The partial homogenization for a rod

Gregory Panasenko

The Partial Homogenization for a Rod

Gregory PANASENKO
University Jean Monnet, Laboratory of Mathematics of the University of Saint Etienne,
23, rue P.Michelon, 42023, St-Etienne,FRANCE,
grigory.panasenko@univ-st-etienne.fr

A great number of applied problems contain small parameters. Normally their presence either in the equation or in the domain makes the numerical implementation more complicated, more time and memory consuming. This issue emphasizes the importance of the asymptotic methods studying the behavior of the solution as the small parameter tends to zero. Nevertheless the asymptotic methods are often related to some cumbersome calculations, or they are not too comprehensible for non-specialists. That is why some special numerical methods taking into account the asymptotic behavior of the solution were developed. One of such ideas has been implemented in the numerical schemes uniform with respect to the small parameter [4],[8],[17] or in some projection numerical methods with a special choice of the projection space basis taking into account the regular part of an asymptotic solution [1],[9],[21], (the idea of projection procedure has been widely used in engineering); in the case of multi-scale problems the ideas of the superelements, of the hierarchic modelling (numerical homogenization) or the two-scale finite element methods is developed (see [20], justified in [6], as well as [2],[3],[7],[22]); another approach is to prescribe some special modified boundary conditions (the so called artificial boundary conditions) in order to increase the accuracy of the approximate solution [10].

In [13],[14] the multi-scale method of partial asymptotic domain decomposition is introduced. This method reduces the problem to a **simplified form on some subdomain of regular asymptotic behavior** of the solution (for example, by means of the dimension reduction) **keeping the initial formulation** on a small part of the domain **where the asymptotic behavior is singular** (for example, where the boundary layers are located). Then these two models are **coupled by some special interface conditions** respecting with great accuracy the asymptotic expansion of the solution. These interface conditions are obtained from some projection procedure in the variational formulation where the projection subspace keeps the asymptotic behavior of the solution **out of a boundary layer zone.** It differs this method from some earlier projection methods "imposing" the regular asymptotic behavior in the whole domain. This difference is especially important, for example, in case of thin domains of complex structure (such as the finite rod structures [11]).

We will apply here this method to the classical homogenization problem simulating physical fields in a composite rod. Homogenization techniques is an effective tool of simulation of macroscopic behavior of microheterogeneous media. Nevertheless, there is a difficulty of the analysis of boundary layers. We propose and justify here the partial homogenization, that keeps the initial equation in some thin neighbourhood of the ends of the rod, homogenizes the equation in the remaining part of the domain and prescribes the appropriate interface conditions for homogenized and non-homogenized parts. This approach allows to avoid the construction of boundary layers.

Consider the model problem in a heterogeneous rod with the Dirichlet boundary condition at the ends of the rod and the Neumann boundary condition on the lateral boundary. We assume that the direction of the rod is the axis x_1; the coefficients do not depend on x_1 whereas f to the contrary depends only on x_1.

Consider the conductivity equation set in a thin rod occupying the parallelepiped $G_\varepsilon = (0,d) \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})^{s-1}$:

$$\sum_{k,j=1}^{s} \frac{\partial}{\partial x_k}(A_{kj}(\frac{x'}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}) = f(x_1), \quad x \in G_\varepsilon,$$

$$\sum_{j=1}^{s} A_{kj}(\frac{x'}{\varepsilon})\frac{\partial u_\varepsilon}{\partial x_j}|_{\{x_k=\pm\frac{\varepsilon}{2}\}\cap\partial G_\varepsilon} = 0,\ k = 2,...,s,\ u_\varepsilon|_{x_1=0} = 0, \quad u_\varepsilon|_{x_1=d} = 0, \tag{2.1}$$

where $x = (x_1, x'),\ x' = (x_2, \dots, x_s),\quad A_{kj}(\xi')$ are functions, satisfying the following conditions:

