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# A 4-choosable Graph that is Not ( $8: 2$ )-choosable 

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#### Abstract

In 1980, Erdős, Rubin and Taylor asked whether for all positive integers $a, b$, and $m$, every $(a: b)$-choosable graph is also $(a m: b m)$-choosable. We provide a negative answer by exhibiting a 4-choosable graph that is not $(8: 2)$-choosable.


Key words and phrases: $(a m: b m)$-conjecture, $(a: b)$-choosable graph, list coloring.
Coloring the vertices of a graph with sets of colors (that is, each vertex is assigned a fixed-size subset of the colors such that adjacent vertices are assigned disjoint sets) is a fundamental notion, which in particular captures fractional colorings. The fractional chromatic number of a graph $G$ can indeed be defined to be the infimum (which actually is a minimum) of the ratios $a / b$ such that, if every vertex of $G$ is replaced by a clique of order $b$ and every edge of $G$ is replaced by a complete bipartite graph between the relevant cliques, then the chromatic number of the obtained graph is at most $a$.

In their seminal work on list coloring, Erdős, Rubin and Taylor [2] raised several intriguing questions about the list version of set coloring. Before stating them, let us review the relevant definitions.

Set coloring. A function that assigns a set to each vertex of a graph is a set coloring if the sets assigned to adjacent vertices are disjoint. For positive integers $a$ and $b \leq a$, an $(a: b)$-coloring of a graph $G$ is a set coloring with range $\binom{\{1, \ldots, a\}}{b}$, i.e., a set coloring that to each vertex of $G$ assigns a $b$-element subset of $\{1, \ldots, a\}$. The concept of $(a: b)$-coloring is a generalization of the conventional vertex coloring. In fact, an ( $a: 1$ )-coloring is exactly an ordinary proper $a$-coloring.

[^0]A list assignment for a graph $G$ is a function $L$ that to each vertex $v$ of $G$ assigns a set $L(v)$ of colors. A set coloring $\varphi$ of $G$ is an $L$-set coloring if $\varphi(v) \subseteq L(v)$ for every $v \in V(G)$. For a positive integer $b$, we say that $\varphi$ is an $(L: b)$-coloring of $G$ if $\varphi$ is an $L$-set coloring and $|\varphi(v)|=b$ for every $v \in V(G)$. If such an $(L: b)$-coloring exists, then $G$ is $(L: b)$-colorable. For an integer $a \geq b$, we say that $G$ is $(a: b)$-choosable if $G$ is $(L: b)$-colorable for every list assignment $L$ such that $|L(v)|=a$ for each $v \in V(G)$. We abbreviate ( $L: 1$ )-coloring, $(L: 1)$-colorable, and ( $a: 1$ )-choosable to $L$-coloring, $L$-colorable, and $a$-choosable, respectively.

Questions and results. It is straightforward to see that if a graph is $(a: b)$-colorable, it is also (am:bm)colorable for every positive integer $m$ : we can simply replace every color in an $(a: b)$-coloring by $m$ new colors. However, this argument fails in the list coloring setting, leading Erdős, Rubin and Taylor [2] to ask whether every $(a: b)$-choosable graph is also $(a m: b m)$-choosable whenever $m \geq 1$. A positive answer to this question is sometimes referred to as "the ( $a m: b m$ )-conjecture". Using the characterization of 2-choosable graphs [2], Tuza and Voigt [4] provided a positive answer when $a=2$ and $b=1$. In the other direction, Gutner and Tarsi [3] demonstrated that if $k$ and $m$ are positive integers and $k$ is odd, then every ( $2 m k: m k$ )-choosable graph is also $2 m$-choosable.

Formulated differently, the question is to know whether every $(a: b)$-choosable graph is also $(c: d)$ choosable whenever $c / d=a / b$ and $c \geq a$. This formulation raises the same question when $c / d>a / b$, which was also asked by Erdős, Rubin and Taylor [2]. About ten years ago, Gutner and Tarsi [3] answered this last question negatively, by studying the $k$ th choice number of a graph for large values of $k$. More precisely, the $k$ th choice number of a graph $G$ is $\operatorname{ch}_{: k}(G)$, the least integer $a$ for which $G$ is ( $a: k$ )-choosable. Their result reads as follows.

Theorem 1 (Gutner \& Tarsi, 2009). Let $G$ be a graph. For every positive real $\varepsilon$, there exists an integer $k_{0}$ such that $\mathrm{ch}_{: ~}(G) \leq k(\chi(G)+\varepsilon)$ for every $k \geq k_{0}$.

