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A 4-choosable Graph that is Not (8:2)-choosable

Zdeněk Dvořák* Xiaolan Hu[†] Jean-Sébastien Sereni[‡]

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Abstract: In 1980, Erdős, Rubin and Taylor asked whether for all positive integers a, b, and m, every (a:b)-choosable graph is also (am:bm)-choosable. We provide a negative answer by exhibiting a 4-choosable graph that is not (8:2)-choosable.

Key words and phrases: (am: bm)-conjecture, (a: b)-choosable graph, list coloring.

Coloring the vertices of a graph with sets of colors (that is, each vertex is assigned a fixed-size subset of the colors such that adjacent vertices are assigned disjoint sets) is a fundamental notion, which in particular captures fractional colorings. The fractional chromatic number of a graph G can indeed be defined to be the infimum (which actually is a minimum) of the ratios a/b such that, if every vertex of G is replaced by a clique of order b and every edge of G is replaced by a complete bipartite graph between the relevant cliques, then the chromatic number of the obtained graph is at most a.

In their seminal work on list coloring, Erdős, Rubin and Taylor [2] raised several intriguing questions about the list version of set coloring. Before stating them, let us review the relevant definitions.

Set coloring. A function that assigns a set to each vertex of a graph is a *set coloring* if the sets assigned to adjacent vertices are disjoint. For positive integers a and $b \le a$, an (a:b)-coloring of a graph G is a set coloring with range $\binom{\{1,\ldots,a\}}{b}$, *i.e.*, a set coloring that to each vertex of G assigns a b-element subset of $\{1,\ldots,a\}$. The concept of (a:b)-coloring is a generalization of the conventional vertex coloring. In fact, an (a:1)-coloring is exactly an ordinary proper a-coloring.

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A *list assignment* for a graph G is a function L that to each vertex v of G assigns a set L(v) of colors. A set coloring φ of G is an L-set coloring if $\varphi(v) \subseteq L(v)$ for every $v \in V(G)$. For a positive integer b, we say that φ is an (L:b)-coloring of G if φ is an L-set coloring and $|\varphi(v)| = b$ for every $v \in V(G)$. If such an (L:b)-coloring exists, then G is (L:b)-colorable. For an integer $a \ge b$, we say that G is (a:b)-choosable if G is (L:b)-colorable for every list assignment L such that |L(v)| = a for each $v \in V(G)$. We abbreviate (L:1)-coloring, (L:1)-colorable, and (a:1)-choosable to L-coloring, L-colorable, and L-colorable.

Questions and results. It is straightforward to see that if a graph is (a:b)-colorable, it is also (am:bm)-colorable for every positive integer m: we can simply replace every color in an (a:b)-coloring by m new colors. However, this argument fails in the list coloring setting, leading Erdős, Rubin and Taylor [2] to ask whether every (a:b)-choosable graph is also (am:bm)-choosable whenever $m \ge 1$. A positive answer to this question is sometimes referred to as "the (am:bm)-conjecture". Using the characterization of 2-choosable graphs [2], Tuza and Voigt [4] provided a positive answer when a=2 and b=1. In the other direction, Gutner and Tarsi [3] demonstrated that if k and m are positive integers and k is odd, then every (2mk:mk)-choosable graph is also 2m-choosable.

Formulated differently, the question is to know whether every (a:b)-choosable graph is also (c:d)-choosable whenever c/d = a/b and $c \ge a$. This formulation raises the same question when c/d > a/b, which was also asked by Erdős, Rubin and Taylor [2]. About ten years ago, Gutner and Tarsi [3] answered this last question negatively, by studying the kth choice number of a graph for large values of k. More precisely, the kth choice number of a graph G is ch:c0, the least integer c1 for which c2 is c3. Their result reads as follows.

Theorem 1 (Gutner & Tarsi, 2009). Let G be a graph. For every positive real ε , there exists an integer k_0 such that $\operatorname{ch}_{k}(G) \leq k(\chi(G) + \varepsilon)$ for every $k \geq k_0$.

As a direct corollary, one deduces that for all integers $m \ge 3$ and $\ell > m$, there exists a graph that is (a:b)-choosable and not $(\ell:1)$ -choosable with $\frac{a}{b} = m$. (To see this, one can for example apply Theorem 1 with $\varepsilon = 1$ to the disjoint union of a clique of order m-1 and a complete balanced bipartite graph with choice number $\ell + 1$.)

