Large Shearing of a Prestressed Tube

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This study is devoted to a prestressed and hyperelastic tube representing a vascular graft subjected to combined deformations. The analysis is carried out for a neo-Hookean response augmented with unidirectional reinforcing that is characterized by a single additional constitutive parameter for strength of reinforcement. It is shown that the stress gradients can be reduced in presence of prestress.

1 Introduction

Mechanical properties are of major importance when selecting a material for the fabrication of small vascular prostheses. The operation and the handing of prostheses vessel by surgeons, on the one part, the design of such grafts, on the other, induce specific loading and particularly boundary or initial conditions. Consequently, the interest in developing a theoretical model to describe the behavior of the prostheses vessel is proved ([1]). In this paper, we consider a thick-walled prestressed tube, hyperelastic, transversely isotropic, and incompressible assimilated to a vessel graft. We give an exact solution of the stress distributions when the tube is subjected to the simultaneous extension, inflation, torsion, azimuthal, and telescopic shears ([2–10]). The first theoretical results, in the case of a silicone tube, indicate that the increase of prestress minimizes the stress gradients due to the effects of the shear.

2 Model Formulation

Consider a nonlinearly elastic opened tube defined by the angle ω_0 (Fig. 1). Let us suppose that the tube undergoes two successive deformations; first, including the closure of the tube which induced residual strains ([11]) and second, including inflation, extension, torsion, azimuthal and telescopic shears. The mapping is described by

$$r=r(R) \quad \theta=\left(\frac{\pi}{\omega_0}\right)\omega+\phi\alpha Z+\Theta(r) \quad z=\lambda\alpha Z+\Delta(r) \quad (1)$$

where (R,ω,Z) and (r,θ,z) are, respectively, the reference and the deformed positions of a material particle in a cylindrical system. ϕ is a twist angle per unloaded length, α and λ are stretch ratios (respectively, for the first and the second deformation), Θ is an angle which defined the azimuthal shear, and Δ is an axial displacement which defined the telescopic shear.

It follows from (1) that the physical components of the deformation gradient $\mathbf{F}$ has the following representation in a cylindrical system:

$$\mathbf{F}=\begin{bmatrix} \dot{r}(R) & 0 & 0 \\ r(R)\dot{\Theta}(r)\dot{r}(R) & \dfrac{r(R)}{R}\dfrac{\pi}{\omega_0} & r\phi\alpha \\ \dot{\Delta}(r)\dot{r}(R) & 0 & \alpha\lambda \end{bmatrix} \quad (2)$$

where the dot denotes the differentiation with respect to the argument.

Incompressibility then requires that $J\equiv\det\mathbf{F}=1$, which upon integration yields

$$r^2=r_i^2+\frac{\omega_0}{\pi\alpha\lambda}(R^2-R_i^2) \quad (3)$$

where R_i and r_i are, respectively, the inner surfaces of the tube in the free and in the loaded configurations (R_e and r_e are the outer surfaces).

The strain energy density per unit undeformed volume for an elastic material, which is locally and transversely isotropic about the $\mathbf{t}(R)$ direction, is given by

$$W=W(I_1,I_2,I_3,I_4,I_5) \quad (4)$$

where

$$I_1=Tr\mathbf{C}, \quad I_2=\tfrac{1}{2}[(Tr\mathbf{C})^2-Tr\mathbf{C}^2], \quad I_3=1,$$
$$I_4=\mathbf{tCt}, \quad I_5=\mathbf{tC}^2\mathbf{t} \quad (5)$$

are the principal invariants of $\mathbf{C}=\overline{\mathbf{F}}\mathbf{F}$ which is the right Cauchy-Green deformation tensor ($\overline{\mathbf{F}}$ is the transpose of $\mathbf{F}$).

The corresponding response equation for the Cauchy stress σ for transversely isotropic incompressible is (see [12])

$$\sigma=-p\mathbf{1}+2[W_1\mathbf{B}-W_2\mathbf{B}^{-1}+I_4W_4\mathbf{T}\otimes\mathbf{T}$$
$$+\mathbf{I}_4W_5(\mathbf{T}\otimes\mathbf{B}\cdot\mathbf{T}+\mathbf{T}\cdot\mathbf{B}\otimes\mathbf{T})] \quad (6)$$

where $\mathbf{B}=\mathbf{F}\overline{\mathbf{F}}$ is the left Cauchy-Green tensor, $\mathbf{1}$ the unit tensor, and p the unknown hydrostatic pressure associated with the incompressibility constraint, $W_i=(\partial W/\partial I_i)$ $(i=1,2,4,5)$ and $\mathbf{T}=(1/\sqrt{I_4})\mathbf{Ft}$.