As a direct corollary, one deduces that for all integers $m \geq 3$ and $\ell>m$, there exists a graph that is $(a: b)$-choosable and not $(\ell: 1)$-choosable with $\frac{a}{b}=m$. (To see this, one can for example apply Theorem 1 with $\varepsilon=1$ to the disjoint union of a clique of order $m-1$ and a complete balanced bipartite graph with choice number $\ell+1$.)

Another related result that should be mentioned here was obtained by Alon, Tuza and Voigt [1]. They proved that for every graph $G$,

$$
\inf \left\{\left.\frac{a}{b} \right\rvert\, G \text { is }(a: b) \text {-choosable }\right\}=\inf \left\{\left.\frac{a}{b} \right\rvert\, G \text { is }(a: b) \text {-colorable }\right\} .
$$

In other words, the fractional choice number of a graph equals its fractional chromatic number.
The purpose of our work is to provide a negative answer to Erdős, Rubin and Taylor's question when $a=4$ and $b=1$.

Theorem 2. There exists a graph $G$ that is 4-choosable, but not $(8: 2)$-choosable.
We build such a graph by incrementally combining pieces with certain properties. Each piece is defined, and its relevant properties established, in the forthcoming lemmas.


Figure 1: The gadget $\left(C, L_{0}\right)$ of Lemma 3 is depicted on the left; the gadget $\left(G_{1}, L_{1}\right)$ of Corollary 4 is depicted on the right.

Gadgets and lemmas. A gadget is a pair ( $G, L_{0}$ ), where $G$ is a graph and $L_{0}$ is an assignment of lists of even size. Given a gadget ( $G, L_{0}$ ), a half-list assignment for $G$ is a list assignment $L$ for $G$ such that $|L(v)|=\left|L_{0}(v)\right| / 2$ for every $v \in V(G)$. Let us start the construction by a key observation on list colorings of 5-cycles.

Lemma 3. Consider the gadget ( $C, L_{0}$ ), presented in Figure 1, where $C=v_{1} v_{2} v_{3} v_{4} v_{5}$ is a 5 -cycle, $L_{0}\left(v_{1}\right)=\{1,2,5,6\}, L_{0}\left(v_{2}\right)=\{1,4,5,6\}, L_{0}\left(v_{3}\right)=L_{0}\left(v_{4}\right)=\{3,4,5,6\}$ and $L_{0}\left(v_{5}\right)=\{2,4,5,6\}$. Then $C$ is $L$-colorable for every half-list assignment $L$ such that $\left|L\left(v_{1}\right) \cap L\left(v_{3}\right)\right| \leq 1$, but C is not $\left(L_{0}: 2\right)$-colorable.

Proof. The first statement is well known, but let us give the easy proof for completeness: since $\mid L\left(v_{1}\right) \cap$ $L\left(v_{3}\right) \mid \leq 1$, we have $\left|L\left(v_{1}\right) \cup L\left(v_{3}\right)\right| \geq 3$, and thus $L\left(v_{1}\right)$ or $L\left(v_{3}\right)$ contains a color $c_{6}$ not belonging to $L\left(v_{2}\right)$. By symmetry, we can assume that $c_{6} \in L\left(v_{1}\right)$. We color $v_{1}$ by $c_{6}$ and then for $i=5,4,3,2$ in order, we color $v_{i}$ by a color $c_{i} \in L\left(v_{i}\right) \backslash\left\{c_{i+1}\right\}$. The resulting coloring is proper-we have $c_{2} \neq c_{6}$, since $c_{6} \notin L\left(v_{2}\right)$.

Suppose now that $C$ has an $\left(L_{0}: 2\right)$-coloring, and for $c \in\{1, \ldots, 6\}$ let $V_{c}$ be the set of vertices of $C$ on which the color $c$ is used. Since two colors are used on each vertex of $C$, we have $\sum_{c=1}^{6}\left|V_{c}\right|=10$. On the other hand, $V_{c}$ is an independent set of a 5 -cycle, and thus $\left|V_{c}\right| \leq 2$ for every color $c$. Furthermore, color 1 only appears in the lists of $v_{1}$ and $v_{2}$, which are adjacent in $C$. It follows that $\left|V_{1}\right| \leq 1$. The situation is similar for color 2 , which appears only in the lists of $v_{1}$ and $v_{5}$, and also for color 3 , which only appears in the lists of $v_{3}$ and $v_{4}$. Consequently, $\sum_{c=1}^{6}\left|V_{c}\right| \leq 3 \cdot 2+3 \cdot 1=9$, which is a contradiction.

