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A LOCALLY COMPACT QUANTUM GROUP ARISING FROM QUANTIZATION OF THE AFFINE GROUP OF A LOCAL FIELD

DAVID JONDREVILLE

ABSTRACT. Using methods coming from non-formal equivariant quantization, we construct in this short note a unitary *dual 2-cocycle* on a discrete family of quotient groups of subgroups of the affine group of a local field (which is not of characteristic 2, nor an extension of \mathbb{Q}_2). Using results of De Commer about Galois objects in operator algebras, we obtain new examples of locally compact quantum groups in the setting of von Neumann algebras.

Keywords: Equivariant quantization, Locally compact quantum groups, Local fields

1 INTRODUCTION

The construction of concrete examples of locally compact quantum groups (LCQG) in the setting of von Neumann algebras [7, 8] is still of utmost importance. The seminal work of De Commer [3] provides a way to construct a new LCGQ from a Galois object on a given LCQG. An important class of Galois objects are associated with 2-cocycles [3, Section 5]. Recall that a *unitary dual 2-cocycle* on a LCQG (\mathcal{N}, Δ) , with dual quantum group $(\hat{\mathcal{N}}, \hat{\Delta})$ (we use the standard notations, see e.g. [3, 10]), is a unitary element $F \in \hat{\mathcal{N}} \bar{\otimes} \hat{\mathcal{N}}$ satisfying the cocycle relation:

$$(1) \quad (1 \otimes F)(\iota \otimes \hat{\Delta})(F) = (F \otimes 1)(\hat{\Delta} \otimes \iota)(F).$$

At the level of an ordinary locally compact group G , the explicit construction of a unitary dual 2-cocycle F is already a non-trivial task. In that case, F is a unitary element of the group von Neumann algebra $W^*(G \times G)$ and the cocycle relation is equivalent (see [10, Section 4]) to the associativity of the following left-equivariant deformed product of the Fourier algebra $A(G)$:

$$f_1 \star f_2(g) = \check{f}_1 \otimes \check{f}_2(\lambda_g \otimes \lambda_g F^*).$$

In the formula above, λ is the left regular representation $\lambda_g(f)(g') = f(g^{-1}g')$ and $f \mapsto \check{f}$ is the standard identification of $A(G)$ with the pre-dual of $W^*(G)$.

Hence, there is a simple strategy to construct dual 2-cocycles on groups. One starts with a left-equivariant and associative product \star on $A(G)$ (or on a dense subalgebra of it), written in term of a distribution (in the sense of Bruhat [2]) K on $G \times G$ as follows:

$$f_1 \star f_2 = \int_{G \times G} K(g_1, g_2) \rho_{g_1}(f_1) \rho_{g_2}(f_2) dg_1 dg_2,$$

where ρ is right regular action $\rho_g(f)(g') = f(g'g)$ and dg is a left-invariant Haar measure. One then considers the following (formal) convolution operator:

$$F := \int_{G \times G} \overline{K(g_1, g_2)} \lambda_{g_1^{-1}} \otimes \lambda_{g_2^{-1}} dg_1 dg_2.$$

All what remains to do then, is to verify that this operator makes sense and is unitary on $L^2(G)$ (for the left-invariant Haar measure). Now, the point is that an associative and left-equivariant product on a group G can be constructed out of a non-formal, G -equivariant quantization map on G . By this we mean a triple $(\Omega, \mathcal{H}, \alpha)$ consisting in a unitary operator Ω from $L^2(G)$ to the Hilbert space $\mathcal{L}^2(\mathcal{H})$ of Hilbert-Schmidt operators on a separable Hilbert space \mathcal{H} , together with an action of G on $\mathcal{B}(\mathcal{H})$ satisfying the covariance relation:

$$\Omega(\lambda_g(f)) = \alpha_g(\Omega(f)), \quad \forall f \in L^2(G).$$

One can then define an associative and left-equivariant product on $L^2(G)$ by transporting the product of $\mathcal{L}^2(\mathcal{H})$ to $L^2(G)$:

$$f_1 \star f_2 := \Omega^*(\Omega(f_1) \Omega(f_2)).$$

In [5], we have constructed such a non-formal, equivariant quantization map and the aim of this short note is to check that the associated candidate for a unitary dual 2-cocycle satisfies this unitary property. The family of groups we consider here are quotient groups of subgroups of the affine group of a local field. To simplify a little this introductory presentation, let us consider the case of the field of p -adic numbers \mathbb{Q}_p with p an odd prime number. Denoting by \mathbb{Z}_p the ring of integers of \mathbb{Q}_p , our groups are unimodular semidirect products of the form $(1 + p^n \mathbb{Z}_p) \ltimes (\mathbb{Q}_p / p^{-n} \mathbb{Z}_p)$. We then have a unitary quantization map (see [5, Proposition 3.9])

$$\Omega : L^2((1 + p^n \mathbb{Z}_p) \ltimes (\mathbb{Q}_p / p^{-n} \mathbb{Z}_p)) \rightarrow \mathcal{L}^2(L^2(1 + p^n \mathbb{Z}_p)),$$

defined for $f \in L^1 \cap L^2((1 + p^n \mathbb{Z}_p) \ltimes (\mathbb{Q}_p / p^{-n} \mathbb{Z}_p))$ by

$$\Omega(f) = p^n \sum_{[t] \in \mathbb{Q}_p / p^{-n} \mathbb{Z}_p} \int_{1 + p^n \mathbb{Z}_p} f(a, [t]) \pi(a, t) \Sigma \pi^*(a, t) da.$$

Here Σ is the (bounded) operator on $L^2(1 + p^n \mathbb{Z}_p)$ given by $\Sigma \varphi(a) = \varphi(a^{-1})$, π is the irreducible unitary representation of $(1 + p^n \mathbb{Z}_p) \ltimes \mathbb{Q}_p$ (a subgroup of the affine group of \mathbb{Q}_p) given by

$$\pi(a, t) \varphi(a_0) := \Psi(a_0^{-1} a t) \varphi(a^{-1} a_0),$$

and Ψ is the basic unitary character of $(\mathbb{Q}_p, +)$ given by

$$\Psi(x) = e^{2i\pi \sum_{n=-k}^{-1} a_n p^n} \quad \text{if } x = \sum_{n=-k}^{\infty} a_n p^n \in \mathbb{Q}_p, \quad a_n \in \{0, 1, \dots, p-1\}.$$

The candidate for the unitary dual 2-cocycle associated with this quantization map is given by:

$$F := p^{2n} \sum_{[t_1], [t_2]} \int \Psi((a_1 - a_1^{-1})t_2 - (a_2 - a_2^{-1})t_1) \lambda_{(a_1, [t_1])^{-1}} \otimes \lambda_{(a_2, [t_2])^{-1}} da_1 da_2,$$

where the sum is over $(\mathbb{Q}_p / p^{-n} \mathbb{Z}_p)^2$ and the integral is over $(1 + p^n \mathbb{Z}_p)^2$.

Our main result here, Theorem 3.4, is that F indeed defines a unitary operator, the 2-cocyclicity being automatic. However, for the sake of completeness, we give a direct proof of 2-cocyclicity in the Appendix. As mentioned earlier, De Commer's results then give an example of a locally compact quantum group in the von Neumann algebraic setting. Moreover, the results of [10] apply too and give a deformation theory for actions of that groups on C^* -algebras.

