

Exact null controllability, complete stabilizability and continuous final observability of neutral type systems

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Outline

- 1 Introduction: the object of investigation
- 2 Exact null controllability and stabilizability
- 3 Exact null controllability and final observability
- 4 Illustrative Examples
- 5 Concluding remarks

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Main objective

Our purpose is to investigate the relation between for neutral type systems the following properties:

- **exact null controllability**: from any initial condition to 0
- **complete stabilizability**: exp. stabilizability with arbitrary decay rate
- **final continuous observability**: continuous obs. of final state

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Recall for finite dimensional linear systems, Kalman relations:

Formal adjoint systems and duality

Observability (Controllability)

$$\begin{cases} \dot{x} &= Ax + Bu \\ y &= Cx \end{cases}$$



Controllability (Observability)

$$\begin{cases} \dot{x} &= A^*x + C^*y \\ u &= B^*x \end{cases}$$

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Pole assignment (Complete stabilizability)

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We have to precise the infinite dimensional setting.

The system and the operator model

We consider then the controlled system described by

The system

$$\dot{z}(t) - A_{-1}\dot{z}(t-1) = \underbrace{\int_{-1}^0 [A_2(\theta)\dot{z}(t+\theta) + A_3(\theta)z(t+\theta)] d\theta}_{Lz_t} + Bu \quad (1)$$

with an output of the forme $y(t) = Cz(t)$ or $y(t) = Cz(t-1)$.

More general form:

$$\frac{d}{dt}[z(t) - Kz_t] = Lz_t + Bu(t)$$

with more general operator K .

$$Kz_t = \int_{-1}^0 d\kappa(\theta)z_t(\theta), \quad \kappa \in B([-1, 0], \mathbb{R}^n).$$

The Operator Model

Delay systems are essential infinite dimensional and may be written as

The operator model

$$\dot{x}(t) = \mathcal{A}x + \mathcal{B}u \quad y(t) = \mathcal{C}x(t)$$

The state vector and the state space are

$$x = (v(t), z_t(\cdot)), \quad M_2(-1, 0; \mathbb{R}^n) = \mathbb{R}^n \times L_2(-1, 0; \mathbb{R}^n),$$

The operators model \mathcal{A} and \mathcal{B} are given by

$$\mathcal{A}x(t) = \begin{pmatrix} Lz_t(\cdot) \\ dz_t(\theta)/d\theta \end{pmatrix}, \quad \mathcal{B}u = \begin{pmatrix} Bu \\ 0 \end{pmatrix}, \quad \mathcal{C}x = Cz(t) \text{ (or } Cz(t-1))$$

where \mathcal{A} is the generator of a C_0 -semigroup $e^{\mathcal{A}t}$ with domain $\mathcal{D}(\mathcal{A})$.

The characteristic matrix

$$\Delta_{\mathcal{A}}(\lambda) = \lambda I - \lambda e^{-\lambda} A_{-1} - \lambda \int_{-1}^0 e^{\lambda s} A_2(s) ds - \int_{-1}^0 e^{\lambda s} A_3(s) ds.$$

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Exact controllability

By **exact controllability** we mean controllability
 From any state in $\mathcal{D}(\mathcal{A})$ to any state in $\mathcal{D}(\mathcal{A})$.

Theorem (Rabah, Sklyar 2007, with Barkhayev 2016)

System (1) is exactly controllable if and only if the following two conditions:

- i) $\text{rank}(\Delta_{\mathcal{A}}(\lambda) \quad B) = n$ for all $\lambda \in \mathbb{C}$,*
- ii) $\text{rank}(\mu I - A_{-1} \quad B) = n$ for all $\mu \in \mathbb{C}$.*

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The time of exact controllability may be determined by the controllability indices of the couple $(A_{-1}), B$.

Exact null controllability : for all states, there is a control which maintains the systems at 0 : $z_T(\cdot) = 0$ (**Krasovskii** problem for delay systems). For abstract systems: to reach 0 from any initial state.

The two notions are equivalent when $e^{\mathcal{A}t}$ is a group ($\det(A_{-1}) \neq 0$).