Corollary 4. Consider the gadget $\left(G_{1}, L_{1}\right)$, presented in Figure 1, where $G_{1}$ consists of a 5-cycle $C=v_{1} v_{2} v_{3} v_{4} v_{5}$ and a path $v_{1} x y v_{3}$, with $L_{1}\left(v_{1}\right)=L_{1}\left(v_{3}\right)=\{1, \ldots, 6\}, L_{1}\left(v_{2}\right)=\{1,4,5,6\}, L_{1}\left(v_{4}\right)=$ $\{3,4,5,6\}, L_{1}\left(v_{5}\right)=\{2,4,5,6\}, L_{1}(x)=\{1,2,3,4\}$ and $L_{1}(y)=\{1,2\}$. Then $G_{1}$ is L-colorable for every half-list assignment $L$ such that $L\left(v_{1}\right)=L\left(v_{3}\right)$, but $G_{1}$ is not $\left(L_{1}: 2\right)$-colorable.

Proof. Let $L$ be a half-list assignment for $G_{1}$. First $L$-color $y$ and $x$ by colors $c_{y} \in L(y)$ and $c_{x} \in L(x) \backslash\left\{c_{y}\right\}$, respectively. Since $c_{x} \neq c_{y}$ and $L\left(v_{1}\right)=L\left(v_{3}\right)$, there exist sets $L^{\prime}\left(v_{1}\right) \subseteq L\left(v_{1}\right) \backslash\left\{c_{x}\right\}$ and $L^{\prime}\left(v_{3}\right) \subseteq$
$L\left(v_{3}\right) \backslash\left\{c_{y}\right\}$ of size two such that $L^{\prime}\left(v_{1}\right) \neq L^{\prime}\left(v_{3}\right)$. Let $L^{\prime}\left(v_{i}\right)=L\left(v_{i}\right)$ for $i \in\{2,4,5\}$. Lemma 3 implies that $C$ is $L^{\prime}$-colorable, which yields an $L$-coloring of $G$.

In an $\left(L_{1}: 2\right)$-coloring, the vertex $y$ would have to be assigned $\{1,2\}$ and $x$ would have to be assigned $\{3,4\}$, and thus the sets of available colors for $v_{1}$ and for $v_{3}$ would have to be $\{1,2,5,6\}$ and $\{3,4,5,6\}$, respectively. However, no such $\left(L_{1}: 2\right)$ coloring of $C$ exists according to Lemma 3.

Next we construct auxiliary gadgets, which will be combined with the gadget from Corollary 4 to deal with the case where $L\left(v_{1}\right) \neq L\left(v_{3}\right)$. Let $G$ be a graph, let $S$ be a subset of vertices of $G$ and $L$ a list assignment for $G$. An $L$-coloring of $S$ is a coloring of the subgraph of $G$ induced by $S$. Moreover, if $S^{\prime}$ is a subset of vertices of $G$ that contains $S$ and $\varphi^{\prime}$ is an $L$-coloring of $S^{\prime}$, then $\varphi^{\prime}$ extends $\varphi$ if $\varphi^{\prime} \mid S=\varphi$. Let $\left(G, L_{0}\right)$ be a gadget, let $v_{1}$ and $v_{3}$ be distinct vertices of $G$, and let $S$ be a set of vertices of $G$ not containing $v_{1}$ and $v_{3}$. The gadget is ( $v_{1}, v_{3}, S$ )-relaxed if every half-list assignment $L$ satisfies at least one of the following conditions.
(i) There exists an $L$-coloring $\psi_{0}$ of $\left\{v_{1}, v_{3}\right\}$ such that every $L$-coloring of $S \cup\left\{v_{1}, v_{3}\right\}$ extending $\psi_{0}$ extends to an $L$-coloring of $G$.
(ii) $L\left(v_{1}\right)=L\left(v_{3}\right)$ and there exists an $L$-coloring $\psi_{0}$ of $S$ such that every $L$-coloring of $S \cup\left\{v_{1}, v_{3}\right\}$ extending $\psi_{0}$ extends to an $L$-coloring of $G$.


Figure 2: The gadget $\left(G_{2}, L_{2}\right)$ of Lemma 5

Lemma 5. Consider the gadget $\left(G_{2}, L_{2}\right)$ presented in Figure 2, where $G_{2}$ consists of a 5 -cycle $C_{2}=$ $v_{1} u_{2} v_{3} u_{4} u_{5}$, a vertex $y_{1}$ adjacent to all vertices of $C_{2}$, a triangle $y_{2} y_{3} y_{4}$, and an edge $y_{1} y_{2}$, with $L_{2}(v)=$ $\{1, \ldots, 6\}$ for every $v \in V\left(C_{2}\right), L_{2}\left(y_{1}\right)=\{1, \ldots, 8\}, L_{2}\left(y_{2}\right)=L_{2}\left(y_{4}\right)=\{1,2,3,4,7,8\}$, and $L_{2}\left(y_{3}\right)=$ $\{1,2,3,4\}$. The gadget is $\left(v_{1}, v_{3},\left\{y_{4}\right\}\right)$-relaxed, and $\varphi\left(y_{4}\right) \cap\{7,8\} \neq \varnothing$ for every $\left(L_{2}: 2\right)$-coloring $\varphi$ of $G_{2}$.