2 PRELIMINARIES

In this section, we first introduce the basic notations and tools we need. We then give a review of the pseudo-differential calculus constructed in [5] and extract a dual 2-cocycle out of the associated \star -product (that is, the composition law of symbols). This pseudo-differential calculus is termed the *p-adic Fuchs calculus* because it imitates Unterberger's Fuchs calculus [14] in the non-Archimedean setting.

2.1 Setting and tools. Let \mathbf{k} be a non-Archimedean local field. In characteristic zero, \mathbf{k} is isomorphic to a finite degree extension of the field of p -adic numbers \mathbb{Q}_p . In positive characteristic, \mathbf{k} is isomorphic to a field of Laurent series $\mathbb{F}_\ell((X))$ with coefficients in a finite field \mathbb{F}_ℓ . We let $|\cdot|_{\mathbf{k}}$ be the ultrametric absolute value, $\mathcal{O}_{\mathbf{k}}$ be the ring of integers of \mathbf{k} (the unit ball), ϖ be a generator of its unique maximal ideal and $\mathcal{O}_{\mathbf{k}}^\times$ be the multiplicative group of units (the unit sphere). We let p be the characteristic of the residue field $\mathcal{O}_{\mathbf{k}}/\varpi\mathcal{O}_{\mathbf{k}}$ and $q = p^f$ its cardinality. The uniformizer ϖ then satisfies $|\varpi|_{\mathbf{k}} = q^{-1}$. For every integer $n \geq 1$, we denote by $U_n := 1 + \varpi^n\mathcal{O}_{\mathbf{k}} \subset \mathcal{O}_{\mathbf{k}}^\times$ the group of higher principal units. Note that U_n is an open and compact subgroup of \mathbf{k}^\times and it acts by dilations on \mathbf{k} . We can therefore consider the semidirect product $G_n := U_n \rtimes \mathbf{k}$, that we view as a subgroup of the affine group $\mathbf{k}^\times \rtimes \mathbf{k}$, with group law

$$(2) \quad (x, t).(x', t') := (xx', x'^{-1}t + t'), \quad x, x' \in \mathbf{k}^\times, t, t' \in \mathbf{k}.$$

The group G_n will be the *covariance group* of our pseudo-differential calculus. The subgroup U_n will play the role of the *configuration space* and the role of the *phase space* will be played by the homogeneous space $X_n := G_n/H_n$, where H_n is the closed subgroup $\{1\} \times \varpi^{-n}\mathcal{O}_{\mathbf{k}} \subset G_n$. Since H_n is normal in G_n , the quotient X_n is endowed with a group structure. *This is on the group X_n that we will construct a unitary dual 2-cocycle.* Of course, X_n is isomorphic to the semidirect product $U_n \rtimes \Gamma_n$, where $\Gamma_n := \mathbf{k}/\varpi^{-n}\mathcal{O}_{\mathbf{k}}$ is discrete and Abelian. As a locally compact group, our phase space X_n is not compact, nor discrete and non-Abelian. We will denote the elements of X_n by pairs $(a, [t])$, where $a \in U_n$ and $[t] := t + \varpi^{-n}\mathcal{O}_{\mathbf{k}} \in \Gamma_n$, so that the group law of X_n is given by

$$(a, [t]).(a', [t']) := (aa', [a'^{-1}t + t']).$$

To simplify the notations, we will frequently denote an element of G_n by $g := (a, t)$ and an element of the quotient group X_n by $[g] := (a, [t])$. We normalize the Haar measure dt of \mathbf{k} so that $\text{Vol}(\mathcal{O}_{\mathbf{k}}) = 1$. Since U_n is open in \mathbf{k}^\times , the Haar measure of U_n is given by the restriction of the Haar measure of \mathbf{k}^\times (which is $|t|_{\mathbf{k}}^{-1}dt$) but since $U_n \subset \mathcal{O}_{\mathbf{k}}^\times$, it is just dt . The G_n -invariant measure of X_n (which is also a Haar measure for the quotient group structure), denoted by $d[g]$, is chosen to be the product of the Haar measure of U_n by the counting measure on Γ_n . In particular, both groups G_n and X_n are unimodular. Once for all, we fix a unitary character Ψ of the additive group \mathbf{k} which is required to be trivial on $\mathcal{O}_{\mathbf{k}}$ but not on $\varpi^{-1}\mathcal{O}_{\mathbf{k}}$. We also deduce by [6, Theorem 23.25] that $\widehat{\Gamma}_n$ is isomorphic to the additive group $\varpi^n\mathcal{O}_{\mathbf{k}}$.

Important ingredients of the p -adic Fuchs calculus, are the square function

$$\sigma : U_n \rightarrow U_n, \quad a \mapsto a^2,$$

and the hyperbolic sine type function

$$\phi : U_n \rightarrow \varpi^n\mathcal{O}_{\mathbf{k}}, \quad a \mapsto a - a^{-1}.$$

The crucial requirement for the p -adic Fuchs calculus to behave well, is that the mappings σ and ϕ are C^1 -homeomorphisms (in the precise sense of [12, p.77]). For this property to hold true, we need to assume that the element $2 \in \mathbf{k}$ belongs to $\mathcal{O}_{\mathbf{k}}^\times$. Equivalently ([5, Proposition 2.1]), we need

to exclude two cases: when \mathbf{k} is of characteristic two and when \mathbf{k} is an extension of \mathbb{Q}_2 . Then, σ and ϕ become surjective isometries of U_n viewed as metric space for the induced ultra-metric of \mathbf{k} . We then define the square root function $U_n \rightarrow U_n$, $a \mapsto a^{\frac{1}{2}}$, as the reciprocal mapping of the square function $\sigma : U_n \rightarrow U_n$. It is then shown in [5, Proposition 2.1] that $|\sigma'|_{\mathbf{k}} = |\phi'|_{\mathbf{k}} = 1$.

Combining [12, p.287] (which only treats the one-dimensional case) and [13, p.49] (which only treats the case of \mathbb{Q}_p^d), we deduce the following substitution formula:

$$\int_V f(t) dt = \int_U f \circ \varphi(t) |\text{Jac}_\varphi(t)|_{\mathbf{k}} dt,$$

where U, V are compact open subsets of \mathbf{k}^d , $f : V \rightarrow \mathbb{C}$ is L^1 and $\varphi : U \rightarrow V$ is a C^1 -homeomorphism such that $\text{Jac}_\varphi(t) \neq 0$ for all $t \in U$. In particular, we will frequently use this formula for $\varphi = \sigma$ and $\varphi = \phi$, which in these cases gives:

$$(3) \quad \int_{U_n} f(\sigma(a)) da = \int_{U_n} f(a) da \quad \text{and} \quad \int_{U_n} h(\phi(a)) da = \int_{\varpi^n \mathcal{O}_{\mathbf{k}}} h(x) dx.$$

