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Abdallah Assi

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THE CANONICAL FAN OF A FORMAL \mathbb{K} -ALGEBRA

ABDALLAH ASSI

ABSTRACT. We associate with an algebra $\mathbf{A} = \mathbb{K}[[f_1, \dots, f_s]] \subseteq \mathbb{K}[[x_1, \dots, x_n]]$ over a field \mathbb{K} a fan called the canonical fan of \mathbf{A} . This generalizes the notion of the standard fan of an ideal.

Keywords: Canonical basis, \mathbb{K} -algebras, affine semigroups

INTRODUCTION

Let \mathbb{K} be a field and let f_1, \dots, f_s be nonzero elements of the ring $\mathbf{F} = \mathbb{K}[[x_1, \dots, x_n]]$ of formal power series in x_1, \dots, x_n over \mathbb{K} . Let $\mathbf{A} = \mathbb{K}[[f_1, \dots, f_s]]$ be the \mathbb{K} -algebra generated by f_1, \dots, f_s . Set $U = \mathbb{R}_+^*$ and let $a \in U^n$. If $a = (a_1, \dots, a_n)$ then a defines a linear form on \mathbb{R}^n which maps $\alpha = (\alpha_1, \dots, \alpha_n) \in \mathbb{R}^n$ to the inner product

$$\langle a, \alpha \rangle = \sum_{i=1}^n a_i \alpha_i$$

of a with α . We denote abusively the linear form by a . Let $\underline{x} = (x_1, \dots, x_n)$ and let $f = \sum c_\alpha \underline{x}^\alpha$ be a nonzero element of \mathbf{F} . We set $\text{Supp}(f) = \{\alpha \mid c_\alpha \neq 0\}$ and we call it the support of f . We set

$$\nu(f, a) = \min\{\langle a, \alpha \rangle \mid \alpha \in \text{Supp}(f)\}$$

and we call it the a -valuation of f . We set by convention $\nu(0, a) = +\infty$. Let

$$\text{in}(f, a) = \sum_{\alpha \in \text{Supp}(f) \mid \langle a, \alpha \rangle = \nu(f, a)} c_\alpha \underline{x}^\alpha.$$

We call $\text{in}(f, a)$ the a -initial form of f . Note that $\text{in}(f, a)$ is a polynomial. This notion can also be defined this way: we associate with f its Newton polyhedron defined to be $\Gamma_+(f) =$ the convex hull in \mathbb{R}_+^n of $\bigcup_{\alpha \in \text{Supp}(f)} \alpha + \mathbb{R}_+^n$. The set of compact faces of $\Gamma_+(f)$ is finite. Let $\{\Delta_1, \dots, \Delta_t\}$ be this set. Given $i \in \{1, \dots, t\}$. We set $f_{\Delta_i} = \sum_{\alpha \in \text{Supp}(f) \cap \Delta_i} c_\alpha \underline{x}^\alpha$. Then $\{\text{in}(f, a) \mid a \in U^n\} = \{f_{\Delta_i} \mid 1 \leq i \leq t\}$.

Let the notations be as above, and let \prec be a well ordering on \mathbb{N}^n . We set $\text{exp}(f, a) = \max_{\prec} \text{Supp}(\text{in}(f, a))$ and $M(f) = c_{\text{exp}(f, a)} \underline{x}^{\text{exp}(f, a)}$. We set $\text{in}(\mathbf{A}, a) = \mathbb{K}[[\text{in}(f, a) \mid f \in \mathbf{A} \setminus \{0\}]]$. We also set $M(\mathbf{A}) = \mathbb{K}[[M(f, a), f \in \mathbf{A} \setminus \{0\}]]$. The set $\text{exp}(\mathbf{A}, a) = \{\text{exp}(f, a) \mid f \in \mathbf{A} \setminus \{0\}\}$ is an affine subsemigroup of \mathbb{N}^n .

If $a \in \mathbb{R}^n$ then $\text{in}(f, a)$ may not be a polynomial, hence $\text{exp}(f)$ is not well defined. If $a = (a_1, \dots, a_n)$ with $a_{i_1} = \dots = a_{i_l} = 0$ then we can avoid this difficulty in completing by the tangent cone order on $(x_{i_1}, \dots, x_{i_l})$. We shall however consider elements in U^n in order to avoid technical definitions and results.

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The aim of this paper is to study the stability of $\text{in}(\mathbf{A}, a)$ and $\text{M}(\mathbf{A}, a)$ when a varies in U^n . Note that $\text{exp}(\mathbf{A}, a)$ is not necessarily finitely generated (see example 12). It becomes so if the length of $\frac{\mathbf{F}}{\mathbf{A}}$ is finite. Under this condition, our main results are the following:

Theorem The set $\{\text{in}(\mathbf{A}, a) | a \in U^n\}$ is a finite set. The same holds for the set $\{\text{M}(\mathbf{A}, a) | a \in U^n\}$.

Theorem Let S be a finitely generated affine semigroup. The set $E_S = \{a \in U^n \mid \text{exp}(\mathbf{A}, a) = S\}$ is a union of convex polyhedral cones and the set of E_S, S defines a fan of U^n .

These results generalize those of [8] for ideals in $\mathbb{K}[x_1, \dots, x_n]$ and [1], [2] for ideals in the ring of differential operators over \mathbb{K} .

The paper is organized as follows. In Section 1. we recall the notion of canonical basis of \mathbf{A} with respect to $a \in U^n$, and we give an algorithm that computes an a -canonical basis starting with a set of generators of \mathbf{A} (see [6] for the case $a = (-1, \dots, -1)$). In Section 2. we prove the finiteness theorem, and in Section 3. we prove the existence of a fan associated with \mathbf{A} .