Proof. Let $L$ be a half-list assignment for $G_{2}$. If not all vertices of $C_{2}$ have the same list, then choose a color $c \in L\left(y_{1}\right) \backslash L\left(y_{2}\right)$, and observe there exists an $L$-coloring of $G_{2}\left[V\left(C_{2}\right) \cup\left\{y_{1}\right\}\right]$ such that $y_{1}$ has

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color $c$. Let $\psi_{0}$ be the restriction of this coloring to $\left\{v_{1}, v_{3}\right\}$. Clearly, every $L$-coloring of $\left\{v_{1}, v_{3}, y_{4}\right\}$ extending $\psi_{0}$ extends to an $L$-coloring of $G_{2}$, and thus (i) holds.

If all the vertices of $C_{2}$ have the same list (and hence in particular $L\left(v_{1}\right)=L\left(v_{3}\right)$ ), then let $c$ be a color in $L\left(y_{1}\right) \backslash L\left(v_{1}\right)$. Observe that there exists an $L$-coloring of $G_{2}\left[\left\{y_{1}, y_{2}, y_{3}, y_{4}\right\}\right]$ such that $y_{1}$ has color $c$. Let $\psi_{0}$ be the restriction of this coloring to $y_{4}$. Again, every $L$-coloring of $\left\{v_{1}, v_{3}, y_{4}\right\}$ extending $\psi_{0}$ extends to an $L$-coloring of $G_{2}$, and thus (ii) holds.

It remains to show that if $\varphi$ is an $\left(L_{2}: 2\right)$-coloring of $G_{2}$ then $\varphi\left(y_{4}\right) \cap\{7,8\} \neq \varnothing$. Suppose, on the contrary, that $\varphi\left(y_{4}\right) \cap\{7,8\}=\varnothing$. It follows that $\varphi\left(y_{4}\right) \cup \varphi\left(y_{3}\right)=\{1,2,3,4\}$, and hence $\varphi\left(y_{2}\right)=\{7,8\}$. As a result, $\varphi\left(y_{1}\right) \subseteq\{1, \ldots, 6\}$ and, by symmetry, we can assume that $\varphi\left(y_{1}\right)=\{5,6\}$. This implies that $\varphi(v) \subseteq\{1,2,3,4\}$ for each $v \in V\left(C_{2}\right)$. In particular, $\varphi \mid V\left(C_{2}\right)$ is a $(4: 2)$-coloring of $C_{2}$, which is a contradiction since the 5-cycle $C_{2}$ has fractional chromatic number 5/2.


Figure 3: The graph $G_{3}$ from Lemma 6 is depicted on the left; the gadget $\left(G_{3}, L_{3}\right)$ is obtained from $\left(G_{2}, L_{2}\right)$ by adding to $G_{3}$ a disjoint copy of the graph depicted on the right (with the corresponding lists) for each $i \in\{1,2\}$, and joining $y_{4}$ to each of $z_{1,1}, z_{1,2}, z_{2,1}$ and $z_{2,2}$.

Lemma 6. Consider the gadget $\left(G_{3}, L_{3}\right)$, obtained from the gadget $\left(G_{2}, L_{2}\right)$ of Lemma 5 as follows (see Figure 3 for an illustration of $G_{3}$ ). The graph $G_{3}$ consists of $G_{2}$ and for $i \in\{1,2\}$, the vertices $z_{i, 1}, \ldots, z_{i, 7}$; the edges $y_{4} z_{i, 1}$ and $y_{4} z_{i, 2}$; the edge $z_{i, j} z_{i, k}$ for every $j$ and every $k$ such that $1 \leq j<k \leq 4$ and $(j, k) \neq$ $(1,2)$; the edges of the triangle $z_{i, 5} z_{i, 6} z_{i, 7}$ and the edge $z_{i, 4} z_{i, 5}$. Let $L_{3}(v)=L_{2}(v)$ for $v \in V\left(G_{2}\right)$, and for $i \in$ $\{1,2\}$ let $L_{3}\left(z_{i, 1}\right)=\{1,2,3,6+i\}, L_{3}\left(z_{i, 2}\right)=\{4,5,6,6+i\}, L_{3}\left(z_{i, 3}\right)=\{1, \ldots, 6\}, L_{3}\left(z_{i, 4}\right)=\{1, \ldots, 8\}$, $L_{3}\left(z_{i, 5}\right)=L_{3}\left(z_{i, 7}\right)=\{1,2,3,4,7,8\}$ and $L_{3}\left(z_{i, 6}\right)=\{1,2,3,4\}$. The gadget $\left(G_{3}, L_{3}\right)$ is $\left(v_{1}, v_{3},\left\{z_{1,7}, z_{2,7}\right\}\right)$ relaxed, and $\varphi\left(z_{1,7}\right)=\{7,8\}$ or $\varphi\left(z_{2,7}\right)=\{7,8\}$ for every $\left(L_{3}: 2\right)$-coloring $\varphi$ of $G_{3}$.