With our choice of normalization for the Haar measure of \mathbf{k} and with the non-trivial previously fixed standard character Ψ (that is constant on $\mathcal{O}_{\mathbf{k}}$ but not on $\varpi^{-1}\mathcal{O}_{\mathbf{k}}$), the Fourier transform

$$(\mathcal{F}_{\mathbf{k}} f)(x) := \int_{\mathbf{k}} f(t) \Psi(xt) dt, \quad f \in L^1(\mathbf{k}) \cap L^2(\mathbf{k}),$$

extends to a unitary operator on $L^2(\mathbf{k})$. Identifying a function $\tilde{f} \in L^1(\Gamma_n)$ with the function $f \in L^1(\mathbf{k})$ invariant under translations in $\varpi^{-n}\mathcal{O}_{\mathbf{k}}$, we therefore have

$$(4) \quad \sum_{[t] \in \Gamma_n} \tilde{f}([t]) = q^{-n} \int_{\mathbf{k}} f(t) dt.$$

In particular, if $f \in L^1(\mathbf{k})$ is invariant by translations in $\varpi^{-n}\mathcal{O}_{\mathbf{k}}$ then $\mathcal{F}_{\mathbf{k}}(f)$ is supported on $\varpi^n \mathcal{O}_{\mathbf{k}}$ and using the identification $\widehat{\Gamma}_n \sim \varpi^n \mathcal{O}_{\mathbf{k}}$, we get

$$(5) \quad \mathcal{F}_{\mathbf{k}} f = q^n \mathcal{F}_{\Gamma_n} \tilde{f},$$

where

$$\mathcal{F}_{\Gamma_n} : L^2(\Gamma_n) \rightarrow L^2(\varpi^n \mathcal{O}_{\mathbf{k}}), \quad f \mapsto \left[x \mapsto \sum_{[t] \in \Gamma_n} f([t]) \Psi(xt) \right],$$

is the Fourier transform on Γ_n .

We denote by $\mathcal{D}(\mathbf{k})$ the space of Bruhat-test functions on \mathbf{k} [2]. Since \mathbf{k} is totally disconnected, $\mathcal{D}(\mathbf{k})$ coincides with the space of locally constant compactly supported functions on \mathbf{k} . Recall also that the Fourier transform $\mathcal{F}_{\mathbf{k}}$ is a homeomorphism of $\mathcal{D}(\mathbf{k})$. Similarly, we let $\mathcal{D}(X_n)$ be the space of Bruhat-test functions X_n , that is the space of functions on X_n which are locally constant in the continuous variable $a \in U_n$ and have finite support in the discrete variable $[t] \in \Gamma_n$.

2.2 The p -adic Fuchs calculus. With Ψ the fixed unitary character of $(\mathbf{k}, +)$, we define the following representation π of the covariance group G_n on the Hilbert space $L^2(U_n)$ of square integrable functions on the configuration space:

$$(6) \quad \pi(a, t)\varphi(a_0) := \Psi(a_0^{-1}at) \varphi(a^{-1}a_0).$$

From Mackey theory [9], any infinite dimensional unitary irreducible representation of G_n is unitarily equivalent to a representation of this form (and they are classified by the orbits of the dual

action of U_n on $\widehat{\mathbf{k}} \simeq \mathbf{k}$). Let then Σ be the group-inversion operator on $L^2(U_n)$ (i.e. the antipode):

$$\Sigma \varphi(a) := \varphi(a^{-1}).$$

Note that Σ is bounded since U_n is unimodular. (More precisely, Σ is unitary and self-adjoint, that is a symmetry.) Consider then the map

$$G_n \rightarrow \mathcal{B}(L^2(U_n)), \quad g \mapsto \pi(g) \Sigma \pi^*(g).$$

Since $\Psi|_{\mathcal{O}_{\mathbf{k}}} = 1$, we observe that the above map is invariant under right-translations in H_n . Therefore, it defines a map

$$\Omega : X_n = G_n/H_n \rightarrow \mathcal{B}(L^2(U_n)),$$

explicitly given by

$$\Omega([g])\varphi(a_0) = \Psi(\phi(aa_0^{-1}t) \varphi(a^2a_0^{-1}).$$

The quantization map underlying the p -adic Fuchs calculus is then defined by

$$\mathbf{\Omega} : L^1(X_n) \rightarrow \mathcal{B}(L^2(U_n)), \quad f \mapsto q^n \int_{X_n} f([g]) \Omega([g]) d[g].$$

By construction, the quantization map is G_n -covariant:

$$(7) \quad \pi(g) \mathbf{\Omega}(f) \pi^*(g) = \mathbf{\Omega}(\lambda_{[g]} f).$$

Remark 2.1. Unterberger's Fuchs calculus is defined in a similar way, with the multiplicative group \mathbb{R}_+^* replacing U_n and with (the connected component of) the affine group $\mathbb{R}_+^* \times \mathbb{R}$ replacing both G_n and X_n . The common point of the two situations, is that \mathbb{R}_+^* and U_n are the largest open subgroups of \mathbb{R}^* and of \mathbf{k}^\times on which the square function is a homeomorphism.

2.3 The \star -product and the dual 2-cocycle. At this stage, the most important property of the quantization map (property requiring that σ and ϕ are homeomorphisms of class C^1), is that $\mathbf{\Omega}$ extends (from $L^1(X_n) \cap L^2(X_n)$) to a surjective isometry from $L^2(X_n)$ to the Hilbert space of Hilbert-Schmidt operators on $L^2(U_n)$ (see [5, Proposition 3.9]). We can therefore transport the associative algebra structure of Hilbert-Schmidt operators to $L^2(X_n)$:

$$\star : L^2(X_n) \times L^2(X_n) \rightarrow L^2(X_n), \quad (f_1, f_2) \mapsto \mathbf{\Omega}^*(\mathbf{\Omega}(f_1) \mathbf{\Omega}(f_2)).$$

It is proven in [5, Proposition 3.13] that for $f_1, f_2 \in L^2(X_n) \cap L^1(X_n)$, the product $f_1 \star f_2 \in L^2(X_n) \cap L^\infty(X_n)$ takes the form

$$f_1 \star f_2([g_0]) = \int_{X_n \times X_n} K([g_0], [g_1], [g_2]) f_1([g_1]) f_2([g_2]) d[g_1] d[g_2],$$

where K is the locally constant and bounded function on X_n^3 given by:

$$K((a_1, [t_1]), (a_2, [t_2]), (a_3, [t_3])) = q^{2n} \Psi(\phi(a_1 a_2^{-1}) t_3 + \phi(a_2 a_3^{-1}) t_1 + \phi(a_3 a_1^{-1}) t_2).$$

By construction, the kernel K is invariant under the diagonal action of G_n on X_n^3 , thus invariant under the diagonal left action of X_n on X_n^3 . Hence (see [5, Proposition 3.14]) we have for $f_1, f_2 \in L^2(X_n) \cap L^1(X_n)$ and with ρ the right regular action:

$$f_1 \star f_2 = \int_{X_n \times X_n} K([e], [g_1], [g_2]) \rho_{[g_1]}(f_1) \rho_{[g_2]}(f_2) d[g_1] d[g_2],$$

with

$$K([e], [g_1], [g_2]) = q^{2n} \Psi(-\phi(a_1) t_2 + \phi(a_2) t_1).$$

Note also that $\mathcal{D}(X_n)$ is stable under \star (see [5, Proposition 3.15]).

