = (2\pi)^{-1}d\theta$  the normalized Lebesgue measure on the circle. The following holds:*

(1) *when  $R = 1$  the dynamic plan  $\mu^*$  in lemma 7.1, i.e. equation (7.1) with*

$$(7.9) \quad \zeta_t^0(\theta) = \sqrt{2} \sin(\sqrt{P^*}t), \quad \zeta_t^1(\theta) = \sqrt{2} |\cos(\sqrt{P^*}t)|, \quad \psi_t^1(\theta) = \begin{cases} \theta & t \leq T/2, \\ \theta + \pi & t > T/2, \end{cases}$$

*as well as the dynamic plan induced by constant speed rotation are minimizers corresponding to the constant pressure  $P^* = (\pi/T)^2$ ;*

(2) *when  $R > 1$  the constant speed rotation is not a minimizer.*

*Proof.* For the first point, observe that from the Euler-Lagrange equations (3.17) the pressure relative to constant speed rotation on  $S_1^1$  is given by

$$(7.10) \quad P^{rot} = \|v\|_g^2 = \frac{\pi^2}{T^2}.$$

This satisfies the hypotheses of corollary 6.5 (see remark 6.2) and therefore the constant rotation is a minimizer. Since the Jacobian stays constant during the rotation, the associated action is given by

$$(7.11) \quad \mathcal{A}^{rot} = \frac{1}{2\pi} \int_0^{2\pi} \int_0^T \|v\|_g^2 dt d\theta = \frac{\pi^2}{T}.$$

On the other hand, by lemma 7.1  $\mu^* \in \mathcal{P}(\Omega)$  is admissible and its action is equal to  $\mathcal{A}(\mu^*) = \pi^2/T$ , independently of  $R$ . Hence  $\mu^*$  is also a minimizer and it must share the same pressure with the constant speed rotation,  $P^{rot} = P^*$ . For the second point, observe that the action for constant speed rotation on  $S_R^1$  is  $\mathcal{A}_R^{rot} = R^2 \mathcal{A}^{rot} > \mathcal{A}(\mu^*)$  whenever  $R > 1$ .  $\square$

**Remark 7.3.** *Theorem 7.2 does not give a complete description of solutions of the generalized CH problem for the rotation on  $S_R^1$ . In particular, when  $R > 1$  we cannot show that our solution is a minimizer since it does not fulfill the hypotheses of the analysis in section 6. In fact for this case, we cannot avoid to deal with the cone singularity and one should produce an alternative version of lemma 6.1 that is valid also for paths reaching the apex.*

**7.2. Collision of peakons and an approximation result.** Before going further with the construction of a generalized solutions on a two-dimensional domain, we need to clarify the connection between the solution presented in theorem 7.2 and diffeomorphisms of the circle. In particular, here we show that if no rotation occurs, the generalized flow in theorem 7.2 can be approximated using linear peakon/anti-peakon collisions. This will serve as a basis to construct a sequence of deterministic flows converging to a non-deterministic minimizer in two dimensions.

Consider the CH equation on the circle  $S_1^1$  with Lagrangian  $\int_0^{2\pi} u^2 + \frac{1}{4}(\partial_\theta u)^2 d\theta$ , where  $u : [0, T] \times S_1^1 \rightarrow \mathbb{R}$  is the Eulerian velocity field. Peakon solutions can be described in terms of momentum  $m = u - \frac{1}{4}\partial_\theta^2 u$  as a linear combination of Dirac delta functions, i.e.

$$(7.12) \quad m(t, \theta) = \sum_{i=1}^N p_i(t) \delta(\theta - \theta_i(t)),$$

where  $p_i : [0, T] \rightarrow \mathbb{R}$  and  $\theta_i : [0, T] \rightarrow S_R^1$  are appropriate functions specifying the momentum carried by the  $i$ th peakon and its location, respectively. The associated velocity field is given by  $u = G * m$  where  $G$  is the Green's function of the operator  $\text{Id} - \frac{1}{4}\partial_\theta^2$ .

The collision of a peakon and an anti-peakon corresponds to the case  $N = 2$ ,  $p_2 = -p_1$ ,  $\theta_2 = 2\pi - \theta_1$ , in which case there exists a finite time  $T^*$  such that as  $t \rightarrow T^*$  collision occurs, i.e.  $\theta_1 = \theta_2$ . A similar behavior occurs for the Lagrangian  $\int_0^{2\pi} \frac{1}{4}(\partial_\theta u)^2 d\theta$ , which corresponds to the Hunter-Saxton equation. In this case, the velocity field is simply given by the linear interpolation of the velocity at  $\theta_1$  and  $\theta_2$  (see figure 1) and the Jacobian of the flow map is piecewise constant. Hence specifying the trajectory  $\theta_1(t)$  uniquely defines the flow. We refer to such a solution

as *linear peakon/anti-peakon* collision. The associated flow on a circle of arbitrary radius  $R$  is described in the following lemma.

**Lemma 7.4.** *For a given  $\epsilon > 0$ , let  $\varphi^\epsilon : [0, T] \times S_R^1 \rightarrow S_R^1$  be the flow map defined in polar coordinates by*

$$(7.13) \quad \varphi_t^\epsilon(0) = 0, \quad \partial_\theta \varphi_t^\epsilon(\theta) = \begin{cases} 1 - \sin\left(\frac{\pi t}{2(T+\epsilon)}\right) & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2}, \\ 1 + \sin\left(\frac{\pi t}{2(T+\epsilon)}\right) & \text{otherwise,} \end{cases}$$

Then the associated action is uniformly bounded and

$$(7.14) \quad \lim_{R \rightarrow 0} \lim_{\epsilon \rightarrow 0} \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}([R\varphi^\epsilon(\theta), \lambda^\epsilon(\theta)]) d\theta = \frac{\pi^2}{16T},$$

where  $\lambda^\epsilon = \sqrt{\text{Jac}(\varphi^\epsilon)}$  and

$$(7.15) \quad \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}([R\varphi^\epsilon(\theta), \lambda^\epsilon(\theta)]) d\theta = \frac{1}{2\pi} \int_0^{2\pi} \int_0^T R^2 (\lambda_t^\epsilon(\theta))^2 \|\dot{\varphi}_t^\epsilon(\theta)\|_g^2 + |\dot{\lambda}_t^\epsilon(\theta)|^2 dt d\theta$$

is the action expressed in polar coordinates.

*Proof.* The result follows by direct computation and by definition of the functional  $\mathcal{A}$  in equation (4.7). Note that the expression for the action in equation (7.15) can be justified by an appropriate change of variables. Specifically, denoting by  $\varphi_R^\epsilon$  the flow map in arc length coordinates  $x \in \mathbb{R}/2\pi R\mathbb{Z}$ , we have  $\theta = x/R$  and

$$(7.16) \quad \varphi_R^\epsilon(x) = R\varphi^\epsilon\left(\frac{x}{R}\right), \quad \partial_x \varphi_R^\epsilon(x) = \partial_\theta \varphi^\epsilon\left(\frac{x}{R}\right).$$

Denoting  $\lambda_R^\epsilon = \sqrt{\partial_x \varphi_R^\epsilon}$ , since  $\rho_0 = (2\pi R)^{-1} dx$  we obtain that the action is given by

$$(7.17) \quad \begin{aligned} \int_{S_R^1} \mathcal{A}([\varphi_R^\epsilon(x), \lambda_R^\epsilon(x)]) d\rho_0(x) &= \frac{1}{2\pi R} \int_0^{2\pi R} \int_0^T ((\lambda_R^\epsilon)_t(x))^2 \|(\dot{\varphi}_R^\epsilon)_t(x)\|_g^2 + |(\dot{\lambda}_R^\epsilon)_t(x)|^2 dt dx \\ &= \frac{1}{2\pi} \int_0^{2\pi} \int_0^T R^2 (\lambda_t^\epsilon(\theta))^2 \|\dot{\varphi}_t^\epsilon(\theta)\|_g^2 + |\dot{\lambda}_t^\epsilon(\theta)|^2 dt d\theta. \end{aligned}$$

□

**Remark 7.5.** *The flow described in lemma 7.4 coincides with a linear peakon/anti-peakon solution of the Hunter-Saxton equation where the momentum is in the form of equation (7.12) and the two peak trajectories are given by*

$$(7.18) \quad \theta_1^\epsilon(t) = \frac{\pi}{2} \left( 1 + \sin\left(\frac{\pi t}{2(T+\epsilon)}\right) \right), \quad \theta_2^\epsilon(t) = \frac{\pi}{2} \left( 3 - \sin\left(\frac{\pi t}{2(T+\epsilon)}\right) \right).$$

The reason why we consider solutions to the Hunter-Saxton equation rather than CH peakons to construct our approximations is related to the way the action in (7.15) scales with the size of the domain. In fact, equation (7.15) suggests that the optimal way to concentrate volume at small scales (i.e. as  $R \rightarrow 0$ ) is captured by solutions to the Hunter-Saxton equation rather than CH.

In figure 2, we give an illustration of the flow defined in equation (7.13) for fixed  $\epsilon$  both in terms of particle trajectories and as a measure on the cone for  $R = 1$  (in which case the cone can be identified with  $\mathbb{R}^2$ ). Note that at collision time the trajectories of particles between the peaks reach the apex of the cone.

In the next lemma we show that the flow describing dense formation of shocks, i.e. linear peakon/anti-peakon collisions, converges to a measure in the same form as the one in lemma 7.1.

**Lemma 7.6.** *Let  $\varphi^\epsilon : [0, T] \times S_R^1 \rightarrow S_R^1$  the flow in lemma 7.4 and for each  $n \in \mathbb{N}$  let  $\hat{\varphi}^n : [0, T] \times S_R^1 \rightarrow S_R^1$  be defined by*

$$(7.19) \quad \hat{\varphi}^n(\theta) := \frac{2\pi}{n} \left\lfloor \frac{\theta n}{2\pi} \right\rfloor + \frac{1}{n} \varphi^{\epsilon_n} \left( n\theta - 2\pi \left\lfloor \frac{\theta n}{2\pi} \right\rfloor \right)$$

with  $\epsilon_n$  being any positive sequence such that  $\epsilon_n \rightarrow 0$ . Then  $\hat{\mu}_n := (\hat{\varphi}^n, \sqrt{\text{Jac}(\hat{\varphi}^n)})_{\#} \rho_0 \rightarrow \hat{\mu}^*$ , where  $\hat{\mu}^*$  is defined by

$$(7.20) \quad \hat{\mu}^* = \frac{1}{2} (\text{Id}, \zeta^0)_{\#} \rho_0 + \frac{1}{2} (\text{Id}, \zeta^1)_{\#} \rho_0,$$

with

$$(7.21) \quad \zeta_t^0(\theta) = \sqrt{2} \sin\left(\frac{\pi t}{4T} + \frac{\pi}{4}\right), \quad \zeta_t^1(\theta) = \sqrt{2} \cos\left(\frac{\pi t}{4T} + \frac{\pi}{4}\right).$$

Moreover,  $\mathcal{A}(\hat{\boldsymbol{\mu}}_n) \rightarrow \mathcal{A}(\hat{\boldsymbol{\mu}}^*) = \pi^2/(16T)$ .