1. PRELIMINARY RESULTS

Let $\mathbf{A} = \mathbb{K}[[f_1, \dots, f_s]]$ be a subalgebra of \mathbf{F} generated by $\{f_1, \dots, f_s\} \subseteq \mathbf{F}$ and let the notations be as above. In particular \prec is a total well ordering on \mathbb{N}^n compatible with sums. Let $a \in U^n$ and consider the total ordering on \mathbb{N}^n defined by:

$$\alpha <_a \alpha' \Leftrightarrow \begin{cases} \langle a, \alpha \rangle < \langle a, \alpha' \rangle \\ \text{or} \\ \langle a, \alpha \rangle = \langle a, \alpha' \rangle \text{ and } \alpha \prec \alpha' \end{cases}$$

The total ordering $<_a$ is compatible with sums in \mathbb{N}^n . We shall use sometimes the notations $\alpha \succ \beta$ for $\beta \prec \alpha$ and $\alpha >_a \beta$ for $\beta <_a \alpha$. We have the following:

Lemma 1. *There doesn't exist infinite sequences $(\alpha_k)_{k \geq 0}$ such that*

$$\alpha_0 >_a \alpha_1 >_a \dots >_a \alpha_k \dots$$

With $\langle a, \alpha_0 \rangle = \langle a, \alpha_k \rangle$ for all $k \geq 1$.

Proof. This is a consequence of Dixon's Lemma, since such a sequence satisfies $\alpha_0 \succ \alpha_1 \succ \dots \succ \alpha_k \succ \dots$ \square

Let $a \in U^n$. Then a defines a filtration on \mathbf{F} : $\mathbf{F} = \sum_{d \geq 0} \mathbf{F}_d$ where \mathbf{F}_d is the \mathbb{K} -vector space generated by $\underline{x}^\alpha, \langle a, \alpha \rangle = d$. Let $U_1 = \mathbb{Q}_+^*$. If $a \in U_1^n$ then, given two indices $d_1 < d_2$, the set of indices d such that $d_1 < d < d_2$ is clearly finite. In particular we get the following:

Lemma 2. *Let $\alpha, \beta \in U_1^n$ and assume that $\alpha >_a \beta$. There doesn't exist infinite sequences $(\alpha_k)_{k \geq 0}$ such that*

$$\alpha >_a \alpha_0 >_a \alpha_1 >_a \dots >_a \beta$$

Proof. The set of $\langle a, \gamma \rangle, \langle a, \alpha \rangle < \langle a, \gamma \rangle < \langle a, \beta \rangle$ is finite. Then the result is a consequence of Lemma 1 \square

Definition 3. *Let $a \in U^n$ and let $f = \sum c_\alpha \underline{x}^\alpha$ be a nonzero element of \mathbf{F} . We say that f is a -homogeneous if $f \in \mathbf{F}_d$ for some d . This is equivalent to $\nu(\alpha, a) = \nu(f, a)$ for all $\alpha \in \text{Supp}(f)$. Note that if $a \notin U_1^n$ then f is a -homogeneous if and only if f is a monomial.*

Definition 4. *Let $a \in U^n$ and let \mathbf{H} be a subalgebra of \mathbf{F} . We say that \mathbf{H} is a -homogeneous if it can be generated by a -homogeneous elements of \mathbf{F} .*

Every nonzero element $f = \sum c_\alpha \underline{x}^\alpha \in \mathbf{F}$ decomposes as $f = \sum_{k \geq d} f_k$, with $f_d = \text{in}(f, a)$ and for all $k > d$, if $f_k \neq 0$ then $f_k \in \mathbf{F}_k$.

Definition 5. Let $a \in U^n$ and let $\{g_1, \dots, g_r\} \subseteq \mathbf{A}$. We say that $\{g_1, \dots, g_r\}$ is an a -canonical basis of \mathbf{A} if $M(\mathbf{A}, a) = \mathbb{K}[[M(g_1, a), \dots, M(g_r, a)]]$. Clearly $\{g_1, \dots, g_r\}$ is an a -canonical basis of \mathbf{A} if and only if $\text{exp}(\mathbf{A}, a)$ is generated by $\{\text{exp}(g_1, a), \dots, \text{exp}(g_r, a)\}$. In this case we write $\text{exp}(\mathbf{A}, a) = \langle \text{exp}(g_1, a), \dots, \text{exp}(g_r, a) \rangle$

An a -canonical basis $\{g_1, \dots, g_r\}$ of \mathbf{A} is said to be minimal if $\{M(g_1, a), \dots, M(g_r, a)\}$ is a minimal set of generators of $M(\mathbf{A}, a)$. It is said to be reduced if the following conditions are satisfied:

- i) $\{g_1, \dots, g_r\}$ is minimal.
- ii) For all $1 \leq i \leq r$, $c_{\text{exp}(g_i, a)} = 1$.
- iii) For all $1 \leq i \leq r$, if $g_i - M(g_i, a) \neq 0$ then $\underline{x}^\alpha \notin \mathbb{K}[[M(g_1, a), \dots, M(g_r, a)]]$ for all $\alpha \in \text{Supp}(g_i - M(g_i, a))$.

Lemma 6. If an a -reduced canonical basis exists, then it is unique.

Proof. Let $F = \{g_1, \dots, g_r\}$ and $G = \{g'_1, \dots, g'_t\}$ be two a -reduced canonical bases of \mathbf{A} . Let $i = 1$. Since $M(g_1, a) \in \mathbb{K}[[M(g'_1, a), \dots, M(g'_t, a)]]$, then $M(g_1, a) = M(g'_1, a)^{l_1} \cdots M(g'_t, a)^{l_t}$ for some $l_1, \dots, l_t \in \mathbb{N}$. Every $M(g'_i, a)$, $i \in \{1, \dots, t\}$ is in $\mathbb{K}[[M(g_1, a), \dots, M(g_r, a)]]$. Then the equation above is possible only if $M(g_1, a) = M(g'_{k_1}, a)$ for some $k_1 \in \{1, \dots, t\}$. This gives an injective map from $\{M(g_1, a), \dots, M(g_r, a)\}$ to $\{M(g'_1, a), \dots, M(g'_t, a)\}$. We construct in the same way an injective map from $\{M(g'_1, a), \dots, M(g'_t, a)\}$ to $\{M(g_1, a), \dots, M(g_r, a)\}$. Hence $r = t$ and both sets are equal. Suppose, without loss of generality that $M(g_i, a) = M(g'_i, a)$ for all $i \in \{1, \dots, r\}$. If $g_i \neq g'_i$ then $M(g_i - g'_i) \in M(\mathbf{A}, a)$ because $g_i - g'_i \in \mathbf{A}$. This contradicts iii). \square

We now recall the division process in \mathbf{A} (see [6] for the tangent cone order $a = (-1, \dots, -1)$ and [4] for $n = 1$).

