



HAL
open science

A 4-choosable graph that is not $(8:2)$ -choosable

Zdeněk Dvořák, Xiaolan Hu, Jean-Sébastien Sereni

► **To cite this version:**

Zdeněk Dvořák, Xiaolan Hu, Jean-Sébastien Sereni. A 4-choosable graph that is not $(8:2)$ -choosable. 2018. hal-01812138v1

HAL Id: hal-01812138

<https://hal.science/hal-01812138v1>

Preprint submitted on 11 Jun 2018 (v1), last revised 9 Dec 2019 (v2)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A 4-choosable graph that is not $(8 : 2)$ -choosable*

Zdeněk Dvořák[†] Xiaolan Hu[‡] Jean-Sébastien Sereni[§]

Abstract

In 1980, Erdős, Rubin and Taylor asked whether for all positive integers a , b , and m , every $(a : b)$ -choosable graph is also $(am : bm)$ -choosable. We provide a negative answer by exhibiting a 4-choosable graph that is not $(8 : 2)$ -choosable.

Coloring the vertices of a graph with sets of colors (that is, each vertex is assigned a fixed-size subset of the colors such that adjacent vertices are assigned disjoint sets) is a fundamental notion, which in particular captures fractional colorings. The fractional chromatic number of a graph G can indeed be defined to be the infimum (which actually is a minimum) of the ratios a/b such that, if every vertex of G is replaced by a clique of order b and every edge of G is replaced by a complete bipartite graph between the relevant cliques, then the chromatic number of the obtained graph is at most a .

In their seminal work on list coloring, Erdős, Rubin and Taylor [2] raised several intriguing questions about the list version of set coloring. Before stating them, let us review the relevant definitions.

Set coloring. A function that assigns a set to each vertex of a graph is a *set coloring* if the sets assigned to adjacent vertices are disjoint. For positive integers a and $b \leq a$, an $(a : b)$ -*coloring* of a graph G is a set coloring with range $\binom{\{1, \dots, a\}}{b}$, *i.e.*, a set coloring that to each vertex of G assigns a b -element subset of $\{1, \dots, a\}$. The concept of $(a : b)$ -coloring is a generalization of the conventional vertex coloring. In fact, an $(a : 1)$ -coloring is exactly an ordinary proper a -coloring.

A *list assignment* for a graph G is a function L that to each vertex v of G assigns a set $L(v)$ of colors. A set coloring φ of G is an *L -set coloring* if $\varphi(v) \subseteq L(v)$ for every $v \in V(G)$. For a positive integer b , we say that φ is an $(L : b)$ -*coloring* of G if φ is an L -set coloring and $|\varphi(v)| = b$ for every $v \in V(G)$. If such an $(L : b)$ -coloring exists, then G is $(L : b)$ -*colorable*. For an integer $a \geq b$, we say that G is $(a : b)$ -*choosable* if G is $(L : b)$ -colorable for every list assignment L

*This project falls within the scope of L.I.A. STRUCO.

[†]Computer Science Institute (CSI) of Charles University, Malostranské náměstí 25, 118 00 Prague, Czech Republic. E-mail: rakdver@iuuk.mff.cuni.cz. Supported by the Center of Excellence – Inst. for Theor. Comp. Sci., Prague (project P202/12/G061 of Czech Science Foundation).

[‡]School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences, Central China Normal University, Wuhan 430079, PR China. E-mail: x1hu@mail.ccnu.edu.cn.

[§]Centre National de la Recherche Scientifique, CSTB (ICube), Strasbourg, France. E-mail: sereni@kam.mff.cuni.cz. This author's work was partially supported by P.H.C. Barande 40625WH.

such that $|L(v)| = a$ for each $v \in V(G)$. We abbreviate $(L : 1)$ -coloring, $(L : 1)$ -colorable, and $(a : 1)$ -choosable to L -coloring, L -colorable, and a -choosable, respectively.

