A maximum entropy network reconstruction of macroeconomic models

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Abstract

In this article the problem of reconstructing the pattern of connection between agents from partial empirical data in a macro-economic model is addressed, given a set of behavioral equations.

This systemic point of view puts the focus on distributional and network effects, rather than time-dependence. Using the theory of complex networks we compare several models to reconstruct both the topology and the flows of money of the different types of monetary transactions, while imposing a series of constraints related to national accounts, and to empirical network sparsity. Some properties of reconstructed networks are compared with their empirical counterpart.

Keywords: econophysics, physics and society, maximum entropy, constraint satisfaction, networks, finance, interfirm, network reconstruction

The state and dynamics of production and financial sectors are affected by interconnection patterns between agents. However these patterns are poorly known at a detailed level because of confidentiality issues. This prevents building risk indicators, that can help preventing the propagation of crises.

Our main motivation here is to infer an ensemble of unobserved multilayer networks in a data-scarce context, taking advantage of available information: first, aggregate public statistics are used to estimate the number and importance of the nodes involved. Then, macro-economic models established by economists provide consistency and behavioral constraints that further reduce the size of the reconstructed ensemble.

A second motivation is to recover empirical regularities from the reconstructed ensemble, that are also available in a few empirical studies, which will help assessing our method. For example, the degrees of firms in buyer-supplier networks have been the subject of many studies, and are hard to measure and to model. They are fundamental properties of the network that influence the dynamical processes that take place on it. The same holds for strengths as functions of degrees. Even though the macro-economic model used below is theoretical, it allows to compute these empirical regularities from the reconstructed ensemble. This paves the way for more detailed models with more sectors (government, central bank, financial services, the environment, etc...), and more refined mechanisms.

A third motivation, not developed in this article and left for future works, will be to explore the consequences of building such a reconstructed ensemble in terms of economic applications: define and estimate risk levels in production networks, study the risk of crisis propagation between the productive and financial sectors, identify hidden clusters among nodes, etc...

To do so our method in this paper builds on advances in the field of data-driven network reconstruction from partial information. While actively investigated in recent years, it remains an open problem. Maximum entropy methods were introduced in statistical physics [1] and were shown to be very useful in studying the properties of network ensembles [2]. Their application to social systems [3, 4, 5, 6], to financial and economic networks is well established [7] and was recently demonstrated by central bankers to be rather accurate [8].

Network theory which deals with the structure and dynamics of network, and the properties of dynamics over networks has been prolific overs past years [9] and seen a rising interest among economists [10]. Theoretical models of supply chain, credit [11], trade, as well as empirical studies have covered various topics such as employment, world trade or ownership control among corporations [12].

However, up to our knowledge, the possibility to include constraints inspired by macroeconomic models

in Maxent network models was little studied so far. In engineering for example, network reconstruction based on physical laws was developed for flow networks [13]. In the present article we propose to introduce such a constraint on the network ensemble, with an unknown topology.

The most direct way to reach that aim would be to extend one of the existing weighted reconstruction methods such as [14, 15], but this extension raises several issues. Rather, we use a two-step method that requires first to estimate the topology of each subnetwork taken independently, from partial empirical data, in the spirit of [16]. This choice stems from the fact that weighted networks estimation is greatly improved when topological information is available. Then, when the topology is reconstructed, network weights can be approximated using various methods, either probabilistic or deterministic.

The highlights of our method to reconstruct an ensemble of networks follow: i) it is probabilistic, ii) relies on Maximum Entropy, iii) enforces non-negativity of weights, iv) allows linear constraints on connections and weights, of the form $A\xi - b = 0$. More specifically :

- it is probabilistic because deterministic methods do not allow sampling random instances, that are necessary to compute average quantities.
- Maximum Entropy makes it possible to respect empirical measurement (e.g. aggregate weights, link density), while maximizing randomness, to avoid biases.
- non-negativity of weights is required by the nature of economic transactions.
- linear constraints in the form $A\xi - b = 0$ incorporates consistency constraints that stems from models built by economists.

Lastly we stress that when detailed micro data is available, a reconstruction error can be computed to assess the accuracy of the method.

In section 1 we present the testbed model for this study, which is inspired by stock-flow consistent (SFC) macroeconomic models, in the restricted linear and steady-state case. In section 2 the corresponding binary network is defined and reconstructed. In section 3, the weights are computed, using several methods. Sections 4 and 5 compare and discuss the results while section 6 concludes.

1. A disaggregated linear model

In this section we describe a toy macroeconomic model, only consisting in a transaction matrix that defines the flows of money between origins and destinations, as shown in Tab. 1. There are no stocks (capital, loans, deposits) nor balance sheet, nor behavioral equations responsible for consumption, capital depreciation. It is a simplification of the aggregated BMW model proposed in [17, §7], an SFC model that introduces private bank money and does not involve a state nor a central bank, and where each sector is represented by a single agent.

However, our model deals with the disaggregated case: the number of households, firms and banks is arbitrary. Households can buy from several firms, get wages from various employers and interest on deposits from several banks. Firms can buy capital goods from many firms, and pay interest on loans to several banks.

To simplify further, and keep the focus on topological and distributional effects, the systems is supposed to be in the steady-state[1] regime. The set of row and column-sums equations in Tab. 1 can be written in the form a linear system:

$$S = \{\xi \ s.t. \ A\xi = b\} \tag{1}$$

Furthermore the system can be written as a function of demands only:

$$\xi = \begin{bmatrix} Cd & Id & WBd & ILd & IDd \end{bmatrix}^T \tag{2}$$

[1] a detailed study in [17] establishes the properties of the BMW model in the transient and steady-state regimes, in the case of the representative model that is when each institutional sector is represented by one agent.

	Households			Firms		Banks	$\sum$
	1	2	3	1	2	1	
Consumption	-C_{d11}	-C_{d21}		C_{s1}			0
		-C_{d22}		C_{s2}			0
	-C_{d13}		-C_{d33}	C_{s3}			0
Investment				-I_{d1}	I_{s1}		0
Wage	WB_{s1}			-WB_{d1}			0
		WB_{s2}		-WB_{d2}			0
			WB_{s3}		-WB_{d3}		0
Interest on loans				-IL_1		IL_1	0
Interest on deposits	ID_1					-ID_1	0
		ID_2				-ID_2	0
			ID_3			-ID_3	0
$\sum$	0	0	0	0	0	0	0

Table 1: Transaction matrix of a disaggregated model with many households, two firms and one bank: agents $nh = 3$, $nf = 2$, $nb = 1$. See Tab. 2 for notations.

with vectors C, I, WB, IL, ID standing for consumption, investment, wage bill, interest on loans, interest on deposits and subscripts d and s referring to demand and supply as summarized in Tab. 2. The vectors are indexed so as to encode all origin-destination information:

$$Cd = \begin{bmatrix} Cd_{1,1} & \dots & Cd_{n_h,1} & Cd_{1,2} & \dots & Cd_{n_h,2} & \dots & Cd_{1,n_f} & \dots & Cd_{n_h,n_f} \end{bmatrix} \tag{3}$$

where Cd_{n_h,n_f} is the amount of consumption goods sold by firm n_f to household n_h.

The same indexing convention is adopted for I, WB, IL, ID, as explained in Appendix A. This leads to the following expression for A and b:

$$A = \begin{bmatrix} -I_{h1} & & I_{h2} & & I_{h3} \\ I_{f1} & I_{f2} & -I_{f3} & -I_{f4} & \\ & & & I_{b1} & -I_{b2} \end{bmatrix}, \; b = 0_{n_b+n_f+n_h} \tag{4}$$

The submatrices I_* reflect the connection pattern among agents, in a way such that each line of A can enforce the constraints in Tab. 1. A is a matrix with elements having values in $\{-1, 0, +1\}$, composed of $2n_f n_h + n_f^2 + n_b(n_f + n_h)$ columns and $n_b + n_f + n_w$ rows. Its sparsity factor is close to 10^{-3} given the values of the parameters. The system $A\xi = b$ is under-determined.