Theorem 7. Let $a \in U_1^n$ and let $\{F_1, \dots, F_s\} \subseteq \mathbb{K}[[\underline{x}]]$. Let F be a nonzero element of $\mathbb{K}[[\underline{x}]]$. There exist $H \in \mathbb{K}[[F_1, \dots, F_s]]$ and $R \in \mathbf{F}$ such that the following conditions hold:

- (1) $F = H + R$
- (2) If $R = \sum_\beta b_\beta \underline{x}^\beta$, then for all $\alpha \in \text{Supp}(R)$, $\underline{x}^\alpha \notin \mathbb{K}[[M(F_1, a), \dots, M(F_s, a)]]$.
- (3) Set $H = \sum_\alpha c_\alpha F_1^{\alpha_1} \cdots F_s^{\alpha_s}$. If $H \neq 0$ then $\text{exp}(F, a) = \max_{<_a} \{\text{exp}(F_1^{\alpha_1} \cdots F_s^{\alpha_s}, a), c_\alpha \neq 0\}$.

Proof. We define the sequences $(F^k)_{k \geq 0}$, $(h^k)_{k \geq 0}$, $(r^k)_{k \geq 0}$ in \mathbf{F} by $F^0 = F$, $h^0 = r^0 = 0$ and $\forall k \geq 0$:

(i) If $M(F^k, a) \in \mathbb{K}[[M(F_1, a), \dots, M(F_s, a)]]$, write $M(F^k, a) = c_\alpha M(F_1, a)^{\alpha_1} \cdots M(F_s, a)^{\alpha_s}$. We set

$$F^{k+1} = F^k - c_\alpha F_1^{\alpha_1} \cdots F_s^{\alpha_s}, \quad h^{k+1} = h^k + c_\alpha F_1^{\alpha_1} \cdots F_s^{\alpha_s}, \quad r^{k+1} = r^k$$

(ii) If $M(F^k, a) \notin \mathbb{K}[[M(F_1, a), \dots, M(F_s, a)]]$, we set

$$F^{k+1} = F^k - M(F^k, a), \quad h^{k+1} = h^k, \quad r^{k+1} = r^k + M(F^k, a)$$

in such a way that for all $k \geq 0$, $\text{exp}(F^k, a) <_a \text{exp}(F^{k+1}, a)$ and $F = F^{k+1} + h^{k+1} + r^{k+1}$. If $F^l = 0$ for some $l \geq 1$ then we set $H = h^l$ and $R = r^l$. We then easily verify that H, R satisfy conditions (1) to (3). Suppose that $\{F^k | k \geq 0\}$ is an infinite set. Note that, by Lemma 1, given $k \geq 1$, if $F^k \neq 0$ then there exists $k_1 > k$ such that $\nu(F^k, a) < \nu(F^{k_1}, a)$. Hence, there exists a subsequence $(F^{j_l})_{l \geq 1}$ such that $\nu(F^{j_1}, a) < \nu(F^{j_2}, a) < \cdots$. In particular, if we set $G = \lim_{k \rightarrow +\infty} F^k$, $H = \lim_{k \rightarrow +\infty} h^k$, and $R = \lim_{k \rightarrow +\infty} r^k$, then $G = 0$, $F = H + R$, and H, R satisfy conditions (1) to (3). This completes the proof. \square

Definition 8. We call the polynomial R of Theorem 7 the a -remainder of the division of F with respect to $\{F_1, \dots, F_s\}$ and we denote it by $R = R_a(F, \{F_1, \dots, F_s\})$.

Suppose that $\{f_1, \dots, f_s\}$ is an a -canonical basis of \mathbf{A} . If $M(f_i, a) \in \mathbb{K}[[M(f_j, a) | j \neq i]]$ for some $1 \leq i \leq s$, then obviously $\{f_j | j \neq i\}$ is also an a -canonical basis of \mathbf{A} , consequently we can get this way a minimal a -canonical basis of \mathbf{A} . Assume that $\{f_1, \dots, f_s\}$ is minimal and let $1 \leq i \leq s$. If $a \in U_1^n$ then, dividing $f = f_i - M(f_i, a)$ by $\{f_1, \dots, f_s\}$, and replacing f_i by $M(f_i, a) + R_a(f_i - M(f_i, a), \{f_1, \dots, f_s\})$, we obtain an a -reduced canonical basis of \mathbf{A} .

The next proposition gives a criterion for a finite set of \mathbf{A} to be an a -canonical basis of \mathbf{A} .

Proposition 9. Let $a \in U_1^n$. The set $\{f_1, \dots, f_s\} \subseteq \mathbf{A}$ is an a -canonical basis of \mathbf{A} if and only if $R_a(f, \{f_1, \dots, f_s\}) = 0$ for all $f \in \mathbf{A}$.

Proof. Suppose that $\{f_1, \dots, f_s\}$ is an a -canonical basis of \mathbf{A} and let $f \in \mathbf{A}$. Let $R = R_a(f, \{f_1, \dots, f_s\})$. If $R \neq 0$ then $M(R, a) \notin M(\mathbf{A}, a)$. This is a contradiction because $R \in \mathbf{A}$. Conversely, suppose that $R_a(f, \{f_1, \dots, f_s\}) = 0$ for all $f \in \mathbf{A}$ and let $f \in \mathbf{A}$. If $M(f, a) \notin \mathbb{K}[[M(f_1, a), \dots, M(f_s, a)]]$ then $M(f, a)$ is a monomial of $R_a(f, \{f_1, \dots, f_s\})$, which is 0. This is a contradiction. \square

The criterion given in Proposition 9 is not effective since we have to divide infinitely many elements of \mathbf{F} . In the following we shall see that it is enough to divide a finite number of elements.

Let $\phi : \mathbb{K}[X_1, \dots, X_s] \mapsto \mathbb{K}[M(f_1, a), \dots, M(f_s, a)]$ be the morphism of rings defined by $\phi(X_i) = M(f_i, a)$ for all $1 \leq i \leq s$. We have the following

Lemma 10. The ideal $\text{Ker}(\phi)$ is a binomial ideal, i.e., it can be generated by binomials.

Proof. Suppose that f_i is monic for all $i \in \{1, \dots, s\}$ and write $M(f_i, a) = x_1^{\theta_1^i} \dots x_n^{\theta_n^i} = \underline{x}^{\theta^i}$.