From (6), the equilibrium equations in the absence of body forces are reduced to

$$\frac{d\sigma_{rr}}{dr}+\frac{\sigma_{rr}-\sigma_{\theta\theta}}{r}=0 \quad (7a)$$

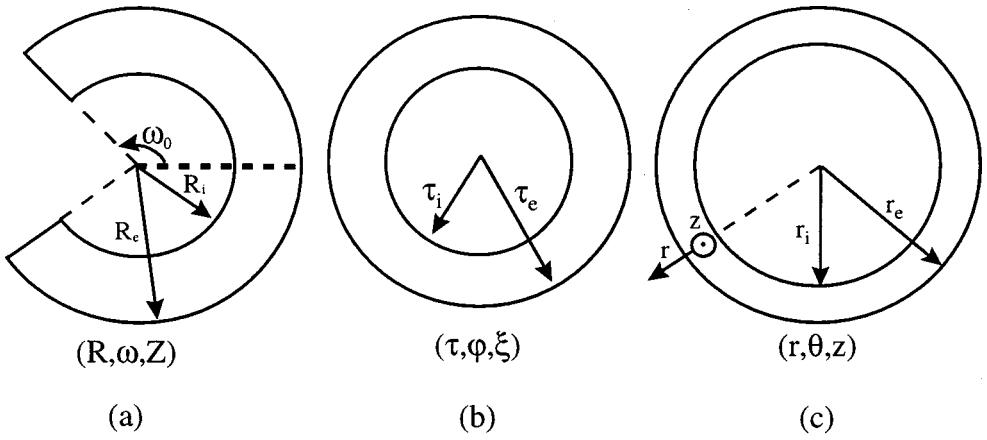

Fig. 1 Cross section of the tube in the stress-free (*a*), unloaded (*b*), and loaded configuration (*c*)

$$\frac{d\sigma_{r\theta}}{dr}+\frac{2\sigma_{r\theta}}{r}=0 \tag{7b}$$

$$\frac{d\sigma_{rz}}{dr}+\frac{\sigma_{rz}}{r}=0. \tag{7c}$$

Suppose that Θ and Δ satisfy the following boundary conditions: (a) $\Theta=\Theta_i$, $\Delta=\Delta_i$ in $r=r_i$ and (b) $\Theta=\Theta_e$, $\Delta=\Delta_e$ in $r=r_e$. Then, a simple computation by integrating (7*b*) and (7*c*) gives the expression of Θ and Δ.

Integrating (7*a*), given the boundary conditions that $\sigma_{rr}(r_i)=-p_i$ and $\sigma_{rr}(r_e)=0$, and taking $\mathbf{t}(R)=t_\omega(R)\mathbf{e}_\omega+t_Z(R)\mathbf{e}_Z$ and using (3) yields the pressure field p:

$$p(r)=p_i+2W_1\left(\frac{R\omega_0}{r\pi\alpha\lambda}\right)^2-2W_2 f(r)+\int_{r_i}^{r}\frac{\sigma_{rr}-\sigma_{\theta\theta}}{s}ds \tag{8a}$$

where

$$f(r)=\dot{\Delta}^2(r)\left[\frac{1}{(\alpha\lambda)^2}+\left(\frac{R\omega_0\phi}{\pi\lambda}\right)^2\right]+\dot{\Theta}^2(r)\left(\frac{R\omega_0}{\pi}\right)^2-2\frac{\dot{\Theta}(r)\dot{\Delta}(r)\phi\omega_0^2}{\alpha\pi}+\left(\frac{r\pi\alpha\lambda}{R\omega_0}\right)^2. \tag{8b}$$