Note that the system eq.(1) is homogeneous and has a trivial solution. For this reason, it will be useful below to define a minimalist nonhomogeneous system:

$$S_1 = \{\xi \; s.t. \; A_1\xi = b_1\} \tag{5}$$

where A_1 and b_1 are such that the consumption of all households is set to the constant value α_0.

How the connection pattern is reconstructed from partial empirical data is the subject of sec. 2.

2. Random network reconstruction and sampling

In order to parametrize the disaggregated model in sec. 1, empirical datasets are necessary. Detailed datasets exist in some particular cases: consumption networks were studied in Latin America and Europe [18, 19]. The buyer/supplier interfirm network in Japan [20, 21], in Estonia [22], in the USA [23] where the distribution of supply chains was modelled by a birth-death process. The ownership network of transnational companies was reconstructed in [12] whith spatial distances as an explanatory variable [24]. Apart from interfirm links, a study of Japanese bank-firms relationships can be found for example in [25].

Variable	Label
loans to firms	L
investment	I
interest on loans	IL
wage bill	WB
interest on workers deposits	ID
consumption of workers	C

Table 2: Labels associated with the different monetary variables, after [17]. The subscripts d and s stand for demand and supply.

However, when detailed transaction databases exist, their access is restricted or paywalled. Only aggregated ones are publicly available for most developed countries.

The topic of input/output relations between industrial sectors has been studied by economists dating back to Leontief [26]. At the aggregate level, this weighted network is densely connected, as remarked in [27], but dense connection is not a property observed at the micro level in available detailed datasets. The same kind of problem arises for other types of networks, which reveals the necessity to perform network reconstruction [7].

Before describing the reconstruction methods used below, let us define some notations: a binary graph G, with at most one edge $e \in E$ between two vertices in $\mathcal{I} \times \mathcal{J}$, is specified by its adjacency matrix $\mathbf{A} = \{a_{ij}\}_{i\in\mathcal{I},j\in\mathcal{J}}$. This covers the unipartite case when $\mathcal{I} = \mathcal{J}$. The degree sequences will be denoted by k. For graph ensembles, p_{ij} is the connection probability between vertices i and j, and $\langle . \rangle$ denotes an average over that ensemble. The n-uple of adjacency matrices that correpond to the model in sec. 1 is $\mathcal{A} = (\mathbf{A}_{cons}, \ldots, \mathbf{A}_{invest})$. Lagrange multipliers in maximum-entropy models will be written in the form x_i, with $i \in \mathcal{I}$.

Popular graph examples include the Erdös-Rényi random graph model (generalized to the bipartite case under the name BiRG, for Bipartite Random Graph [28]), and the configuration model (CM, see [7, 2.2.2]). The latter defines an maximally random ensemble of graphs such that the degree k_i of each vertex is constrained to experimental values, that is in the undirected case: $\forall i \in \mathcal{I}, \langle k_i \rangle = k_i$. However such detailed information is not available in our case. To go round this difficulty the notion of fitness g_i was put forward in the litterature [7, p.82]. The hypothesis made is that "the probability for any two nodes to interact can be explicitly written in terms of non-structural quantities", which suggests to write the Lagrange multiplier in the form $x_i = f(g_i)$, where g_i are "non-structural quantities" such as the trade volume between countries.

Node-specific fitnesses of the form x_i can be used, for example in the bilinear model $p_{ij} = x_i x_j$ that is associated to the sparse hypothesis [2, 29].

The following functional form was introduced in [16] to reconstruct the World Trade Web using the Gross Domestic Product (GDP) of various countries as an explanatory variable:

$$p_{ij} \quad = \quad \frac{z\ x_i x_j}{1 + z\ x_i x_j} \tag{6}$$

This form is an example of "fitness-induced configuration model" (FiCM), that makes it possible to respect both the maximum likelihood criterion and the total empirical number of links, or equivalently the overall link density of the network, as noted in [30].

Many extensions were debated to cover for example weighted, directed, bipartite graphs. In the next section, dyadic terms such as the distance d_{ij} in the gravity model $p_{ij} = \frac{x_i x_j}{d_{ij}^2}$, where the respective contributions of nodes i and j can't be disentangled, will be introduced.

2.1. Application to a disaggregated SFC model

In sec. 1 an example of disaggregated economic model was considered. In sec. 2, reconstruction strategies were discussed. In the present section, the nature of the graph built according to the model in sec. 1 is

explained, as well as the probabilistic models used to reconstruct it from partial data. Section 2.2 describes the necessary data.

Let G be the graph associated to the transactions of the disaggregated model. It is composed of various subgraphs with heterogeneous properties, due to the diversity of the transactions involved, as summarized in Tab. 3, where all subgraphs are binary and do not include any self-loop (a similar graph representing assets and liabilities may be written, following [31]).

G is a multipartite, multilayer network. The population of agents is divided into three categories (banks, firms, households). Each type of transaction corresponds to one layer. This type of network, while common for social networks [32, Tab.2], is not often found in economic empirical studies that tend to focus on individual layers, one at a time (finance, interbank, production, consumption, ...). Furthermore, imposing a constraint over the network (here $A\xi = b$) is used in engineering to reflect conservation laws (mass, energy), or in statistical mechanics of networks, for example to study motif-constrained ensembles (e.g. 2-stars, see [2]), but is not commonly found in economic network models.

Transaction type	Graph type	Nodes	Edge $i \to j$ or $i \leftrightarrow j$ present if:
Investment of firms	unipartite directed	firms	firm i is selling capital goods to firm j
Consumption of households	bipartite undirected	firms, households	firm i is selling consumption goods to household j
Wages	bipartite undirected	firms, households	firm i pays a wage to household j
Interests on loans	bipartite undirected	banks, firms	bank i gets interests from firm j
Interests on deposits	bipartite undirected	banks, households	bank i pays interests to household j

Table 3: Graph type for all subgraphs of the full transaction graph G. All subgraphs are binary.

Following the methods explained in sec. 2, we need to specify the probability model p_{ij} for each type of transaction using the different ingredients available (node-specific or dyadic terms, ...). Not all subgraphs could be modelled as FiCM maximum-entropy networks, due to the availability of data to fit the models. In that case, BiRG networks were used.

Let us first remark that in a few cases, the available empirical data impose a community structure to the graph. For example, the demography of firms in public datasets is given at an aggregate sectorial level. The same is true for intermediate consumption and the investment in capital goods of firms (see sec. 2.2). As a consequence, the probability matrix $\{p_{ij}\}$ that firm investment occurs between any two firms i and j will in the simplest model be a block matrix with terms taken in $\{p_{s_i,s_j}\}$, the probability matrix that a transaction occurs between sectors s_i and s_j, as represented in Fig. 1(a). The same is true for consumption of households, because the source of final consumption of households is known at the level of industrial sectors.

The different functional forms corresponding to the transactions of this simplified block model are summarized in Tab. 4. z is a free parameter which value is set independently for each transaction using maximum likelihood, as will be seen below. Dyadic factors are noted $d_{s_i s_j}$ and node-specific factors are noted x_i. Using node-specific rather than dyadic factors in FiCM is also a matter of data availability, as will be seen in sec. 2.2.

The first two rows in Tab. 4 are associated with FiCM networks which definitions were recalled in sec. 2. In the case of consumption, all households are considered homogeneous. This explains the difference in the form of p_{ij} between the first two lines.