$$(i)\exists\kappa_0 > 0,\ \forall\xi' \in [-\frac{1}{2}, \frac{1}{2}]^{s-1}, \forall\eta \in I\!R^s, \eta = (\eta_1, \dots, \eta_s), \quad \sum_{k,j=1}^{s} A_{kj}(\xi')\eta_j\eta_k \geq \kappa_0 \sum_{i=1}^{s} \eta_i^2$$

$$(ii)\quad \forall\xi' \in [-\frac{1}{2}, \frac{1}{2}]^{s-1}, k, j \in \{1, \dots, s\}, \quad A_{kj}(\xi') = A_{jk}(\xi'),$$

f is a C^∞ - regular function.

Let d be a finite number, multiple of ε.

Assume that A_{kj} are piecewise smooth functions in the sense [5]. Then there exists a unique solution to this problem (see [5]). So we assume that f varies only in direction of the rod whereas the coefficients, to the contrary, vary only in directions orthogonal to the rod.

This function u_ε satisfies the following variational formulation.

Let $H^1_{0,(0,d)}(G_\varepsilon)$ be a space that is completion of the space of $C^\infty-$ regular functions vanishing if $x_1 = 0$ or $x_1 = d$ with respect to the norm $H^1((0,d) \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})^{s-1})$; then u_ε is sought as function of $H^1_{0,(0,d)}(G_\varepsilon)$ satisfying

$$J(u,\varphi) = \int_{G_\varepsilon} \left\{ \sum_{k,j=1}^{s} A_{kj}(\frac{x'}{\varepsilon}) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_k} + f\varphi \right\} dx \ = \ 0, \quad \forall \varphi \in H^1_{0,(0,d)}(G_\varepsilon). \tag{2.2}$$

0.1 The structure of asymptotic expansion of the solution

The asymptotic expansion of the solution to problem (2.1) has a form ([12])

$$u_\varepsilon^{a\ (K)} = u_{BLO}^{(K)}(x_1, \frac{x}{\varepsilon}) + u_{BLd}^{(K)}(x_1, \frac{(x_1 - d, x')}{\varepsilon}) +$$

$$+ \sum_{l=0}^{K+1} \varepsilon^l N_l(\frac{x'}{\varepsilon}) D_1^l v_\varepsilon^{(K)}(x_1), \tag{2.3}$$

$$v_\varepsilon^{(K)}(x_1) = \sum_{j=0}^{K+1} \varepsilon^j v_j(x_1), \tag{2.4}$$

where the "boundary layers" $u_{BLO}^{(K)}$ and $u_{BLd}^{(K)}$ are exponentially decaying functions such that for all $x_1 \in [0,d]$, $|u_{BLO}^{(K)}(x_1,\xi)|, \ |u_{BLd}^{(K)}(x_1,\xi)| \ \le \ C_1 e^{-C_2|\xi_1|}$, $C_1, C_2 > 0, \ \ C_1, C_2$ do not depend on ε; $N_l(\xi')$ are solutions of the sequence of cell problems in $Q = (-\frac{1}{2}, \frac{1}{2})^{s-1}$:

$$L_{\xi\xi} N_1 = -\sum_{k=2}^{s} \frac{\partial}{\partial \xi_k} A_{k1}, \ \xi \in Q,$$

$$\sum_{j=2}^{s} A_{kj} \frac{\partial}{\partial \xi_j} N_1 = -A_{k1}, \xi \in \partial Q, \xi_k = \pm\frac{1}{2}, k = 2, ..., s, \tag{2.5}$$

$$L_{\xi\xi}N_l = -\sum_{k=2}^{s} \frac{\partial}{\partial \xi_k}(A_{k1}N_{l-1}) -$$

$$-\sum_{k=2}^{s} A_{1k}\frac{\partial N_{l-1}}{\partial \xi_k} - A_{11}N_{l-2} +$$

$$+ < \sum_{k=2}^{s} A_{1k}\frac{\partial N_{l-1}}{\partial \xi_k} + A_{11}N_{l-2} >, \ \xi \in Q,$$