Another related result that should be mentioned here was obtained by Alon, Tuza and Voigt [1]. They proved that for every graph G,

$$\inf \left\{ \frac{a}{b} \mid G \text{ is } (a:b)\text{-choosable} \right\} = \inf \left\{ \frac{a}{b} \mid G \text{ is } (a:b)\text{-colorable} \right\}.$$

In other words, the fractional choice number of a graph equals its fractional chromatic number.

The purpose of our work is to provide a negative answer to Erdős, Rubin and Taylor's question when a = 4 and b = 1.

Theorem 2. There exists a graph G that is 4-choosable, but not (8:2)-choosable.

We build such a graph by incrementally combining pieces with certain properties. Each piece is defined, and its relevant properties established, in the forthcoming lemmas.

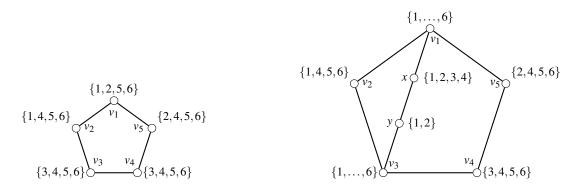


Figure 1: The gadget (C, L_0) of Lemma 3 is depicted on the left; the gadget (G_1, L_1) of Corollary 4 is depicted on the right.

Gadgets and lemmas. A *gadget* is a pair (G, L_0) , where G is a graph and L_0 is an assignment of lists of even size. Given a gadget (G, L_0) , a *half-list assignment* for G is a list assignment L for G such that $|L(v)| = |L_0(v)|/2$ for every $v \in V(G)$. Let us start the construction by a key observation on list colorings of 5-cycles.

Lemma 3. Consider the gadget (C, L_0) , presented in Figure 1, where $C = v_1 v_2 v_3 v_4 v_5$ is a 5-cycle, $L_0(v_1) = \{1, 2, 5, 6\}$, $L_0(v_2) = \{1, 4, 5, 6\}$, $L_0(v_3) = L_0(v_4) = \{3, 4, 5, 6\}$ and $L_0(v_5) = \{2, 4, 5, 6\}$. Then C is L-colorable for every half-list assignment L such that $|L(v_1) \cap L(v_3)| \le 1$, but C is not $(L_0 : 2)$ -colorable.

Proof. The first statement is well known, but let us give the easy proof for completeness: since $|L(v_1) \cap L(v_3)| \le 1$, we have $|L(v_1) \cup L(v_3)| \ge 3$, and thus $L(v_1)$ or $L(v_3)$ contains a color c_6 not belonging to $L(v_2)$. By symmetry, we can assume that $c_6 \in L(v_1)$. We color v_1 by c_6 and then for i = 5, 4, 3, 2 in order, we color v_i by a color $c_i \in L(v_i) \setminus \{c_{i+1}\}$. The resulting coloring is proper—we have $c_2 \ne c_6$, since $c_6 \notin L(v_2)$.

Suppose now that C has an $(L_0:2)$ -coloring, and for $c \in \{1,\ldots,6\}$ let V_c be the set of vertices of C on which the color c is used. Since two colors are used on each vertex of C, we have $\sum_{c=1}^6 |V_c| = 10$. On the other hand, V_c is an independent set of a 5-cycle, and thus $|V_c| \le 2$ for every color c. Furthermore, color 1 only appears in the lists of v_1 and v_2 , which are adjacent in C. It follows that $|V_1| \le 1$. The situation is similar for color 2, which appears only in the lists of v_1 and v_5 , and also for color 3, which only appears in the lists of v_3 and v_4 . Consequently, $\sum_{c=1}^6 |V_c| \le 3 \cdot 2 + 3 \cdot 1 = 9$, which is a contradiction.

Corollary 4. Consider the gadget (G_1,L_1) , presented in Figure 1, where G_1 consists of a 5-cycle $C = v_1v_2v_3v_4v_5$ and a path v_1xyv_3 , with $L_1(v_1) = L_1(v_3) = \{1,...,6\}$, $L_1(v_2) = \{1,4,5,6\}$, $L_1(v_4) = \{3,4,5,6\}$, $L_1(v_5) = \{2,4,5,6\}$, $L_1(x) = \{1,2,3,4\}$ and $L_1(y) = \{1,2\}$. Then G_1 is L-colorable for every half-list assignment L such that $L(v_1) = L(v_3)$, but G_1 is not $(L_1:2)$ -colorable.