As explained in the introduction, the natural candidate for a unitary dual 2-cocycle F on the group X_n , is given by the following convolution operator, initially defined as a quadratic form on $\mathcal{D}(X_n \times X_n)$:

$$(8) \quad F := q^{2n} \int_{X_n \times X_n} \Psi(\phi(a_1)t_2 - \phi(a_2)t_1) \lambda_{[g_1]^{-1}} \otimes \lambda_{[g_2]^{-1}} d[g_1] d[g_2].$$

The goal of the next section is to prove that F preserves the Bruhat space $\mathcal{D}(X_n \times X_n)$ and extends to a unitary operator on $L^2(X_n)$. Once this will be proven, the associativity of \star on $\mathcal{D}(X_n)$ will immediately imply the 2-cocycle relation for F :

$$(1 \otimes F)(\text{Id} \otimes \hat{\Delta}(F)) = (F \otimes 1)(\hat{\Delta}(F) \otimes \text{Id}),$$

where $\hat{\Delta}$ is the coproduct of the group von Neumann algebra $W^*(X_n)$, defined on the generators by $\hat{\Delta}(\lambda_{[g]}) = \lambda_{[g]} \otimes \lambda_{[g]}$.

3 UNITARITY OF THE DUAL 2-COCYCLE

We start with an important factorization property:

Proposition 3.1. *Let $\Xi : U_n \times \varpi^n \mathcal{O}_{\mathbf{k}} \times U_n \times \varpi^n \mathcal{O}_{\mathbf{k}} \rightarrow U_n \times \varpi^n \mathcal{O}_{\mathbf{k}} \times U_n \times \varpi^n \mathcal{O}_{\mathbf{k}}$ be the continuous map defined by:*

$$\Xi(a_1, x_1, a_2, x_2) := (\phi^{-1}(a_2^{-1}x_2)a_1, x_1, \phi^{-1}(-a_1^{-1}x_1)a_2, x_2),$$

and let T_{Ξ} be the continuous operator on $\mathcal{D}(U_n \times \varpi^n \mathcal{O}_{\mathbf{k}} \times U_n \times \varpi^n \mathcal{O}_{\mathbf{k}})$ given by:

$$T_{\Xi}f = f \circ \Xi.$$

Then, the dual 2-cocycle F factorizes as:

$$F = (\text{Id} \otimes \mathcal{F}_{\Gamma_n}^{-1} \otimes \text{Id} \otimes \mathcal{F}_{\Gamma_n}^{-1}) T_{\Xi} (\text{Id} \otimes \mathcal{F}_{\Gamma_n} \otimes \text{Id} \otimes \mathcal{F}_{\Gamma_n}).$$

Proof. Take $f \in \mathcal{D}(X_n \times X_n)$ and let $[g'_j] = (a'_j, [t'_j]) \in X_n$, $j = 1, 2$. Then we have:

$$\begin{aligned} & Ff([g'_1], [g'_2]) \\ &= q^{2n} \int_{X_n \times X_n} \Psi(-\phi(a_2)t_1) \Psi(\phi(a_1)t_2) f((a_1, [t_1]).(a'_1, [t'_1]); (a_2, [t_2]).(a'_2, [t'_2])) d[g_1]d[g_2] \\ &= q^{2n} \int_{X_n \times X_n} \Psi(-\phi(a_2)t_1) \Psi(\phi(a_1)t_2) f_1(a_1a'_1, [a_1^{-1}t_1 + t'_1]; a_2a'_2, [a_2^{-1}t_2 + t'_2]) d[g_1]d[g_2]. \end{aligned}$$

The change of variables $[t_1] \mapsto [a'_1(t_1 - t'_1)]$ and $[t_2] \mapsto [a'_2(t_2 - t'_2)]$ leads to:

$$\begin{aligned} Ff([g'_1], [g'_2]) &= q^{2n} \int_{U_n \times U_n} \Psi(\phi(a_2)a'_1t'_1) \Psi(-\phi(a_1)a'_2t'_2) \\ &\quad \times (\text{Id} \otimes \mathcal{F}_{\Gamma_n} \otimes \text{Id} \otimes \mathcal{F}_{\Gamma_n})f(a_1a'_1, -a'_1\phi(a_2); a_2a'_2, a'_2\phi(a_1)) da_1 da_2. \end{aligned}$$

The second formula in (3) then gives:

$$\begin{aligned} Ff([g'_1], [g'_2]) &= q^{2n} \int_{\varpi^n \mathcal{O}_{\mathbf{k}} \times \varpi^n \mathcal{O}_{\mathbf{k}}} \bar{\Psi}(x_1t'_1) \bar{\Psi}(x_2t'_2) \\ &\quad \times (\text{Id} \otimes \mathcal{F}_{\Gamma_n} \otimes \text{Id} \otimes \mathcal{F}_{\Gamma_n})f(\phi^{-1}(a_2^{-1}x_2)a'_1, x_1; \phi^{-1}(-a_1^{-1}x_1)a'_2, x_2) dx_1 dx_2, \end{aligned}$$

which is the announced formula. \square

Since the operator of composition by a continuous function preserves the space of locally constant functions, we deduce:

Corollary 3.2. *The dual 2-cocycle F is continuous on the Bruhat space $\mathcal{D}(X_n \times X_n)$. In particular, F also makes sense as a densely defined operator affiliated with the group von Neumann algebra $W^*(X_n \times X_n)$.*

Next, we come to our main technical result:

Lemma 3.3. *The map Ξ defined in Proposition 3.1 is a homeomorphism of class C^1 with $|\text{Jac}_\Xi|_{\mathbf{k}} = 1$.*

Proof. The main difficulty is to show that Ξ is bijective. Due to the specific form of the map Ξ , it is equivalent to prove that for any $(a'_1, x'_1, a'_2, x'_2) \in U_n \times \varpi^n \mathcal{O}_{\mathbf{k}} \times U_n \times \varpi^n \mathcal{O}_{\mathbf{k}}$, there exists a unique $(a_1, a_2) \in U_n \times U_n$ such that

$$(9) \quad \Xi(a_1, x'_1, a_2, x'_2) = (a'_1, x'_1, a'_2, x'_2).$$

The relation (9) give the (equivalent) systems of equations:

$$\begin{cases} \phi^{-1}(a_2^{-1} x'_2) a_1 = a'_1 \\ \phi^{-1}(-a_1^{-1} x'_1) a_2 = a'_2 \end{cases} \Leftrightarrow \begin{cases} a_2^{-1} x'_2 = \phi(a'_1 a_1^{-1}) \\ -a_1^{-1} x'_1 = \phi(a'_2 a_2^{-1}) \end{cases} \Leftrightarrow \begin{cases} a_1^{-1} a_1^2 + a_2^{-1} x'_2 a_1 - a'_1 = 0 \\ a_2^{-1} a_2^2 - a_1^{-1} x'_1 a_2 - a'_2 = 0 \end{cases}.$$

Note first that this system is invariant under the dilations:

$$(a_1, a_2, ; a'_1, a'_2, x'_1, x'_2) \mapsto (\alpha_1 a_1, \alpha_2 a_2, ; \alpha_1 a'_1, \alpha_2 a'_2, \alpha_1 x'_1, \alpha_2 x'_2), \quad \alpha_1, \alpha_2 \in U_n.$$