Questions and results. It is straightforward to see that if a graph is $(a : b)$ -colorable, it is also $(am : bm)$ -colorable for every positive integer m : we can simply replace every color in an $(a : b)$ -coloring by m new colors. However, this argument fails in the list coloring setting, leading Erdős, Rubin and Taylor [2] to ask whether every $(a : b)$ -choosable graph is also $(am : bm)$ -choosable whenever $m \geq 1$. A positive answer to this question is sometimes referred to as “the $(am : bm)$ -conjecture”. Using the characterization of 2-choosable graphs found in *loc. cit.*, Tuza and Voigt [4] provided a positive answer when $a = 2$ and $b = 1$. In the other direction, Gutner and Tarsi [3] demonstrated that if k and m are positive integers and k is odd, then every $(2mk : mk)$ -choosable graph is also $2m$ -choosable.

Formulated differently, the question is to know whether every $(a : b)$ -choosable graph is also $(c : d)$ -choosable whenever $c/d = a/b$ and $c \geq a$. This formulation raises the same question when $c/d > a/b$, which was also asked by Erdős, Rubin and Taylor [2]. About ten years ago, Gutner and Tarsi [3] answered this last question negatively, by studying the k th choice number of a graph for large values of k . More precisely, the k th choice number of a graph G is $\text{ch}_{:k}(G)$, the least integer a for which G is $(a : k)$ -choosable. Their result reads as follows.

Theorem 1 (Gutner & Tarsi, 2009). *Let G be a graph. For every positive real ϵ , there exists an integer k_0 such that $\text{ch}_{:k}(G) \leq k(\chi(G) + \epsilon)$ for every $k \geq k_0$.*

As a direct corollary, one deduces that for all integers $m \geq 3$ and $\ell > m$, there exists a graph that is $(a : b)$ -choosable and not $(\ell : 1)$ -choosable with $\frac{a}{b} = m$. (To see this, one can for example apply Theorem 1 with $\epsilon = 1$ to the disjoint union of a clique of order $m - 1$ and a complete balanced bipartite graph with choice number $\ell + 1$.)

Another related result that should be mentioned here was obtained by Alon, Tuza and Voigt [1]. They proved that for every graph G ,

$$\inf \left\{ \frac{a}{b} \mid G \text{ is } (a : b)\text{-choosable} \right\} = \inf \left\{ \frac{a}{b} \mid G \text{ is } (a : b)\text{-colorable} \right\}.$$

In other words, the fractional choice number of a graph equals its fractional chromatic number.

The purpose of our work is to provide a negative answer to Erdős, Rubin and Taylor’s question when $a = 4$ and $b = 1$.

Theorem 2. *There exists a graph G that is 4-choosable, but not $(8 : 2)$ -choosable.*

We build such a graph by incrementally combining pieces with certain properties. Each piece is defined, and its relevant properties established, in the forthcoming lemmas.

Gadgets and lemmas. A *gadget* is a pair (G, L_0) , where G is a graph and L_0 is an assignment of lists of even size. A *half-size list assignment* for the gadget is a list assignment L for G such that $|L(v)| = |L_0(v)|/2$ for every $v \in V(G)$. Let us start the construction by a key observation on list colorings of 5-cycles.

Lemma 3. Consider the gadget (C, L_0) , where $C = v_1v_2v_3v_4v_5$ is a 5-cycle, $L_0(v_1) = \{1, 2, 5, 6\}$, $L_0(v_2) = \{1, 4, 5, 6\}$, $L_0(v_3) = L_0(v_4) = \{3, 4, 5, 6\}$ and $L_0(v_5) = \{2, 4, 5, 6\}$. Then C is L -colorable for every half-list assignment L such that $|L(v_1) \cap L(v_3)| \leq 1$, but C is not $(L_0 : 2)$ -colorable.

Proof. The first statement is well known, but let us give the easy proof for completeness: since $|L(v_1) \cap L(v_3)| \leq 1$, we have $|L(v_1) \cup L(v_3)| \geq 3$, and thus $L(v_1)$ or $L(v_3)$ contains a color c_6 not belonging to $L(v_2)$. By symmetry, we can assume that $c_6 \in L(v_1)$. We color v_1 by c_6 and then for $i = 5, 4, 3, 2$ in order, we color v_i by a color $c_i \in L(v_i) \setminus \{c_{i+1}\}$. The resulting coloring is proper—we have $c_2 \neq c_6$, since $c_6 \notin L(v_2)$.