Proof. Let *L* be a half-list assignment for G_1 . First *L*-color *y* and *x* by colors $c_y \in L(y)$ and $c_x \in L(x) \setminus \{c_y\}$, respectively. Since $c_x \neq c_y$ and $L(v_1) = L(v_3)$, there exist sets $L'(v_1) \subseteq L(v_1) \setminus \{c_x\}$ and $L'(v_3) \subseteq L(v_3)$

 $L(v_3) \setminus \{c_y\}$ of size two such that $L'(v_1) \neq L'(v_3)$. Let $L'(v_i) = L(v_i)$ for $i \in \{2,4,5\}$. Lemma 3 implies that C is L'-colorable, which yields an L-coloring of G.

In an $(L_1:2)$ -coloring, the vertex y would have to be assigned $\{1,2\}$ and x would have to be assigned $\{3,4\}$, and thus the sets of available colors for v_1 and for v_3 would have to be $\{1,2,5,6\}$ and $\{3,4,5,6\}$, respectively. However, no such $(L_1:2)$ coloring of C exists according to Lemma 3. \square

Next we construct auxiliary gadgets, which will be combined with the gadget from Corollary 4 to deal with the case where $L(v_1) \neq L(v_3)$. Let G be a graph, let S be a subset of vertices of G and L a list assignment for G. An L-coloring of S is a coloring of the subgraph of G induced by S. Moreover, if S' is a subset of vertices of G that contains S and φ' is an L-coloring of S', then φ' extends φ if $\varphi'|S = \varphi$. Let (G, L_0) be a gadget, let v_1 and v_3 be distinct vertices of G, and let S be a set of vertices of G not containing v_1 and v_3 . The gadget is (v_1, v_3, S) -relaxed if every half-list assignment L satisfies at least one of the following conditions.

- (i) There exists an *L*-coloring ψ_0 of $\{v_1, v_3\}$ such that every *L*-coloring of $S \cup \{v_1, v_3\}$ extending ψ_0 extends to an *L*-coloring of *G*.
- (ii) $L(v_1) = L(v_3)$ and there exists an L-coloring ψ_0 of S such that every L-coloring of $S \cup \{v_1, v_3\}$ extending ψ_0 extends to an L-coloring of G.

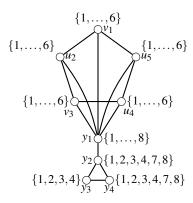


Figure 2: The gadget (G_2, L_2) of Lemma 5

Lemma 5. Consider the gadget (G_2, L_2) presented in Figure 2, where G_2 consists of a 5-cycle $C_2 = v_1u_2v_3u_4u_5$, a vertex y_1 adjacent to all vertices of C_2 , a triangle $y_2y_3y_4$, and an edge y_1y_2 , with $L_2(v) = \{1, \ldots, 6\}$ for every $v \in V(C_2)$, $L_2(y_1) = \{1, \ldots, 8\}$, $L_2(y_2) = L_2(y_4) = \{1, 2, 3, 4, 7, 8\}$, and $L_2(y_3) = \{1, 2, 3, 4\}$. The gadget is $(v_1, v_3, \{y_4\})$ -relaxed, and $\varphi(y_4) \cap \{7, 8\} \neq \emptyset$ for every $(L_2 : 2)$ -coloring φ of G_2 .

Proof. Let *L* be a half-list assignment for G_2 . If not all vertices of C_2 have the same list, then choose a color $c \in L(y_1) \setminus L(y_2)$, and observe there exists an *L*-coloring of $G_2[V(C_2) \cup \{y_1\}]$ such that y_1 has

color c. Let ψ_0 be the restriction of this coloring to $\{v_1, v_3\}$. Clearly, every L-coloring of $\{v_1, v_3, y_4\}$ extending ψ_0 extends to an L-coloring of G_2 , and thus (i) holds.

If all the vertices of C_2 have the same list (and hence in particular $L(v_1) = L(v_3)$), then let c be a color in $L(y_1) \setminus L(v_1)$. Observe that there exists an L-coloring of $G_2[\{y_1, y_2, y_3, y_4\}]$ such that y_1 has color c. Let ψ_0 be the restriction of this coloring to y_4 . Again, every L-coloring of $\{v_1, v_3, y_4\}$ extending ψ_0 extends to an L-coloring of G_2 , and thus (ii) holds.