The mean number of links $\langle L \rangle = \sum_i \sum_{j \neq i} p_{ij}$ in the undirected case is constrained and should be equal to the observed one $L = \sum_i \sum_{j \neq i} a_{ij}$, which is given by empirical observation. Under the chosen models,

Transaction type	Model type	p_{ij}	Indices	Constraint
Investment of firms	FiCM	$\frac{z\ d_{s_i s_j}}{1+z\ d_{s_i s_j}}$	$(i,j) \in [1,nf]^2$	$d_{s_i s_j}$: propensity of sector s_i to sell capital goods to s_j
Consumption of households	FiCM	$\frac{z\ x_{s_i}}{1+z\ x_{s_i}}$	$(i,j) \in [1,nf] \times [1,nh]$	x_{s_i}: propensity of sector s_i to sell consumption goods to households
Wages	BiRG	$\frac{L}{nf\ nh}$	$(i,j) \in [1,nf] \times [1,nh]$	$\langle L \rangle = L^*$, or $\langle k \rangle = k^*$
Interests on loans	BiRG	$\frac{L}{nb\ nf}$	$(i,j) \in [1,nb] \times [1,nf]$	$\langle L \rangle = L^*$, or $\langle k \rangle = k^*$
Interests on deposits	BiRG	$\frac{L}{nb\ nh}$	$(i,j) \in [1,nb] \times [1,nh]$	$\langle L \rangle = L^*$, or $\langle k \rangle = k^*$

Table 4: Block model for each transaction type. All subgraphs are binary. z is a free parameter which value is set independently for each transaction using maximum likelihood.

Transaction type	Model type	p_{ij}	Indices	Constraint
Investment of firms	FiCM	$\frac{z a_i a_j\ d_{s_i s_j}}{1+z a_i a_j\ d_{s_i s_j}}$	$(i,j) \in [1,nf]^2$	a_i: propensity of firm i to play a role in investment, either as a buyer or a seller.
Wages	FiCM	$\frac{z a_i x_{s_i}}{1+z a_i x_{s_i}}$	$(i,j) \in [1,nf] \times [1,nh]$	a_i: see investment. x_{s_i}: propensity of sector s_i to attract the workforce

Table 5: Random fitness model for transactions that differ from the block model in Tab. 4.

given empirical fitnesses and L, the values of the remaining free parameters z for each network is set using maximum-likelihood procedure, for the probabilities p_{ij} to be fully specified. In the case of the FiCM model for investments, this leads to solving the one-dimensional nonlinear equation in z:

$$L = \sum_{i<j} \frac{z\ x_i x_j}{1 + z\ x_i x_j} \tag{7}$$

which can be done approximately using standard numerical methods. One can then sample independently the edges, and obtain random network samples for each transaction. An example is given in Fig. 2(d).

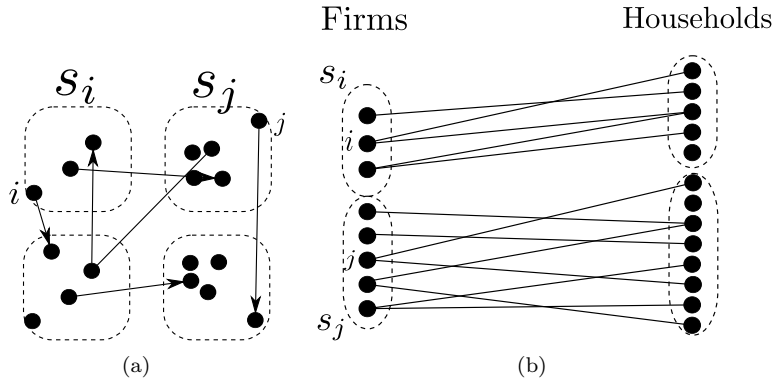


Figure 1: Simplified network of (a) investments between firms i and j belonging to different sectors s_i and s_j ; (b) wages.

In Tab. 4 the last three rows refer to a BiRG network (which definition is recalled in sec. 2) such that only the average number of links is constrained. This type of model is not appropriate if local information

is available, but this is not the case for the examined transactions. The edge probability is uniform and has the simple expression $p_{ij} = \frac{L}{N_1 N_2}$ where N_1 and N_2 are the number of vertices in each layer of the bipartite network. The average number of links $\langle L \rangle = L^*$ constraint can be equivalently expressed as a mean degree constraint, either on one subset of the nodes or the other since these networks are bipartite with a fixed number of vertices. Furthermore, for the BiRG networks:

- Wages: this network (firms on one hand and households on the other) can be subdivided into different independent bipartite networks that correspond to industrial sectors as shown in Fig. 1(b). Firms and households are randomly assigned to a sector depending on business demography data (i.e. number of firms and number of employees in each sector). Then the connection probabilty is set such that the mean degree on the households side equals the empirical mean: $\forall i \in [1, nh],\ \langle k_i \rangle = \bar{k}^{wage}$
- Interests on loans: the degree of this network (banks and firms) is uniform on the firms side, and equals its empirical mean: $\forall i \in [1, nf],\ \langle k_i \rangle = \bar{k}^{loans}$
- Interests on deposits: the degree of this network (banks and households) is uniform on the households side, and equals its empirical mean: $\forall i \in [1, nh],\ \langle k_i \rangle = \bar{k}^{deposit}$

In the block model, for some subnetworks, all firms in a given economic sector have the same connection probability with firms in another sector. This approach is useful but not realistic. A simple way to introduce diversity among sectors is to use random fitnesses sampled from a specific distribution. Instead of writing p_{ij} in the investment network in the form $\frac{z\ d_{s_i s_j}}{1+z\ d_{s_i s_j}}$, one may write:

$$p_{ij} \quad = \quad \frac{z a_i a_j\ d_{s_i s_j}}{1 + z a_i a_j\ d_{s_i s_j}} \tag{8}$$

where a_i are firm-specific terms sampled from some distribution that can be fit to empirical values. In this article a continuous uniform distribution $X = \mathcal{U}_{[0,1]}$ is chosen, but power law distributions will be considered in further works. The wage network is modified accordingly, and can be seen as an FiCM network determined by the firm-specific fitness a_i in eq.(8) and a sector-specific fitness x_{s_i} that reflects the sectorial assignment of the workforce in empirical data, keeping the household sector homogeneous:

$$p_{ij} \quad = \quad \frac{z a_i x_{s_i}}{1 + z a_i x_{s_i}} \tag{9}$$

The specification of this random fitness model is summarized in Tab.5. Other propositions to go beyond the limits of this formalization will be discussed in sec.5

2.2. Empirical data and fitnesses

Most datasets are extracted from Eurostat databases [33] as summarized in Tab. B.10. The figures mentionned in this table concern the demography of businesses and are not used directly to initialize simulations, but are downscaled to permit computation in reasonable time. The agricultural sector, being separated from business demography data, is not included in this study but will be added in further works.

The structure of supply, use and input-output tables is explained in Appendix B, along with the notations for the Eurostat sector aggregates. Each correspond to a given country, and a given year, but these mentions are dropped for simplicity.

These datasets are used to compute the fitnesses $d_{i,j}$ and x_i in Tab. 4. $d_{s_i s_j}$ quantifies the propensity of sector s_i to sell capital goods to s_j. Since this information is not available directly in public datasets, our proposition is to use the volume of intermediate consumption by industry in the use table as a proxy for $d_{s_i s_j}$. It is a dyadic factor depending specifically on the couple (s_i, s_j).