$$\sum_{j=2}^{s} A_{kj}\frac{\partial}{\partial \xi_j}N_1 = -A_{k1}, \xi \in \partial Q, \xi_k = \pm\frac{1}{2}, k = 2, ..., s,$$

$$l \geq 2, \tag{2.6}$$

here $<>= \int_Q d\xi'$, $L_{\xi\xi} = \sum_{k,j=2}^{s} \frac{\partial}{\partial \xi_k}(A_{kj}(\xi')\frac{\partial}{\partial \xi_j})$; $< N_l >= 0$ for $l > 0$, $N_0 = 1$, $D_1 = \frac{\partial}{\partial x_1}$.

These cell problems are set in variational formulation in $H^1(Q)$. This sequence of problems is solved recurrently for $l = 1, 2, \ldots, K+1$.

Functions v_j are defined in another sequence of problems (see [12]) but we do not need this information for the further developments.

0.2 The partial asymptotic decomposition of a heterogeneous domain

Let us apply the method of asymptotic partial decomposition of domain [13],[14] for problem (2.1) (in formulation (2.2)).

Consider the subspace $H_{\varepsilon,\delta,\ dec}$ of the space $H^1_{0,(0,d)}(G_\varepsilon)$ that consists of all functions u of $H^1_{0,(0,d)}(G_\varepsilon)$ having for all $x_1 \in [\delta, d-\delta]$ the following presentation in the form of Bakhvalov's ansatz [5]:

$$u(x) = \sum_{l=0}^{K+1} \varepsilon^l N_l(\frac{x'}{\varepsilon}) D_1^l v, \tag{2.7}$$

where $v \in H^{K+2}((\delta, d-\delta))$.

So every $u \in H_{\varepsilon,\delta,dec}$ is related by (2.7) to some $v \in H^{K+2}((\delta, d-\delta))$, so that we can consider a pair (u, v).

Remark 2.1. Asymptotic solution (2.3),(2.4) can be modified in such a way that it belongs to $H_{\varepsilon,\delta,\ dec}$. Indeed, let us multiply the boundary layer function $u_{BLO}^{(K)}(x, \frac{x}{\varepsilon})$ by $\chi(\frac{x_1 - C_2\varepsilon(K+2)|ln(\varepsilon)|}{\varepsilon})$, where $\chi(t)$ is equal to 1 for $t \leq 0$, and it is equal to zero for $t \geq 1$, passing smoothly from 1 to 0 on the segment [0,1]; χ does not depend on ε; in the same way let us multiply $u_{BLd}^{(K)}(x, \frac{(x_1-d,x')}{\varepsilon})$ by $\chi(\frac{x_1-d+C_2\varepsilon(K+2)|ln(\varepsilon)|}{\varepsilon})$.

Now, so modified $u_\varepsilon^{a\ (K)}$ will be kept $O(\varepsilon^{K+1})$ close to the exact solution in the norm $H^1(G_\varepsilon)$ The same will be the order of discrepancy in (2.2). And this function belongs to $H_{\varepsilon,\delta,\ dec}$ for $\delta \geq (C_2+1)\varepsilon(K+2)|ln(\varepsilon)|$. Denote in what follows $u^{(K)}$ so modified as $u_\varepsilon^{a\ (K)}$. We will omit as well the subscript ε in $v_\varepsilon^{(K)}$.