It remains to show that if φ is an $(L_2:2)$ -coloring of G_2 then $\varphi(y_4) \cap \{7,8\} \neq \emptyset$. Suppose, on the contrary, that $\varphi(y_4) \cap \{7,8\} = \emptyset$. It follows that $\varphi(y_4) \cup \varphi(y_3) = \{1,2,3,4\}$, and hence $\varphi(y_2) = \{7,8\}$. As a result, $\varphi(y_1) \subseteq \{1,\ldots,6\}$ and, by symmetry, we can assume that $\varphi(y_1) = \{5,6\}$. This implies that $\varphi(v) \subseteq \{1,2,3,4\}$ for each $v \in V(C_2)$. In particular, $\varphi(V(C_2))$ is a (4:2)-coloring of C_2 , which is a contradiction since the 5-cycle C_2 has fractional chromatic number 5/2.

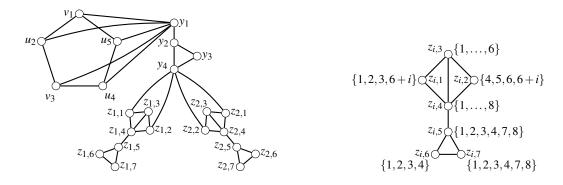


Figure 3: The graph G_3 from Lemma 6 is depicted on the left; the gadget (G_3, L_3) is obtained from (G_2, L_2) by adding to G_3 a disjoint copy of the graph depicted on the right (with the corresponding lists) for each $i \in \{1, 2\}$, and joining y_4 to each of $z_{1,1}$, $z_{1,2}$, $z_{2,1}$ and $z_{2,2}$.

Lemma 6. Consider the gadget (G_3, L_3) , obtained from the gadget (G_2, L_2) of Lemma 5 as follows (see Figure 3 for an illustration of G_3). The graph G_3 consists of G_2 and for $i \in \{1, 2\}$, the vertices $z_{i,1}, \ldots, z_{i,7}$; the edges $y_4z_{i,1}$ and $y_4z_{i,2}$; the edge $z_{i,j}z_{i,k}$ for every j and every k such that $1 \le j < k \le 4$ and $(j,k) \ne (1,2)$; the edges of the triangle $z_{i,5}z_{i,6}z_{i,7}$ and the edge $z_{i,4}z_{i,5}$. Let $L_3(v) = L_2(v)$ for $v \in V(G_2)$, and for $i \in \{1,2\}$ let $L_3(z_{i,1}) = \{1,2,3,6+i\}$, $L_3(z_{i,2}) = \{4,5,6,6+i\}$, $L_3(z_{i,3}) = \{1,\ldots,6\}$, $L_3(z_{i,4}) = \{1,\ldots,8\}$, $L_3(z_{i,5}) = L_3(z_{i,7}) = \{1,2,3,4,7,8\}$ and $L_3(z_{i,6}) = \{1,2,3,4\}$. The gadget (G_3,L_3) is $(v_1,v_3,\{z_{1,7},z_{2,7}\})$ -relaxed, and $\varphi(z_{1,7}) = \{7,8\}$ or $\varphi(z_{2,7}) = \{7,8\}$ for every $(L_3:2)$ -coloring φ of G_3 .

Proof. Let L be a half-list assignment for G_3 . The gadget $(G_2, L_3|V(G_2))$ is $(v_1, v_3, \{y_4\})$ -relaxed by Lemma 5. Suppose first that (i) holds for the restriction of L to G_2 (with $S = \{y_4\}$), and let ψ_0 be the corresponding L-coloring of $\{v_1, v_3\}$. For $i \in \{1, 2\}$, if $L(z_{i,1}) \cap L(z_{i,2}) \neq \emptyset$, then let c_i be a color in $L(z_{i,1}) \cap L(z_{i,2})$. Otherwise, $|L(z_{i,1}) \cup L(z_{i,2})| = 4 > |L(z_{i,3})|$, and thus we can choose a color $c_i \in (L(z_{i,1}) \cup L(z_{i,2})) \setminus L(z_{i,3})$. Let c be a color in $L(y_4) \setminus \{c_1, c_2\}$. By (i) for G_2 , we know that ψ_0 extends to an L-coloring ψ of G_2 such that $\psi(y_4) = c$. If $L(z_{i,1}) \cap L(z_{i,2}) \neq \emptyset$, then color both $z_{i,1}$ and $z_{i,2}$ by c_i , otherwise color one of them by c_i and the other one by an arbitrary color from its list that is different