The fitness value x_{s_i} in Tab. 4 is supposed to quantify the propensity of sector s_i to sell consumption goods to households. While household consumption could in fact be separated into different subgroups (see [19]), we keep it aggregated here for simplicity. The proposed proxy comes in the form of a dot-product:

$$x_{s_i} \quad \propto \quad \sum_{p \in [1,P]} sup[p, s_i] \times use^{fin}[p] \tag{10}$$

where $sup[p, s]$ is the value of product $p \in [1, P]$ produced by sector $s \in S$, and $use^{fin}[p]$ is the value of product $p \in [1, P]$ consumed by households as a final use. This factor weighs the proximity between the supply profile of sector s_i and the final use profile of the household sector. It is normalized to 1 across sectors.

Average degrees of networks typically are not included in public datasets, but rather in private ones. However their magnitude can be looked for in the literature in order to parametrize both FiCM and BiRG models in sec. 2.1:

- the average number of edges per vertex for the interfirm network has been studied in the USA for the period 1979-2002, and has a average value of 1.06 according to [23]. In the case of Japan, the estimated value is close to 5 [21]. We believe these estimations aggregate both intermediate consumption of firms, and acquisition of capital goods by firms. However we consider they are valuable proxies for the mean link number $\langle L \rangle$ in the investment network.
- the average number of suppliers for households can be estimated from recent studies at the individual level [18]. We set it to 20 in experiments below.
- the average number of jobs per household $\bar{k}^{wage}$ is set to 1.

These choices are further discussed in sec.5.

2.3. Network properties

In this section we analyze the properties of some of the network models discussed in sec. 2.1. The availability of a probabilistic model allows us to compute analytically the moments of some higher-order topological properties [7], so as to compare it to stylized facts found in empirical studies. The programs used to generate the figures are publicly available[2]. The figures dealt with in this section concern the Czech Republic in 2010.

In sec. 2.1, the expression of the connection probabilities was given for all transaction networks. Fig. 2(a) illustrates the behavior of p_{s_i,s_j} in the case of the investment block network, proxied by the interfirm consumption network in this article. Almost all sectors are primarily connected to themselves. Four aggregated sectors stand out in terms of probability magnitude: B-E, F, G-I, and M-N (see definitions in Tab. B.11). This can be related to the number of firms in each sector, jointly represented with probabilities in Fig. 2(c). In first approximation the latter can be considered as symmetric, but this is not strictly verified at a finer scale as shown in Fig. 2(b), where $\{|p_{s_i,s_j} - p_{s_j,s_i}|\}$ is represented, as well as the sign of the difference. For example industry (B-E) sells more to construction (F) than the opposite. A randomly generated network corresponding to p_{ij} is shown in Fig. 2(d) and illustrates these facts.

The network of households' consumption is simpler, as defined in sec. 2.1, because the connection probability p_{ij} doesn't depend on j. Since the household sector is homogeneous, the matrix p_{ij} can be summed-up by its cross-section along the firms' axis, as shown in Fig. 3. Note the importance of sector K ("financial and insurance activities").

The degree is another classical indicator of network topology. Theoretical expectations of $k_{in}(j)$ and $k_{out}(i)$ are directly available from p_{ij}, computing $\sum_i p_{ij}$ and $\sum_j p_{ij}$. The variance can be computed using the fact that independent Bernoulli variables are sampled along rows and columns. By a central limit argument[3], the pdf can be approximated for high degrees by a normal law. Fig. 4(a-b) show sample degrees as functions of theoretical degree with approximate standard deviations plotted as error bars. It can be noticed that, consistently with the block hypothesis, all firms in a given sector have the same degree. Furthermore the four prominent sectors are the same as in Fig. 2.

Then we consider another classical high-order topological indicator, the average nearest neighbors degrees. For each agent, it measures the average degree of its neighbors.

[2] `https://gitlab.com/hazaa/sfc_proba`

[3] more specifically the de Moivre-Laplace theorem.

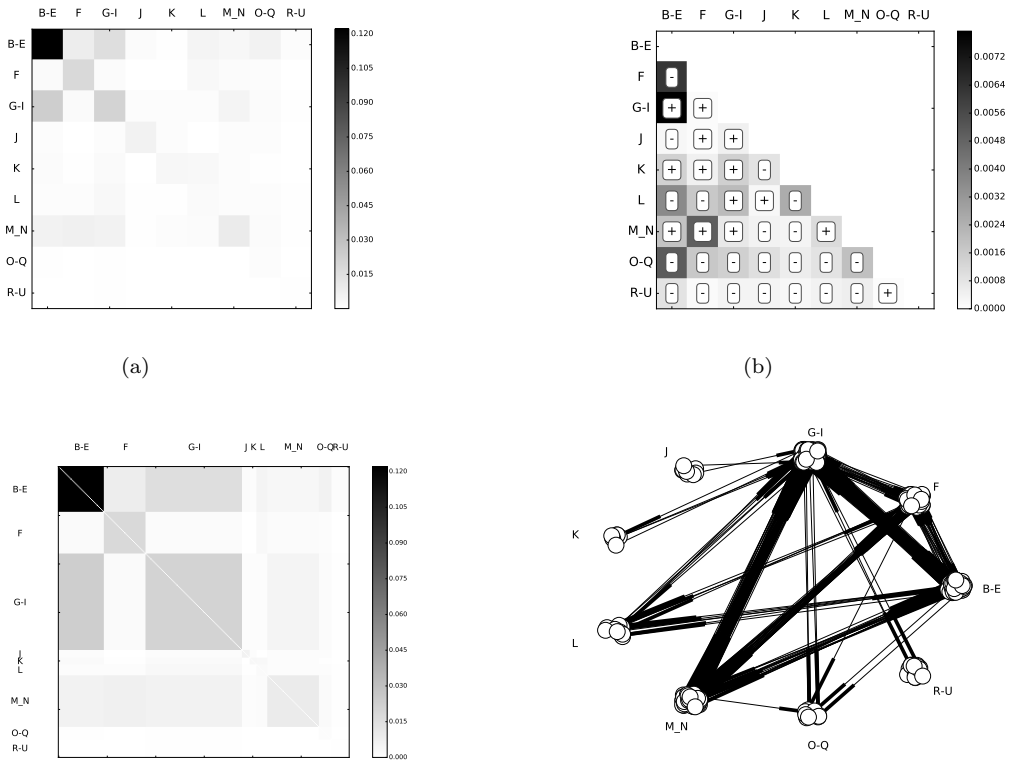


Figure 2: Properties of the investment block network, unipartite and directed. (a) sector-level connection probability matrix p_{s_i,s_j} (b) lower triangle part of the matrix $\{|p_{s_i,s_j} - p_{s_j,s_i}|\}$ with the corresponding sign in the bounding box (c) probability matrix p_{ij} with $nf = 300$ (d) random network sample.