Let $\delta = \hat{K}\varepsilon \mid ln\ \varepsilon \mid$, where $\hat{K}$ is some positive number independent of ε. Consider the partially decomposed variational problem

$$J(u,\varphi) = 0, \quad \forall \varphi \in H_{\varepsilon,\delta,dec}. \tag{2.8}$$

If u_d is its solution then for any $K \in (0,\infty)$, there exist $\hat{K}$ such that if $\delta = \hat{K}\ \varepsilon \mid ln\ \varepsilon \mid$ then the estimate holds ([14])

$$\|u_\varepsilon - u_d\|_{H^1(G_\varepsilon)} = O(\varepsilon^{K+1}). \tag{2.9}$$

In the parallepiped $(\delta, d-\delta) \times (-\frac{\varepsilon}{2}, \frac{\varepsilon}{2})^{s-1}$ u_d is presented by formula (2.7) as well as

$$\varphi = \sum_{l=0}^{K+1} \varepsilon^l N_l(\frac{x'}{\varepsilon}) D_1^l w,$$

$$w \in H^{K+2}((\delta, d-\delta)). \tag{2.10}$$

Calculate integrals

$$I_1(u,\varphi) \ = \ \int_{(\delta,d-\delta)\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})^{s-1}} \{\sum_{k,j=1}^{s} A_{kj}(\frac{x'}{\varepsilon})\frac{\partial u}{\partial x_j}.\frac{\partial \varphi}{\partial x_k}\}dx$$

and

$$I_2(\varphi) \ = \ \int_{(\delta,d-\delta)\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})^{s-1}} f(x_1)\varphi dx$$

for $u, \varphi \in H^{K+2}_{\varepsilon,\delta,dec}$, satisfying (2.7) and (2.10). We get

$$I_1(u,\varphi) \ = \ \varepsilon^{s-1} \int_{(\delta,d-\delta)} \sum_{l,m=1}^{K+2} \varepsilon^{l+m-2}\ \tilde{h}_{l,m} D_1^l v D_1^m w dx_1,$$

$$I_2(\varphi) \;=\; \varepsilon^{s-1}\int_{(\delta,d-\delta)} f(x_1)w(x_1)dx_1,$$

where

$$\tilde{h}_{l,m} = \sum_{k,j=1}^{s} < A_{kj}(\xi')(\frac{\partial}{\partial \xi_j}N_l + \delta_{j1}N_{l-1})(\frac{\partial}{\partial \xi_k}N_m + \delta_{k1}N_{m-1}) >$$

defined by the solutions of cell problems (2.5),(2.6). The terms containing derivatives $\frac{\partial}{\partial \xi_k}$ are dropped if $m = K+2$, and $\frac{\partial}{\partial \xi_j}$ are dropped if $l = K+2$; these coefficients were introduced for the problem stated in the whole space in [37]. This representation holds true due to the separation of variables x' and x_1 in (2.7) and (2.10) and because $< N_l >= 0$ for all $l > 0$.

Below ε is a fixed positive number small enough.

Consider space $H_{\varepsilon,\delta,dec}\times H^{K+2}((\delta,d-\delta))$ supplied by the following inner product for

$$(u,v),(\varphi,w) \in H_{\varepsilon,\delta,dec}\times H^{K+2}((\delta,d-\delta))$$

related by (2.7) and (2.10):

$$\begin{aligned} a(u,v;\varphi,w) \;&=\; \int_{G_{BL}}\{\sum_{k,j=1}^{s} A_{kj}(\frac{x'}{\varepsilon})\frac{\partial u}{\partial x_j}.\frac{\partial \varphi}{\partial x_k}\}dx \;+ \\ &+\; \varepsilon^{s-1}\int_{(\delta,d-\delta)}\sum_{l,m=1}^{K+2}\varepsilon^{l+m-2}\;\tilde{h}_{l,m}D_1^l v D_1^m w dx_1 \;= \\ &=\; \int_{G_\varepsilon}\{\sum_{k,j=1}^{s} A_{kj}(\frac{x'}{\varepsilon})\frac{\partial u}{\partial x_j}.\frac{\partial \varphi}{\partial x_k}\}dx \end{aligned} \tag{2.11}$$

where

$$G_{BL} = (0,\delta)\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})^{s-1}\cup(d-\delta,d)\times(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})^{s-1}.$$

Varying $w \in H^{K+2}((\delta,d-\delta))$ in (2.8) we prove that v satisfies on $(\delta,d-\delta)$ the ordinary differential equation

$$\sum_{l,m=1}^{K+2}\varepsilon^{l+m-2}\;(-1)^m\tilde{h}_{l,m}D_1^{l+m}v(x_1) + f(x_1) = 0,\;\; x_1 \in (\delta,d-\delta). \tag{2.12}$$

Applying now estimate (2.9) we get the following theorem.