Complete stabilizability

By the way we consider the relation with the notion of complete stabilizability for the same system (1): **exponential stabilizability with arbitrary decay rate**:

$$\forall \omega \in \mathbb{R}, \exists \mathcal{F} : \|e^{(\mathcal{A} + \mathcal{B}\mathcal{F})t}\| \leq Me^{\omega t}, \quad t \geq 0$$

by an appropriate feedback

$$u(t) = F_{-1}\dot{z}(t-1) + \underbrace{\int_{-1}^0 [F_2(\theta)\dot{z}(t+\theta) + F_3(\theta)z(t+\theta)] d\theta}_{Fz_t} \quad (2)$$

Note that this feedback is not bounded.

Abstract result

System in Hilbert spaces $\dot{x} = \mathcal{A}x + \mathcal{B}u$, where \mathcal{B} is bounded and $u = \mathcal{F}x$ with \mathcal{F} bounded.

Group: Exact contr. & $e^{\mathcal{A}t}$ \Leftrightarrow Complete stabiliz. (Zabczyk 1976)

Onto +: Exact contr. & $e^{\mathcal{A}t}$ \Leftrightarrow Compl. stabz. (Rabah et al. 1997)

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Boundedness may be replaced by **admissibility** (or \mathcal{A} -boundedness) by a result obtained in 1997 by Curtain et al. (Curtain, Logemann, Townley and Zwart). This result may be applied to neutral type systems, but can also be obtained directly for such systems.

Problem

Is exact **null** controllability with (un)bounded control operator \mathcal{B} and (un)bounded feedback \mathcal{F} equivalent to complete stabilizability? What additional conditions are needed?

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There exist very rapid semigroups:

$$\forall \omega > 0, \exists M_\omega > 0, \quad \|S(t)\| \leq M_\omega e^{-\omega t}, \quad (3)$$

They are “**Completely stabilizable**” without feedback control but, may be or not null controllable in the following sense:

$$\forall x_0, \quad \exists T : \quad S(t)x_0 = 0, \quad t \geq T.$$

Examples

Example 1

$X = L_2(0, +\infty)$, the semigroup is $S(t)f(x) = e^{-\frac{t^2}{2} - xt}f(x + t)$.
The spectrum $\sigma(\mathcal{A}) = \emptyset$. We have in fact $M_\omega \geq ke^\omega$

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This system is not *null controllable* if $B = 0$ (no control action): there are initial conditions f such that $S(t)f \neq 0$ for any $t \geq 0$.

Example 2

In the space $L_2(0, 1)$, consider the semigroup

$$S(t)f(x) = \begin{cases} f(x+t) & 0 \leq t+x \leq 1, \\ 0 & t+x > 1. \end{cases}$$

We have also $\sigma(S(t)) = \{0\}$ and then the spectrum of the infinitesimal generator is empty: $\sigma(\mathcal{A}) = \emptyset$.

But, for any initial function $f \in L_2(0, 1)$, we have $S(t)f(x) = 0$ for $t > 2$. This means that $S(t) = 0$, for all $t > 2$. Then, for any control operator \mathcal{B} , the corresponding system is exactly null controllable at time $T > 2$ with the trivial control $u = 0$.

Results for neutral type systems

Global Conjecture

The following statements are equivalent:

- 1 System (1) is exactly null controllable,
- 2 The following two conditions hold
 - i) $\text{rank} \begin{pmatrix} \Delta_{\mathcal{A}}(\lambda) & B \end{pmatrix} = n$ for all $\lambda \in \mathbb{C}$,
 - ii) $\text{rank} \begin{pmatrix} \mu I - A_{-1} & B \end{pmatrix} = n$ for all $\mu \in \mathbb{C}$, $\mu \neq 0$,
- 3 System (1) is completely stabilizable by a feedback law of the form

$$u(t) = F_{-1}\dot{z}(t-1) + Fz_t(\cdot), \quad (4)$$

where $Fz_t(\cdot) = \int_{-1}^0 [F_2(\theta)\dot{z}(t+\theta) + F_3(\theta)z(t+\theta)] d\theta$.

Proved results

Scheme

$$1) \xrightarrow{A} 2) \xleftrightarrow{B} 3)$$

Step A: in a classical way, following idea by Metelskii-Miniuk (2006). We obtain instead of condition $i)$ the equivalent condition:

$$\text{rank} \begin{pmatrix} B & A_{-1}B & \cdots & A_{-1}^{n-1}B \end{pmatrix} = \text{rank} \begin{pmatrix} B & A_{-1}B & \cdots & A_{-1}^{n-1}B & A_{-1}^n \end{pmatrix}$$

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Step B: condition 2 allows to move an infinite number of eigenvalues, condition 1 a finite number.