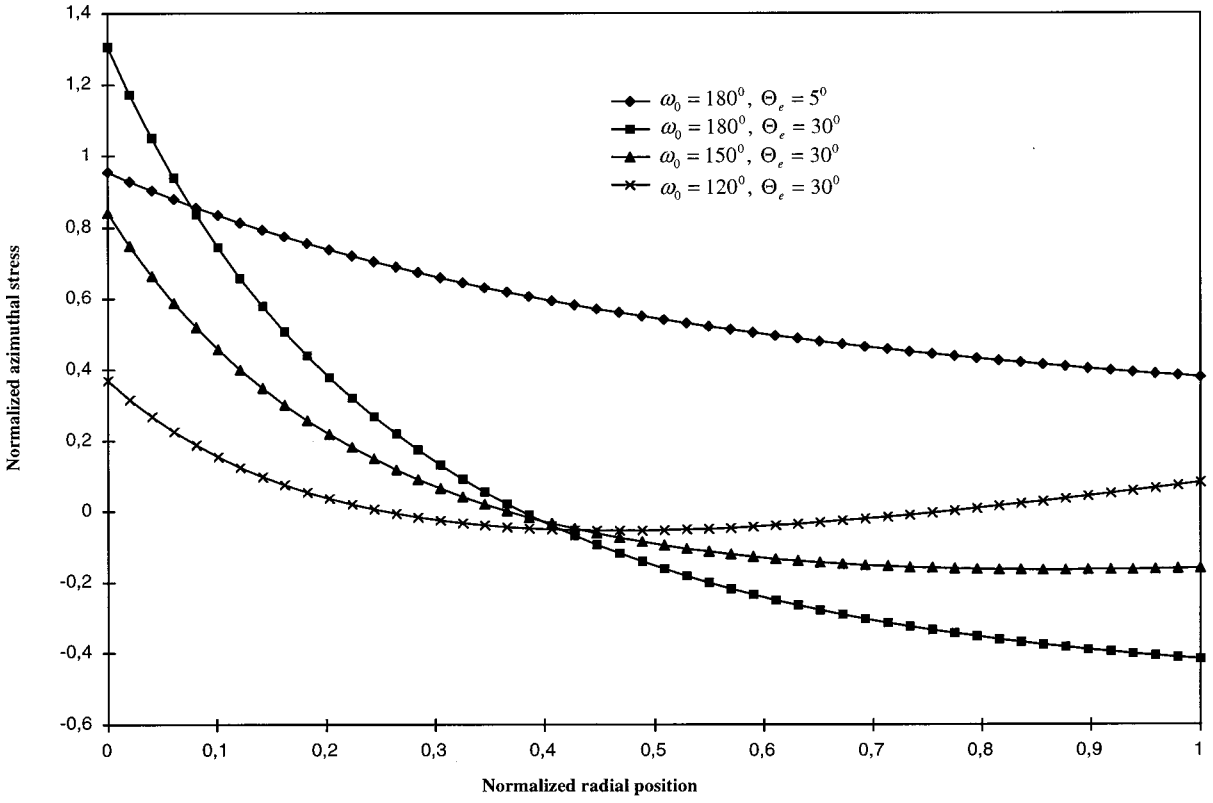

Fig. 2 Azimuthal stresses distribution inside the wall without fibers (stresses normalized by $\sigma_{r\theta}(r_e)$, $\mu=0.166$ Mpa, $p_i=0.0133$ Mpa, $\tau_i=2$ mm, $\tau_e=3$ mm)