Let F be a polynomial of $\text{Ker}(\phi)$ and write $F = M_1 + \dots + M_p$ where M_i is a monomial for all $i \in \{1, \dots, p\}$. We shall prove by induction on p , that F is a finite sum of binomials, each of them is in $\text{Ker}(\phi)$. Write $M_i = b_i X_1^{\beta_1^i} \dots X_s^{\beta_s^i}$. If $p = 2$ then F is a binomial. Suppose that $p \geq 3$. We have $\phi(M_1) = b_1 M(f_1, a)^{\beta_1^1} \dots M(f_s, a)^{\beta_s^1}$, which is a monomial in x_1, \dots, x_n . Write $\phi(M_1) = b_1 x_1^{\theta_1^1} \dots x_n^{\theta_n^1}$. Since $\phi(F) = 0$ then $\phi(M_i) = b_i x_1^{\theta_1^i} \dots x_n^{\theta_n^i}$ for some $i \in \{1, \dots, p\}$. whence $\phi(M_1 - \frac{b_1}{b_i} M_i) = 0$. Write

$$F = M_1 - \frac{b_1}{b_i} M_i + (b_i + \frac{b_1}{b_i}) M_i + \sum_{j \neq 1, i} M_j$$

If $F_1 = (b_i + \frac{b_1}{b_i}) M_i + \sum_{j \neq 1, i} M_j$ then the cardinality of monomials of F_1 is at most $p - 1$. By induction, F_1 is a sum of binomials, each of them is in $\text{Ker}(\phi)$. Consequently the same holds for F . \square

Let $\bar{S}_1, \dots, \bar{S}_m$ be a system of generators of $\text{Ker}(\phi)$, and assume, by Lemma 10, that $\bar{S}_1, \dots, \bar{S}_m$ are binomials in $\mathbb{K}[X_1, \dots, X_s]$. Assume that f_1, \dots, f_s are monic with respect to $<_a$. For all $1 \leq i \leq m$, we can write $S_i(X_1, \dots, X_s) = X_1^{\alpha_1^i} \dots X_s^{\alpha_s^i} - X_1^{\beta_1^i} \dots X_s^{\beta_s^i}$. Let $S_i = \bar{S}_i(f_1, \dots, f_s)$. We have $S_i = f_1^{\alpha_1^i} \dots f_s^{\alpha_s^i} - f_1^{\beta_1^i} \dots f_s^{\beta_s^i}$, and $\exp(S_i) >_a \exp(f_1^{\alpha_1^i} \dots f_s^{\alpha_s^i}) = \exp(f_1^{\beta_1^i} \dots f_s^{\beta_s^i})$

Proposition 11. Let $a \in U_1^n$. With the notations above, the following conditions are equivalent:

- (1) The set $\{f_1, \dots, f_s\}$ is an a -canonical basis of \mathbf{A} .
- (2) For all $i \in \{1, \dots, m\}$, $R_a(S_i, \{f_1, \dots, f_s\}) = 0$.

Proof. (1) implies (2) by Proposition 9.

(2) \implies (1): We shall prove that $R_a(f, \{f_1, \dots, f_s\}) = 0$ for all $f \in \mathbf{A}$. Let f be a nonzero element of \mathbf{A} and let $R = R_a(f, \{f_1, \dots, f_s\})$. Then $R \in \mathbf{A}$. If $R \neq 0$ then $M(R, a) \in M(\mathbf{A}, a)$. Write

$$R = \sum_{\theta} c_{\theta} f_1^{\theta_1} \cdots f_s^{\theta_s}$$

and let $\alpha = \inf_{\theta, c_{\theta} \neq 0} (\exp(f_1^{\theta_1} \cdots f_s^{\theta_s}, a))$. Since $\exp(R, a) \notin \exp(\mathbf{A}) = \langle \exp(f_1, a), \dots, \exp(f_s, a) \rangle$ then $\exp(R, a) >_a \alpha$. Let $\{\theta^1, \dots, \theta^l\}$ be such that $\alpha = \exp(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}, a)$ for all $i \in \{1, \dots, l\}$ (such a set is clearly finite). If $\sum_{i=1}^l c_{\theta^i} M(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}, a) \neq 0$, then $\exp(R, a) \in \langle \exp(f_1, a), \dots, \exp(f_s, a) \rangle$, which is a contradiction. Hence, $\sum_{i=1}^l c_{\theta^i} M(f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}, a) = 0$, and consequently $\sum_{i=1}^l c_{\theta^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i}$ is an element of $\ker(\phi)$. In particular

$$\sum_{i=1}^l c_{\theta^i} X_1^{\theta_1^i} \cdots X_s^{\theta_s^i} = \sum_{k=1}^m \lambda_k \bar{S}_k$$

with $\lambda_k \in \mathbb{K}[X_1, \dots, X_s]$ for all $k \in \{1, \dots, m\}$. Whence

$$\sum_{i=1}^l c_{\theta^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i} = \sum_{k=1}^m \lambda_k(f_1, \dots, f_s) S_k.$$

From the hypothesis $R_a(S_k, \{f_1, \dots, f_s\}) = 0$ for all $k \in \{1, \dots, m\}$. Hence there is an expression of S_k of the form $S_k = \sum_{\beta^k} c_{\beta^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$ with $\exp(f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}, a) >_a \exp(S_k, a)$. Replacing $\sum_{i=1}^l c_{\theta^i} f_1^{\theta_1^i} \cdots f_s^{\theta_s^i}$ by $\sum_{k=1}^m \lambda_k(f_1, \dots, f_s) \sum_{\beta^k} c_{\beta^k} f_1^{\beta_1^k} \cdots f_s^{\beta_s^k}$ in the expression of R , we can rewrite R as

$$R = \sum_{\theta'} c_{\theta'} f_1^{\theta'_1} \cdots f_s^{\theta'_s}$$

with $\alpha_1 = \inf_{\theta', c_{\theta'} \neq 0} (\exp(f_1^{\theta'_1} \cdots f_s^{\theta'_s}, a)) >_a \alpha$. Then we restart with this representation. We construct this way an infinite sequence $\exp(R, a) >_a \dots >_a \alpha_1 >_a \alpha$, which contradicts Lemma 2. \square