Theorem 2.1. *The estimate holds true*

$$\|u_\varepsilon - u_d\|_{H^1(G_\varepsilon)} = O(\varepsilon^{K+1}). \tag{2.13}$$

The solution of problem (2.8) satisfies equation (2.1) for $x \in G_{BL}$ and boundary conditions (2.1) for $x_1 = 0$ and for $x_1 = d$, it satisfies as well equation (2.12) for v_d and the $2K+4$ interface conditions for $x_1 = \delta$ and for $x_1 = d - \delta$. Let us give these conditions for $x_1 = \delta$; for $x_1 = d - \delta$ they are the same. Firstly, u_d and v satisfy conditions (2.7) for $x_1 = \delta$ (and for $x_1 = d - \delta$). And then there are $K+2$ interface conditions that are the consequences of the integration by parts and of the equating the coefficients of $D_1^r w(\delta), r = 0, ..., K+1$:

$$\varepsilon^{1-s} \int_{(-\frac{\varepsilon}{2},\frac{\varepsilon}{2})^{s-1}} \sum_{j=1}^{s} A_{1j}(\frac{x'}{\varepsilon}) \frac{\partial u_d}{\partial x_j}|_{x_1=\delta} N_r(\frac{x'}{\varepsilon}) dx' =$$

$$= \sum_{l=1}^{K+2} \sum_{m=r+1}^{K+2} \varepsilon^{l+m-2} (-1)^{m-r-1} \tilde{h}_{l,m} D_1^{l+m-r-1} v_d(\delta). \tag{2.14}$$

Consider an example $K = 0$. We have then

$$a(u, v; \varphi, w) = \int_{G_{BL}} \sum_{k,j=1}^{s} A_{kj}(\frac{x'}{\varepsilon}) \frac{\partial u}{\partial x_j} \frac{\partial \varphi}{\partial x_k} dx +$$

$$+ \int_{G_I} \sum_{l,m=1}^{2} \varepsilon^{l+m-2} \tilde{h}_{lm} \frac{\partial^l v}{\partial x_1^l} \frac{\partial^m w}{\partial x_1^m} dx, \tag{2.19}$$

where

$$G_I = G_\varepsilon \backslash \bar{G}_{BL},$$

$$\tilde{h}_{11} = \hat{A}_{11} = \sum_{j=1}^{s} < A_{1j} \frac{\partial (N_1 + \xi_1)}{\partial \xi_j} >;$$

$$\tilde{h}_{12} = \tilde{h}_{21} = \sum_{j=1}^{s} < A_{1j} \frac{\partial (N_1 + \xi_1)}{\partial \xi_j} N_1 >;$$

$$\tilde{h}_{22} = < A_{11} N_1^2 > .$$

Integrating identity (2.8) by parts and taking into account that w is an arbitrary function of the space $H^2((\delta, d-\delta))$, we see that v is a solution to the equation of the fourth order (if $\tilde{\tilde{h}}_{22} \neq 0$)

$$\varepsilon^2 \tilde{h}_{22} \frac{\partial^4 v}{\partial {x_1}^4} - \tilde{h}_{11} \frac{\partial^2 v}{\partial x_1^2} = -f(x_1), \quad x_1 \in (\delta, d-\delta) \tag{2.20}$$

with the interface conditions following from the equations

$$\int_{\{x_1=\delta, x_2,\dots,x_s \in [-\varepsilon/2, \varepsilon/2]\}} \{\tilde{h}_{11} \frac{\partial v}{\partial x_1} w(\delta) + (\varepsilon^2 \tilde{h}_{22} \frac{\partial^2 v}{\partial x_1^2} + \varepsilon \tilde{h}_{12} \frac{\partial v}{\partial x_1}) \frac{\partial w}{\partial x_1}(\delta) - \varepsilon^2 \tilde{h}_{22} \frac{\partial^3 v}{\partial x_1^3} w(\delta)\} dx' =$$