Hence, we get the equivalent system of coupled quadratic equations in term of the homogeneous variables $u_j := a_j'^{-1} a_j \in U_n$ and parameters $X_j := a_j'^{-1} x_j \in \varpi^n \mathcal{O}_{\mathbf{k}}$, $j = 1, 2$:

$$(10) \quad \begin{cases} u_1^2 + X_2 u_2^{-1} u_1 - 1 = 0 \\ u_2^2 - X_1 u_1^{-1} u_2 - 1 = 0 \end{cases}.$$

Note also that this system is invariant under the transformation

$$(11) \quad (u_1, u_2, X_1, X_2) \mapsto (u_2, u_1, -X_2, -X_1).$$

The discriminant of the first equation in (10) (treating $u_2 \in U_n$ as a parameter) is given by

$$\Delta_1 := 4 + \frac{X_2^2}{u_2^2} = 4 \left(1 + \frac{X_2^2}{4u_2^2} \right).$$

Since $X_2 \in \varpi^n \mathcal{O}_{\mathbf{k}}$, $u_2 \in U_n \subset \mathcal{O}_{\mathbf{k}}^\times$ and since 4 is a unit too, we have:

$$1 + \frac{X_2^2}{4u_2^2} \in 1 + \varpi^{2n} \mathcal{O}_{\mathbf{k}} \subset 1 + \varpi^n \mathcal{O}_{\mathbf{k}} = U_n,$$

which therefore possesses a (unique) square root in U_n . Since 4 possesses exactly two square roots in \mathbf{k} (which are ± 2), Δ_1 also possesses exactly two square roots in \mathbf{k} . Hence, the first equation of the quadratic system possesses exactly two solutions in \mathbf{k} namely:

$$Y_{\pm} = \pm \left(1 + \frac{X_2^2}{4u_2^2} \right)^{\frac{1}{2}} - \frac{X_2}{2u_2} \in \pm U_n + \varpi^n \mathcal{O}_{\mathbf{k}} = \pm 1 + \varpi^n \mathcal{O}_{\mathbf{k}}.$$

The solution Y_- is not admissible because it does not belong to U_n as it should be. Hence, we get only one solution, namely

$$(12) \quad u_1 = \left(1 + \frac{X_2^2}{4u_2^2}\right)^{\frac{1}{2}} - \frac{X_2}{2u_2} \in U_n.$$

Substituting this expression for u_1 in the second equation of the system (10) yields the relation:

$$X_1 = \left(\left(1 + \frac{X_2^2}{4u_2^2}\right)^{\frac{1}{2}} - \frac{X_2}{2u_2}\right)(u_2 - u_2^{-1}),$$

which is equivalent to:

$$(13) \quad X_1 + \frac{X_2}{2u_2}(u_2 - u_2^{-1}) = \left(1 + \frac{X_2^2}{4u_2^2}\right)^{\frac{1}{2}}(u_2 - u_2^{-1}).$$

Squaring the relation (13) and multiplying it by u_2^2 give, in term of the variable $U := u_2^2 \in U_n$, the following quadratic equation:

$$(14) \quad U^2 - (2 + X_1X_2 + X_1^2)U + (1 + X_1X_2) = 0.$$

The discriminant of equation (14) is:

$$\Delta_2 = X_1^2(4 + (X_1 + X_2)^2).$$

Since

$$1 + \left(\frac{X_1}{2} + \frac{X_2}{2}\right)^2 \in 1 + \varpi^{2n}\mathcal{O}_{\mathbf{k}} \subset 1 + \varpi^n\mathcal{O}_{\mathbf{k}} = U_n,$$

Δ_2 possesses exactly two square roots in \mathbf{k} , given by

$$\pm 2X_1 \left(1 + \left(\frac{X_1}{2} + \frac{X_2}{2}\right)^2\right)^{\frac{1}{2}} \in \varpi^n\mathcal{O}_{\mathbf{k}},$$

and thus, the equation (14) possesses exactly two solutions:

$$(15) \quad U_{\pm} = 1 + \frac{X_1X_2}{2} + \frac{X_1^2}{2} \pm X_1 \left(1 + \left(\frac{X_1}{2} + \frac{X_2}{2}\right)^2\right)^{\frac{1}{2}}.$$

We need to prove that only one solution is admissible and this not that easy since here both U_+ and U_- belong to U_n . To this aim, note first that we can assume without loss of generality that $X_1 \neq 0$ (if $X_1 = 0$ there is no ambiguity). We then evaluate the relation (13) in U_{\pm} and multiply it by U_{\pm} to get:

$$X_1U_{\pm} = \left(\left(U_{\pm} + \frac{X_2^2}{4}\right)^{\frac{1}{2}} - \frac{X_2}{2}\right)(U_{\pm} - 1).$$

Since

$$U_{\pm} - 1 = \frac{X_1X_2}{2} + \frac{X_1^2}{2} \pm X_1 \left(1 + \left(\frac{X_1}{2} + \frac{X_2}{2}\right)^2\right)^{\frac{1}{2}},$$

we get, after simplification by $X_1 \neq 0$, the condition:

$$U_{\pm} = \left(\left(U_{\pm} + \frac{X_2^2}{4}\right)^{\frac{1}{2}} - \frac{X_2}{2}\right) \left(\frac{X_2}{2} + \frac{X_1}{2} \pm \left(1 + \left(\frac{X_1}{2} + \frac{X_2}{2}\right)^2\right)^{\frac{1}{2}}\right).$$

The first factor in the above expression belongs to U_n while the second factor belongs to $\pm 1 + \varpi^n\mathcal{O}_{\mathbf{k}}$. This condition thus excludes U_- . Hence, $u_2^2 = U_+$, which only gives one solution since the square root function is bijective on U_n :

$$u_2 = \left(1 + \frac{X_1X_2}{2} + \frac{X_1^2}{2} + X_1 \left(1 + \left(\frac{X_1}{2} + \frac{X_2}{2}\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

By the symmetry (11) of the system, we also get:

$$u_1 = \left(1 + \frac{X_1 X_2}{2} + \frac{X_2^2}{2} - X_2 \left(1 + \left(\frac{X_1}{2} + \frac{X_2}{2}\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}.$$

Finally, in term of the original variables and parameters, we get:

$$\begin{aligned} a_2 &= a'_2 \left(1 + \frac{x'_1 x'_2}{2a'_1 a'_2} + \frac{x_1'^2}{2a_1'^2} + \frac{x'_1}{a'_1} \left(1 + \left(\frac{x'_1}{2a'_1} + \frac{x'_2}{2a'_2}\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}, \\ a_1 &= a'_1 \left(1 + \frac{x'_1 x'_2}{2a'_1 a'_2} + \frac{x_2'^2}{2a_2'^2} - \frac{x'_2}{a'_2} \left(1 + \left(\frac{x'_1}{2a'_1} + \frac{x'_2}{2a'_2}\right)^2\right)^{\frac{1}{2}}\right)^{\frac{1}{2}}. \end{aligned}$$

This shows that Ξ is one-to-one and onto and the explicit expressions above entail that it is moreover a homeomorphism of class C^1 .