Suppose now that C has an $(L_0 : 2)$ -coloring, and for $c \in \{1, \dots, 6\}$ let V_c be the set of vertices of C on which the color c is used. Since two colors are used on each vertex of C , we have $\sum_{c=1}^6 |V_c| = 10$. On the other hand, V_c is an independent set of a 5-cycle, and thus $|V_c| \leq 2$ for every color c . Furthermore, color 1 only appears in the lists of v_1 and v_2 , which are adjacent in C . It follows that $|V_1| \leq 1$. The situation is similar for color 2, which appears only in the lists of v_1 and v_5 , and also for color 3, which only appears in the lists of v_3 and v_4 . Consequently, $\sum_{c=1}^6 |V_c| \leq 3 \cdot 2 + 3 \cdot 1 = 9$, which is a contradiction. \square

Corollary 4. Consider the gadget (G_1, L_1) , where G_1 consists of a 5-cycle $C = v_1v_2v_3v_4v_5$ and a path v_1xv_3 , with $L_1(v_1) = L_1(v_3) = \{1, \dots, 6\}$, $L_1(v_2) = \{1, 4, 5, 6\}$, $L_1(v_4) = \{3, 4, 5, 6\}$, $L_1(v_5) = \{2, 4, 5, 6\}$, $L_1(x) = \{1, 2, 3, 4\}$ and $L_1(y) = \{1, 2\}$. Then G_1 is L -colorable for every half-list assignment L such that $L(v_1) = L(v_3)$, but G_1 is not $(L_1 : 2)$ -colorable.

Proof. Let L be a half-list assignment for G_1 . First L -color y and x by colors $c_y \in L(y)$ and $c_x \in L(x) \setminus \{c_y\}$, respectively. Since $c_x \neq c_y$ and $L(v_1) = L(v_3)$, there exist sets $L'(v_1) \subseteq L(v_1) \setminus \{c_x\}$ and $L'(v_3) \subseteq L(v_3) \setminus \{c_y\}$ of size two such that $L'(v_1) \neq L'(v_3)$. Let $L'(v_i) = L(v_i)$ for $i \in \{2, 4, 5\}$. Lemma 3 implies that C is L' -colorable, which yields an L -coloring of G .

In an $(L_1 : 2)$ -coloring, the vertex y would have to be assigned $\{1, 2\}$ and x would have to be assigned $\{3, 4\}$, and thus the sets of available colors for v_1 and for v_3 would have to be $\{1, 2, 5, 6\}$ and $\{3, 4, 5, 6\}$, respectively. However, no such $(L_1 : 2)$ coloring of C exists according to Lemma 3. \square

Next we construct auxiliary gadgets, which will be combined with the gadget from Corollary 4 to deal with the case where $L(v_1) \neq L(v_3)$. Let G be a graph, let S be a subset of vertices of G and L a list assignment for G . An L -coloring of S is a coloring of the subgraph of G induced by S . Moreover, if S' is a subset of vertices of G that contains S and φ' is an L -coloring of S' , then φ' extends φ if $\varphi'|_S = \varphi$. Let (G, L_0) be a gadget, let v_1 and v_3 be distinct vertices of G , and let S be a set of vertices of G not containing v_1 and v_3 . The gadget is (v_1, v_3, S) -relaxed if every half-list assignment L satisfies at least one of the following conditions.

- (i) There exists an L -coloring ψ_0 of $\{v_1, v_3\}$ such that every L -coloring of $S \cup \{v_1, v_3\}$ extending ψ_0 extends to an L -coloring of G .
- (ii) $L(v_1) = L(v_3)$ and there exists an L -coloring ψ_0 of S such that every L -coloring of $S \cup \{v_1, v_3\}$ extending ψ_0 extends to an L -coloring of G .