from c. There are at least two colors in $L(z_{i,4})$ distinct from the colors of $z_{i,1}$ and $z_{i,2}$, choose such a color c'_i so that $L(z_{i,5}) \setminus \{c'_i\} \neq L(z_{i,6})$. Color $z_{i,4}$ by c'_i and extend the coloring to $z_{i,3}$, which is possible by the choice of c_i . Observe that any L-coloring of $z_{i,7}$ extends to an L-coloring of the triangle $z_{i,5}z_{i,6}z_{i,7}$ where the color of $z_{i,5}$ is not c'_i . We conclude that (G_3, L_3) with the half-list assignment L satisfies (i).

Suppose next that (ii) holds for the restriction of L to G_2 (with $S = \{y_4\}$), and let ψ'_0 be the corresponding L-coloring of y_4 . For $i \in \{1,2\}$, greedily extend ψ'_0 to an L-coloring of $z_{i,1}, \ldots, z_{i,7}$ in order, and let ψ_0 be the restriction of the resulting coloring to $\{z_{1,7}, z_{2,7}\}$. Observe that (G_3, L_3) with the half-list assignment L satisfies (ii).

Finally, let φ be an $(L_3:2)$ -coloring of G_3 . Lemma 5 implies that $\varphi(y_4) \cap \{7,8\} \neq \emptyset$. By symmetry, we can assume that $7 \in \varphi(y_4)$. It follows that $\varphi(z_{1,1}) \subset \{1,2,3\}$ and $\varphi(z_{1,2}) \subset \{4,5,6\}$, and thus $\varphi(z_{1,1}) \cup \varphi(z_{1,2}) \cup \varphi(z_{1,3}) = \{1,\ldots,6\}$. Consequently, $\varphi(z_{1,4}) = \{7,8\}$, and $\varphi(z_{1,5})$ is a subset of $\{1,2,3,4\}$. This yields that $\varphi(z_{1,5}) \cup \varphi(z_{1,6}) = \{1,2,3,4\}$, and therefore $\varphi(z_{1,7}) = \{7,8\}$.

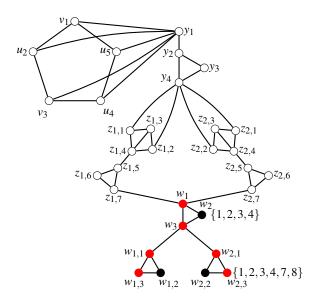


Figure 4: The graph G_4 from Lemma 7: the filled vertices are those added to G_3 to form G_4 ; the red ones have list $\{1,2,3,4,7,8\}$ while the black ones have list $\{1,2,3,4\}$.

Lemma 7. Consider the gadget (G_4, L_4) obtained from the gadget (G_3, L_3) of Lemma 6 as follows (see Figure 4 for an illustration of G_4). The graph G_4 consists of G_3 ; the three triangles $w_1w_2w_3$ and $w_{i,1}w_{i,2}w_{i,3}$ for $i \in \{1,2\}$; and the edges $z_{1,7}w_1$, $z_{2,7}w_1$, $w_3w_{1,1}$ and $w_3w_{2,1}$. Let $L_4(v) = L_3(v)$ for $v \in V(G_3)$, and for $i \in \{1,2\}$, let $L_4(w_1) = L_4(w_3) = L_4(w_{i,1}) = L_4(w_{i,3}) = \{1,2,3,4,7,8\}$ and $L_4(w_2) = L_4(w_{i,2}) = \{1,2,3,4\}$. The gadget is $(v_1,v_3,\{w_{1,3},w_{2,3}\})$ -relaxed, and $\varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7,8\}$ for every $(L_4:2)$ -coloring φ of G_4 .