The characterization given in Proposition 11 suggests an algorithm that construct, starting with a set of generators of \mathbf{A} , an a -canonical basis of \mathbf{A} . However, such a canonical basis can be infinite as it is shown in the following example:

Example 12. (see [9]) Let $\mathbf{A} = \mathbb{K}[[x + y, xy, xy^2]]$ and let $a = (2, 1)$. Then $M(x + y, a) = x$, $M(xy, a) = xy$, and $M(xy^2, a) = xy^2$. The kernel of the map:

$$\phi : \mathbb{K}[X, Y, Z] \longmapsto \mathbb{K}[x, y], \phi(X) = x, \phi(Y) = xy, \phi(Z) = xy^2$$

is generated by $\bar{S}_1 = XZ - Y^2$. Hence $S = (x + y)xy^2 - x^2y^2 = -xy^3 = R_a(-xy^3, \{x + y, xy, xy^2\})$. Then xy^3 is a new element of the a -canonical basis of \mathbf{A} . If we restart with the representation $\mathbf{A} = \mathbb{K}[[x + y, xy, xy^2, xy^3]]$, then a new element, xy^4 , will be added to the a -canonical basis of \mathbf{A} . In fact, xy^n is an element of the minimal reduced a -canonical basis of \mathbf{A} for all $n \geq 1$. In particular the a -canonical basis of \mathbf{A} is infinite.

In the following we shall assume that the length $l(\frac{\mathbf{F}}{\mathbf{A}})$ is finite. This guarantees the finiteness of a canonical basis. Under this hypothesis, using the results above, we get the following algorithm:

Algorithm. Let $\mathbf{A} = \mathbb{K}[[f_1, \dots, f_s]]$ and let $a \in U_1^n$. Let $\{\bar{S}_1, \dots, \bar{S}_m\}$ be a set of generators of the map ϕ of Proposition 10 and let $S_i = \bar{S}_i(f_1, \dots, f_s)$ for all $i \in \{1, \dots, m\}$.

- (1) If $R_a(S_i, \{f_1, \dots, f_s\}) = 0$ for all $i \in \{1, \dots, m\}$ then $\{f_1, \dots, f_s\}$ is an a -canonical basis of \mathbf{A} .
- (2) If $R = R_a(S_i, \{f_1, \dots, f_s\}) \neq 0$ for some $i \in \{1, \dots, m\}$ then we set $R = f_{s+1}$ and we restart with $\{f_1, \dots, f_r, f_{r+1}\}$. Note that in this case, we have $\langle \exp(f_1, a), \dots, \exp(f_s, a) \rangle \subset \langle \exp(f_1, a), \dots, \exp(f_s, a), \exp(f_{s+1}, a) \rangle \subseteq \exp(\mathbf{A}, a)$. By hypothesis, $l(\frac{\mathbf{F}}{\mathbf{A}}) < +\infty$, hence, after a finite number of operations, we get an a -canonical basis of \mathbf{A} .

2. A FINITENESS THEOREM

Let the notations be as in Section 1. In particular $\mathbf{A} = \mathbb{K}[[f_1, \dots, f_s]]$ with $\{f_1, \dots, f_s\} \subseteq \mathbf{F}$. We shall assume that $l(\frac{\mathbf{F}}{\mathbf{A}}) < +\infty$. The aim of this section is to prove that the set of $\text{in}(A, a), a \in U^n$ is finite. We first recall this result when $\mathbf{A} = \mathbb{K}[f]$ then we prove some preliminary results which will also be used later in the paper.

Lemma 13. *Let f be a nonzero element of $\mathbb{K}[[x_1, \dots, x_n]]$. The set $\{M(f, a), a \in U^n\}$ (resp. $\{\text{in}(f, a), a \in U^n\}$) is finite.*

Proof. Write $f = \sum_{\alpha} c_{\alpha} \underline{x}^{\alpha}$ and let $E = \cup_{\alpha \in \text{Supp}(f)} \alpha + \mathbb{N}^n$. Then $E + \mathbb{N}^n \subseteq E$, and consequently there exists a finite set of E , say $\{\alpha_1, \dots, \alpha_r\}$, such that $E = \cup_{i=1}^r \alpha_i + \mathbb{N}^n$. By definition $\alpha_i \in \text{Supp}(f)$ for all $i \in \{1, \dots, r\}$. If we choose the α_i 's such that $\alpha_i \notin \cup_{j \neq i} \alpha_j + \mathbb{N}^n$, then $\{\exp(A, a), a \in U^n\} = \{\alpha_1, \dots, \alpha_r\}$. This proves our assertion. □

Lemma 14. *Let $a \neq b$ be two elements of U^n . Let $\{g_1, \dots, g_r\}$ be an a -reduced canonical basis of \mathbf{A} . If $M(\mathbf{A}, a) = M(\mathbf{A}, b)$ then $\{g_1, \dots, g_r\}$ is also a b -reduced canonical basis of \mathbf{A} .*

Proof. Let $i \in \{1, \dots, r\}$ and write $g_i = M(g_i, a) + \sum c_{\beta} \underline{x}^{\beta}$ where for all β , if $c_{\beta} \neq 0$, then $\underline{x}^{\beta} \notin M(\mathbf{A}, a)$. Since $M(\mathbf{A}, a) = M(\mathbf{A}, b)$ then for all β , if $c_{\beta} \neq 0$, then $\underline{x}^{\beta} \notin M(\mathbf{A}, a)$. This implies that $M(g_i, a) = M(g_i, b)$. Hence $\{g_1, \dots, g_r\}$ is a b -canonical basis of \mathbf{A} , and the same argument shows that this basis is also reduced. □

Lemma 15. *Let $a \neq b$ be two elements of U^n . If $M(\mathbf{A}, a) \neq M(\mathbf{A}, b)$, then $M(\mathbf{A}, a) \not\subseteq M(\mathbf{A}, b)$.*