*Proof.* For simplicity, we prove the result for  $R = 1$  but the argument presented here applies for any  $R > 0$ . Let  $\mathcal{F}$  be any bounded Lipschitz functional on  $\Omega$  with Lipschitz constant  $L$ . We need to check that

$$(7.22) \quad \lim_{n \rightarrow +\infty} \int_{\Omega} \mathcal{F}(z) d\hat{\boldsymbol{\mu}}_n(z) = \int_{\Omega} \mathcal{F}(z) d\hat{\boldsymbol{\mu}}^*(z).$$

Denoting by  $\hat{\lambda}^n = \sqrt{\text{Jac}(\hat{\varphi}^n)}$  and by  $\hat{\lambda}^{\epsilon_n} = \sqrt{\text{Jac}(\hat{\varphi}^{\epsilon_n})}$ , we observe that

$$(7.23) \quad \begin{aligned} \int_{\Omega} \mathcal{F}(z) d\hat{\boldsymbol{\mu}}_n(z) &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}([\hat{\varphi}^n(\theta), \hat{\lambda}^n(\theta)]) d\theta \\ &= \frac{1}{2\pi} \sum_{i=0}^{n-1} \int_0^{2\pi/n} \mathcal{F}\left(\left[\frac{2\pi i}{n} + \frac{1}{n} \varphi^{\epsilon_n}(n\theta), \lambda^{\epsilon_n}(n\theta)\right]\right) d\theta, \end{aligned}$$

and similarly,

$$(7.24) \quad \int_{\Omega} \mathcal{F}(z) d\hat{\boldsymbol{\mu}}^*(z) = \frac{1}{4\pi} \sum_{i=0}^{n-1} \int_0^{2\pi/n} \mathcal{F}\left(\left[\frac{2\pi i}{n} + \theta, \zeta^0(\theta)\right]\right) + \mathcal{F}\left(\left[\frac{2\pi i}{n} + \theta, \zeta^1(\theta)\right]\right) d\theta.$$

We consider separately each integral in the sums in equation (7.23) and (7.24). Rescaling the integrals in  $\theta$  and using Lipschitz continuity of  $\mathcal{F}$ , we observe that the result is proven if

$$(7.25) \quad \lim_{n \rightarrow +\infty} \frac{1}{n} \sum_{i=0}^{n-1} \left| \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}\left(\left[\frac{2\pi i}{n} + \frac{1}{n} \varphi^{\epsilon_n}(\theta), \lambda^{\epsilon_n}(\theta)\right]\right) d\theta - I_i^n \right| = 0,$$

where

$$(7.26) \quad I_i^n = \frac{1}{2} \mathcal{F}\left(\left[\frac{2\pi i}{n}, \zeta^0\left(\frac{2\pi i}{n}\right)\right]\right) + \frac{1}{2} \mathcal{F}\left(\left[\frac{2\pi i}{n}, \zeta^1\left(\frac{2\pi i}{n}\right)\right]\right).$$

For any fixed sufficiently large  $n$ , we need to provide an appropriate bound for each term in the sum in equation (7.25). For any integer  $i$  with  $0 \leq i \leq n-1$ , we have

$$(7.27) \quad \begin{aligned} E^{i,n} &:= \left| \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}\left(\left[\frac{2\pi i}{n} + \frac{1}{n} \varphi^{\epsilon_n}(\theta), \lambda^{\epsilon_n}(\theta)\right]\right) d\theta - I_i^n \right| \\ &\leq \frac{1}{2\pi} \left| \int_0^{2\pi} \mathcal{F}\left(\left[\frac{2\pi i}{n} + \frac{1}{n} \varphi^{\epsilon_n}(\theta), \lambda^{\epsilon_n}(\theta)\right]\right) d\theta - \int_0^{2\pi} \mathcal{F}\left(\left[\frac{2\pi i}{n}, \lambda^{\epsilon_n}(\theta)\right]\right) d\theta \right| \\ &\quad + \left| \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}\left(\left[\frac{2\pi i}{n}, \lambda^{\epsilon_n}(\theta)\right]\right) d\theta - I_0^n \right| := E_0^n + E_1^n. \end{aligned}$$

Observe that for  $\alpha \in [0, \pi/2]$ ,  $\sqrt{1 - \sin(\alpha)} = \sqrt{2} \cos(\alpha/2 + \pi/4)$  and  $\sqrt{1 + \sin(\alpha)} = \sqrt{2} \sin(\alpha/2 + \pi/4)$  therefore

$$(7.28) \quad \lambda^{\epsilon_n}(\theta) = \sqrt{\partial_{\theta} \varphi_t^{\epsilon_n}(\theta)} = \begin{cases} \sqrt{2} \cos\left(\frac{\pi t}{4(T+\epsilon_n)} + \frac{\pi}{4}\right) & \text{if } \frac{\pi}{2} < \theta < \frac{3\pi}{2}, \\ \sqrt{2} \sin\left(\frac{\pi t}{4(T+\epsilon_n)} + \frac{\pi}{4}\right) & \text{otherwise.} \end{cases}$$

Since  $\lambda^{\epsilon_n}$  is piecewise constant in  $\theta$  we can write

$$(7.29) \quad \frac{1}{2\pi} \int_0^{2\pi} \mathcal{F}\left(\left[\frac{2\pi i}{n}, \lambda^{\epsilon_n}(\theta)\right]\right) d\theta = \frac{1}{2} \mathcal{F}\left(\left[\frac{2\pi i}{n}, \lambda^{\epsilon_n}(0)\right]\right) + \frac{1}{2} \mathcal{F}\left(\left[\frac{2\pi i}{n}, \lambda^{\epsilon_n}(\pi)\right]\right).$$

Comparing the expression for  $\zeta^0$  and  $\zeta^1$  with that of  $\lambda^{\epsilon_n}$  and using the fact that  $\mathcal{F}$  is Lipschitz we obtain  $E_1^n \leq C(\epsilon_n)$ , where  $C(\epsilon_n) > 0$  is a constant depending on  $\epsilon_n$  and  $L$  such that  $C(\epsilon_n) \rightarrow 0$  as  $n \rightarrow +\infty$ . A similar argument holds for  $E_0^n$  and therefore we can find a constant  $C_n$  independent of  $i$  such that  $E^{i,n} \leq C_n$  and  $C_n \rightarrow 0$  as  $n \rightarrow +\infty$ . This implies equation (7.25).

Finally, convergence of the action is a consequence of lemma 7.4. In particular, it is immediate to verify that  $\mathcal{A}(\hat{\boldsymbol{\mu}}^*) = \pi^2/(16T)$ . Moreover, by the same reasoning as in the proof of lemma 7.4 and the change of variables in equation (7.23) we obtain that the action  $\mathcal{A}(\hat{\boldsymbol{\mu}}_n)$  is given by

$$(7.30) \quad \begin{aligned} \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A}([\hat{\varphi}^n(\theta), \hat{\lambda}^n(\theta)]) \, d\theta &= \frac{1}{2\pi} \sum_{i=0}^{n-1} \int_0^{2\pi/n} \mathcal{A} \left( \left[ \frac{1}{n} \varphi^{\epsilon_n}(n\theta), \lambda^{\epsilon_n}(n\theta) \right] \right) \, d\theta \\ &= \frac{1}{2\pi} \int_0^{2\pi} \mathcal{A} \left( \left[ \frac{1}{n} \varphi^{\epsilon_n}(n\theta), \lambda^{\epsilon_n}(n\theta) \right] \right) \, d\theta. \end{aligned}$$

Therefore, the limit of  $\mathcal{A}(\hat{\boldsymbol{\mu}}_n)$  for  $n \rightarrow +\infty$  is the same to that in equation (7.14).  $\square$

In figure 3, we give an illustration of the flow defined in equation (7.19) for fixed  $n$  both in terms of particle trajectories and as a measure on the cone for  $R = 1$ . It can be seen that convergence towards the measure  $\boldsymbol{\mu}^*$  defined in lemma 7.6 is due to the appearance of fast oscillations in the Jacobian together with the fact that particles tend to stay still as  $n \rightarrow +\infty$ .

We can use the flows defined in lemma 7.6 to construct a sequence that converges to the generalized flow  $\boldsymbol{\mu}^*$  in theorem 7.2 but where no rotation occurs. The construction consists in concatenating in time the flows in lemma 7.6 so that a small portion of the domain stretches and then return to its original size. This is shown in figure 4. The convergence result is stated explicitly in the following proposition.

**Proposition 7.7.** *Let  $\hat{\varphi}^n : [0, T] \times S_R^1 \rightarrow S_R^1$  be the sequence defined in lemma 7.6 and for each  $n \in \mathbb{N}$  let  $\varphi^n : [0, T] \times S_R^1 \rightarrow S_R^1$  be defined by  $\varphi_t^n = \varphi_{T-t}^n$  and*

$$(7.31) \quad \varphi_t^n(\theta) = \begin{cases} \hat{\varphi}_{T-4t}^n((\hat{\varphi}_T^n)^{-1}(\theta)) & \text{if } t \leq \frac{T}{4}, \\ \hat{\varphi}_{4t-T}^n((\hat{\varphi}_T^n)^{-1}(\theta) + \frac{\pi}{n}) - \frac{\pi}{n} & \text{if } \frac{T}{4} < t \leq \frac{T}{2}. \end{cases}$$

Then  $\boldsymbol{\mu}_n := (\varphi^n, \sqrt{\text{Jac}(\varphi^n)})_{\#} \rho_0$  can be rescaled to a sequence  $\tilde{\boldsymbol{\mu}}_n \rightarrow \boldsymbol{\mu}^*$ , where  $\boldsymbol{\mu}^*$  is defined as in equation (7.20) with

$$(7.32) \quad \zeta_t^0(\theta) = \sqrt{2} \sin\left(\frac{\pi t}{T}\right), \quad \zeta_t^1(\theta) = \sqrt{2} \left| \cos\left(\frac{\pi t}{T}\right) \right|.$$

Moreover,  $\mathcal{A}(\boldsymbol{\mu}_n) \rightarrow \mathcal{A}(\boldsymbol{\mu}^*) = \pi^2/T$ .