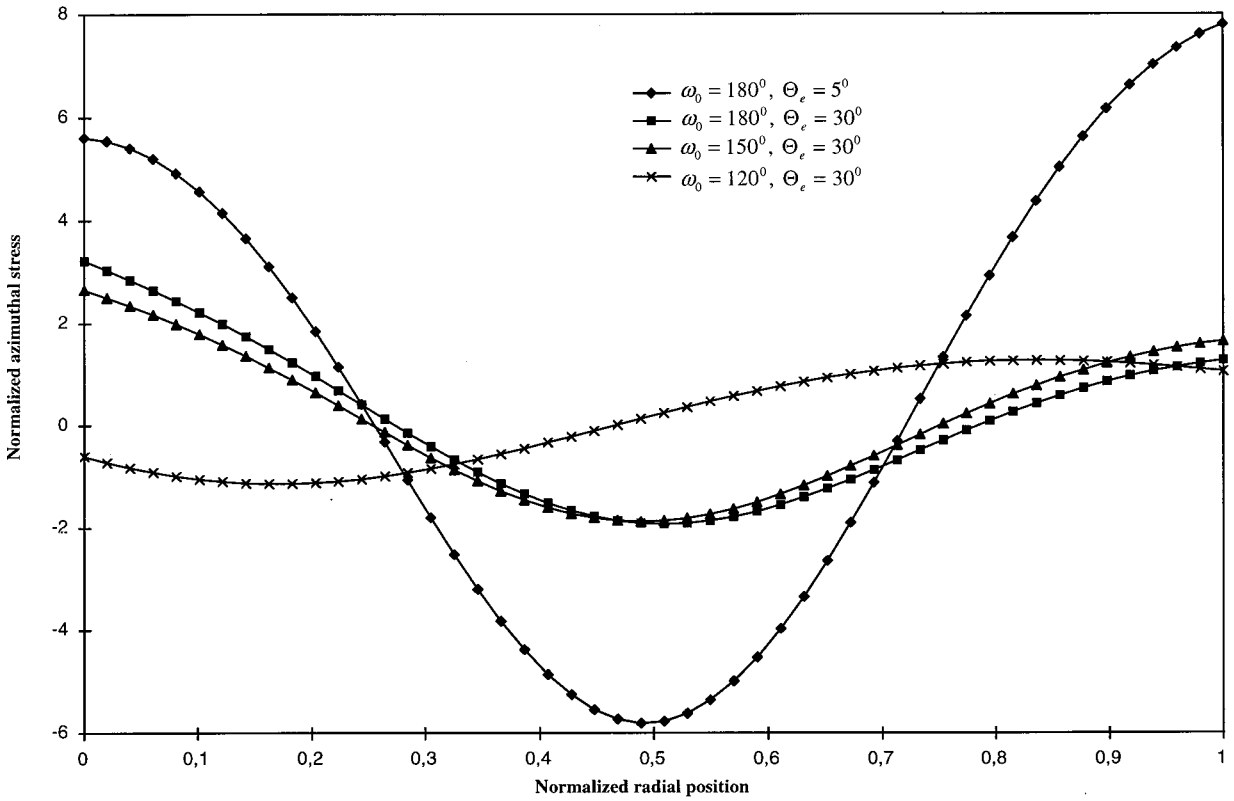

Fig. 3 Azimuthal stresses distribution inside the wall with fibers (stresses normalized by $\sigma_{r\theta}(r_e)$, $\mu = 0.166$ Mpa, $E_f = 10$ Mpa, $p_i = 0.0133$ Mpa, $\tau_i = 2$ mm, $\tau_e = 3$ mm)

The expressions of Θ, Δ, and p determine all the components of the Cauchy stress tensor σ.

3 Results

To illustrate the response of the proposed model, we use the extended Mooney Rivlin strain energy function which represents the behavior of a prosthesis ([13]) constituted of a silicone matrix and textile fibers,

$$W = W(I_1, I_4) = \frac{\mu}{2}(I_1 - 3) + \frac{E_f}{8}(I_4 - 1)^2, \tag{9}$$

where μ is the shear modulus of the isotropic matrix at infinitesimal deformations and E_f is the elastic modulus of the fibers.

The local tangent vector of the fibers is chosen here as $\mathbf{t}(R) = \cos\gamma(R)\mathbf{e}_\omega + \sin\gamma(R)\mathbf{e}_Z$ that represent a helical distribution of fibers ([1]).

From Eqs. (7*b*), (7*c*) and using (3) it easily follows that the expressions of Θ and Δ are

$$\Theta(r) = (\Theta_e - \Theta_i)\frac{\log\left[\dfrac{r}{r_i\sqrt{1 + k(r^2 - r_i^2)}}\right]}{\log\left[\dfrac{r_e}{r_i\sqrt{1 + k(r_e^2 - r_i^2)}}\right]} + \Theta_i \tag{10}$$

$$\Delta(r) = (\Delta_e - \Delta_i)\frac{\log[1 + k(r^2 - r_i^2)]}{\log[1 + k(r_e^2 - r_i^2)]} + \Delta_i \tag{11}$$

where $k = \pi\alpha\lambda / R_i^2\omega_0$.

As an illustrative result, we focus our attention only when the tube is submitted to azimuthal shear strain. Figure 2 shows the distribution of circumferential stresses generated by applied external azimuthal strain at a given pressure when taking into account the effects of such residual stresses. We show clearly that a decrease in ω_0 angle helps to distribute stresses in the loaded state when the shear is important. This result does not change qualitatively when varying the pressure p_i.

Furthermore, the particular effects of the presence of fibers have been examined with a linear distribution of fiber orientation within the data range $\gamma(R_i) = -40$ deg and $\gamma(R_e) = 40$ deg. As illustrated in Fig. 3, it is shown here that the effects of the azimuthal shear upon the distribution of the circumferential stresses within the wall become significant. When the tube is prestressed, the stresses are also distributed. Clearly these results will be able to help the design and fabrication of a small vascular prosthesis ([1]).

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