Proof. Let L be a half-list assignment for G_4 . The gadget $(G_3, L_4|V(G_3))$ is $(v_1, v_3, \{z_{1,7}, z_{2,7}\})$ -relaxed by Lemma 6. Suppose first that (i) holds for the restriction of L to G_3 (with $S = \{z_{1,7}, z_{2,7}\}$), and

let ψ_0 be the corresponding L-coloring of $\{v_1, v_3\}$. Choose a color $c_1 \in L(z_{1,7})$ so that $L(w_1) \setminus \{c_1\} \neq L(w_2)$. If $c_1 \in L(w_1)$, then choose $c_2 \in L(z_{2,7}) \setminus (L(w_1) \setminus \{c_1\})$, otherwise choose $c_2 \in L(z_{2,7})$ so that $L(w_1) \setminus \{c_2\} \neq L(w_2)$. Choose a color $c_3 \in L(w_3)$ so that $L(w_{i,1}) \setminus \{c_3\} \neq L(w_{i,2})$ for $i \in \{1,2\}$. By (i) for G_3 , there exists an L-coloring of G_3 extending ψ_0 and assigning c_i to $z_{i,7}$ for each $i \in \{1,2\}$. Color w_3 by c_3 and observe that the L-coloring can be extended to w_1 and w_2 thanks to the choice of c_1 and c_2 . Moreover, the choice of c_3 ensures that for each $i \in \{1,2\}$, we can color $w_{i,3}$ with any color in $L(w_{i,3})$ and further extend the coloring to $w_{i,1}$ and $w_{i,2}$. We conclude that (G_4, L_4) with the half-list assignment L satisfies (i).

Suppose next that (ii) holds for the restriction of L to G_4 (with $S = \{z_{1,7}, z_{2,7}\}$), and let ψ'_0 be the corresponding L-coloring of $\{z_{1,7}, z_{2,7}\}$. Greedily extend ψ'_0 to an L-coloring of w_1, w_2, w_3 , and $w_{i,1}, w_{i,2}, w_{i,3}$ for $i \in \{1,2\}$ in order, and let ψ_0 be the restriction of the resulting coloring to $\{w_{1,3}, w_{2,3}\}$. Observe that (G_4, L_4) with the half-list assignment L satisfies (ii).

Finally, let φ be an $(L_4:2)$ -coloring of G_4 . By Lemma 6 and by symmetry, we can assume that $\varphi(z_{1,7}) = \{7,8\}$. Consequently, $\varphi(w_1) \subset \{1,2,3,4\}$, and thus $\varphi(w_1) \cup \varphi(w_2) = \{1,2,3,4\}$, which yields that $\varphi(w_3) = \{7,8\}$. We conclude analogously that $\varphi(w_{1,3}) = \{7,8\} = \varphi(w_{2,3})$.

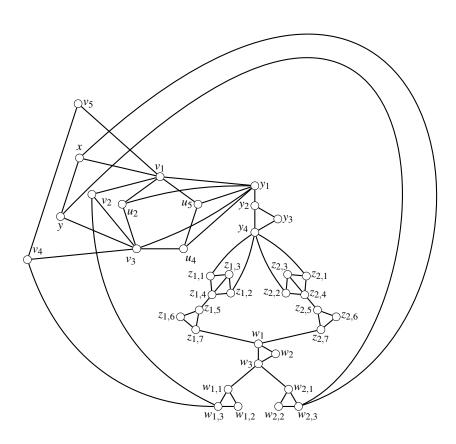


Figure 5: The graph G_5 from Lemma 8.

We can now combine (G_1, L_1) with (G_4, L_4) to obtain a gadget (G_5, L_5) that is L-colorable from each half-list assignment L, but not $(L_5 : 2)$ -colorable.

Lemma 8. Consider the gadget (G_5, L_5) obtained from the gadgets (G_1, L_1) of Corollary 4 and (G_4, L_4) of Lemma 7 as follows (see Figure 5 for an illustration of G_5). The graph G_5 is obtained from the union of the graphs G_1 and G_4 (intersecting in $\{v_1, v_3\}$) by adding the edges $w_{1,3}v_2, w_{1,3}v_4, w_{2,3}x$ and $w_{2,3}y$. Let $L_5(v) = L_4(v)$ for $v \in V(G_4)$, $L_5(v) = L_1(v)$ for $v \in V(G_1) \setminus \{v_2, v_4, x, y\}$, and $L_5(v) = L_1(v) \cup \{7, 8\}$ for $v \in \{v_2, v_4, x, y\}$. Then G_5 is L-colorable for every half-list assignment L, but not $(L_5: 2)$ -colorable.