$$\int_{\{x_1=\delta, x_2,\dots,x_s \in [-\varepsilon/2, \varepsilon/2]\}} \sum_{j=1}^{s} A_{ij} \frac{\partial \bar{u}_d}{\partial x_j} (w(\delta) + \varepsilon N_1(\frac{x}{\varepsilon}) \frac{\partial w}{\partial x_1}(\delta)) dx'$$

and

$$\bar{u}_d(\delta, x_2, \dots, x_s) = v(\delta) + \varepsilon N_1(\frac{x}{\varepsilon}) \frac{\partial v}{\partial x_1}(\delta);$$

i.e. for $x_1 = \delta$ denoting $S_\delta = \{x_1 = \delta, x_2, \dots, x_s \in [-\varepsilon/2, \varepsilon/2]\}$, we get

$$\left\{ \begin{array}{l} \tilde{h}_{11} \frac{\partial v}{\partial x_1} - \varepsilon^2 \tilde{h}_{22} \frac{\partial^3 v}{\partial x_1^3} = \frac{1}{\varepsilon^{s-1}} \int_{S_\delta} \sum_{j=1}^{s} A_{1j} \frac{\partial \bar{u}_d}{\partial x_j} dx', \\ \varepsilon^2 \tilde{h}_{22} \frac{\partial^2 v}{\partial x_1^2} + \varepsilon \tilde{h}_{12} \frac{\partial v}{\partial x_1} = \frac{1}{\varepsilon^{s-1}} \int_{S_\delta} \sum_{j=1}^{s} A_{1j} \frac{\partial \bar{u}_d}{\partial x_j} N_1(\frac{x}{\varepsilon}) dx', \\ \bar{u}_d(\delta, x_2, \dots, x_s) = v(\delta) + \varepsilon N_1(\frac{\delta}{\varepsilon}, \frac{x_2}{\varepsilon}, \dots; \frac{x_s}{\varepsilon}) \frac{\partial v}{\partial x_1}(\delta). \end{array} \right. \tag{2.21}$$

We have the same interface conditions on the surface $S_{d-\delta} = \{x_1 = d-\delta,\ x_2, \dots, x_s \in [-\varepsilon/2, \varepsilon/2]\}$; δ in these conditions have to be replaced by $d-\delta$. In G_{BL} we keep equation (2.1) for $\bar{u}_d$. So, (2.1) in G_{BL} for $\bar{u}_d$, (2.20) in G_I for v, and interface conditions (2.21) on S_δ and on $S_{d-\delta}$ constitute the differential version of partially homogenized problem (2.8) for $K = 0$. Estimates (2.13) give the error of order $O(\varepsilon)$. This estimate improves the standard estimate for the difference of the exact solution and of the first order asymptotic solution without the boundary layer corrector that is of order $O(\sqrt{\varepsilon})$.

Conclusion

The partial homogenization (as well as the MAPDD) can be interpreted as a multi-scale model coupling the homogenized (macroscopic) description in the internal main part of the domain and the microscopic zoom in the domain of the location of the boundary layers. The estimates of the main theorems justify these methods. They are applied to the partial differential equations in thin domains as well as to the homogenization problems.

Acknowledgements

This work is supported by the Region Rhône-Alpes grant "Modélisation multi-échelle des nanostructures" 04 022487 01.

References.

[1] D.N.Arnold, A.L.Madureira. Asymptotic estimates of hierarchical modeling. *Mathematical Models and Methods in Applied Sciences,* 13(9):1329-1350, 2003.

[2] I. Babuska, L.Li. Hierarchic modelling of plates. *Computers and Structures,* 40:419-430, 1991.