*Proof.* The rescaling to be performed in order to obtain the sequence  $\tilde{\boldsymbol{\mu}}_n$  is given by

$$(7.33) \quad \tilde{\boldsymbol{\mu}}_n = \text{dil}_{r_{T/4}, 2} \boldsymbol{\mu}_n.$$

In fact, by lemma 4.5,  $\tilde{\boldsymbol{\mu}}_n$  is concentrated on paths such that  $r_{T/4} = 1$ . Then, the result can be deduced as a consequence of lemmas 7.6 and 7.4.  $\square$

**Remark 7.8.** *The maps defined by equation (7.31) are piecewise smooth in space since their Jacobian is piecewise constant with a finite number of discontinuities. However, using a regularization argument, it is not difficult to see that one can also construct a sequence of smooth diffeomorphisms satisfying proposition 7.7. For this it is sufficient to repeat the construction above using a regularized version of the linear peakon/anti-peakon collision, which can be defined by convolution of the flow map with a sequence of positive symmetric mollifiers.*

Proposition 7.7 allows us to infer that generalized flows charging paths starting and ending at the apex should arise in the limit from dense formation of shocks. With this interpretation in mind, one can deduce that it is thanks to the generalized description that such solutions can occur. In fact, it is unlikely that, for example, the generalized flow in theorem 7.2 can be approached by diffeomorphisms of the circle. This is because even if particles are concentrated in a small portion of the domain, they cannot rotate on  $S_R^1$  following a deterministic flow without the rest of the particles to rotate as well. On the other hand, in the generalized setting particles can cross each other and there is no topological impediment. This suggests that our formulation cannot be a tight relaxation of the  $H(\text{div})$  (i.e.  $H^1$ ) geodesic problem on the diffeomorphism group of the circle. However, in the next section we will see that adding just one dimension, there still exist pathological solutions like the one considered here but these are not spurious anymore and are rather implied by the smooth formulation.

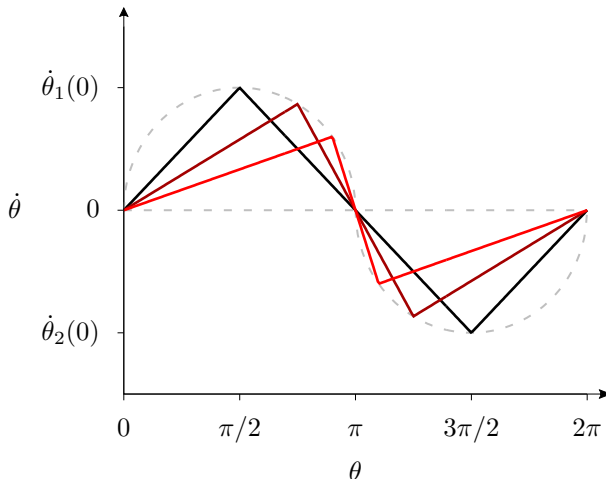


FIGURE 1. Velocity field evolution for the linear peakon/anti-peakon solution of the Hunter-Saxton equation.

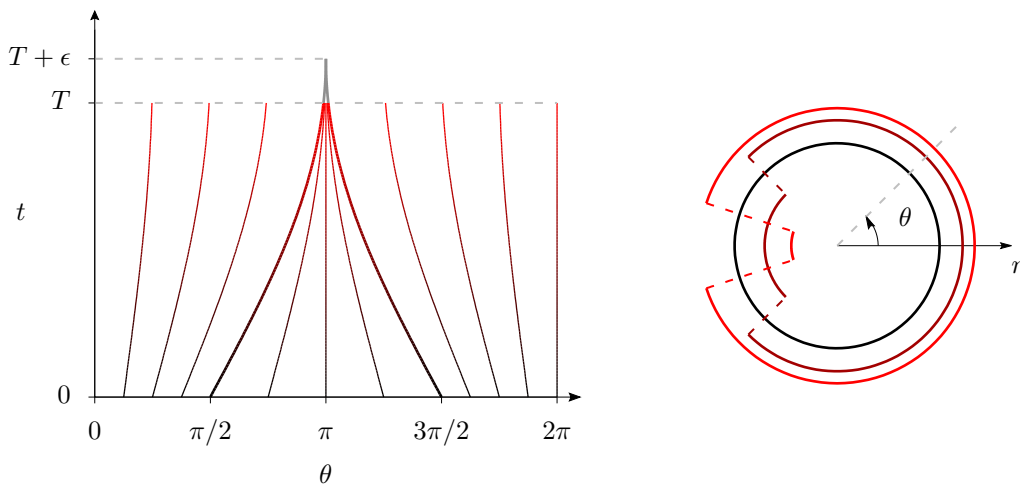


FIGURE 2. Particle trajectories  $t \mapsto \varphi_t^\epsilon(\theta)$  for the linear peakon/anti-peakon solution (left) and support of fixed time marginals for the measure  $(\varphi^\epsilon, \sqrt{\text{Jac}(\varphi^\epsilon)})_\# \rho_0$  (right).

**7.3. Construction of a generalized solution on the torus.** We now consider the generalized CH problem on the torus  $T_{1,R}^2 := S_1^1 \times S_R^1$ , with one of the two radii set to one. We consider as boundary condition a uniform twist of the torus in which each particle rotates of half of the length on both circles. For this specific boundary condition we can construct a generalized minimizer using the construction of the previous section which realizes smaller action than the constant speed rotation. This is made precise in the following theorem.

**Theorem 7.9.** *Consider the generalized CH problem on  $T_{1,R}^2$  with coupling constraint given by uniform rotation on both circles by half of the circles length, i.e. in polar coordinates  $h : (\theta, \phi) \in \mathbb{R}^2 / (2\pi\mathbb{Z})^2 \rightarrow (\theta + \pi, \phi + \pi)$  so that  $\text{Jac}(h) = 1$ . Denote by  $\rho_0 = (2\pi)^{-2} d\theta d\phi$  the normalized Lebesgue measure on the torus. Then, the dynamic plan  $\mu^*$  in lemma 7.1, i.e. equation (7.1) with*

(7.34)

$$\zeta_t^0(\theta, \phi) = \sqrt{2} \sin(\sqrt{P^*}t), \quad \zeta_t^1(\theta, \phi) = \sqrt{2} |\cos(\sqrt{P^*}t)|, \quad \psi_t^1(\theta, \phi) = \begin{cases} (\theta, \phi) & t \leq T/2, \\ (\theta + \pi, \phi + \pi) & t > T/2, \end{cases}$$

where  $P^* = (\pi/T)^2$ , is a minimizer, whereas the constant speed rotation is not a minimizer.

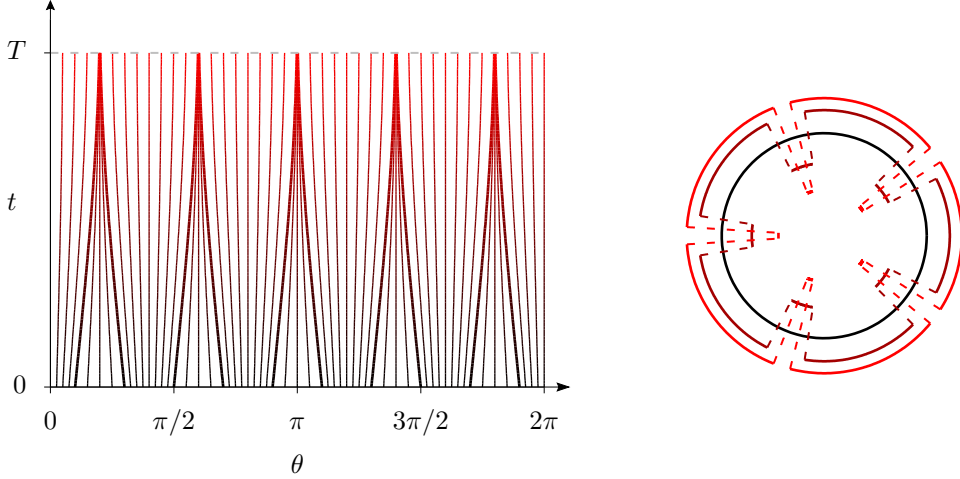


FIGURE 3. Particle trajectories  $t \mapsto \hat{\varphi}_t^n(\theta)$  relative to the map constructed in proposition 7.6 (left) and support of fixed time marginals for the measure  $(\hat{\varphi}^n, \sqrt{\text{Jac}(\hat{\varphi}^n)})_{\#} \rho_0$  (right), for  $n = 5$ .

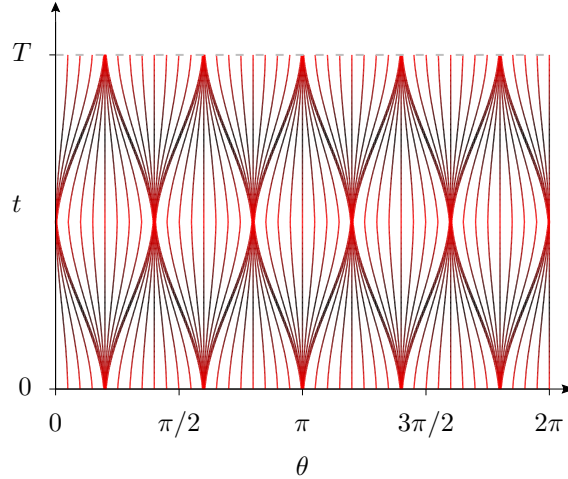


FIGURE 4. Particle trajectories  $t \mapsto \varphi_t^n(\theta)$  relative to the map constructed in proposition 7.7 for  $n = 5$ .