Proof. Let $L$ be a half-list assignment for $G_{3}$. The gadget $\left(G_{2}, L_{3} \mid V\left(G_{2}\right)\right)$ is $\left(v_{1}, v_{3},\left\{y_{4}\right\}\right)$-relaxed by Lemma 5. Suppose first that (i) holds for the restriction of $L$ to $G_{2}$ (with $S=\left\{y_{4}\right\}$ ), and let $\psi_{0}$ be the corresponding $L$-coloring of $\left\{v_{1}, v_{3}\right\}$. For $i \in\{1,2\}$, if $L\left(z_{i, 1}\right) \cap L\left(z_{i, 2}\right) \neq \varnothing$, then let $c_{i}$ be a color in $L\left(z_{i, 1}\right) \cap L\left(z_{i, 2}\right)$. Otherwise, $\left|L\left(z_{i, 1}\right) \cup L\left(z_{i, 2}\right)\right|=4>\left|L\left(z_{i, 3}\right)\right|$, and thus we can choose a color $c_{i} \in\left(L\left(z_{i, 1}\right) \cup L\left(z_{i, 2}\right)\right) \backslash L\left(z_{i, 3}\right)$. Let $c$ be a color in $L\left(y_{4}\right) \backslash\left\{c_{1}, c_{2}\right\}$. By (i) for $G_{2}$, we know that $\psi_{0}$ extends to an $L$-coloring $\psi$ of $G_{2}$ such that $\psi\left(y_{4}\right)=c$. If $L\left(z_{i, 1}\right) \cap L\left(z_{i, 2}\right) \neq \varnothing$, then color both $z_{i, 1}$ and $z_{i, 2}$ by $c_{i}$, otherwise color one of them by $c_{i}$ and the other one by an arbitrary color from its list that is different
from $c$. There are at least two colors in $L\left(z_{i, 4}\right)$ distinct from the colors of $z_{i, 1}$ and $z_{i, 2}$, choose such a color $c_{i}^{\prime}$ so that $L\left(z_{i, 5}\right) \backslash\left\{c_{i}^{\prime}\right\} \neq L\left(z_{i, 6}\right)$. Color $z_{i, 4}$ by $c_{i}^{\prime}$ and extend the coloring to $z_{i, 3}$, which is possible by the choice of $c_{i}$. Observe that any $L$-coloring of $z_{i, 7}$ extends to an $L$-coloring of the triangle $z_{i, 5} z_{i, 6} z_{i, 7}$ where the color of $z_{i, 5}$ is not $c_{i}^{\prime}$. We conclude that $\left(G_{3}, L_{3}\right)$ with the half-list assignment $L$ satisfies (i).

Suppose next that (ii) holds for the restriction of $L$ to $G_{2}$ (with $S=\left\{y_{4}\right\}$ ), and let $\psi_{0}^{\prime}$ be the corresponding $L$-coloring of $y_{4}$. For $i \in\{1,2\}$, greedily extend $\psi_{0}^{\prime}$ to an $L$-coloring of $z_{i, 1}, \ldots, z_{i, 7}$ in order, and let $\psi_{0}$ be the restriction of the resulting coloring to $\left\{z_{1,7}, z_{2,7}\right\}$. Observe that $\left(G_{3}, L_{3}\right)$ with the half-list assignment $L$ satisfies (ii).