Important remark : to move the infinite spectrum we use the feedback F_{-1} : $\sigma(A_{-1} + BF_{-1}) = \{0\}$. It may occur that we obtain a generator such that $\sigma(\mathcal{A}) = \emptyset$. In this case all the solutions of neutral type systems are small. Then they vanish at a finite time. We obtain **exact null controllability without control**. Else, it is not proved that complete stabilizability implies null exact controllability.

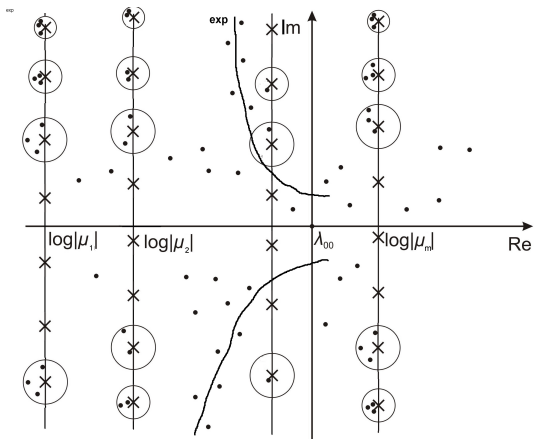


Figure: Spectrum of \mathcal{A} ($\det A_{-1} = 0$)

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The considered problems

Exact controllability (observability)

Exact controllability : the reachability set \mathcal{R}_T from 0-function is $\mathcal{D}(\mathcal{A})$ (all states $z_T(\cdot)$ are reachable from all states $z_0(\cdot)$).

Continuous observability : the operator

$$\mathcal{K}_T : \mathcal{D}(\mathcal{A}) \longrightarrow L_2(0, T; \mathbb{R}^p), \quad x_0 \longmapsto \mathcal{K}_T x_0 = \mathcal{C} e^{\mathcal{A}t} x_0.$$

is boundedly invertible (the initial state is continuously observable).

Exact null controllability (final observability)

Exact null controllability : for all states, there is a control which maintains the systems at 0 : $z_T(\cdot) = 0$ (**Krasovskii** problem).

Continuous final observability : the final state $z_T(\cdot)$ is continuously observable.

Short bibliography

The initial problem of null controllability and observability for delay systems: Krasovskii (1960-th), Krasovskii-Kurzhanskii (1966), Sverdlovsk and Minsk schools in 70-th and 80-th.

In North America : Banks-Jacob-Langenhop, Delfour-Mitter, Karrakchou (later in Morocco), Manitius, O'Connor, Tarn, Olbrot, . . . , etc.

Null controllability, approximate and spectral controllability for neutral systems: The polish school (Olbrot, Bartosiewicz, etc.)

Neutral systems with distributed delays in control and observation and duality: Salamon (1984),

Neutral type systems with discrete commensurate delays :
Metel'skiĭ-Minyuk (2006 –), Khartovskii et al. (2013)

Exact controllability and observability

We obtained the following

Duality (Rabah-Sklyar, Syst. Contr. Letters 2016, MTNS 2014)

Continuous initial observability of the original systems with

$$\dot{z}(t) - A_{-1}z(t-1) = Lz_t \quad y(t) = Cz(t-1) \quad \text{on } [0, T]$$

is equivalent to exact controllability of the system

$$\dot{z}(t) - A_{-1}^*z(t-1) = L^T z_t + C^* u(t) \quad \text{at time } T$$

This is verified iff

- (i) For all $\lambda \in \mathbb{C}$, $\text{rank}(\Delta_{\mathcal{A}}^*(\lambda) \quad C^*) = n$, where $\Delta_{\mathcal{A}}^*(\lambda)$ is the characteristic matrix of the dual systems.
- (ii) For all $\lambda \in \mathbb{C}$, $\text{rank}(\lambda I - A_{-1}^* \quad C^*) = n$,

For a group ($\det A_{-1} \neq 0$), then $y(t) = Cz(t-1) \iff y(t) = Cz(t)$

Exact **null** controllability and **final** observability

Theorem

The final state of system

$$\dot{z}(t) - A_{-1}z(t-1) = Lz_t \quad y(t) = Cz(t-1) \quad \text{on } [0, T]$$

is continuously observable : $\int_0^T \|\mathcal{C}e^{\mathcal{A}t}x_0\|_{\mathbb{R}^p}^2 dt \geq \delta^2 \|e^{\mathcal{A}T}x_0\|_{M_2}^2$
if and only if the adjoint system

$$\dot{z}(t) - A_{-1}^*z(t-1) = L^T z_t + C^*u(t))$$

is exactly null-controllable at time T .