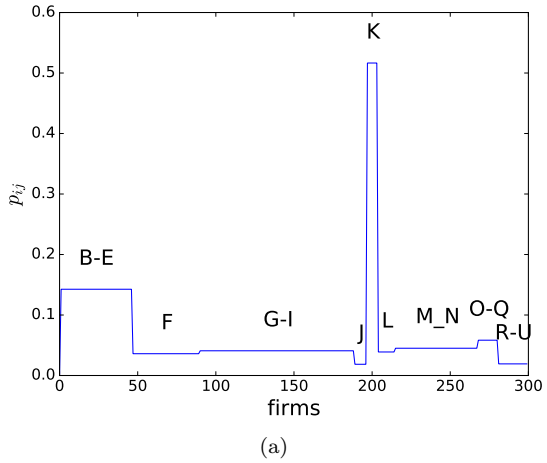
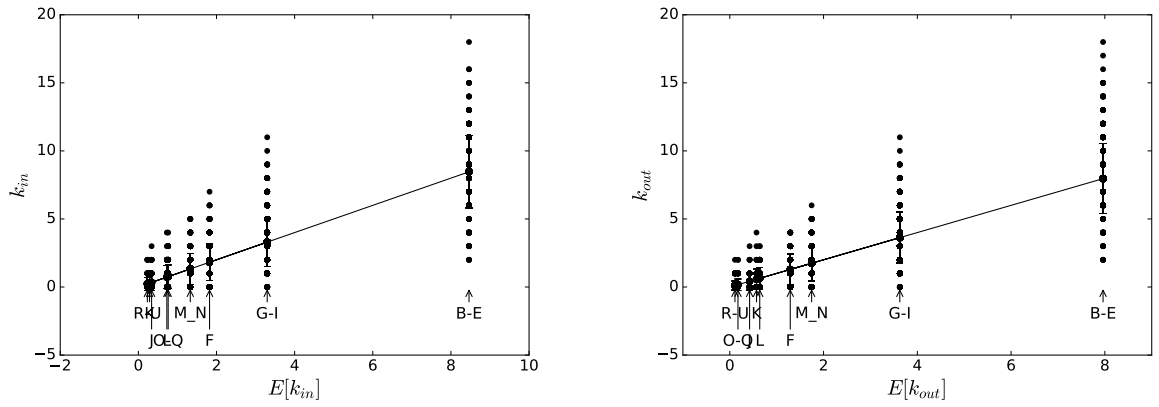


Figure 3: Properties of the households' consumption block network, bipartite and undirected.



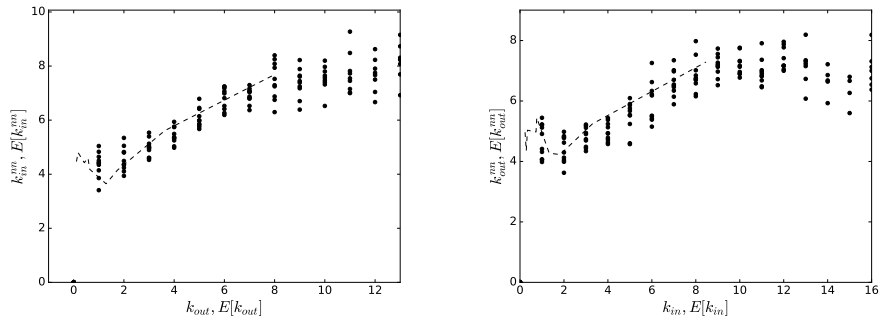
(a) (b)

Figure 4: Degree of the investment block network. Dots are randomly sampled graphs, lines represent theoretical approximation of the expectation. Errobar represent ± 1 standard deviation (a) in-degree (b) out-degree. $nf = 300$

$$k_{in}^{nn}(i) = \frac{\sum_{j=1} a_{ij} k_{in}(j)}{k_{out}(i)} \tag{11}$$

$$k_{out}^{nn}(j) = \frac{\sum_{i=1} a_{ij} k_{out}(i)}{k_{in}(j)} \tag{12}$$

Knowing p_{ij}, the expectation of these indicators can be approximated [34, eq.(34-35)]. Using these results, in Fig. 5 we compare randomly sampled values and their theoretical couterpart. Both agree concerning the assortative nature or the network, on the buyers and suppliers sides: suppliers with a small (resp. high) out-degree k_{out} in Fig. 5(a) tend to be connected to suppliers buyers with a small k_{in}^{nn} (resp. high). The same is true for buyers with respect to in-degree. This is a consequence of the dense conection inside the first four sectors (that account for 85% of all links in this model), be it with inter or intra-sector links. It contrasts with the disassortative nature of buyers-suppliers networks found in recent empirical studies such as the interfirm payment network in Estonia [22], Italy [35, §2], and Japan [36, 3.1].

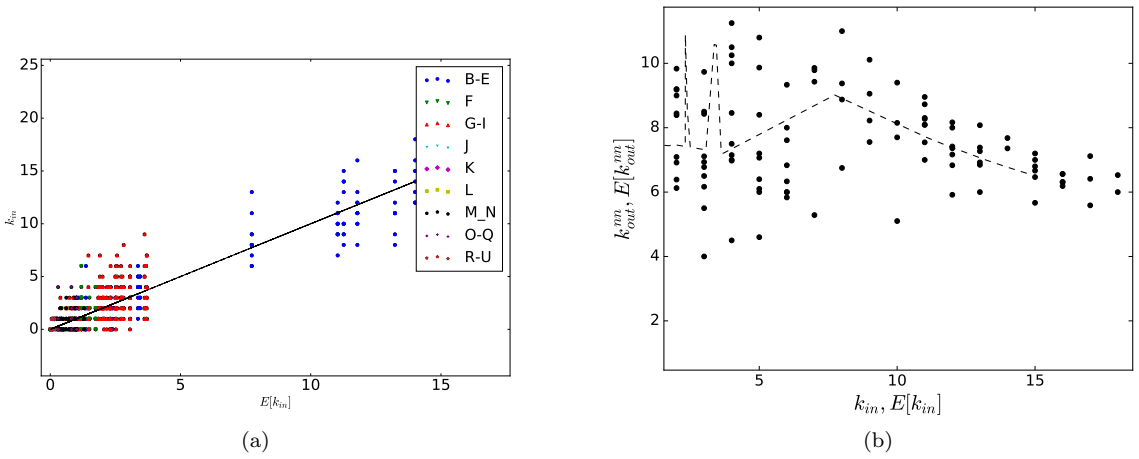


(a) (b)

Figure 5: Nearest network degree of the investment block network. Dots are randomly sampled graphs, lines represent theoretical approximations.

The random fitness model depicted in sec. 2.1 and summarized in Tab. 5 was proposed to avoid such

inconsistencies. In Fig. 6(a) the in-degree distribution of the investment network shows that some mixing was succesfully introduced among firms belonging to different sectors, unlike in the case of block model in Fig.4 where all firms of a given sector share the same degree. Similarly, the investment network is no longer assortative in the case of the random fitness network as shown by Fig. 6(b). It can be noticed that in that case, the probability distribution of degrees of a fitness model can be computed, following [37, eq. (1-2)].



(a) (b)

Figure 6: Random fitness investment network. (a) in-degree; (b) nearest-neighbour degree. $nf = 300$

While in this section only topological features were examined, the topic of average money flows will also be dealt with below.

3. Estimation of marginal probabilities of monetary flows for a particular network

In sec. 2 only the topological properties of the model inspired by sec. 1 were examined. The present section deals with the values of flows ξ associated to vertices given a specific topology. Moreover, we are interested in the relationship between the solutions and the topological properties of networks. The n-uple of adjacency matrices $\mathcal{A}_{SFC} = (\mathbf{A}_{cons}, \ldots, \mathbf{A}_{invest})$ can be temporarily considered as a quenched variable: it is sampled once from the statistical ensemble $\{\mathcal{A}_{SFC}\}$ subject to empirical constraints, and then considered fixed.

The linear time-invariant underdetermined problem in eq. (1) has the following properties: A has full rank and is such that $m < n$ with m the number of rows and n the number of columns. In that case, AA^T is invertible, the Moore-Penrose pseudoinverse writes $A^\dagger = A^T(AA^T)^{-1}$, and coincides with the least-square solution of eq. (1):

$$\xi \quad = \quad A^T(AA^T)^{-1}b \tag{13}$$

This result can be related to probabilistic approaches: Bayesian and maximum entropy methods were applied to flow networks in [13], using the unbounded Gaussian prior $\mathcal{N}(\mu, \Sigma)$. The authors get the following expression for the posterior average [4] :

$$\langle\xi\rangle = \mu + \Sigma A^T(A\Sigma A^T)^{-1}(b - A\mu) \tag{14}$$

With the simplifying homoscedasticity assumptions $\Sigma = \sigma I_n$, this solution can be compared to the one obtained with the Moore-Penrose pseudoinverse in eq.(13). However, several problems arise: firstly, eq.(1)

[4] to do so the likelihood is used to enforce the constraint in eq.(1) $p(b|\xi) = \delta(b - A\xi)$, then using the Gaussian expression of the delta function yields $-2\ln p(b|\xi) \propto \lim_{\Sigma_A \to 0}(b - A\xi)^T\Sigma_A^{-1}(b - A\xi)$

has a trivial solution $x = 0$ because the system is homogeneous. It is thus necessary to turn to the nonhomogeneous system eq.(5) to obtain nontrivial solutions. Secondly, negative solutions [5] may be found for this system, which limits the interest of this method.