Finally, let $\varphi$ be an $\left(L_{3}: 2\right)$-coloring of $G_{3}$. Lemma 5 implies that $\varphi\left(y_{4}\right) \cap\{7,8\} \neq \varnothing$. By symmetry, we can assume that $7 \in \varphi\left(y_{4}\right)$. It follows that $\varphi\left(z_{1,1}\right) \subset\{1,2,3\}$ and $\varphi\left(z_{1,2}\right) \subset\{4,5,6\}$, and thus $\varphi\left(z_{1,1}\right) \cup$ $\varphi\left(z_{1,2}\right) \cup \varphi\left(z_{1,3}\right)=\{1, \ldots, 6\}$. Consequently, $\varphi\left(z_{1,4}\right)=\{7,8\}$, and $\varphi\left(z_{1,5}\right)$ is a subset of $\{1,2,3,4\}$. This yields that $\varphi\left(z_{1,5}\right) \cup \varphi\left(z_{1,6}\right)=\{1,2,3,4\}$, and therefore $\varphi\left(z_{1,7}\right)=\{7,8\}$.


Figure 4: The graph $G_{4}$ from Lemma 7: the filled vertices are those added to $G_{3}$ to form $G_{4}$; the red ones have list $\{1,2,3,4,7,8\}$ while the black ones have list $\{1,2,3,4\}$.

Lemma 7. Consider the gadget $\left(G_{4}, L_{4}\right)$ obtained from the gadget $\left(G_{3}, L_{3}\right)$ of Lemma 6 as follows (see Figure 4 for an illustration of $G_{4}$ ). The graph $G_{4}$ consists of $G_{3}$; the three triangles $w_{1} w_{2} w_{3}$ and $w_{i, 1} w_{i, 2} w_{i, 3}$ for $i \in\{1,2\}$; and the edges $z_{1,7} w_{1}, z_{2,7} w_{1}, w_{3} w_{1,1}$ and $w_{3} w_{2,1}$. Let $L_{4}(v)=L_{3}(v)$ for $v \in$ $V\left(G_{3}\right)$, and for $i \in\{1,2\}$, let $L_{4}\left(w_{1}\right)=L_{4}\left(w_{3}\right)=L_{4}\left(w_{i, 1}\right)=L_{4}\left(w_{i, 3}\right)=\{1,2,3,4,7,8\}$ and $L_{4}\left(w_{2}\right)=$ $L_{4}\left(w_{i, 2}\right)=\{1,2,3,4\}$. The gadget is $\left(v_{1}, v_{3},\left\{w_{1,3}, w_{2,3}\right\}\right)$-relaxed, and $\varphi\left(w_{1,3}\right)=\varphi\left(w_{2,3}\right)=\{7,8\}$ for every $\left(L_{4}: 2\right)$-coloring $\varphi$ of $G_{4}$.

Proof. Let $L$ be a half-list assignment for $G_{4}$. The gadget $\left(G_{3}, L_{4} \mid V\left(G_{3}\right)\right)$ is $\left(v_{1}, v_{3},\left\{z_{1,7}, z_{2,7}\right\}\right)$-relaxed by Lemma 6. Suppose first that (i) holds for the restriction of $L$ to $G_{3}$ (with $S=\left\{z_{1,7}, z_{2,7}\right\}$ ), and
let $\psi_{0}$ be the corresponding $L$-coloring of $\left\{v_{1}, v_{3}\right\}$. Choose a color $c_{1} \in L\left(z_{1,7}\right)$ so that $L\left(w_{1}\right) \backslash\left\{c_{1}\right\} \neq$ $L\left(w_{2}\right)$. If $c_{1} \in L\left(w_{1}\right)$, then choose $c_{2} \in L\left(z_{2,7}\right) \backslash\left(L\left(w_{1}\right) \backslash\left\{c_{1}\right\}\right)$, otherwise choose $c_{2} \in L\left(z_{2,7}\right)$ so that $L\left(w_{1}\right) \backslash\left\{c_{2}\right\} \neq L\left(w_{2}\right)$. Choose a color $c_{3} \in L\left(w_{3}\right)$ so that $L\left(w_{i, 1}\right) \backslash\left\{c_{3}\right\} \neq L\left(w_{i, 2}\right)$ for $i \in\{1,2\}$. By (i) for $G_{3}$, there exists an $L$-coloring of $G_{3}$ extending $\psi_{0}$ and assigning $c_{i}$ to $z_{i, 7}$ for each $i \in\{1,2\}$. Color $w_{3}$ by $c_{3}$ and observe that the $L$-coloring can be extended to $w_{1}$ and $w_{2}$ thanks to the choice of $c_{1}$ and $c_{2}$. Moreover, the choice of $c_{3}$ ensures that for each $i \in\{1,2\}$, we can color $w_{i, 3}$ with any color in $L\left(w_{i, 3}\right)$ and further extend the coloring to $w_{i, 1}$ and $w_{i, 2}$. We conclude that $\left(G_{4}, L_{4}\right)$ with the half-list assignment $L$ satisfies (i).