If $\det A_{-1} \neq 0$, then observability and final observability are equivalent.

The state and the adjoint operators

The state operator

$$\mathcal{A} \begin{pmatrix} v \\ \varphi(\cdot) \end{pmatrix} = \begin{pmatrix} \varphi(0) - A_{-1}\varphi(-1) \\ d\varphi(\theta)/d\theta \end{pmatrix},$$

with the domain

$$\mathcal{D}(\mathcal{A}) = \{(v, \varphi(\cdot)) : \varphi(\cdot) \in H^1, \quad v = \varphi(0) - A_{-1}\varphi(-1)\},$$

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while the adjoint operator is

$$\mathcal{A}^* \begin{pmatrix} w \\ \psi(\cdot) \end{pmatrix} = \begin{pmatrix} (A_2^*(0)w + \psi(0)) \\ -\frac{d[\psi(\theta) + A_2^*(\theta)w]}{d\theta} + A_3^*(\theta)w \end{pmatrix},$$

with the domain $\mathcal{D}(\mathcal{A}^*) =$

$$\{(w, \psi(\cdot))\} = \begin{cases} \psi(\theta) + A_2^*(\theta)w & \in H^1, \\ A_{-1}^*(A_2^*(0)w + \psi(0)) & = \psi(-1) + A_2^*(-1)w \end{cases}$$

Dual systems: adjoint system seems to be of different nature

The original system : **courant state** $z(t + \theta) = z_t(\theta)$

$$\underbrace{\begin{pmatrix} v(t) \\ z_t(\theta) \end{pmatrix}}_{x(t)} : \quad \dot{z}(t) - A_{-1}z(t-1) = \underbrace{\int_{-1}^0 [A_2(\theta)\dot{z}_t(\theta) + A_3(\theta)z_t(\theta)] d\theta}_{Lz_t}$$

The adjoint system : **courant state** $w(t + \theta) = w_t(\theta)$

$$\underbrace{\begin{pmatrix} w(t) \\ \psi_t(\theta) \end{pmatrix}}_{x^T(t)} : \quad \dot{w}(t+1) - A_{-1}^*w(t) = \underbrace{\int_{-1}^0 [A_2^*(\theta)\dot{w}_{t+1}(\theta) + A_3^*(\theta)w_{t+1}(\theta)] d\theta}_{L^T w_{t+1}}$$

There a shift $t \leftrightarrow t + 1$ and a “castling” for the courant state !

Criterion of final observability?

The analysis of duality and the condition of exact null controllability give:

Theorem

Exact final observability implies

- 1 $\text{Ker} \begin{pmatrix} \Delta_{\mathcal{A}}(\lambda) \\ C \end{pmatrix} = \{0\}$ for all $\lambda \in \mathbb{C}$,
- 2 $\text{Ker} \begin{pmatrix} \lambda I - A_{-1} \\ C \end{pmatrix} = \{0\}$ for all $\lambda \in \mathbb{C}$, $\lambda \neq 0$.

or, which is equivalent:

- 1 For all $\lambda \in \mathbb{C}$, $\text{rank} \begin{pmatrix} \Delta_{\mathcal{A}}^*(\lambda) & C^* \end{pmatrix} = n$,
- 2 For all $\lambda \in \mathbb{C}$, $\lambda \neq 0$, $\text{rank} \begin{pmatrix} \lambda I - A_{-1}^* & C^* \end{pmatrix} = n$.

These conditions are conjectured to be sufficient. It is true for several neutral type systems.

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Discrete delay systems

To illustrate our results and hypothesis we give here 3 examples.

- The first one shows that for continuous observability a **delay in the output is needed** if the semigroup is not a group.
- The second one is taken from the paper by Metelskiy & Minyuk (2006) and it is shown that in fact we have the very exact controllability **not only exact null controllability**.
- The last example **illustrates our Conjecture** on equivalence between exact controllability and complete stabilizability.

All examples are given in the form of a system with one discrete delay:

$$\dot{z}(t) = A_{-1}z(t-1) + A_0z(t) + A_1z(t-1) + Bu(t). \quad (5)$$

Example 3.