Adding inequalities to eq.(1), it can be considered as a Constraint Satisfaction Problem (CSP):

$$S = \{\xi \ s.t. \ A\xi = b, \xi_0 \leq \xi \leq \xi_1\} \tag{15}$$

Various properties of the set of solution vectors ξ can be studied numerically: as examplified in the field of metabolic networks research, samples can be generated in a uniform [38, 39] or non-uniform way [40]. This type of method was applied already to SFC macro models in [41]. Distributional properties were observed in the partially aggregated case in [42]. Unfortunately, sampling methods can't be applied directly in the present case because of the dimension of the problem. The positivity constraint can also be dealt with by the Expectation Propagation algorithm [43] which yields an analytic approximation of the marginal probability distribution $P(\xi_i)$, using truncated Gaussian priors. But according to preliminary experiments the dimension of our problem is too large for existing implementations.

Other algorithms are then necessary to handle the constrained high-dimensional case. As for metabolic network analysis [44, §12.5], linear programming or basis pursuit can be used. It the objective function to be minimized has the form $1^T\xi$, sparse solutions can be found numerically. This property is interesting for example to force high unemployment, but is too restrictive otherwise. Nonnegative Least Square (NNLS) numerically solves the problem:

$$\underset{\xi}{\operatorname{argmin}} \| \ A\xi - b\|, s.t. \ \xi \geq 0 \tag{16}$$

which is equivalent to a quadratic programming problem, and which solution can be efficiently approximated [45, §23] for large problems.

3.1. Properties of flows

In this section we present numerical NNLS solutions to the nonhomogeneous problem:

$$\underset{\xi}{\operatorname{argmin}} \| \ A_1\xi - b_1\|, s.t. \ \xi \geq 0 \tag{17}$$

with A_1, b_1 defined in eq.(5). The networks that specify A_1 are sampled from the ensemble defined by the random fitness model in sec. 2.1. Due to computational constraints, the size of the networks will be limited to $n_b = 3, \ n_f = 100, \ n_h = 1000$, resulting in a system with more than 2.10^5 unknowns.

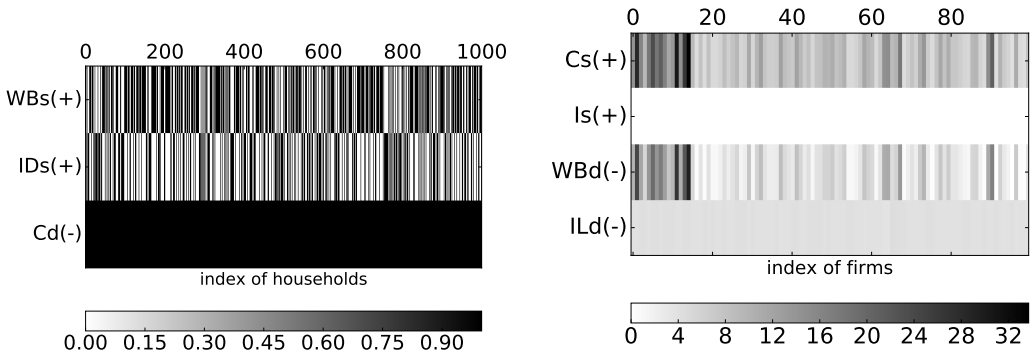
Fig. 7(a) represents the budget of all households, with a fixed consumption demand α_0 imposed by A_1 and b_1, as a minimalist way to ensure nonhomogeneity. The population of agents can be clustered in two groups: their consumption is either financed by wage bills or by interests on deposits, and this two values are negatively correlated. Furthermore the values taken by WBs and IDs are highly clustered

Fig. 7(b) represents the budget of firms. Income is mainly composed of consumption supply to households, which is much larger than investment supply to firms. Expense includes wage bills and interests on loans, the former dominating the latter. We note also that WBd and Cs are positively correlated. There is a sectorial dependence of Cs that can be observed more in detail in Fig.8

3.2. Relationship between topological properties, stocks and flows in the block model

In the present section n_{iter} networks are sampled from the ensemble defined by the random fitness model. For each of them, the approximate NNLS solution to eq.(17) is computed. Then the values of ξ corresponding to various economic transactions are compared to the topological features of the related subnetworks established in sec. 2.3.

[5] for numerical reasons, the solution is not computed directly from the expression of the pseudoinverse, instead sparse least-norm solvers should be used.



(a) (b)

Figure 7: Budget of (a) households; (b) firms. $nb = 3,\ nf = 100,\ nh = 1000$

The consumption C_s supplied to households by firms is shown in Fig. 8 with respect to the out-degree of firms, that is the number of their customers among households. The distribution of sectors along the x-axis is well clustered and reflects the distribution of the connection probability, seen in Fig. 3, for example the high probability of connection of sector K. However, this clustering effect is attenuated by the uniform random fitness model in comparison to the block model defined in sec. 2.1. Given the sector, it seems that C_s is uniformly distributed on some interval $[u, v]$. Comparing sectors B-E and K, v does not appear to be a linear function of the sector's connection probability p_{s_i}.

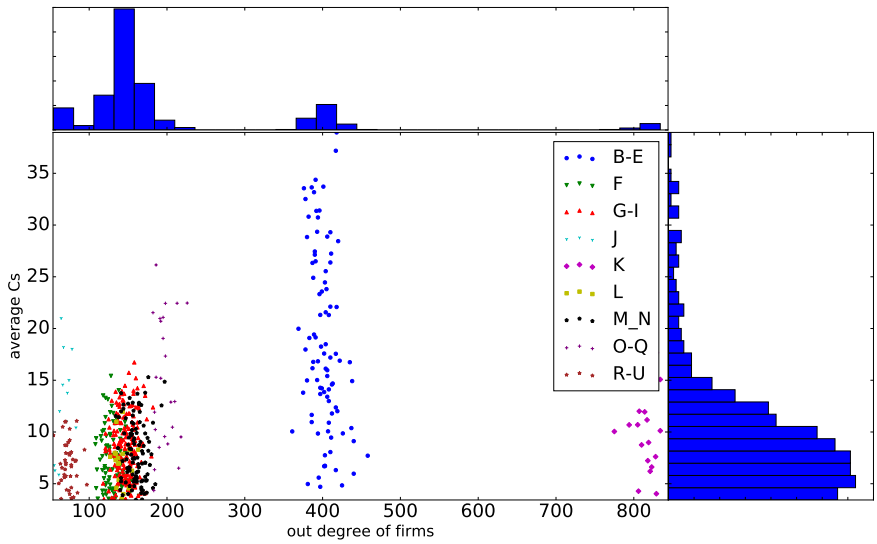


Figure 8: Consumption supply C_s from firms to households and out-degree of the consumption network. (center) Each dot corresponds to a specific firm and represents its out-degree, C_s computed using NNLS, and its sector; (top) marginal histogram of out-degree; (right) marginal histogram of C_s. $n_b = 3, n_f = 50, n_h = 300, n_{iter} = 10$

The average consumption C_d demanded by households is represented in Fig. 9 with respect to the in-degree of households, that is the number of firms a given household is buying from. It can be verified that households form a homogeneous group in the block model and the random fitness models. Furthermore

they can't be associated to an industrial sector in a one-to-one map. The clustering around the value of C_d imposed by the rhs b_1 in eq.(17) is clearly observed.