Proof. Assume that $M(\mathbf{A}, a) \subseteq M(\mathbf{A}, b)$ and let $\{g_1, \dots, g_r\}$ be a b -reduced canonical basis of \mathbf{A} . By hypothesis, there is $1 \leq i \leq r$ such that $M(g_i, b) \in M(\mathbf{A}, b) \setminus M(\mathbf{A}, a)$. Write $g_i = M(g_i, b) + \sum c_{\beta} \underline{x}^{\beta}$. For all β , if $c_{\beta} \neq 0$, then $\underline{x}^{\beta} \notin M(\mathbf{A}, b)$, hence $\underline{x}^{\beta} \notin M(\mathbf{A}, a)$, which implies that $M(g_i, a) \notin M(\mathbf{A}, a)$ ($M(\mathbf{A}, a) \subseteq M(\mathbf{A}, b)$). This is a contradiction because $g_i \in \mathbf{A}$. □

Corollary 16. *If $a \in U^n \setminus U_1^n$ then there exists $b \in U_1^n$ such that $M(\mathbf{A}, a) = M(\mathbf{A}, b)$.*

Proof. Let $\{g_1, \dots, g_r\}$ be an a -reduced canonical basis of \mathbf{A} . By hypothesis, there exists $\epsilon > 0$ such that for all $i \in \{1, \dots, r\}$ and for all $b \in B(a, \epsilon)$, $M(g_i, a) = M(g_i, b)$ (where $B(a, \epsilon)$ denotes the ball of ray ϵ centered at a). Take $b \in U^n \cap B(a, \epsilon)$. We have $M(\mathbf{A}, a) \subseteq M(\mathbf{A}, b)$. By Lemma 15, $M(\mathbf{A}, a) = M(\mathbf{A}, b)$ □

Corollary 17. *Let $a \neq b$ be two elements of U^n . If $\text{in}(\mathbf{A}, a) \neq \text{in}(\mathbf{A}, b)$, then $\text{in}(\mathbf{A}, a) \not\subseteq \text{in}(\mathbf{A}, b)$*

Proof. We shall prove that if $\text{in}(\mathbf{A}, a) \subseteq \text{in}(\mathbf{A}, b)$ then $\text{in}(\mathbf{A}, a) = \text{in}(\mathbf{A}, b)$. Let to this end $\{g_1, \dots, g_r\}$ be an a -reduced canonical basis of \mathbf{A} and let $i \in \{1, \dots, r\}$. Write $\text{in}(g_i, a) = M_1 + \dots + M_t$ where M_j is b -homogeneous for all $i \in \{1, \dots, t\}$. By hypothesis $\text{in}(g_i, a) \in \text{in}(\mathbf{A}, b)$, hence $M_j \in \text{in}(\mathbf{A}, b)$ for all $j \in \{1, \dots, t\}$. Suppose that $M(g_i, a)$ is a monomial of M_1 . We have $M_1 \in \text{in}(\mathbf{A}, b)$. But M_1 is also a -homogeneous. It follows that $M(M_1, b) = M(M_1, a) = M(g_i, a)$,

in particular $M(g_i, a) \in M(\mathbf{A}, b)$. This proves that $M(\mathbf{A}, a) \subseteq M(\mathbf{A}, b)$. By Lemma 15 we have $M(\mathbf{A}, a) = M(\mathbf{A}, b)$ and $\{g_1, \dots, g_r\}$ is also a b -reduced canonical basis of \mathbf{A} . Finally $\text{in}(\mathbf{A}, a)$ is generated by $\{\text{in}(g_1, a), \dots, \text{in}(g_s, a)\}$ (resp. $\text{in}(\mathbf{A}, b)$ is generated by $\{\text{in}(g_1, b), \dots, \text{in}(g_r, b)\}$). Now the argument above shows that for all $i \in \{1, \dots, r\}$, $\text{in}(g_i, a) = \text{in}(g_i, b)$. This proves the equality. \square

Remark 18. 1. Let $a \neq b$ be two elements of U^n . The proof of Corollary 17 implies that if $\text{in}(\mathbf{A}, a) = \text{in}(\mathbf{A}, b)$ and if $\{g_1, \dots, g_r\}$ is an a -reduced canonical basis of \mathbf{A} then $\{g_1, \dots, g_r\}$ is also a b -reduced canonical basis of \mathbf{A} .

2. Corollary 17 implies the following: if $a \in U^n \setminus U_1^n$ then there exists $b \in U_1^n$ such that $\text{in}(\mathbf{A}, a) = M(\mathbf{A}, a) = M(\mathbf{A}, b) = \text{in}(\mathbf{A}, b)$.

We can now state and prove the following finiteness theorem:

Theorem 19. Let $A = \mathbb{K}[[f_1, \dots, f_s]]$ and let the notations be as above. The set $M(A) = \{M(A; a) | a \in U^n\}$ is finite. In particular the set $I(A) = \{\text{in}(\mathbf{A}, a) | a \in U^n\}$ is finite.

Proof. We need only to prove that $M(A)$ is a finite set. Assume that M is infinite. By Lemma 13 there is an infinite set $U_1 = \{a_1, a_2, \dots\}$ in U^n such that for all $1 \leq k \leq s$ and for all $a \in U_1$, $M(f_k, a) = m_k$ where m_k is a nonzero monomial of f_k . Let $J_1 = \mathbb{K}[[m_1, \dots, m_s]]$: $J_1 \subseteq M(\mathbf{A}, a)$ for all $a \in U_1$. Obviously $J_1 \neq M(\mathbf{A}, a_1)$ (otherwise, $M(\mathbf{A}, a_1) \subset M(\mathbf{A}, a_2)$, for example. This contradicts Lemma 15). We claim that there is $f_{s+1} \in \mathbf{A}$ such that for all $\beta \in \text{Supp}(f_{s+1})$, $\underline{x}^\beta \notin J_1$. Let to this end $m \in M(\mathbf{A}, a_1) \setminus J_1$ and let $f \in \mathbf{A}$ such that $M(f, a_1) = m$. Obviously $m \notin J_1$. We set $f_{s+1} = R_{a_1}(f, \{f_1, \dots, f_s\})$. By lemma 13, there is monomial m_{s+1} of f_{s+1} and an infinite subset $U_2 \subseteq U_1$ such that for all $a \in U_2$, $M(f_{s+1}, a) = m_{s+1}$.