*Proof.* Consider the functional  $\pi_\theta : \Omega(T_{1,R}^2) \rightarrow \Omega(S_1^1)$  defined by

$$(7.35) \quad \pi_\theta(z) := (t \in [0, T] \mapsto [\theta_t, r_t] \in \mathcal{C}),$$

for any  $z = (t \in [0, T] \mapsto [(\theta_t, \phi_t), r_t] \in \mathcal{C})$ . In other words,  $\pi_\theta$  applies at each time the canonical projection on the circle of unit radius. We observe that for any admissible dynamic plan  $\mu \in \mathcal{P}(\Omega(T_{1,R}^2))$  for the generalized CH problem on the torus,

$$(7.36) \quad \mu_\theta := \pi_{\theta\#} \mu \in \mathcal{P}(\Omega(S_1^1))$$

is admissible for the generalized CH problem on  $S_1^1$  with boundary conditions associated with the map  $h_\theta : \theta \in \mathbb{R}/2\pi\mathbb{Z} \rightarrow \theta + \pi$ . In fact, if for example  $\mu$  satisfies the homogeneous marginal constraint with respect to the normalized measure  $(2\pi)^{-2} d\theta d\phi$ , then also  $\mu_\theta$  satisfies the same constraint since for any  $t \in [0, T]$  and  $f \in C^0(S_1^1)$ ,

$$(7.37) \quad \int_{\Omega(S_1^1)} f(\theta_t) r_t^2 d\mu_\theta(z) = \int_{\Omega(T_{1,R}^2)} f(\theta_t) r_t^2 d\mu(z) = \frac{1}{2\pi} \int_{S_1^1} f(\theta) d\theta,$$

and similarly for the coupling constraint. The problem on  $S_1^1$  admits a non-deterministic minimizer, which was given in theorem 7.2 and we denote it by  $\mu_\theta^*$ . Then, we have for any admissible  $\mu \in \mathcal{P}(\Omega(T_{1,R}^2))$ ,

$$(7.38) \quad \mathcal{A}(\mu) \geq \mathcal{A}(\mu_\theta) \geq \mathcal{A}(\mu_\theta^*) = \frac{\pi^2}{T}.$$

However, by lemma 7.1, the dynamic plan  $\mu^*$  defined by equation (7.34) satisfies  $\mathcal{A}(\mu^*) = \mathcal{A}(\mu_\theta^*)$  and so it must be a minimizer. On the other hand, the action for constant speed rotation is given by  $\mathcal{A}^{rot} = \pi^2(R^2 + 1)/T$  and therefore such a solution cannot be a minimizer since  $R > 0$ .  $\square$

**7.4. Approximation of a generalized minimizer on the torus.** The generalized minimizer in theorem 7.9 is very similar to its one-dimensional counterpart of theorem 7.2. Importantly, however, the extra dimension gives us enough flexibility to produce deterministic approximations, which is the main result of this section. Such approximations will be similar in spirit to those presented in the one-dimensional case. In brief, using again peakon/anti-peakon collisions we will be able to reach the final configuration by moving two complementary subsets of the domain at different times, when they occupy a small volume.

**Theorem 7.10.** *Let  $\mu^*$  and  $h$  be the minimizer and the coupling, respectively, defined in theorem 7.9 on the torus  $M = T_{1,R}^2$ . There exists a sequence of continuous flow maps  $\varphi^n : [0, T] \times M \rightarrow M$ ,  $n \in \mathbb{N}$ , such that for every  $t \in [0, T]$ ,  $\varphi_t^n : M \rightarrow M$  is smooth almost everywhere, and*

- for all  $n \in \mathbb{N}$ ,  $\varphi_0^n = \text{Id}$  and  $\varphi_T^n = h$ ;
- the sequence  $\mu_n := (\varphi^n, \sqrt{\text{Jac}(\varphi^n)})_{\#} \rho_0$  can be rescaled to a sequence  $\tilde{\mu}_n \rightarrow \mu^*$ ;
- $\mathcal{A}(\tilde{\mu}_n) \rightarrow \mathcal{A}(\mu^*)$ .

*Proof.* For simplicity, we prove the result for  $R = 1$  but the argument presented here applies for any  $R > 0$ . In addition, performing an appropriate change of variables, one can easily verify that it is sufficient to prove the theorem with  $h : (\theta, \phi) \in \mathbb{R}^2 / (2\pi\mathbb{Z})^2 \rightarrow (\theta, \phi + \pi)$  and  $\mu^*$  defined as in equation (7.34), but with  $\psi^1$  defined by

$$(7.39) \quad \psi_t^1(\theta, \phi) = \begin{cases} (\theta, \phi) & t \leq T/2, \\ (\theta, \phi + \pi) & t > T/2. \end{cases}$$

For each  $n \in \mathbb{N}$ , the map  $\varphi^n$  will be constructed using two basic flows. The first is defined as follows. Fix a sequence  $\epsilon_n = \epsilon_0/n^3$ ,  $n \in \mathbb{N}$ , where  $\epsilon_0$  is a sufficiently small constant. Moreover, for any  $\epsilon > 0$  consider the set  $B_\epsilon \subset S_1^1$  defined by

$$(7.40) \quad B_\epsilon := \bigcup_{i=0}^{n-1} \left[ \frac{\pi}{n}(2i+1) - \frac{\epsilon}{2}, \frac{\pi}{n}(2i+1) + \frac{\epsilon}{2} \right],$$

and let  $\phi_{rot}^n : S_1^1 \rightarrow S_1^1$  such that  $0 \leq \phi_{rot}^n \leq \pi$ ,  $\phi_{rot}^n(\theta) = \pi$  for all  $\theta \in B_{\epsilon_n}$  and  $\phi_{rot}^n(\theta) = 0$  for all  $\theta \in S_1^1 \setminus B_{2\epsilon_n}$ . For  $k = 0, 1$ , we let  $\varphi_{rot}^{k,n} : [0, \sqrt{\epsilon_n}] \times T_{1,1}^2 \rightarrow T_{1,1}^2$  be the flow defined by

$$(7.41) \quad (\varphi_{rot}^{0,n})_t(\theta, \phi) := \left( \theta, \phi + \frac{t}{\sqrt{\epsilon_n}} \phi_{rot}^n(\theta) \right), \quad (\varphi_{rot}^{1,n})_t(\theta, \phi) := \left( \theta, \phi + \frac{t}{\sqrt{\epsilon_n}} (\pi - \phi_{rot}^n(\theta)) \right).$$

Consider now the flow  $\hat{\varphi}^n$  defined in equation (7.19), with  $\epsilon_n$  defined as above. With a slight abuse of notation, we will also denote by  $\hat{\varphi}^n$  its canonical extension to the torus which leaves the  $\phi$  coordinate fixed. Moreover, for any  $\alpha \in \mathbb{R}/2\pi\mathbb{Z}$  denote by  $R_\alpha^\theta : T_{1,1}^2 \rightarrow T_{1,1}^2$  the map  $R_\alpha^\theta(\theta, \phi) := (\theta + \alpha, \phi)$ . Then, we define the flow  $\varphi_{exp}^{0,n} : [\sqrt{\epsilon_n}, T/2] \times T_{1,1}^2 \rightarrow T_{1,1}^2$  by

$$(7.42) \quad (\varphi_{exp}^{0,n})_t(\theta, \phi) := \begin{cases} \hat{\varphi}_{a_n(t)}^n((\hat{\varphi}_T^n)^{-1}(\theta, \phi)) & \text{if } t \leq \frac{T}{4}, \\ R_{-\pi/n}^\theta \circ \hat{\varphi}_{4t-T}^n \left( R_{\pi/n}^\theta \circ (\hat{\varphi}_T^n)^{-1}(\theta, \phi) \right) & \text{if } \frac{T}{4} < t \leq \frac{T}{2}, \end{cases}$$

where  $a_n(t) := T(T - 4t)(T - 4\sqrt{\epsilon_n})^{-1}$ . Note that setting  $\epsilon_n = 0$  this flow coincides with the canonical extension to the torus of the flow in equation (7.31). Similarly,

$$(7.43) \quad \varphi_{exp}^{1,n} := R_{-\pi/n}^\theta \circ \varphi_{exp}^{0,n} \circ R_{\pi/n}^\theta.$$

We construct the sequence  $\varphi^n$  by glueing together the maps  $\varphi_{rot}^{k,n}$  and  $\varphi_{exp}^{k,n}$  so that for each  $n \in \mathbb{N}$  the final flow consists of four stages: in the first,  $n$  stripes of the domain rotate while the rest of the domain stays put as prescribed by  $\varphi_{rot}^{0,n}$ ; in the second, the stripes expand up to a symmetric configuration in which the rest of the domain occupies stripes of the same size, as



prescribed by  $\varphi_{exp}^{0,n}$ ; in the third, the rest of the points rotate as prescribed by  $\varphi_{rot}^{1,n}$ ; finally, we use  $\varphi_{exp}^{1,n}$  to compress the stripes to their original size. More precisely,

$$(7.44) \quad \varphi_t^n := \begin{cases} (\hat{\varphi}_{rot}^{0,n})_t & \text{if } t \leq \sqrt{\epsilon_n}, \\ (\hat{\varphi}_{exp}^{0,n})_t \circ (\hat{\varphi}_{rot}^{0,n})_{\sqrt{\epsilon_n}} & \text{if } \sqrt{\epsilon_n} < t \leq \frac{T}{2}, \\ (\hat{\varphi}_{rot}^{1,n})_{t-T/2} \circ (\hat{\varphi}_{exp}^{0,n})_{T/2} \circ (\hat{\varphi}_{rot}^{0,n})_{\sqrt{\epsilon_n}} & \text{if } \frac{T}{2} < t \leq \frac{T}{2} + \sqrt{\epsilon_n}, \\ (\hat{\varphi}_{exp}^{1,n})_{t-T/2} \circ (\hat{\varphi}_{rot}^{1,n})_{\sqrt{\epsilon_n}} \circ (\hat{\varphi}_{exp}^{0,n})_{T/2} \circ (\hat{\varphi}_{rot}^{0,n})_{\sqrt{\epsilon_n}} & \text{if } \frac{T}{2} + \sqrt{\epsilon_n} < t \leq T. \end{cases}$$

A graphical representation of this flow is given in figure 7.4 for  $n = 1$  (so that we have only one stripe) and in the original coordinates (so that the boundary conditions are those associated with double rotation).

Note that the flow defined in equation (7.44) is very similar to the one defined in proposition 7.7, whose canonical extension to the torus will be denoted by  $\varphi^{0,n}$ . As in proposition 7.7, we define again the rescaled measure  $\tilde{\mu}_n$  using equation (7.33). This means that for any Lipschitz continuous bounded functional  $\mathcal{F} : \Omega \rightarrow \mathbb{R}$ ,

$$(7.45) \quad \int_{\Omega} \mathcal{F}(z) d\tilde{\mu}_n(z) = \frac{1}{4\pi^2} \int_{T_{1,1}^2} \mathcal{F} \left( \left[ \varphi^n \circ (\varphi_{T/4}^n)^{-1}(\theta, \phi), \bar{\lambda}^n(\theta, \phi) \right] \right) d\theta d\phi,$$

where

$$(7.46) \quad \bar{\lambda}_t^n := \left( \frac{\text{Jac}(\varphi_t^n)}{\text{Jac}(\varphi_{T/4}^n)} \right)^{1/2} \circ (\varphi_{T/4}^n)^{-1} = \left( \text{Jac}(\varphi_t^n \circ (\varphi_{T/4}^n)^{-1}) \right)^{1/2}.$$