[3] I. Babuska, C. Schwab. A posteriori error estimation for the hierarchic models of elliptic boundary value problems on thin domains. *SIAM J. Numer.Anal.,* 33:221-246, 1996.

[4] N.S. Bakhvalov. On optimization of the methods of solution of boundary value problems with boundary layers.*Zh. Vyc.Mat.i Mat.Fis.(ZVMMF)*,9(4):841-859, 1969.

[5] N.S. Bakhvalov, G.P.Panasenko. *Homogenization: Averaging Processes in Periodic Media*, Moscow: Nauka, 1984; English translation: Dordrecht etc.: Kluwer, 1989.

[6] R.Z.Dautov. On a version of the finite element method in the domains with periodic structure. *Differentsialnye Uravneniia (Differential Equations),* 21(7):1155-1164, 1985.

[7] T.Y. Hou, X.H.Wu, Z.Q.Cai. Convergence of a multi-scale finite element method for elliptic problems with rapidly oscillating coefficients. *Mathematics of Computations,* 68:913-943, 1999.

[8] A.M.Ilyin. Difference scheme for a differential equation with small parameter in the higher derivative. *Mat. Zametki,* 6(2),237-248, 1969.

[9] L.V.Kantorovich, V.I.Krylov. *Approximate Methods of Higher Analysis*, Noordhoff, Groningen, 1958.

[10] S.A. Nazarov, and A.S. Zorin. Two-term asymptotics of the problem on longitudinal deformation of a plate with clamped edge. *Computer mechanics of solids,* 2:10-21, 1991 (Russian).

[12] G.P. Panasenko. Asymptotic solutions of the elasticity theory system of equations for lattice and skeletal structures. *Math.Sb.*,183(1):89-113,

1992 (in Russian). English transl. by *AMS in Russian Acad. Sci. Sbornik Math.* 75(1):85-110, 1993.

[13] G.P. Panasenko. Asymptotic analysis of bar systems.I.*Russian Journal of Math. Physics*, 2(3):325-352, 1994; II. *Russian Journal of Math. Physics*, 4(1):87-116, 1996.

[14] G.P. Panasenko. Method of asymptotic partial decomposition of domain. *Mathematical Models and Methods in Applied Sciences,* 8(1):139-156 ,1998.

[15] G.P. Panasenko. Method of asymptotic partial decomposition of rod structures. *International Journal of Computational, Civil and Structural Engineering (Begel House Publ.)* 1(2):57-70, 2000.

[16] G.P. Panasenko. *Multi-scale Modelling for Structures and Composites*, Springer, 2005.

[17] G.I.Shishkin. *Network Approximations for Singularly Perturbed Elliptic and Parabolic Problems.* Ekaterinburg:Russian Academy of Sciences, Ural Department, 1992.

[18] S.M. Shugrin. Coupling of one-dimensional and two-dimensional models of water flow. *Water Resources*, 5 : 5-15, 1987 (in Russian).

[19] V.P. Smyshlyaev, K.D. Cherednichenko. On derivation of "strain gradient" effects in the overall behaviour of periodic heterogeneous media. *J. Mech. Phys. Solids*, 48:1325-1357, 2000.

[20] L.G.Strakhovskaya, R.P.Fedorenko. On one version of the finite element method. *Zh.Vyc.Mat.i Mat.Fis.(ZVMMF),* 4:950-960, 1979.

[21] M.Vogelius, I.Babuska. On a dimensional reduction method. I.(The optimal selection of basis functions)*Mathematics of Computations*, 37(155):31-46, 1981;II.(Some approximation-theoretic results)*Mathematics of Computations*, 37(155):47-68, 1981; III. (A-posteriori error estimation and an adaptive approach)*Mathematics of Computations*, 37:361-384, 1981.

[22] M.Vogelius, Papanicolaou. A projection method applied to diffusion in a periodic structure. *SIAM J. Appl. Math.*, 42:1302-1322, 1982.