It remains to compute the absolute value of the Jacobian of Ξ . Since for every $a \in U_n$, $\phi'(a) = 1 + a^{-2}$, we have

$$(\phi^{-1})'(x) = \left(1 + (\phi^{-1}(x))^{-2}\right)^{-1}, \quad \forall x \in \varpi^n \mathcal{O}_{\mathbf{k}}.$$

As a consequence, we get

$$\begin{aligned} \text{Jac}_{\Xi}(a_1, x_1, a_2, x_2) &= \phi^{-1}(-a_1^{-1}x_1) \phi^{-1}(a_2^{-1}x_2) \\ &\quad + a_1^{-1}a_2^{-1} \left(1 + (\phi^{-1}(-a_1^{-1}x_1))^{-2}\right)^{-1} \left(1 + (\phi^{-1}(a_2^{-1}x_2))^{-2}\right)^{-1} x_1 x_2. \end{aligned}$$

The first term clearly belongs to U_n . For the second term, note first that

$$1 + (\phi^{-1}(\pm a_j^{-1}x_j))^{-2} \in 1 + U_n = 2 + \varpi^n \mathcal{O}_{\mathbf{k}} \subset \mathcal{O}_{\mathbf{k}}^{\times}.$$

Hence in the second term, all the factors preceding $x_1 x_2$ are units. Hence the second term belongs to $\varpi^{2n} \mathcal{O}_{\mathbf{k}}$. This proves that Jac_{Ξ} takes values in $U_n + \varpi^{2n} \mathcal{O}_{\mathbf{k}} = U_n$, from which the claim follows. \square

An immediate consequence of Lemma 3.3 is that the operator T_{Ξ} is unitary on $L^2(U_n \times \varpi^n \mathcal{O}_{\mathbf{k}} \times U_n \times \varpi^n \mathcal{O}_{\mathbf{k}})$. Since the partial Fourier transform $\text{Id} \otimes \mathcal{F}_{\Gamma_n}$ is unitary from $L^2(X_n)$ to $L^2(U_n \times \varpi^n \mathcal{O}_{\mathbf{k}})$, Proposition 3.1 immediately implies:

Theorem 3.4. *Let \mathbf{k} be a local field which is not of characteristic 2, nor an extension of \mathbb{Q}_2 . Then, the dual 2-cocycle F given in (8) is a unitary operator on $L^2(X_n \times X_n)$.*

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A PROOF OF 2-COCYCLICITY

Our goal here is to provide a direct proof that the (unitary) convolution operator:

$$F = q^{2n} \int_{X_n \times X_n} \Psi(\phi(a_1)t_2 - \phi(a_2)t_1) \lambda_{[g_1]^{-1}} \otimes \lambda_{[g_2]^{-1}} d[g_1] d[g_2]$$

satisfies the 2-cocycle relation:

$$(1 \otimes F)(\text{Id} \otimes \hat{\Delta}(F)) = (F \otimes 1)(\hat{\Delta}(F) \otimes \text{Id}).$$

For convenience, we shall work with the adjoint of F , given on $\mathcal{D}(X_n \times X_n)$ by:

$$F^* := q^{2n} \int_{X_n^2} \overline{\Psi}(\phi(a_1)t_2 - \phi(a_2)t_1) \lambda_{[g_1]} \otimes \lambda_{[g_2]} d[g_1] d[g_2].$$

In term of F^* , the cocycle relation reads:

$$(\text{Id} \otimes \hat{\Delta}(F^*))(1 \otimes F^*) = (\hat{\Delta}(F^*) \otimes \text{Id})(F^* \otimes 1).$$

By density of $\mathcal{D}(X_n \times X_n \times X_n)$ in $L^2(X_n \times X_n \times X_n)$, it is enough to prove this relation on test functions. For $f \in \mathcal{D}(X_n \times X_n \times X_n)$ and $[g], [g'], [g''] \in X_n$, we have:

$$\begin{aligned} (\text{Id} \otimes \hat{\Delta}(F^*))f([g], [g'], [g'']) &= q^{2n} \int_{X_n^2} \Psi(\phi(a_2)t_1) \overline{\Psi}(\phi(a_1)t_2) \\ &\quad \times f(aa_1^{-1}, [t - a_1a^{-1}t_1]; a'a_2^{-1}, [t' - a_2a'^{-1}t_2]; a''a_2^{-1}, [t'' - a_2a''^{-1}t_2]) d[g_1]d[g_2], \\ (\hat{\Delta}(F^*) \otimes \text{Id})f([g], [g'], [g'']) &= q^{2n} \int_{X_n^2} \Psi(\phi(a_2)t_1) \overline{\Psi}(\phi(a_1)t_2) \\ &\quad \times f(aa_1^{-1}, [t - a_1a^{-1}t_1]; a'a_1^{-1}, [t' - a_1a'^{-1}t_1]; a''a_2^{-1}, [t'' - a_2a''^{-1}t_2]) d[g_1]d[g_2]. \end{aligned}$$

Since f is compactly supported, the integrals above are, for fixed $[g], [g'], [g''] \in X_n$, absolutely convergent. To simplify a little the notations, we shall let \mathcal{F}_{Γ_n} be the partial Fourier transform:

$$\mathcal{F}_{\Gamma_n} : L^2(U_n \times \Gamma_n) \rightarrow L^2(U_n \times \hat{\Gamma}_n) \simeq L^2(U_n \times \varpi^{-n}\mathcal{O}_{\mathbf{k}}).$$

The dilation $t_1 \mapsto -aa_1^{-1}t_1$ followed by the translation $t_1 \mapsto t_1 - t$ in the first expression and the dilation $t_2 \mapsto -a''a_2^{-1}t_2$ followed by the translation $t_2 \mapsto t_2 - t''$ in the second, entail:

$$\begin{aligned} (\text{Id} \otimes \hat{\Delta}(F^*))f([g], [g'], [g'']) &= q^{2n} \int_{U_n \times X_n} \Psi(aa_1^{-1}\phi(a_2)t) \overline{\Psi}(\phi(a_1)t_2) \\ &\quad \times (\mathcal{F}_{\Gamma_n} \otimes 1 \otimes 1)f(aa_1^{-1}, -aa_1^{-1}\phi(a_2); a'a_2^{-1}, [t' - a_2a'^{-1}t_2]; a''a_2^{-1}, [t'' - a_2a''^{-1}t_2]) da_1d[g_2], \\ (\hat{\Delta}(F^*) \otimes \text{Id})f([g], [g'], [g'']) &= q^{2n} \int_{U_n \times X_n} \overline{\Psi}(a''a_2^{-1}\phi(a_1)t'') \Psi(\phi(a_2)t_1) \\ &\quad \times (1 \otimes 1 \otimes \mathcal{F}_{\Gamma_n})f(aa_1^{-1}, [t - a_1a^{-1}t_1]; a'a_1^{-1}, [t' - a_1a'^{-1}t_1]; a''a_2^{-1}, a''a_2^{-1}\phi(a_1)) da_2d[g_1]. \end{aligned}$$