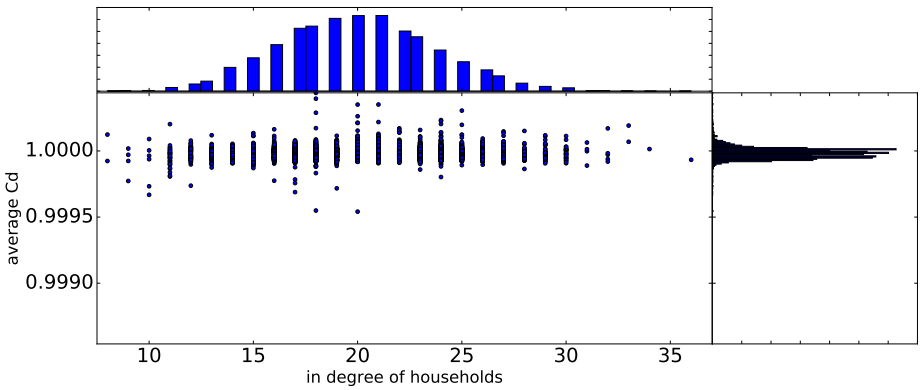


Figure 9: Consumption demand C_d by households and in-degree of the consumption network. (center) Each dot corresponds to a specific household and represents its average C_d computed using NNLS and its in-degree ; (top) marginal histogram of in-degree ; (right) marginal histogram of C_d. $n_b = 3, n_f = 100, n_h = 1000, n_{iter} = 10$

In sec. 3.1, it was remarked that the investment I_s between firms had a residual level compared to other quantities. As a first explanation, it can be noticed that the number of inter-firm links is very low compared to firm-household links. This problem is likely to disappear when adding capital depreciation equations that constrain the investment level, as in the BMW model.

4. Comparison to other methods

In sec. 3, numerical NNLS solutions to the problem in eq.(17) were computed, as the network topology was sampled from a probabilistic FiCM model. In this section, our approach, noted FiCM+NNLS, is compared to other existing works.

Firstly, the "degree-corrected gravity model" (FiCM+dCGM) is a two-step method that builds on an FiCM model [46, 47], inspired by the gravity model, and corrected in order to set a given level of sparsity, and to apply a constraint on the degree sequence. The value of weights placed on edges is:

$$w_{ij} \quad = \quad \frac{a_{ij}}{W}(z^{-1} + s_i s_j) \tag{18}$$

where W is a normalization factor, $s_i = \sum_j w_{ij}$ and $s_j = \sum_i w_{ij}$ are the nodes' strengths. For the sake of comparison, W_i, W_j and W are computed below using the NNLS solution.

Secondly, FiCM+NNLS will be compared to the Bayesian method in [13] that results in eq.(14). All methods give w_{ij} conditioned on the knowledge of the connection probabilities a_{ij} of the subgraphs. Even though FiCM+dcGM and FiCM+NNLS build probabilistic models of topologies, expressed by p_{ij}, they compute connection weights conditionaly on a particular realization of a_{ij}, as explained above. Similarly, the Bayesian method in [13] computes the mean $\langle \xi \rangle$ using a closed-form expression that needs a_{ij} to be known. We will be interested by several features of the obtained solutions:

- do they return negative money flows (that are not compatible with the desired behavior) ? This is quantified by the percentage of negative coefficients over non-zero coefficients in ξ.
- to what extend do they respect the linear system of equations in eq.(15) ? This is quantified by a relative error defined by $100 \times \frac{\|A_1 \xi - b_1\|_1}{\|\xi\|_1}$.

Tab. 6 summarizes a computational comparison based on repreating $n = 100$ times each method, and estimating the sample value of the error indicators. As expected, since FiCM+dcGM is not designed to impose the linear constraint $A_1\xi = b_1$, the relative error rate obtained is much higher than the value obtained with the other methods. FiCM+NNLS and the Bayesian algorithm yield a satisfactory relative residual error below 1%. Then, the rate of negative coefficients is compared, and we find a value as high as 21% for the Bayesian method. This was anticipated in sec. 3, but is not acceptable in our context. Potential solutions to this issue will be discussed in sec. 5.

Some other features can be remarked: first the FiCM+dcGM and Bayesian method yield a closed-form expression for w_{ij} conditioned on a_{ij}, unlike FiCM+NNLS. There are method, such as the ECM [14], that outputs both $P(a_{ij} = a)$ and $P(w_{ij} = w)$ in a closed-form way, but with the same drawback as FiCM+dcGM since there is no way to impose the condition $A_1\xi = b_1$.

To conclude this section, FiCM+NNLS stands as the only method among the considered ones that respects both non-negativity of coefficients and the linear system of equations $A_1\xi = b_1$. In the following section, other possible improvements or research directions will be discussed.

Method	FiCM+dcGM	FiCM+NNLS	Bayesian
Input	Fitnesses x_i, y_j, number of links L, W_i, W_j for each subnetwork.	Fitnesses x_i, y_j, number of links L for each subnetwork. A_1, b_1 corresponding to a_{ij}	Network topology a_{ij}. A_1, b_1 corresponding to a_{ij}
Probabilistic model of topology p_{ij}	Yes	Yes	No
Closed-form expression of w_{ij}	Yes, conditioned on a_{ij}.	No	Yes, conditioned on a_{ij}.
Relative error %	15.9	0.18	0.15
Negative coefficients %	0.	0.	21.1
Refs	[47]	Present article	[13]
Comments	High error since $A_1\xi = b_1$ is not taken into account.	No pdf is available for w_{ij}.	Outputs negative coefficients. Estimation can be augmented with observed values.

Table 6: Comparison of ensemble reconstruction methods. Relative error is defined by $100 \times \frac{\|A_1\xi - b_1\|_1}{\|\xi\|_1}$ and averaged over $n = 100$ trials. Negative coefficients % is the percentage of negative coefficients over non-zero coeffecients of ξ.

5. Discussion

The main idea developed in this article is to build a systemic macroeconomic model able to reproduce topological features, with respect to a given theoretical behavior. This explains the choice of a minimalist set of behavioral equations, that may be extended to include other features of SFC models. The interplay between stocks (wealth of households M, stock of loans L, capital stock of firms K) and flows was not examined here. It is known from empirical studies that the network of interfirm sales should be disassortative [35, §2] but we could not observe realistic values for Is as explained in sec. 3.2. Apart from linear models, nonlinearities may be introduced, following the example given in engineering [48].

In sec. 2.1, several topological models were proposed for each transaction subnetwork. It was noticed that some were not consistent with empirical evidence. Among FiCM networks, the block model for investment and consumption of firms was modified to account for heterogeneity among sectors. A uniform random fitness model was proposed and can be extended to power-laws, building on studies of firm size [49, 50]. The scope could be extended beyond national limits, following [51]. The network of household consumption should incorporate more empirical data and notably the different socioeconomic positions [19]. The BiRG

networks may be replaced by appropriate FiCM networks. The network of bank loans to firms should use central bank data [52, 3.2]. The network of wages could benefit from aggregate constraints such that the compensation of employees in national accounts [33, 5.1]. The investment network, which was proxied by firm consumption, could be enhanced using a better data source. Agriculture, which was not excluded in the network model for convenience, should be included.