Let $J_2 = \mathbb{K}[[m_1, \dots, m_s, m_{s+1}]]$: $J_1 \subset J_2$. The same process applied to $\{f_1, \dots, f_{s+1}\}$, J_2 and U_2 will construct $m_{s+2} \notin J_2$, $f_{s+2} \in \mathbf{A}$, and an infinite subset $U_3 \subseteq U_2$ such that for all $a \in U_3$, $M(f_{s+2}, a) = m_{s+2}$. We get this way an infinite increasing sequence $J_1 \subset J_2 \subset J_3 \subset \dots$ and for all i , there is $a_i \in U^n$ such that $J_i \subseteq M(\mathbf{A}, a_i)$. This is a contradiction because $l(\frac{\mathbf{F}}{\mathbf{A}})$ is finite. \square

Definition 20. The set $\{g_1, \dots, g_r\}$ of \mathbf{A} which is an a -canonical basis of \mathbf{A} for all $a \in U^n$, is called the universal canonical basis of \mathbf{A} .

3. THE NEWTON FAN

Let $\mathbf{A} = \mathbb{K}[[f_1, \dots, f_s]]$ and let the notations be as in Section 2. In this section we aim to study the stability of $\text{exp}(\mathbf{A}, a)$ and $\text{in}(\mathbf{A}, a)$ when a vary in U^n . Let S be a finitely generated affine semigroup of \mathbb{N}^n . Let

$$E_S = \{a \in U^n \mid \text{exp}(\mathbf{A}, a) = S\}$$

We have the following:

Theorem 21. There exists a partition \mathcal{P} of U^n into convex rational polyherdal cones such that for all $\sigma \in \mathcal{P}$, $\text{exp}(\mathbf{A}, a)$ and $\text{in}(\mathbf{A}, a)$ do not depend on $a \in \sigma$.

In order to prove Theorem 21 we start by fixing some notations. Let S be a finitely generated affine semigroup of \mathbb{N}^n and let $a \in E_S$. Let $\{g_1, \dots, g_r\}$ be the a -reduced canonical basis of \mathbf{A} . By Lemma 15, Lemma 17, and Remark 18, $\{g_1, \dots, g_r\}$ is also the b -reduced canonical basis of \mathbf{A} for all $b \in E_S$. Denote by \sim the equivalence relation on U^n defined from $\{g_1, \dots, g_r\}$ by

$$a \sim b \iff \text{in}(g_i, a) = \text{in}(g_i, b) \quad \text{for all } i \in \{1, \dots, r\},$$

Proposition 22. \sim defines on U^n a finite partition into convex rational polyhedral cones and E_S is a union of a part of these cones.

Proof. Let $c, d \in U^n$ such that $c \sim d$ and let e in the segment $[c, d]$, also let $\theta \in [0, 1]$ such that $e = \theta c + (1 - \theta)d$. Then $a \in U^n$ and $\text{in}(g_i, e) = \text{in}(g_i, c) = \text{in}(g_i, d)$ by an immediate verification. Moreover, $c \sim t \cdot c$ for all $c \in U^n$ and $t > 0$. Therefore the equivalence classes are convex rational polyhedral cones (the rationality results from Corollary 16 and Remark 18). On the other hand, if $c \sim d$ and $c \in E_S$, then $d \in E_S$. This proves that E_S is a union of classes for \sim , the number of classes being finite by Theorem 19 \square

Proof of Theorem 21 We define \mathcal{P} in the following way: for each S we consider the restriction \mathcal{P}_S on E_S of the above partition, and then \mathcal{P} is the finite union of the \mathcal{P}_S 's. On each cone of the partition, $\text{in}(\mathbf{A}, a)$ and $\text{exp}(\mathbf{A}, a)$ are fixed. Conversely assume that $\text{in}(\mathbf{A}, a)$ is fixed and let b such that $\text{in}(\mathbf{A}, a) = \text{in}(\mathbf{A}, b)$. By Corollary 17 and Remark 18, an a -reduced canonical basis $\{g_1, \dots, g_r\}$ of \mathbf{A} is also a b -reduced canonical basis of \mathbf{A} . Moreover, $\text{in}(g_i, a) = \text{in}(g_i, b)$ and $\text{exp}(\mathbf{A}, a) = \text{exp}(\mathbf{A}, b)$. This ends the proof of the theorem except for the convexity of E_S proved below.

Lemma 23. E_S is a convex set: if $a \neq b \in E_S$ then $[a, b] \subseteq E_S$

Proof. Let $a, b \in E_S$ and let $\lambda \in]0, 1[$. Let $\{g_1, \dots, g_r\}$ be an a (and then b) reduced canonical basis of \mathbf{A} . Let $i \in \{1, \dots, r\}$ and set $M = \text{in}(g_i, a)$. Write $M = M_1 + \dots + M_t$ where M_k is b -homogeneous for all $k \in \{1, \dots, t\}$ and $\nu(M_1, b) > \nu(M_k, b)$ for all $k \in \{2, \dots, t\}$. We have $\nu(g_i, a) = \nu(M_1, a) = \nu(M_k, a)$ and $\nu(M_1, b) > \nu(M_k, b)$ for all $k \in \{2, \dots, t\}$. This implies that $\nu(g_i, \theta a + (1 - \theta)b) = \nu(M_1, \theta a + (1 - \theta)b) > \nu(M_k, \theta a + (1 - \theta)b)$ for all $k \in \{2, \dots, t\}$, hence $\text{exp}(g_i, \theta a + (1 - \theta)b) = \text{exp}(g_i, a) = \text{exp}(g_i, b)$. In particular $\text{exp}(\mathbf{A}, a) \subseteq \text{exp}(\mathbf{A}, \theta a + (1 - \theta)b)$. By Lemma 15 we get the equality. This proves that $\text{exp}(\mathbf{A}, \theta a + (1 - \theta)b) = S$. \square

In the following we shall give some precisions about the partition above. Obviously if S_1 and S_2 are two distinct finitely generated affine semigroups and if $a \in E_{S_1}$ and $b \in E_{S_2}$ then $\text{in}(\mathbf{A}, a) \neq \text{in}(\mathbf{A}, b)$ and $\text{exp}(\mathbf{A}, a) \neq \text{exp}(\mathbf{A}, b)$, hence, by Lemma 15 and Corollary 17, neither $\text{in}(\mathbf{A}, a) \subseteq \text{in}(\mathbf{A}, b)$ nor $\text{in}(\mathbf{A}, b) \subseteq \text{in}(\mathbf{A}, a)$, and the same conclusion holds for $\text{exp}(\mathbf{A}, a)$ and $\text{exp}(\mathbf{A}, b)$.