For $a, a', a'', a_1, a_2 \in U_n$ and $[t], [t'], [t''] \in \Gamma_n$ fixed, we shall consider the functions

$$\begin{aligned} \varphi_{a,a',a'',a_1,a_2,[t],[t'],[t'']}([t_2]) &:= \\ &\quad (\mathcal{F}_{\Gamma_n} \otimes 1 \otimes 1)f(aa_1^{-1}, -aa_1^{-1}\phi(a_2); a'a_2^{-1}, [t' - a_2a'^{-1}t_2]; a''a_2^{-1}, [t'' - a_2a''^{-1}t_2]), \\ \psi_{a,a',a'',a_1,a_2,[t],[t']}([t_1]) &:= \\ &\quad (1 \otimes 1 \otimes \mathcal{F}_{\Gamma_n})f(aa_1^{-1}, [t - a_1a^{-1}t_1]; a'a_1^{-1}, [t' - a_1a'^{-1}t_1]; a''a_2^{-1}, a''a_2^{-1}\phi(a_1)). \end{aligned}$$

In term of these functions, we have

$$(16) \quad (\text{Id} \otimes \hat{\Delta}(F^*))f([g], [g'], [g'']) = q^{2n} \int_{U_n^2} \Psi(aa_1^{-1}\phi(a_2)t) (\mathcal{F}_{\Gamma_n} \varphi_{a,a',a'',a_1,a_2,[t],[t'],[t'']})(-\phi(a_1)) da_1da_2,$$

$$(17) \quad (\hat{\Delta}(F^*) \otimes \text{Id})f([g], [g'], [g'']) = q^{2n} \int_{U_n^2} \overline{\Psi}(a''a_2^{-1}\phi(a_1)t'') (\mathcal{F}_{\Gamma_n} \psi_{a,a',a'',a_1,a_2,[t],[t']})(\phi(a_2)) da_1da_2.$$

In order to compute $(\text{Id} \otimes \hat{\Delta}(F^*))(1 \otimes F^*)f$ and $(\hat{\Delta}(F^*) \otimes \text{Id})(F^* \otimes 1)f$, we will substitute f by $(1 \otimes F^*)f$ in (16) and f by $(F^* \otimes 1)f$ in (17). To do that, we first need a convenient formula for F^*h , with $h \in \mathcal{D}(X_n \times X_n)$. Using (partially) the factorization given in Proposition (3.1), we get:

$$F^*h([g_1], [g_2]) = q^{2n} \int_{U_n^2} \Psi(a_1 a_3^{-1} \phi(a_4) t_1) \bar{\Psi}(a_2 a_4^{-1} \phi(a_3) t_2) \\ \times (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n}) h(a_1 a_3^{-1}, -a_1 a_3^{-1} \phi(a_4); a_2 a_4^{-1}, a_2 a_4^{-1} \phi(a_3)) da_3 da_4.$$

From this, we deduce:

$$(18) \quad (\mathcal{F}_{\Gamma_n} \otimes 1 \otimes 1)(1 \otimes F^*)f(aa_1^{-1}, -aa_1^{-1} \phi(a_2); a' a_2^{-1}, [t' - a_2 a'^{-1} t_2]; a'' a_2^{-1}, [t'' - a_2 a''^{-1} t_2]) \\ = q^{2n} \int_{U_n^2} \Psi(a' a_2^{-1} a_3^{-1} \phi(a_4)(t' - a_2 a'^{-1} t_2)) \bar{\Psi}(a'' a_2^{-1} a_4^{-1} \phi(a_3)(t'' - a_2 a''^{-1} t_2)) \\ \times (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n}) f(aa_1^{-1}, -aa_1^{-1} \phi(a_2); a' a_2^{-1} a_3^{-1}, -a' a_2^{-1} a_3^{-1} \phi(a_4); \\ a'' a_2^{-1} a_4^{-1}, a'' a_2^{-1} a_4^{-1} \phi(a_3)) da_3 da_4,$$

$$(19) \quad (1 \otimes 1 \otimes \mathcal{F}_{\Gamma_n})(F^* \otimes 1)f(aa_1^{-1}, [t - a_1 a^{-1} t_1]; a' a_1^{-1}, [t' - a_1 a'^{-1} t_1]; a'' a_2^{-1}, a_2^{-1} a'' \phi(a_1)) \\ = q^{2n} \int_{U_n^2} \Psi(aa_1^{-1} a_3^{-1} \phi(a_4)(t - a_1 a^{-1} t_1)) \bar{\Psi}(a' a_1^{-1} a_4^{-1} \phi(a_3)(t' - a_1 a'^{-1} t_1)) \\ \times (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n}) f(aa_1^{-1} a_3^{-1}, -aa_1^{-1} a_3^{-1} \phi(a_4); a' a_1^{-1} a_4^{-1}, a' a_1^{-1} a_4^{-1} \phi(a_3); \\ a'' a_2^{-1}, a'' a_2^{-1} \phi(a_1)) da_3 da_4.$$

Using that $\phi(a_3) a_4^{-1} - \phi(a_4) a_3^{-1} = \phi(a_3 a_4^{-1})$, (18) and (19) are respectively given by:

$$q^{2n} \int_{U_n^2} \Psi(\phi(a_3 a_4^{-1}) t_2) \Psi(a' a_2^{-1} a_3^{-1} \phi(a_4) t') \bar{\Psi}(a'' a_2^{-1} a_4^{-1} \phi(a_3) t'') (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n}) f \\ (aa_1^{-1}, -aa_1^{-1} \phi(a_2); a' a_2^{-1} a_3^{-1}, -a' a_2^{-1} a_3^{-1} \phi(a_4); a'' a_2^{-1} a_4^{-1}, a'' a_2^{-1} a_4^{-1} \phi(a_3)) da_3 da_4,$$

and by

$$q^{2n} \int_{U_n^2} \Psi(\phi(a_3 a_4^{-1}) t_1) \Psi(aa_1^{-1} a_3^{-1} \phi(a_4) t) \bar{\Psi}(a' a_1^{-1} a_4^{-1} \phi(a_3) t') (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n}) f \\ (aa_1^{-1} a_3^{-1}, -aa_1^{-1} a_3^{-1} \phi(a_4); a' a_1^{-1} a_4^{-1}, a' a_1^{-1} a_4^{-1} \phi(a_3); a'' a_2^{-1}, a'' a_2^{-1} \phi(a_1)) da_3 da_4.$$

By the dilation $a_3 \mapsto a_4 a_3$ and the change of variable $x_3 = \phi(a_3)$, (18) and (19) become:

$$q^{2n} \int_{\varpi^n \mathcal{O}_{\mathbf{k}} \times U_n} \Psi(x_3 t_2) \Psi(a' a_2^{-1} a_4^{-1} \phi(a_4) (\phi^{-1}(x_3))^{-1} t') \bar{\Psi}(a'' a_2^{-1} a_4^{-1} \phi(\phi^{-1}(x_3) a_4) t'') \\ \times (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n}) f(aa_1^{-1}, -aa_1^{-1} \phi(a_2); a' a_2^{-1} a_4^{-1} (\phi^{-1}(x_3))^{-1}, -a' a_2^{-1} a_4^{-1} \phi(a_4) (\phi^{-1}(x_3))^{-1}; \\ a'' a_2^{-1} a_4^{-1}, a'' a_2^{-1} a_4^{-1} \phi(\phi^{-1}(x_3) a_4)) dx_3 da_4,$$

and

$$q^{2n} \int_{\varpi^n \mathcal{O}_{\mathbf{k}} \times U_n} \Psi(x_3 t_1) \Psi(aa_1^{-1} a_4^{-1} \phi(a_4) (\phi^{-1}(x_3))^{-1} t) \bar{\Psi}(a' a_1^{-1} a_4^{-1} \phi(\phi^{-1}(x_3) a_4) t') \\ \times (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n}) f(aa_1^{-1} a_4^{-1} (\phi^{-1}(x_3))^{-1}, -aa_1^{-1} a_4^{-1} \phi(a_4) (\phi^{-1}(x_3))^{-1}; \\ a' a_1^{-1} a_4^{-1}, a' a_1^{-1} a_4^{-1} \phi(\phi^{-1}(x_3) a_4); a'' a_2^{-1}, a'' a_2^{-1} \phi(a_1)) dx_3 da_4.$$