The experimental analysis of the reconstructed networks' properties in sec. 2.3 and 3.2 consists in a comparison to basic stylized facts in the empirical litterature. This validation, while necessary is not sufficient and should be extended in two possible directions: either with a comparison to detailed micro-data from empirical studies, of from ABM simulation where the real topology is known.

The two-step method devised here first needs to reconstruct the topology from economic data. Then, weights are approximated using an numerical method. Those weights are then expected to reflect the economic data used in the first step, which can be validated experimentally (but is not guaranteed by construction). A single-step method similar to [14] and able to cope with the constraint in eq. (1), could provide such a guarantee. Some recent works [53] in that direction question the relevant form for the entropy. Also, maximum entropy methods with inequality constraint should be taken into account.

The role of time may be questionned, as it is expected to play no role in this article, on the opposite of ABM that can deal with growth and transient phenomena. Small fluctuations in the vicinity of the data points used to learn the subnetworks can be discussed, but it is unlikely that radical change in the network structure can be modelled. Stability issues close to the steady-state can be examined using the theory of dynamics on networks [9, 18.2],[54]. More importantly, it must be stressed that the concept of steady-state [55] has been debated during several decades in the field of ecological economics. The benefits of using our approach in that context will be examined.

A detailed analysis of our results in the light of economic knowledge is needed. For example the influence of the macroeconomic parameters on the network structure may be compared to what is expected by existing economic theories, and empirical observations.

6. Conclusion

In this article we propose an intermediate model between ABM and complex networks, able to reflect topological features, heterogeneity and interaction, with theoretical properties that are easier to establish than in the ABM case. This comes at the cost of losing time-dependence.

Data-driven economic network reconstruction methods do not include so far constraints stemming from macroeconomic models. In this article we propose to introduce such a constraint, that induces a specific distribution for the weights of the network. To do so, we use a two-step method that requires first to estimate the topology of each subnetwork taken independently, then to estimate network weights. The first step can be skipped if a detailed empirical description of the network topology is known.

Building on the fitness-induced configuration model we defined several connection probabilities to model the topology of subnetworks representing various economic transactions. These models respect the empirical link density found in empirical studies and were fit to national accounts empirical data.

In the future we will extend the methods developed here in several direction: first, instead of a two-step approach, we plan to get the parameters of the full network in just one step. Furthermore, reconstructed networks will be compared to a ground truth obtained from empirical or simulated data. More detailed economic models, possibly non-linear, can be studied.

Lastly we stress that the results obtained can be used by practitioners in the ABM or SFC communities that are interested in network and distributional phenomena in the steady state.

Appendix A. Notations

In this section the notations of sec. 1 are explicited.

$$
\begin{aligned}
WBd &= \begin{bmatrix} WBd_{1,1} & \dots & WBd_{n_h,1} & WBd_{1,2} & \dots & WBd_{n_h,2} & \dots & WBd_{1,n_f} & \dots & WBd_{n_h,n_f} \end{bmatrix} \\
IDd &= \begin{bmatrix} IDd_{1,1} & \dots & IDd_{n_h,1} & IDd_{1,2} & \dots & IDd_{n_h,2} & \dots & IDd_{1,n_b} & \dots & IDd_{n_h,n_b} \end{bmatrix} \\
ILd &= \begin{bmatrix} ILd_{1,1} & \dots & ILd_{n_h,1} & ILd_{1,2} & \dots & ILd_{n_h,2} & \dots & ILd_{1,n_b} & \dots & ILd_{n_h,n_b} \end{bmatrix} \\
Id &= \begin{bmatrix} Id_{1,1} & \dots & Id_{n_f,1} & Id_{1,2} & \dots & Id_{n_f,2} & \dots & Id_{1,n_f} & \dots & Id_{n_f,n_f} \end{bmatrix}
\end{aligned}
\tag{A.1}
$$

where WBd_{n_h,n_f} is the wage obtained by household n_h from firm n_f, IDd_{n_h,n_b} is the amount of interest obtained by household n_h from bank n_b, ILd_{n_h,n_b} is the amount of interest paid by firm n_h to bank n_b, $Id_{n_{f,1},n_{f,2}}$ is the investment paid by firm $n_{f,1}$ to firm $n_{f,2}$.

Appendix B. Eurostat national accounts databases

All tables in this section are built by national accountants for a given country, and a given year, but these mentions are dropped for clarity reasons.

The production matrix forms a part of the supply table. For each category of products in the rows, it displays the value of the production, grouped by industry type in the columns. To simplify, only the production matrix is shown in Tab. B.7 while the others components of the supply table are dropped (no imports, trade and transport margins, taxes less subsidies on products) [33, Tab. 4.3, §4.1]. The value of product $p \in [1, P]$ produced by sector $s \in S$ will be noted $sup[p, s]$.

	Output of industries			
Industries	Agriculture	...	Other services	Total domestic output
Products	1	...	n	
Products of agriculture		...		
⋮		...		Σ
Products of other services		...		
Total		Σ		

Table B.7: Production matrix, that constitutes the first quadrant of the supply table.

As explained in [33, §5.1] "a use table shows the use of goods and services by product and by type of use for intermediate consumption by industry, final consumption expenditure, gross capital formation or exports". Only the quadrant named "Final uses" will be used here, and more particularly two columns, "Final consumption expenditure by households" and "Gross fixed capital formation". This is summarized in Tab.B.8. Since our model does not involve intermediate consumption, the corresponding part of the use table is not exploited here. The value of product $p \in [1, P]$ consumed by households as a final use is noted $use^{fin}[p]$.

The column "Gross fixed capital formation" can in fact be disaggregated by investing industry [33, Fig 5.1 p.125], and is called the Investment matrix. The amount of fixed capital of product $p \in [1, P]$ formed by the industrial sector $s \in S$ is written $use^{cap}[p, s]$.

The rows of the industry-by-industry input-output table in Tab.B.9 explain how the production of a given sector is sent to other sectors. The columns show the different inputs of a given sector.

Tab. B.10 summarizes the data sources used in this article.

Appendix C. Acknowledgements

We thank two anonymous reviewers for their comments that helped improving the manuscript. Open-source software were used to perform this research: Python, Scipy, LaTeX, Matplotlib.

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	Final uses		
Products	Final consumption expenditure by households	Gross fixed capital formation ...	Total
Products of agriculture	...		
⋮	...		Σ
Products of other services	...		
Total	Σ		

Table B.8: Final uses matrix, that constitutes the second quadrant of the use table.

	Agri.	Industry	Services	Uses
Agriculture	10	34	10	
Industry	20	152	40	Σ
Services	10	72	20	
Output		Σ		

Table B.9: Input-output matrix, industry-by-industry.

Dataset label	Dataset definition	Input for transaction networks	Corresponding variable or fitness
bd_9ac_l_form_r2	business demography by legal form	Investment, consumption, wage	nf, nh by sector
naio_10_cp15, naio_10_cp16, naio_10_cp1750	supply,use, input-output tables	Investment	$d_{s_i s_j}$
		Consumption	x_{s_i}

Table B.10: Eurostat datasets used to parametrize random networks

Label	Definition
A	agriculture, forestry and fishing
B	mining and quarrying
C	manufacturing
D	electricity, gas, steam and air conditioning supply
E	water supply; sewerage, waste management and remediation activities
F	construction
G	wholesale and retail trade; repair of motor vehicles and motorcycles
H	transportation and storage
I	accommodation and food service activities
J	information and communication
K	financial and insurance activities
L	real estate activities
M	professional, scientific and technical activities
N	administrative and support service activities
O	public administration and defence; compulsory social security
P	education
Q	human health and social work activities
R	arts, entertainment and recreation
S	other service activities
T	activities of households as employers or for own use
U	activities of extraterritorial organisations and bodies

Table B.11: Definition of sectors

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