Next we shall characterize open cones of the partition with maximal dimension. Let f be a nonzero element of $\mathbb{K}[[x_1, \dots, x_n]]$ and let $a \in U^n$. We say that $\text{in}(f, a)$ is multihomogeneous if $\text{in}(f, a)$ is b -homogeneous for all $b \in U^n$. This is equivalent to saying that $\text{in}(f, a)$ is a monomial.

Definition 24. Let $a \in U^n$. We say that $\text{in}(\mathbf{A}, a)$ is a multihomogeneous algebra if it is generated by multihomogeneous elements. Note that in this case, If $g \in \text{in}(\mathbf{A}, a)$ then every monomial of g is also in $\text{in}(\mathbf{A}, a)$.

Lemma 25. Let $a \in U^n$ and let $\{g_1, \dots, g_r\}$ be an a -reduced canonical basis of \mathbf{A} . Then $\text{in}(\mathbf{A}, a)$ is a multihomogeneous algebra if and only if $\text{in}(g_i, a)$ is a monomial for all $i \in \{1, \dots, r\}$.

Proof. We only need to prove the if part. Let $i \in \{1, \dots, r\}$ and write $\text{in}(g_i, a) = M_1 + \dots + M_t$ with $\text{exp}(g_i, a) = \text{exp}(M_1, a)$. Assume that $t > 1$. Since $\text{in}(\mathbf{A}, a)$ is multihomogeneous then $M_i \in \text{in}(\mathbf{A}, a)$ for all $i \in \{2, \dots, t\}$. But $\{g_1, \dots, g_r\}$ is reduced. This is a contradiction. Hence $t = 1$ and $\text{in}(g_i, a)$ is a monomial. \square

Proposition 26. The set of $a \in U^n$ for which $\text{in}(\mathbf{A}, a)$ is multihomogeneous defines the open cones of dimension n of \mathcal{P} .

Proof. Let $a \in U^n$ and let $\{g_1, \dots, g_r\}$ be an a -reduced canonical basis of \mathbf{A} . For all $i \in \{1, \dots, r\}$, $\text{in}(g_i, a)$ is a monomial, hence there exists $\epsilon > 0$ such that $\text{in}(g_i, b) = \text{in}(g_i, a)$ for all $b \in B(a, \epsilon)$ (where $B(a, \epsilon)$ is the ball centered at a of ray ϵ). This proves, by Lemma 15 and Corollary 17, that $\{g_1, \dots, g_r\}$ is also a b -reduced canonical basis of \mathbf{A} for all $b \in B(a, \epsilon)$. Conversely, if a is in an open

cone of E_S for some S , then for all b in a neighbourhood, an a -reduced canonical basis $\{g_1, \dots, g_r\}$ of \mathbf{A} is also a b -reduced canonical basis of \mathbf{A} . This implies that $\text{in}(g_i, a)$ is a monomial. This proves our assertion. \square

Definition 27. \mathcal{P} is called the standard fan of \mathbf{A} .

Remark 28. (1) Although we do not have a proof for the existence of a fan for subalgebras \mathbf{A} with $l(\frac{\mathbf{F}}{\mathbf{A}})$ not necessarily finite, we think that this fan does exist. This is true of course if $\exp(\mathbf{A}, a)$ is finitely generated for all $a \in U^n$, but it is not easy to verify this condition a priori.

(2) (see [4]) Suppose that $n = 1$, i.e. $\mathbf{A} = \mathbb{K}[[f(t), \dots, f_s(t)]] \subseteq \mathbb{K}[[t]]$. Then $X_1 = f_1(t), \dots, X_s = X_s(t)$ represents the expansion of a curve in \mathbb{K}^s near the origin. In this case, $\text{in}(\mathbf{A}, a)$ does not depend on $a \in U^n$ and if the parametrization is primitive then $l(\frac{\mathbb{K}[[t]]}{\mathbf{A}}) < +\infty$. If $s = 2$ then $\text{in}(\mathbf{A}, a)$ is a free numerical semigroup and the arithmetic of this semigroup contains a lot of information about the singularity of the curve at the origin.

(3) (see also [5]) If $\mathbf{A} = \mathbb{K}[f_1, \dots, f_s]$ is a subalgebra of $\mathbf{P} = \mathbb{K}[x_1, \dots, x_n]$ then Theorem 19 and Theorem 21 remain valid when we vary $a \in \mathbb{R}_+^n$, under the assumption that $l(\frac{\mathbf{P}}{\mathbf{A}}) < +\infty$ (note that in this case, $\text{in}(f, a)$ is a polynomial for all $a \in \mathbb{R}_+^n$ and for all $f \in \mathbf{P}$).

Example 29. Let $f(X_1, \dots, X_n, Y) \in \mathbb{K}[[X_1, \dots, X_n]][Y]$ and suppose that f has a parametrization of the form $X_1 = t_1^{e_1}, \dots, X_n = t_n^{e_n}, Y = Y(t_1, \dots, t_n) \in \mathbb{K}[[t_1, \dots, t_n]]$ (for instance, this is true if f is a quasi-ordinary polynomial, i.e. the discriminant of f is of the form $X_1^{N_1} \cdots X_s^{N_s}(c + \phi(X_1, \dots, X_s))$ with $c \in \mathbb{K}^*$ and $\phi(0, \dots, 0) = 0$). Then $\frac{\mathbb{K}[[X_1, \dots, X_n]][Y]}{f} \simeq \mathbf{A} = \mathbb{K}[[t_1^{e_1}, \dots, t_n^{e_n}, Y(t_1, \dots, t_n)]]$. In this case, $(e_1, 0, \dots, 0), \dots, (0, \dots, 0, e_n)$ belong to $\exp(\mathbf{A}, a)$ for all $a \in U^n$. Moreover, $\exp(\mathbf{A}, a)$ is a free finitely generated affine semigroup in the sense of [3]. In this case, $l(\frac{\mathbf{F}}{\mathbf{A}})$ need not to be finite, but results of Theorem 19 and Theorem 21 are valid.

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UNIVERSITÉ D'ANGERS, MATHÉMATIQUES, 49045 ANGERS CEDED 01, FRANCE
E-mail address: assi@math.univ-angers.fr