Let us introduce, for fixed $a, a', a'', a_1, a_2, a_4 \in U_n$ and $[t], [t'], [t''] \in \Gamma_n$, the following functions

$$\begin{aligned} \tilde{\varphi}_{a,a',a'',a_1,a_2,a_4,[t],[t']}(x_3) &:= \Psi(a'a_2^{-1}a_4^{-1}\phi(a_4)(\phi^{-1}(x_3))^{-1}t') \bar{\Psi}(a''a_2^{-1}a_4^{-1}\phi(\phi^{-1}(x_3)a_4)t'') \\ &\times (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n})f(aa_1^{-1}, -aa_1^{-1}\phi(a_2); a'a_2^{-1}a_4^{-1}(\phi^{-1}(x_3))^{-1}, -a'a_2^{-1}a_4^{-1}\phi(a_4)(\phi^{-1}(x_3))^{-1}; \\ &\quad a''a_2^{-1}a_4^{-1}, a''a_2^{-1}a_4^{-1}\phi(\phi^{-1}(x_3)a_4)), \\ \tilde{\psi}_{a,a',a'',a_1,a_2,a_4,[t],[t']}(x_3) &:= \Psi(aa_1^{-1}a_4^{-1}\phi(a_4)(\phi^{-1}(x_3))^{-1}t) \bar{\Psi}(a'a_1^{-1}a_4^{-1}\phi(\phi^{-1}(x_3)a_4)t') \\ &\times (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n})f(aa_1^{-1}a_4^{-1}(\phi^{-1}(x_3))^{-1}, -aa_1^{-1}a_4^{-1}\phi(a_4)(\phi^{-1}(x_3))^{-1}; \\ &\quad a'a_1^{-1}a_4^{-1}, a'a_1^{-1}a_4^{-1}\phi(\phi^{-1}(x_3)a_4); a''a_2^{-1}, a''a_2^{-1}\phi(a_1)). \end{aligned}$$

Using the isomorphism $\hat{\Gamma}_n \sim \varpi^n \mathcal{O}_{\mathbf{k}}$, we then see that the integral over $x_3 \in \varpi^n \mathcal{O}_{\mathbf{k}}$ can be rewritten as an inverse Fourier transform, and we get:

$$\begin{aligned} (\mathcal{F}_{\Gamma_n} \otimes 1 \otimes 1)(1 \otimes F^*)f(aa_1^{-1}, -aa_1^{-1}\phi(a_2); a'a_2^{-1}, [t' - a_2a'^{-1}t_2]; a''a_2^{-1}, [t'' - a_2a''^{-1}t_2]) \\ = \int_{U_n} \mathcal{F}_{\Gamma_n}^{-1} \tilde{\varphi}_{a,a',a'',a_1,a_2,a_4,[t],[t']}([-t_2]) da_4, \\ (1 \otimes 1 \otimes \mathcal{F}_{\Gamma_n})(F^* \otimes 1)f(aa_1^{-1}, [t - a_1a^{-1}t_1]; a'a_1^{-1}, [t' - a_1a'^{-1}t_1]; a''a_2^{-1}, a_2^{-1}a''\phi(a_1)) \\ = \int_{U_n} \mathcal{F}_{\Gamma_n}^{-1} \tilde{\psi}_{a,a',a'',a_1,a_2,a_4,[t],[t']}([-t_1]) da_4. \end{aligned}$$

Plugging these expressions in (16) and (17), we get after the cancellation of the Fourier transforms:

$$\begin{aligned} (\text{Id} \otimes \hat{\Delta}(F^*))(1 \otimes F^*)f([g], [g'], [g'']) \\ = \int_{U_n^3} \Psi(aa_1^{-1}\phi(a_2)t) \Psi(a'a_1^{-1}a_2^{-1}a_4^{-1}\phi(a_4)t') \bar{\Psi}(a''a_2^{-1}a_4^{-1}\phi(a_1a_4)t'') (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n})f \\ (aa_1^{-1}, -aa_1^{-1}\phi(a_2); a'a_1^{-1}a_2^{-1}a_4^{-1}, -a'a_1^{-1}a_2^{-1}a_4^{-1}\phi(a_4); a''a_2^{-1}a_4^{-1}, a''a_2^{-1}a_4^{-1}\phi(a_1a_4)) da_1 da_2 da_4. \end{aligned}$$

and

$$\begin{aligned} (\hat{\Delta}(F^*) \otimes \text{Id})(F^* \otimes 1)f([g], [g'], [g'']) \\ = \int_{U_n^3} \Psi(aa_2a_1^{-1}a_4^{-1}\phi(a_4)t) \bar{\Psi}(a'a_1^{-1}a_4^{-1}\phi(a_4a_2^{-1})t') \bar{\Psi}(a''a_2^{-1}\phi(a_1)t'') (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n})f \\ (aa_2a_1^{-1}a_4^{-1}, aa_2a_1^{-1}a_4^{-1}\phi(a_4^{-1}), a'a_1^{-1}a_4^{-1}, a'a_1^{-1}a_4^{-1}\phi(a_4a_2^{-1}); a''a_2^{-1}, a''a_2^{-1}\phi(a_1)) da_1 da_2 da_4. \end{aligned}$$

The last point is to show that the two above expressions are actually the same. Performing the dilations $a_2 \mapsto a_4a_2$ and $a_1 \mapsto a_2a_1$, the last expression becomes:

$$\begin{aligned} (\hat{\Delta}(F^*) \otimes \text{Id})(F^* \otimes 1)f([g], [g'], [g'']) \\ = \int_{U_n^3} \Psi(aa_1^{-1}\phi(a_4)t) \Psi(a'a_1^{-1}a_2^{-1}a_4^{-1}\phi(a_2)t') \bar{\Psi}(a''a_2^{-1}a_4^{-1}\phi(a_1a_2)t'') (\mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n} \otimes \mathcal{F}_{\Gamma_n})f \\ (aa_1^{-1}, -aa_1^{-1}\phi(a_4), a'a_1^{-1}a_2^{-1}a_4^{-1}, -a'a_1^{-1}a_2^{-1}a_4^{-1}\phi(a_2); a''a_2^{-1}a_4^{-1}, a''a_2^{-1}a_4^{-1}\phi(a_1a_2)) da_1 da_2 da_4, \end{aligned}$$

which, after the relabelling $a_2 \leftrightarrow a_4$, is exactly the formula we found for $(\text{Id} \otimes \hat{\Delta}(F^*))(1 \otimes F^*)f$.

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