Note that equation (7.45) is a direct consequence of the definition of the dilation map and the change of variables formula. Due to proposition 7.7 and the way  $\varphi^n$  is constructed, to prove the convergence  $\tilde{\mu}_n \rightarrow \mu^*$ , it is sufficient to focus on the interval  $[0, T/2]$  and check that  $I^n \rightarrow 0$ , where

$$(7.47) \quad I^n := \int_{T_{1,1}^2} \sup_{t \in [0, T/2]} d_C([\varphi_t^n \circ (\varphi_{T/4}^n)^{-1}, \bar{\lambda}_t^n], [\varphi_t^{0,n} \circ (\varphi_{T/4}^{0,n})^{-1}, \bar{\lambda}_t^{0,n}]) d\theta d\phi$$

and  $\bar{\lambda}_t^{0,n} := \left( \text{Jac}(\varphi_t^{0,n} \circ (\varphi_{T/4}^{0,n})^{-1}) \right)^{1/2}$ . Because of the similar structure of the flows  $\varphi^n$  and  $\varphi^{0,n}$ ,  $I^n$  reduces to

$$(7.48) \quad I^n = \int_{T_{1,1}^2} \sup_{t \in [0, T/4]} d_C([\varphi_t^n \circ (\varphi_{T/4}^n)^{-1}, \bar{\lambda}_t^n], [\varphi_t^{0,n} \circ (\varphi_{T/4}^{0,n})^{-1}, \bar{\lambda}_t^{0,n}]) d\theta d\phi.$$

Let  $A_\epsilon := \varphi_{T/4}^{0,n}(B_\epsilon \times S_1^1)$ , for any  $\epsilon > 0$ . We decompose  $I^n = I^{0,n} + I^{1,n}$  where  $I^{0,n}$  and  $I^{1,n}$  are the integrals over  $A_{2\epsilon_n}$  and  $T_{1,1}^2 \setminus A_{2\epsilon_n}$ , respectively. Define for  $0 \leq a < b \leq T/4$

$$(7.49) \quad I_{a,b}^{0,n} := \int_{A_{2\epsilon_n}} \sup_{t \in [a,b]} d_C([\varphi_t^n \circ (\varphi_{T/4}^n)^{-1}, \bar{\lambda}_t^n], [\varphi_t^{0,n} \circ (\varphi_{T/4}^{0,n})^{-1}, \bar{\lambda}_t^{0,n}]) d\theta d\phi.$$

We have  $I^n \leq I_{0,\sqrt{\epsilon_n}}^{0,n} + I_{\sqrt{\epsilon_n}, T/4}^{0,n} + I^{1,n}$ . By continuity of the flow maps it is easy to verify that  $I_{\sqrt{\epsilon_n}, T/4}^{0,n} \rightarrow 0$  and  $I^{1,n} \rightarrow 0$  as  $n \rightarrow +\infty$ . On the other hand, by construction  $0 < \bar{\lambda}^n, \bar{\lambda}^{0,n} < \sqrt{2}$ . Therefore, by the triangular inequality

$$(7.50) \quad \begin{aligned} I_{0,\sqrt{\epsilon_n}}^{0,n} &\leq \int_{A_{2\epsilon_n}} \sup_{t \in [0,\sqrt{\epsilon_n}]} (\bar{\lambda}_t^n + \bar{\lambda}_t^{0,n}) d\theta d\phi, \\ &\leq 4\pi n \epsilon_n (2\sqrt{2}) + \int_{B_{\pi/n} \times S_1^1} \sup_{t \in [0,\sqrt{\epsilon_n}]} (\bar{\lambda}_t^n + \bar{\lambda}_t^{0,n}) d\theta d\phi, \end{aligned}$$

where on the second line, we decomposed the integral over the part of  $A_{2\epsilon_n}$  that gets stretched and the part that gets compressed under  $\varphi^n \circ (\varphi_{T/4}^n)^{-1}$  for  $t \in [0, \sqrt{\epsilon_n}]$ . In particular, the integrand in the second line tends to 0 as  $n \rightarrow +\infty$ , which yields  $I^n \rightarrow 0$ . A similar argument can be applied on the interval  $[T/2, T]$ , which proves that  $\tilde{\mu}_n \rightarrow \mu^*$ .

In order to prove convergence of the action, in view of lemma 7.4, it is sufficient to show

$$(7.51) \quad \int_{T_{1,1}^2} \mathcal{A}([\varphi_{rot}^{k,n}(\theta, \phi), 1]) d\theta d\phi \rightarrow 0,$$

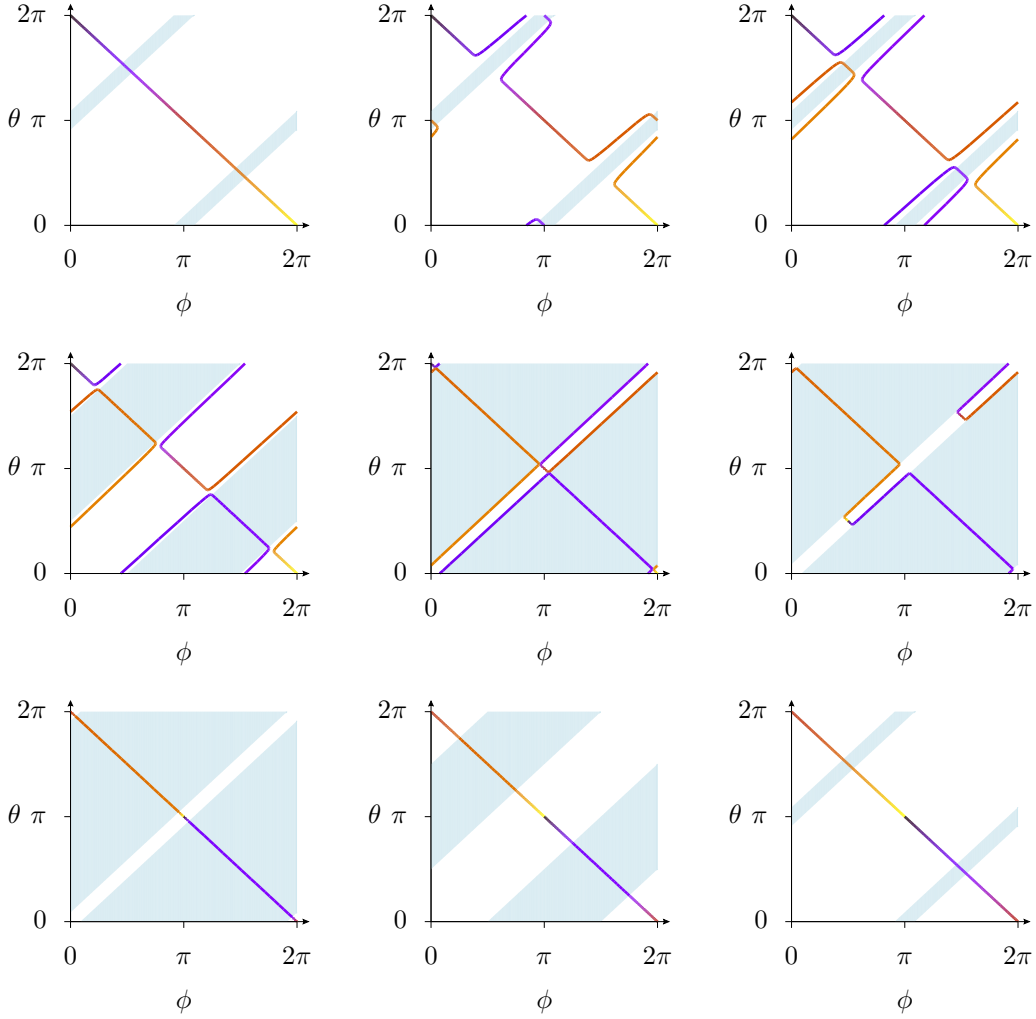


FIGURE 5. Particle trajectories for the flow  $\varphi^n$  in (7.44) for  $n = 1$  (in appropriate coordinates to determine double rotation on the torus). We use different colors to label particles and follow their motion. The stripes indicate the particles in between the peaks' peaks. In the limit, the trajectories of such particles lifted to the cone will start and end at the apex.

for  $k = 0, 1$ , as  $n \rightarrow +\infty$ , where the action is computed over the time interval  $[0, \sqrt{\epsilon_n}]$ . For all  $n \in \mathbb{N}$ , under the flow  $\varphi_{rot}^{k,n}$  only points with  $\theta \in B_{2\epsilon_n}$  rotate with velocity bounded by  $\pi/\sqrt{\epsilon_n}$ ; hence

$$(7.52) \quad \int_{T_{1,1}^2} \mathcal{A}([\varphi_{rot}^{k,n}(\theta, \phi), 1]) \, d\theta \, d\phi \leq 2\pi(n2\epsilon_n) \frac{\pi^2}{\sqrt{\epsilon_n}} = \sqrt{\frac{\epsilon_0}{n}} 4\pi^3,$$

which concludes the proof.  $\square$

**Remark 7.11.** *As for the one-dimensional case (see remark 7.11), the maps defined by equation (7.44) are piecewise smooth in space since their Jacobian is piecewise constant with a finite number of discontinuities. Also in this case, it is sufficient to repeat the construction above using a regularized version of the linear peakon/anti-peakon collision, to obtain a sequence of smooth diffeomorphisms satisfying theorem 7.10.*

## 8. DISCRETE GENERALIZED SOLUTIONS

There are two main obstacles in translating problem 4.2 to the discrete setting. On one hand, we need to make computations on an unbounded domain; on the other, we need to be able

to single out a representative for the equivalence class of minimizers with respect to rescaling. However, if one is interested in simulating solutions that are not singular (see definition 4.11), it is appropriate to enforce the strong coupling constraint in (4.6) instead of (4.8). Hence, if we substitute  $\mathcal{C}$  by  $\mathcal{C}_R$  for a fixed  $R > 1$  and use the strong coupling constraint in the generalized CH problem, we obtain a modified formulation that is able to reproduce a particular class of solutions, which includes all deterministic solutions with bounded Jacobian. In this section we describe a numerical algorithm based on entropy regularization and Sinkhorn algorithm that solves such a modified formulation. Our scheme is based on similar methods for the incompressible Euler equations developed in [30, 6, 5]. We also provide some numerical results illustrating the behavior of generalized CH flows.