System (5) with

$$A_0 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_1 = 0, \quad A_{-1} = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

It is easy to see that, for all $\lambda \in \mathbb{C}$,

$$\text{rank} \begin{pmatrix} \Delta_{\mathcal{A}}(\lambda) & B \end{pmatrix} = \begin{pmatrix} \lambda & -\lambda e^{-\lambda} - 1 & 0 \\ 0 & \lambda & 1 \end{pmatrix} = 2.$$

Moreover, for all $\lambda \in \mathbb{C}$, $\text{rank}(\lambda I - A_{-1} \quad B) = n$, then the system is **exactly controllable (not only to zero)**. The transposed system

$$\begin{cases} \dot{z}_1(t) &= 0 \\ \dot{z}_2(t) &= \dot{z}_1(t-1) + z_1(t) \end{cases}$$

is continuously observable with the output $y = z_2(t-1)$ but not with $y(t) = z_2(t)$.

Example 4.

The following system was given for exact null controllability and continuous final observability by Metelskiy & Minyuk (2006):

$$A_0 = 0, \quad A_1 = \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad A_{-1} = \begin{pmatrix} 0 & -1 \\ 0 & 1 \end{pmatrix}.$$

In fact, for this system the initial condition is exactly observable by the output

$$y = Cz(t-1), \quad C = (1 \quad 0),$$

and the transposed system is exactly controllable because, for all $\lambda \in \mathbb{C}$,

$$\text{rank} (\lambda I - A_{-1}^* \quad C^*) = \text{rank} \begin{pmatrix} \lambda & 0 & 1 \\ 1 & \lambda - 1 & 0 \end{pmatrix} = 2.$$

However, the initial system is **not exactly observable by the output** $y = Cz(t) = z_1(t)$, because the initial function $z_0(\theta)$, $\theta \in [0, 1]$ cannot be determined.

Example 5.

$$A_0 = \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad A_1 = 0, \quad A_{-1} = \begin{pmatrix} 1 & 0 \\ 0 & 0 \end{pmatrix}, \quad B = \begin{pmatrix} 1 \\ 0 \end{pmatrix}.$$

We have for all $\lambda \in \mathbb{C}$, $\text{rank} \begin{pmatrix} \lambda I - A_0 & B \\ \Delta_{\mathcal{A}}(\lambda) & B \end{pmatrix} = \begin{pmatrix} \lambda - \lambda e^{-\lambda} & 0 & 1 \\ -1 & \lambda & 0 \end{pmatrix} = 2$,
and for all complex $\lambda \neq 0$,

$$\text{rank} \begin{pmatrix} \lambda I - A_{-1} & B \end{pmatrix} = \begin{pmatrix} \lambda - 1 & 0 & 1 \\ 0 & \lambda & 0 \end{pmatrix} = 2.$$

The system is **exactly null controllable** (our results and Metelskii-Minyuk), **completely stabilizable** and the transposed system

$$\begin{cases} \dot{z}_1(t) &= z_2(t), \\ \dot{z}_2(t) &= \dot{z}_1(t-1). \end{cases}$$

is continuously finally observable **by the output** $y = z_1(t-1)$.

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- 3 Exact null controllability and final observability
- 4 Illustrative Examples
- 5 Concluding remarks**

Concluding remarks

For a large class of neutral type systems with distributed delays we **tried** to obtain the equivalence between null exact controllability and complete stabilizability (this is characterized).

A possible solution of the conjecture will be given later (talk by Pavel Barkhayev).

By duality we obtain condition of final continuous observability. **These considerations may be also used for detectability with arbitrary decay rate.**

But it is important to consider more general neutral term: **with several delays and with distributed delay.**

Main publications in this subject

- R. Rabah–G. Sklyar, Exact controllability of neutral type systems, **SIAM J. Control and Optim.** (2007).
- R. Rabah–G. Sklyar–P. Barkhayev, Stability and stabilizability for neutral type systems, **ESAIM-COCV** (2012).
- K. Sklyar–R. Rabah–G. Sklyar, Spectrum and eigenvectors assignment, **SIAM J. Control and Optim.** (2015).
- R. Rabah–G. Sklyar, Observability and controllability for neutral type systems, **Syst. and Control Letters** (2016).
- R. Rabah–G. Sklyar– P. Barkhayev, On exact controllability of neutral type systems, **Ukrain. Math. Journal** (2016). In english and russian.
- R. Rabah–G. Sklyar– P. Barkhayev, this paper, **J. Appl. Math. Comp. Sci.** (2017), No 3.

Thank you for your attention.