Suppose next that (ii) holds for the restriction of $L$ to $G_{4}$ (with $S=\left\{z_{1,7}, z_{2,7}\right\}$ ), and let $\psi_{0}^{\prime}$ be the corresponding $L$-coloring of $\left\{z_{1,7}, z_{2,7}\right\}$. Greedily extend $\psi_{0}^{\prime}$ to an $L$-coloring of $w_{1}, w_{2}, w_{3}$, and $w_{i, 1}, w_{i, 2}$, $w_{i, 3}$ for $i \in\{1,2\}$ in order, and let $\psi_{0}$ be the restriction of the resulting coloring to $\left\{w_{1,3}, w_{2,3}\right\}$. Observe that ( $G_{4}, L_{4}$ ) with the half-list assignment $L$ satisfies (ii).

Finally, let $\varphi$ be an ( $L_{4}: 2$ )-coloring of $G_{4}$. By Lemma 6 and by symmetry, we can assume that $\varphi\left(z_{1,7}\right)=\{7,8\}$. Consequently, $\varphi\left(w_{1}\right) \subset\{1,2,3,4\}$, and thus $\varphi\left(w_{1}\right) \cup \varphi\left(w_{2}\right)=\{1,2,3,4\}$, which yields that $\varphi\left(w_{3}\right)=\{7,8\}$. We conclude analogously that $\varphi\left(w_{1,3}\right)=\{7,8\}=\varphi\left(w_{2,3}\right)$.


Figure 5: The graph $G_{5}$ from Lemma 8.

We can now combine $\left(G_{1}, L_{1}\right)$ with $\left(G_{4}, L_{4}\right)$ to obtain a gadget $\left(G_{5}, L_{5}\right)$ that is $L$-colorable from each half-list assignment $L$, but not ( $L_{5}: 2$ )-colorable.

Lemma 8. Consider the gadget $\left(G_{5}, L_{5}\right)$ obtained from the gadgets $\left(G_{1}, L_{1}\right)$ of Corollary 4 and $\left(G_{4}, L_{4}\right)$ of Lemma 7 as follows (see Figure 5 for an illustration of $G_{5}$ ). The graph $G_{5}$ is obtained from the union of the graphs $G_{1}$ and $G_{4}$ (intersecting in $\left\{v_{1}, v_{3}\right\}$ ) by adding the edges $w_{1,3} v_{2}, w_{1,3} v_{4}, w_{2,3} x$ and $w_{2,3} y$. Let $L_{5}(v)=L_{4}(v)$ for $v \in V\left(G_{4}\right), L_{5}(v)=L_{1}(v)$ for $v \in V\left(G_{1}\right) \backslash\left\{v_{2}, v_{4}, x, y\right\}$, and $L_{5}(v)=L_{1}(v) \cup\{7,8\}$ for $v \in\left\{v_{2}, v_{4}, x, y\right\}$. Then $G_{5}$ is L-colorable for every half-list assignment $L$, but not ( $L_{5}: 2$ )-colorable.

Proof. Let $L$ be a half-list assignment for $G_{5}$. The gadget $\left(G_{4}, L_{5} \mid V\left(G_{4}\right)\right)$ is $\left(v_{1}, v_{3},\left\{w_{1,3}, w_{2,3}\right\}\right)$-relaxed by Lemma 7. Suppose first that (i) holds for the restriction of $L$ to $G_{4}$ (with $S=\left\{w_{1,3}, w_{2,3}\right\}$ ), and let $\psi_{0}$ be the corresponding $L$-coloring of $\left\{v_{1}, v_{3}\right\}$. Greedily extend $\psi_{0}$ to an $L$-coloring $\psi$ of $G_{1}$. Choose $c_{1} \in L\left(w_{1,3}\right) \backslash\left\{\psi\left(v_{2}\right), \psi\left(v_{4}\right)\right\}$ and $c_{2} \in L\left(w_{2,3}\right) \backslash\{\psi(x), \psi(y)\}$. By (i) for $G_{4}$, there exists an $L$-coloring of $G_{4}$ that extends $\psi_{0}$ and assigns to $w_{i, 3}$ the color $c_{i}$ for each $i \in\{1,2\}$. This yields, together with $\psi$, an $L$-coloring of $G_{5}$.