**8.1. Discrete formulation.** We set  $M = [0, 1]^d$  and consider a uniform discretization with points  $\{x_i\}_{i=1}^{N_x}$ , and a discretization of the interval  $(0, R]$  with points  $\{r_i\}_{i=1}^{N_r}$  such that  $r_j = 1$  for a fixed  $j \in \{1, \dots, N_r\}$ . These induce a discretization of the cone with points  $\{z_i\}_{i=1}^N$  where  $N = N_x N_r$ . Similarly, we also consider a uniform discretization  $\{t_i\}_{i=1}^K$  of  $[0, T]$ . Generalized flows are then replaced by a coupling arrays  $\boldsymbol{\mu} \in (\mathbb{R}_{\geq 0}^N)^K$ . Note that we can incorporate the boundary condition  $\lambda_0 = 1$  by reducing the dimension of  $\boldsymbol{\mu}$ . In particular, we now denote by  $\pi_x$  and  $\pi_r$  the canonical projections from  $M \times (0, R]$  to  $M$  and  $(0, R]$  respectively. We use the same notation to indicate the maps  $\pi_x : \{1, \dots, N\} \rightarrow \{1, \dots, N_x\}$  and  $\pi_r : \{1, \dots, N\} \rightarrow \{1, \dots, N_r\}$  mapping directly the discretization indices. Then, we set for any  $\{j_1, \dots, j_K\} \in \{1, \dots, N\}^K$ ,

$$(8.1) \quad \boldsymbol{\mu}_{j_1, \dots, j_K} = \mathbb{1}_{\{\pi_r(z_{j_1})=1\}} \tilde{\boldsymbol{\mu}}_{\pi_x(j_1), j_2, \dots, j_K},$$

where  $\mathbb{1}$  is the indicator function and  $\tilde{\boldsymbol{\mu}} \in \mathbb{R}_{\geq 0}^{N_x} \times (\mathbb{R}_{\geq 0}^N)^{K-1}$ . We denote by  $\Pi_0$  the set of couplings satisfying (8.1). The marginal at a given time  $t_k$  is a discrete measure on  $M \times (0, R]$ . We denote this by  $S_k(\boldsymbol{\mu}) \in \mathbb{R}_{\geq 0}^N$ , and it is defined as follows:

$$(8.2) \quad [S_k(\boldsymbol{\mu})]_j = \sum_{j_1, \dots, j_{k-1}, j_{k+1}, \dots, j_K} \boldsymbol{\mu}_{j_1, \dots, j_{k-1}, j, j_{k+1}, \dots, j_K}.$$

We denote by  $M_n : \mathbb{R}_{\geq 0}^N \rightarrow \mathbb{R}_{\geq 0}^N$  the  $n$ th moment taken in the radial direction, i.e.

$$(8.3) \quad M_n[A]_i = \sum_{j, \pi_x(j)=i} \pi_r(z_j)^n A_j.$$

Hence the constraint in (4.4) becomes

$$(8.4) \quad M_2[S_k(\boldsymbol{\mu})]_i = 1/N_x.$$

Moreover, we denote by  $\Pi$  the set of admissible coupling arrays,

$$(8.5) \quad \Pi = \{\boldsymbol{\mu} \in \Pi_0; \forall i, M_2[S_k(\boldsymbol{\mu})]_i = 1/N_x\}.$$

The constraint on the coupling between time 0 and  $T$  can be enforced weakly by including it directly in the cost, which is given by the following array

$$(8.6) \quad C_{j_1, \dots, j_K} = \frac{K-1}{T} \sum_{k=1}^{K-1} d_C(z_{j_k}, z_{j_{k+1}})^2 + \alpha d_C(z_{j_K}, (h(\pi_x(z_{j_1})), \sqrt{\text{Jac}(h)}))^2,$$

where  $\alpha > 0$  is a parameter. The regularized discrete problem is then,

$$(8.7) \quad \min_{\boldsymbol{\mu} \in \Pi} \langle C, \boldsymbol{\mu} \rangle - \epsilon E(\boldsymbol{\mu}),$$

where  $\epsilon > 0$  is another parameter and  $E(\boldsymbol{\mu})$  is the entropy of the coupling defined by

$$(8.8) \quad E(\boldsymbol{\mu}) = -\langle \boldsymbol{\mu}, \log(\boldsymbol{\mu}) - 1 \rangle.$$

Problem (8.7) can be solved by means of alternating projections which consist in enforcing recursively the marginal constraints at the different time levels. In particular, we consider the following augmented functional

$$(8.9) \quad \min_{\boldsymbol{\mu}} \langle C, \boldsymbol{\mu} \rangle - \epsilon E(\boldsymbol{\mu}) - \sum_{i,k} p_i^k (M_2[S_k(\boldsymbol{\mu})]_i - 1/N_x),$$

where  $p^k \in \mathbb{R}^{N_x}$  for all  $k \in \{1, \dots, K\}$ . From (8.9) we obtain

$$(8.10) \quad \boldsymbol{\mu}_{j_1, \dots, j_K} = e^{-\frac{C_{j_1, \dots, j_K}}{\epsilon}} e^{\sum_k p_{\pi_x(j_k)}^k r_{\pi_r(j_k)}^2}.$$

Enforcing the constraint at time level  $n$  allows us to solve for  $p^n$  given the set  $\{p^k\}_{k \neq n}$ . This amounts to solving the following nonlinear equation for all  $i \in \{1, \dots, N_x\}$ ,

$$(8.11) \quad \sum_j B_{i,j} e^{p_i^n r_j^2} r_j^2 = 1/N_x,$$

where

$$(8.12) \quad B = S_n \left[ e^{-\frac{c_{j_1, \dots, j_K}}{\epsilon} e^{\sum_{k, k \neq n} p_{\pi_x(j_k)}^k r_{\pi_r(j_k)}^2}} \right].$$

Due to the structure of the cost, we only need to store two arrays  $D^0, D^1 \in \mathbb{R}^N \times \mathbb{R}^N$ , given by

$$(8.13) \quad D_{i,j}^0 = d_{\mathcal{C}}(z_i, z_j)^2, \quad D_{i,j}^1 = d_{\mathcal{C}}(z_i, (h(\pi_x(z_j)), \sqrt{\text{Jac}(h)}))^2.$$

**8.2. Numerical results: from CH to Euler.** We now present some numerical results illustrating the behavior of generalized solutions of the CH problem and their relation to generalized incompressible Euler solutions. We consider two types of couplings to define the boundary conditions: a classical deterministic coupling, which we use to illustrate the emergence of discontinuities in the flow map, and a generalized coupling that obliges particles to cross each other so that the solution is not deterministic. For both cases, the domain will be the one-dimensional interval  $M = [0, 1]$  and  $T = 1$ .

*A peakon-like solution.* Consider the continuous map  $h : [0, 1] \rightarrow [0, 1]$ , defined by

$$(8.14) \quad h(x) = \begin{cases} 1.4x & \text{if } x \leq 0.5, \\ 0.6x + 0.4 & \text{if } x > 0.5. \end{cases}$$

We use this map to define the coupling on the cone as in equation (4.6). We compute the solution using the algorithm presented in the previous section with  $N_x = 40$ ,  $N_r = 41$ ,  $0.55 \leq r \leq 1.45$ ,  $K = 35$ ,  $\alpha = 40$ ,  $\epsilon = 5 \cdot 10^{-4}$ . In figure 6 we show the evolution of the transport plan on the domain  $M$  given by  $(e_{0,t_k}^M)_{\#} \mu \in \mathcal{P}(M^2)$ , where  $e_{0,t_k}^M(z) := (x_0, x_{t_k})$ , for selected times. In figure 7 we show the evolution of the marginals on the cone given by  $(e_{t_k})_{\#} \mu \in \mathcal{P}(\mathcal{C})$  for the same times. We remark that the dynamic plan is approximately deterministic since there is very little diffusion of the mass in the domain, which is at least partially due to the entropic regularization. In addition the discontinuity in the Jacobian of the coupling map propagates to the whole solution, which resembles a peakon with the discontinuity point corresponding to the peak of the peakon.

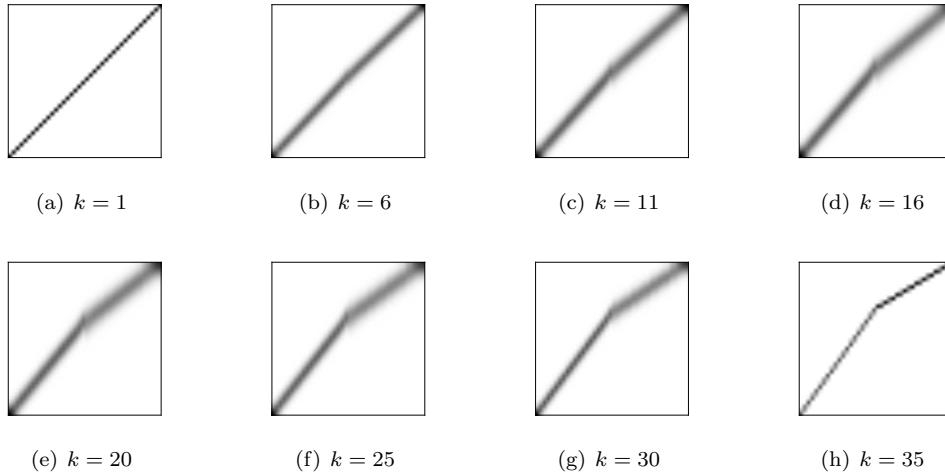


FIGURE 6. Transport couplings  $(e_{0,t_k}^M)_{\#} \mu$  on  $M \times M$  for the peakon-like solution associated with the boundary conditions specified by the map in equation (8.14).

*A non-deterministic solution.* The homogeneous marginal constraint allows us to consider very general couplings even defined by non-injective maps or maps that do not preserve the local

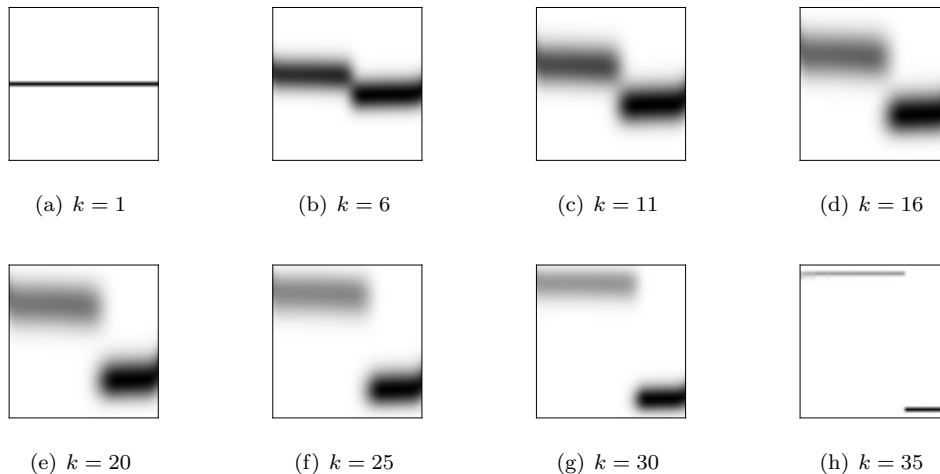


FIGURE 7. Fixed time marginals  $(e_{t_k})_{\#}\mu$  on the cone section  $M \times [r_{min}, r_{max}]$  ( $r_{min} = 0.55$ ,  $r_{max} = 1.45$ ) for the peakon-like solution associated with the boundary conditions specified by the map in equation (8.14).