Proof. Let L be a half-list assignment for G_5 . The gadget $(G_4, L_5|V(G_4))$ is $(v_1, v_3, \{w_{1,3}, w_{2,3}\})$ -relaxed by Lemma 7. Suppose first that (i) holds for the restriction of L to G_4 (with $S = \{w_{1,3}, w_{2,3}\}$), and let ψ_0 be the corresponding L-coloring of $\{v_1, v_3\}$. Greedily extend ψ_0 to an L-coloring ψ of G_1 . Choose $c_1 \in L(w_{1,3}) \setminus \{\psi(v_2), \psi(v_4)\}$ and $c_2 \in L(w_{2,3}) \setminus \{\psi(x), \psi(y)\}$. By (i) for G_4 , there exists an L-coloring of G_4 that extends ψ_0 and assigns to $w_{i,3}$ the color c_i for each $i \in \{1,2\}$. This yields, together with ψ , an L-coloring of G_5 .

Suppose next that (ii) holds for the restriction of L to G_4 (with $S = \{w_{1,3}, w_{2,3}\}$), and let ψ_0 be the corresponding L-coloring of $\{w_{1,3}, w_{2,3}\}$. Note that $L(v_1) = L(v_3)$ in this case. Corollary 4 implies that G_1 has an L-coloring ψ such that $\psi(v_2) \in L(v_2) \setminus \{\psi_0(w_{1,3})\}$, $\psi(v_4) \in L(v_4) \setminus \{\psi_0(w_{1,3})\}$, $\psi(x) \in L(x) \setminus \{\psi_0(w_{2,3})\}$, and $\psi(y) \in L(y) \setminus \{\psi_0(w_{2,3})\}$. By (ii), the restriction of $\psi \cup \psi_0$ to $\{v_1, v_3, w_{1,3}, w_{2,3}\}$ extends to an L-coloring of G_4 , which together with ψ gives an L-coloring of G_5 .

It remains to show that G_5 is not $(L_5:2)$ -colorable. If φ were an $(L_5:2)$ -coloring of G_5 , then by Lemma 7 we would have $\varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7,8\}$, and thus the restriction of φ to G_1 would be an $(L_1:2)$ -coloring, thereby contradicting Corollary 4.

The final graph. We are now in a position to prove Theorem 2 by simply using a standard construction to ensure uniform lengths of lists.

Proof of Theorem 2. Let G be a graph and L' an assignment of lists of size 8 obtained as follows. Let K be a clique with vertices r_1, \ldots, r_4 , and let $L'(r_1) = \cdots = L'(r_4) = \{9, \ldots, 16\}$. For every (L': 2)-coloring ψ of K, let G_{ψ} be a copy of the graph G_5 from the gadget (G_5, L_5) of Lemma 8, and for each vertex $v \in V(G_{\psi})$ such that $|L_5(v)| = 2k$ with $k \in \{2,3\}$, we add the edges vr_1, \ldots, vr_{4-k} and let $L'(v) = L_5(v) \cup \bigcup_{i=1}^{4-k} \psi(r_i)$. If G had an (L': 2)-coloring G, then letting G be the restriction of G to G would be an G would be an G to G thereby contradicting Lemma 8.

Consider now a list assignment L for G such that |L(v)| = 4 for every $v \in V(G)$. Let φ be any L-coloring of K. For each (L':2)-coloring ψ of K, let L_{ψ} be the list assignment for G_{ψ} obtained by, for each $v \in V(G_{\psi})$, removing the colors of neighbors in K according to φ , and possibly removing further colors to ensure that $|L_{\psi}(v)| = |L_5(v)|/2$. By Lemma 8, the graph G_{ψ} has an L_{ψ} -coloring. The union of these colorings and φ yields an L-coloring of G.

Concluding remarks. It follows from Theorem 2 that for each integer $a \ge 4$, there exists a graph G_a that is a-choosable but not (2a:2)-choosable—if we have such a graph G_a , taking the disjoint union of $\binom{2(a+1)}{2}$ copies of G_a and adding a vertex adjacent to all other vertices yields G_{a+1} , by an argument analogous to the list uniformization procedure used for the proof of Theorem 2. It is natural to ask

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whether there exists a graph that is 3-choosable but not (6:2)-choosable. We believe this to be the case; in particular, Corollary 4 only requires lists of size at most 6. However, it does not seem easy to construct a gadget that satisfies the properties stated in Lemma 5 without using a vertex with a list of size 8.

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