Suppose next that (ii) holds for the restriction of $L$ to $G_{4}$ (with $S=\left\{w_{1,3}, w_{2,3}\right\}$ ), and let $\psi_{0}$ be the corresponding $L$-coloring of $\left\{w_{1,3}, w_{2,3}\right\}$. Note that $L\left(v_{1}\right)=L\left(v_{3}\right)$ in this case. Corollary 4 implies that $G_{1}$ has an $L$-coloring $\psi$ such that $\psi\left(v_{2}\right) \in L\left(v_{2}\right) \backslash\left\{\psi_{0}\left(w_{1,3}\right)\right\}, \psi\left(v_{4}\right) \in L\left(v_{4}\right) \backslash\left\{\psi_{0}\left(w_{1,3}\right)\right\}, \psi(x) \in$ $L(x) \backslash\left\{\psi_{0}\left(w_{2,3}\right)\right\}$, and $\psi(y) \in L(y) \backslash\left\{\psi_{0}\left(w_{2,3}\right)\right\}$. By (ii), the restriction of $\psi \cup \psi_{0}$ to $\left\{v_{1}, v_{3}, w_{1,3}, w_{2,3}\right\}$ extends to an $L$-coloring of $G_{4}$, which together with $\psi$ gives an $L$-coloring of $G_{5}$.

It remains to show that $G_{5}$ is not $\left(L_{5}: 2\right)$-colorable. If $\varphi$ were an $\left(L_{5}: 2\right)$-coloring of $G_{5}$, then by Lemma 7 we would have $\varphi\left(w_{1,3}\right)=\varphi\left(w_{2,3}\right)=\{7,8\}$, and thus the restriction of $\varphi$ to $G_{1}$ would be an ( $L_{1}: 2$ )-coloring, thereby contradicting Corollary 4.

The final graph. We are now in a position to prove Theorem 2 by simply using a standard construction to ensure uniform lengths of lists.

Proof of Theorem 2. Let $G$ be a graph and $L^{\prime}$ an assignment of lists of size 8 obtained as follows. Let $K$ be a clique with vertices $r_{1}, \ldots, r_{4}$, and let $L^{\prime}\left(r_{1}\right)=\cdots=L^{\prime}\left(r_{4}\right)=\{9, \ldots, 16\}$. For every $\left(L^{\prime}: 2\right)$ coloring $\psi$ of $K$, let $G_{\psi}$ be a copy of the graph $G_{5}$ from the gadget $\left(G_{5}, L_{5}\right)$ of Lemma 8, and for each vertex $v \in V\left(G_{\psi}\right)$ such that $\left|L_{5}(v)\right|=2 k$ with $k \in\{2,3\}$, we add the edges $v r_{1}, \ldots, v r_{4-k}$ and let $L^{\prime}(v)=L_{5}(v) \cup \bigcup_{i=1}^{4-k} \psi\left(r_{i}\right)$. If $G$ had an $\left(L^{\prime}: 2\right)$-coloring $\varphi$, then letting $\psi$ be the restriction of $\varphi$ to $K$, the restriction of $\varphi$ to $G_{\psi}$ would be an ( $\left.L_{5}: 2\right)$-coloring of $G_{5}$, thereby contradicting Lemma 8.

Consider now a list assignment $L$ for $G$ such that $|L(v)|=4$ for every $v \in V(G)$. Let $\varphi$ be any $L$-coloring of $K$. For each ( $L^{\prime}: 2$ )-coloring $\psi$ of $K$, let $L_{\psi}$ be the list assignment for $G_{\psi}$ obtained by, for each $v \in V\left(G_{\psi}\right)$, removing the colors of neighbors in $K$ according to $\varphi$, and possibly removing further colors to ensure that $\left|L_{\psi}(v)\right|=\left|L_{5}(v)\right| / 2$. By Lemma 8, the graph $G_{\psi}$ has an $L_{\psi}$-coloring. The union of these colorings and $\varphi$ yields an $L$-coloring of $G$.

Concluding remarks. It follows from Theorem 2 that for each integer $a \geq 4$, there exists a graph $G_{a}$ that is $a$-choosable but not ( $2 a: 2$ )-choosable-if we have such a graph $G_{a}$, taking the disjoint union of $\binom{2(a+1)}{2}$ copies of $G_{a}$ and adding a vertex adjacent to all other vertices yields $G_{a+1}$, by an argument analogous to the list uniformization procedure used for the proof of Theorem 2. It is natural to ask

## A 4-Choosable Graph that is Not (8:2)-CHOOSABLE

whether there exists a graph that is 3 -choosable but not $(6: 2)$-choosable. We believe this to be the case; in particular, Corollary 4 only requires lists of size at most 6 . However, it does not seem easy to construct a gadget that satisfies the properties stated in Lemma 5 without using a vertex with a list of size 8 .

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