orientation of the domain. Measure-preserving maps provide a special example since these were used by Brenier to define boundary conditions for generalized incompressible Euler flows. In fact if  $h$  is measure-preserving, i.e.  $h_{\#}\rho_0 = \rho_0$ , then we can use as coupling

$$(8.15) \quad \gamma = [(\text{Id}, 1), (h, 1)]_{\#}\rho_0.$$

The existence of a generalized solution of the CH problem in this case is a direct consequence of the existence result proved by Brenier in [7]; this is because generalized Euler solution can be easily lifted to admissible solutions of our formulation concentrated on paths  $z$  with  $r_t = 1$  for every time. Here, we take  $h : [0, 1] \rightarrow [0, 1]$  to be the map

$$(8.16) \quad h(x) = 1 - x,$$

which can only be realized by a non-deterministic plan. We compute the discrete solution associated with such boundary conditions with  $N_x = 40$ ,  $N_r = 41$ ,  $0.6 \leq r \leq 1.4$ ,  $K = 35$ ,  $\alpha = 40$ ,  $\epsilon = 5 \cdot 10^{-4}$ . As before, we show the evolution of the transport plan on the domain  $M$  given by  $(e_{0,t_k}^M)_{\#}\mu \in \mathcal{P}(M^2)$  in figure 8. In figure 9 we show the evolution of the marginals on the cone given by  $(e_{t_k})_{\#}\mu \in \mathcal{P}(\mathcal{C})$ . The transport plan evolution is remarkably similar to that of the incompressible Euler equation for the same coupling (see, e.g., [6]). However, the two do not coincide as it is evident from the marginals on the cone in figure 9. In the case of incompressible Euler, these marginals are concentrated on  $r = 1$  for every time, i.e. the transport plan remains measure-preserving during the evolution. This is clearly not the case for the generalized CH solution, for which also the Jacobian appears to be non-deterministic.

## 9. OUTLOOK

There are several natural questions that were not addressed in this paper and that we reserve to future work:

- *Tight relaxation.* Brenier's relaxation of incompressible Euler is not tight in two dimensions but it is in three dimensions due to the work of Shnirelman [33]. It is an open question whether a similar result holds for the generalized problem studied in this paper. The approximation results in section 7 suggest that this is the case. In particular, we conjecture that our formulation is a tight relaxation of the  $H(\text{div})$  geodesic boundary value problem in dimension  $d \geq 2$ .

As for the generalized Euler solutions, a better understanding of the structure of minimizing generalized flows is of theoretical interest:

- *Occurrence of singular solutions.* In this paper we did not fully characterize the emergence of singular solutions. Even for the case of rotation on the circle or on the torus,

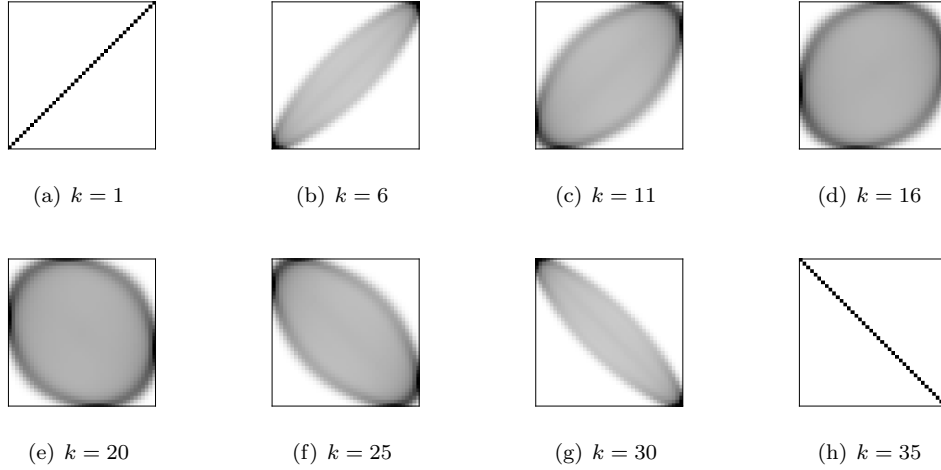


FIGURE 8. Transport couplings  $(e_{0,t_k}^M)_{\#}\mu$  on  $M \times M$  for the non-deterministic solution associated to the boundary conditions specified by the map in equation (8.16).

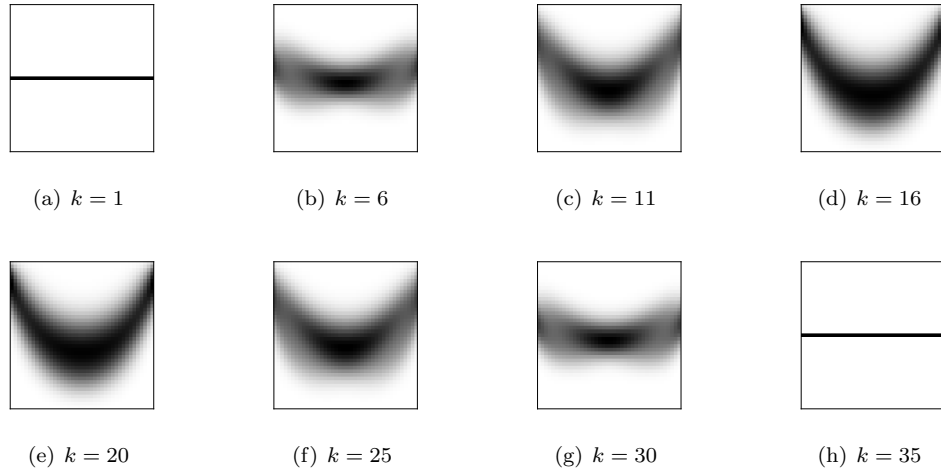


FIGURE 9. Fixed time marginals  $(e_{t_k})_{\#}\mu$  on the cone section  $M \times [r_{min}, r_{max}]$  ( $r_{min} = 0.6, r_{max} = 1.4$ ) for the non-deterministic associated with the boundary conditions specified by the map in equation (8.16).

for example, we did not prove that these are the unique minimizers for the problem. In addition, such examples suggest that singular solutions appear whenever particles' displacement is sufficiently large. It would be interesting to give a full characterization in this direction, specifying when solutions are singular in terms of the boundary conditions and the dimension and geometry of the base space  $M$ ;

- *Regularity of the pressure.* Brenier's result on the existence and uniqueness of the pressure in incompressible Euler was subsequently improved by Ambrosio and Figalli [1] in terms of regularity of the pressure field. It is natural to ask whether such a result can be extended to the generalized CH problem. This question is related to the previous one, due to the fact that a sufficiently regular pressure field can prevent the occurrence of singular solutions as it can be deduced from the proofs in section 6.

Addressing these theoretical questions will also guide the development of numerical schemes which are better suited to the formulation considered in this paper than methods based on

entropic regularization. A viable alternative in this context is given by semi-discrete methods, whose use for the generalized CH problem will also be studied in future work.

#### ACKNOWLEDGMENTS

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#### APPENDIX A. PROOF OF LEMMA 4.3

*Proof.* Here we prove that the homogeneous marginal constraint can be enforced at each time rather than in integral form as in equation (4.9).

First, we prove that the constraint in equation (4.9) implies the one in equation (4.11). In order to show this, for any fixed  $t^* \in [0, T]$  and  $f \in C^0(M)$ , consider the following functionals

$$(A.1) \quad \mathcal{F}(z) := r_{t^*}^2 f(x_{t^*}), \quad \mathcal{F}_n(z) := \int_0^T r_t^2 f(x_t) \delta_{n,t^*}(t) dt,$$

where  $\delta_{n,t^*} : [0, T] \rightarrow \mathbb{R}$ ,  $n \in \mathbb{N}$ , is a Dirac sequence of continuous functions converging to  $\delta_{t^*}$ . Then for any  $z \in \Omega$ ,  $\mathcal{F}_n(z) \rightarrow \mathcal{F}(z)$  as  $n \rightarrow +\infty$ . Moreover, using Jensen's inequality,

$$(A.2) \quad \begin{aligned} \mathcal{F}_n(z) &\leq \|f\|_{C^0} \int_0^T r_t^2 \delta_{n,t^*} dt \\ &\leq 2\|f\|_{C^0} \left( r_0^2 + \int_0^T (r_t - r_0)^2 \delta_{n,t^*} dt \right) \\ &\leq 2\|f\|_{C^0} \left( r_0^2 + \int_0^T r_t^2 dt \int_0^T t \delta_{n,t^*} dt \right) \\ &\leq 2\|f\|_{C^0} (r_0^2 + T\mathcal{A}(z)). \end{aligned}$$

The right-hand side is  $\mu$ -integrable since  $\mathcal{A}(\mu) < +\infty$  and because of the coupling constraint. Hence, we get the result by the dominated convergence theorem.

Similarly, if  $f \in C^0([0, T] \times M)$ , we take

$$(A.3) \quad \mathcal{F}(z) := \int_0^T f(t, x_t) r_t^2 dt, \quad \mathcal{F}_n(z) := \frac{T}{K} \sum_{k=0}^K f(t_k, x_{t_k}) r_{t_k}^2,$$

where  $t_k := kT/K$ . Then for any  $z \in \Omega$ ,  $\mathcal{F}_n(z) \rightarrow \mathcal{F}(z)$  as  $n \rightarrow +\infty$ . Moreover,

$$(A.4) \quad \begin{aligned} \mathcal{F}_n(z) &\leq 2\|f\|_{C^0} \left( r_0^2 + \frac{T}{K} \sum_{k=1}^K (r_{t_k} - r_0)^2 \right) \\ &\leq 2\|f\|_{C^0} \left( r_0^2 + \frac{T}{K} \sum_{k=1}^K t_k \int_0^{t_k} r_t^2 dt \right) \\ &\leq 2\|f\|_{C^0} (r_0^2 + T^2 \mathcal{A}(z)), \end{aligned}$$

and we can apply again the dominated convergence theorem to conclude the proof.  $\square$

#### APPENDIX B. PROOF OF LEMMA 6.1

*Proof.* Throughout this proof, all metric operations are performed with respect to the cone metric  $g_{\mathcal{C}}$ , so to simplify the notation we will simply use  $\|\cdot\|$  for the norm and  $\langle \cdot, \cdot \rangle$  for the inner product on  $T\mathcal{C}$ . Moreover, given a vector field  $u$  on the cone and a curve  $t \mapsto p(t) \in \mathcal{C}$ ,  $\nabla_t u(p(t)) := \nabla_{\dot{p}(t)} u(p(t))$  is the covariant derivative of  $u$  at  $p(t)$  with respect to the vector  $\dot{p}(t)$ .

Given a smooth solution  $(\varphi, \lambda)$  and a fixed  $x \in M$ , let  $z^* = [x^*, r^*] \in \Omega$  be the curve defined by  $x^* : t \rightarrow x_t^* := \varphi_t(x)$  and  $r^* : t \rightarrow r_t^* := \lambda_t(x)$ . We want to show that for any curve  $z \in AC^2([0, T]; \mathcal{C})$  such that  $z \neq z^*$ ,  $z_0 = z_0^*$  and  $z_T = z_T^*$ , we have  $\mathcal{B}(z) > \mathcal{B}(z^*)$ . We proceed in two steps: first we show that the inequality holds when  $z$  is smooth and when the geodesics



between  $z_t^*$  and  $z_t$  are smooth for all  $t \in [0, T]$ ; then we derive sufficient conditions for which the inequality holds also for curves  $z$  which are farther away from  $z^*$ .

Let  $s \in [0, 1] \mapsto c(t, s) \in \mathcal{C}$  be a family of geodesics parameterized by  $t \in [0, T]$  such that  $c(t, 0) = z_t^*$  and  $c(t, 1) = z_t$ . In order for such geodesics to be smooth we need to assume

$$(B.1) \quad d_M(x_t^*, x_t) < \pi, \quad \forall t \in [0, T].$$

Let  $J(t, s) := \partial_t c(t, s)$ , which is a Jacobi field when restricted to any geodesic  $c(t, \cdot)$  for any fixed  $t \in [0, T]$ . Moreover,  $J(t, 0) = \dot{z}_t^*$  and  $J(t, 1) = \dot{z}_t$ . Hence we want to show that

$$(B.2) \quad \int_0^T \|J(t, 0)\|^2 - \Psi_p(t, c(t, 0)) dt \leq \int_0^T \|J(t, 1)\|^2 - \Psi_p(t, c(t, 1)) dt.$$

Let  $C := \sup_{t \in [0, T]} \sup_{x \in M} \|\text{Hess } \Psi_p\|$ . The Taylor expansion of  $\Psi_p(t, c(s, t))$  with respect to  $s$  at  $s = 0$  yields

$$(B.3) \quad \Psi_p(t, c(t, 1)) - \Psi_p(t, c(t, 0)) - \langle \nabla \Psi_p(t, c(t, 0)), \partial_s c(t, 0) \rangle \leq \frac{C}{2} \int_0^1 \|\partial_s c(t, s)\|^2 ds.$$

Since  $\partial_s c(t, s) = 0$  at  $t = 0$  and  $t = T$ , by the Poincaré inequality we also have

$$(B.4) \quad \int_0^T \|\partial_s c(t, s)\|^2 dt \leq \frac{T^2}{\pi^2} \int_0^T |\partial_t \|\partial_s c(t, s)\||^2 dt \leq \frac{T^2}{\pi^2} \int_0^T \|\nabla_t \partial_s c(t, s)\|^2 dt.$$

Let  $\dot{J}(t, s) := \nabla_s \partial_t c(t, s)$  and exchanging the order of derivatives in the equation above we obtain

$$(B.5) \quad \int_0^T \|\partial_s c(t, s)\|^2 dt \leq \frac{T^2}{\pi^2} \int_0^T \|\dot{J}(t, s)\|^2 dt.$$

Integrating over  $[0, T]$  equation (B.3) and using equation (B.5) we get

$$(B.6) \quad \int_0^T \Psi_p(t, c(t, 1)) - \Psi_p(t, c(t, 0)) - \langle \nabla \Psi_p(t, c(t, 0)), \partial_s c(t, 0) \rangle dt \leq \frac{CT^2}{2\pi^2} \int_0^1 \|\dot{J}(t, s)\|^2 ds.$$

Consider the term involving the gradient of  $\Psi_p$ . Substituting  $\nabla \Psi_p(t, c(t, 0)) = -2\nabla_t \dot{z}_t^* = -2\nabla_t J(t, 0)$ , integrating by parts in  $t$ , and exchanging the order of derivatives for this term yields

$$(B.7) \quad \int_0^T \Psi_p(t, c(t, 1)) - \Psi_p(t, c(t, 0)) - 2\langle J(t, 0), \dot{J}(t, 0) \rangle dt \leq \frac{CT^2}{2\pi^2} \int_0^1 \|\dot{J}(t, s)\|_{gc}^2 ds.$$

Let  $f(s) := \int_0^T \|J(t, s)\|^2 dt$ , then

$$(B.8) \quad f'(0) = \int_0^T 2\langle J(t, 0), \dot{J}(t, 0) \rangle dt,$$

and

$$(B.9) \quad \begin{aligned} f(1) - f(0) - f'(0) &= \int_0^1 (1-s)f''(s) ds \\ &= \int_0^1 \int_0^T 2(1-s)(\|\dot{J}(t, s)\|^2 + \langle J(t, s), \nabla_s \dot{J}(t, s) \rangle) dt ds \\ &\geq \int_0^1 \int_0^T 2(1-s)\|\dot{J}(t, s)\|^2 dt ds, \end{aligned}$$

where the last inequality is due to the fact that for a Jacobi field  $J(t, s)$ ,

$$(B.10) \quad \nabla_s \dot{J}(t, s) = -R(J(t, s), \partial_s c(t, s))\partial_s c(t, s),$$

where  $R$  is the Riemann tensor, which for any tangent vectors  $X$  and  $Y$  at the same point on the cone over a flat manifold satisfies  $\langle X, R(X, Y)Y \rangle \leq 0$ . Moreover since the Jacobi fields are finite dimensional and  $[0, T] \times M$  is compact, there exists a constant  $C_0 > 0$  such that

$$(B.11) \quad f(1) - f(0) - f'(0) \geq \frac{C_0}{2} \int_0^1 \int_0^T \|J(t, s)\|^2 dt ds.$$

Combining this with (B.7) and rearranging terms we obtain

$$(B.12) \quad \left( \frac{C_0}{2} - \frac{CT^2}{2\pi^2} \right) \int_0^1 \int_0^T \|J(t, s)\|^2 dt ds + \int_0^T \|J(t, 0)\|^2 - \Psi_p(t, c(t, 0)) dt \\ \leq \int_0^T \|J(t, 1)\|^2 - \Psi_p(t, c(t, 1)) dt.$$

Because of the inequality (6.3), shows that  $z^*$  is minimizing among all paths  $z \in \Omega$  which satisfy (B.1) and it is unique when the inequality is strict. Note that when  $M = S_1^1$ , the circle of unit radius, we can identify  $\mathcal{C}$  with  $\mathbb{R}^2$  and condition (B.1) is not necessary. Furthermore, since geodesics are straight lines with constant speed, from equation (B.9) we find  $C_0 = 2$ . This concludes the proof for the case  $M = S_1^1$ .

Now, assume that for all  $x \in M$ ,  $d_{\mathcal{C}}(z_{t_0}, z_{t_1}) \leq \epsilon$ , for all  $t_0, t_1 \in [0, T]$ . Let

$$(B.13) \quad B_\delta := \bigcap_{t \in [0, T]} \{q \in \mathcal{C}; d_{\mathcal{C}}(q, z_t^*) \leq \delta\},$$

and take  $\epsilon < \delta := \frac{r_{min}}{2}$ , where  $r_{min} := \min_{(t,x) \in [0, T] \times M} \lambda_t(x)$ . For any  $q \in B_\delta$  and any  $t \in [0, T]$  the geodesic path between  $q$  and  $z_t^*$  cannot pass through the apex, since otherwise the distance between the two points should be at least equal to  $r_{min}$ . In other words, we must have  $d_M(q, z_t^*) < \pi$  and the path  $z^*$  is minimizing among all paths  $z \in \Omega$  contained in  $B_\delta$ . Moreover, the geodesic path from  $z_0^*$  to  $z_T^*$  is also included in  $B_\delta$ . Consider the following quantity

$$(B.14) \quad E(\delta, q, T^*) := \inf_{p \in \partial B_\delta / \mathcal{C}(\partial M)} \left\{ \inf_{z \in AC^2([0, T^*]; \mathcal{C})} \left\{ \int_0^{T^*} \|\dot{z}_t\|^2 - \Psi_p(t, z_t) dt; z_0 = q \in B_\delta, z_T = p \right\} \right\},$$

which is the infimum action over the interval  $[0, T^*]$  among paths starting at a point  $q \in B_\delta$  and reaching its boundary  $\partial B_\delta$  (but not points on  $\partial M$ ) at time  $T^*$ . Given any path  $z$  such that  $z_0 = z_0^*$  and  $z_T = z_T^*$  not contained in  $B_\delta$ , we have

$$(B.15) \quad \mathcal{B}(z) \geq \inf_{T_1 + T_2 \leq T} (E(\delta, z_0^*, T_1) + E(\delta, z_T^*, T_2)),$$

and we want to show that  $\mathcal{B}(z) > \mathcal{B}(z^*)$ . We have

$$(B.16) \quad E(\delta, z_0^*, T_1) \geq \inf_p \inf_z \int_0^{T_1} \|\dot{z}_t\|^2 dt - (r_{max} + \delta)^2 CT_1 \\ \geq \frac{(\delta - \epsilon)^2}{T_1} - (r_{max} + \delta)^2 CT_1,$$

where  $C := \sup_{(t,x) \in [0, T] \times M} |P(t, x)|$  and  $r_{max} := \max_{(t,x) \in [0, T] \times M} \lambda_t(x)$ . Hence, by equation (B.15),

$$(B.17) \quad \mathcal{B}(z) \geq \frac{4(\delta - \epsilon)^2}{T} - (r_{max} + \delta)^2 CT.$$

On the other hand, we can deduce an upper bound for  $\mathcal{B}(z^*)$  using the geodesic path  $z^g$  between  $z_0^*$  and  $z_T^*$ , yielding

$$(B.18) \quad \mathcal{B}(z) \leq \int \|\dot{z}_t^g\|^2 dt + r_{max}^2 CT \leq \frac{\epsilon^2}{T} + r_{max}^2 CT.$$

Therefore we find the following sufficient condition for optimality of the path  $z^*$ :

$$(B.19) \quad [r_{max}^2 + (r_{max} + \delta)^2] CT \leq \frac{4(\delta - \epsilon)^2}{T} - \frac{\epsilon^2}{T}.$$

The right-hand side is positive if  $\epsilon < 2\delta/3$ . Hence taking  $\epsilon = \delta/2$  and substituting  $\delta = \frac{r_{min}}{2}$ ,

$$(B.20) \quad \left[ r_{max}^2 + \left( r_{max} + \frac{r_{min}}{2} \right)^2 \right] CT \leq \frac{3r_{min}^2}{8T}.$$

This is the same as equation (6.5). For uniqueness we only need to substitute the inequality in (B.20) by a strict one, which concludes the proof.  $\square$

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