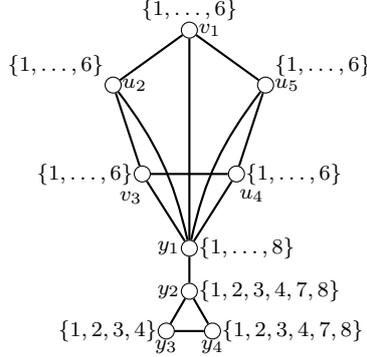


Figure 1: The gadget (G_2, L_2) of Lemma 5

Lemma 5. Consider the gadget (G_2, L_2) presented in Figure 1, where G_2 consists of a 5-cycle $C_2 = v_1u_2v_3u_4u_5$, a vertex y_1 adjacent to all vertices of C_2 , a triangle $y_2y_3y_4$, and an edge y_1y_2 , with $L_2(v) = \{1, \dots, 6\}$ for every $v \in V(C_2)$, $L_2(y_1) = \{1, \dots, 8\}$, $L_2(y_2) = L_2(y_4) = \{1, 2, 3, 4, 7, 8\}$, and $L_2(y_3) = \{1, 2, 3, 4\}$. The gadget is $(v_1, v_3, \{y_4\})$ -relaxed, and $\varphi(y_4) \cap \{7, 8\} \neq \emptyset$ for every $(L_2 : 2)$ -coloring φ of G_2 .

Proof. Let L be a half-list assignment for G_2 . If not all vertices of C_2 have the same list, then choose a color $c \in L(y_1) \setminus L(y_2)$, and observe there exists an L -coloring of $G_2[V(C_2) \cup \{y_1\}]$ such that y_1 has color c . Let ψ_0 be the restriction of this coloring to $\{v_1, v_3\}$. Clearly, every L -coloring of $\{v_1, v_3, y_4\}$ extending ψ_0 extends to an L -coloring of G , and thus (i) holds.

If all the vertices of C_2 have the same list (and hence in particular $L(v_1) = L(v_3)$), then let c be a color in $L(y_1) \setminus L(v_1)$. Observe that there exists an L -coloring of $G_2[\{y_1, y_2, y_3, y_4\}]$ such that y_1 has color c . Let ψ_0 be the restriction of this coloring to y_4 . Again, every L -coloring of $\{v_1, v_3, y_4\}$ extending ψ_0 extends to an L -coloring of G , and thus (ii) holds.

It remains to show that if φ is an $(L_2 : 2)$ -coloring of G_2 then $\varphi(y_4) \cap \{7, 8\} \neq \emptyset$. Suppose, on the contrary, that $\varphi(y_4) \cap \{7, 8\} = \emptyset$. It follows that $\varphi(y_4) \cup \varphi(y_3) = \{1, 2, 3, 4\}$, and hence $\varphi(y_2) = \{7, 8\}$. As a result, $\varphi(y_1) \subseteq \{1, \dots, 6\}$ and, by symmetry, we can assume that $\varphi(y_1) = \{5, 6\}$. This implies that $\varphi(v) \subseteq \{1, 2, 3, 4\}$ for each $v \in V(C_2)$. In particular, $\varphi|_{V(C_2)}$ is a $(4 : 2)$ -coloring of C_2 , which is a contradiction since the 5-cycle C_2 has fractional chromatic number $5/2$. \square

Lemma 6. Consider the gadget (G_3, L_3) , obtained from the gadget (G_2, L_2) of Lemma 5 as follows (see Figure 2 for an illustration of G_3). The graph G_3 consists of G_2 and for $i \in \{1, 2\}$, the vertices $z_{i,1}, \dots, z_{i,7}$; the edges $y_4z_{i,1}$ and $y_4z_{i,2}$; the edge $z_{i,j}z_{i,k}$ for every j and every k such that $1 \leq j < k \leq 4$ and $(j, k) \neq (1, 2)$; the edges of the triangle $z_{i,5}z_{i,6}z_{i,7}$ and the edge $z_{i,4}z_{i,5}$. Let $L_3(v) = L_2(v)$ for $v \in V(G_2)$, and for $i \in \{1, 2\}$ let $L_3(z_{i,1}) = \{1, 2, 3, 6 + i\}$, $L_3(z_{i,2}) = \{4, 5, 6, 6 + i\}$, $L_3(z_{i,3}) = \{1, \dots, 6\}$, $L_3(z_{i,4}) = \{1, \dots, 8\}$, $L_3(z_{i,5}) = L_3(z_{i,7}) = \{1, 2, 3, 4, 7, 8\}$ and $L_3(z_{i,6}) = \{1, 2, 3, 4\}$. The gadget (G_3, L_3) is

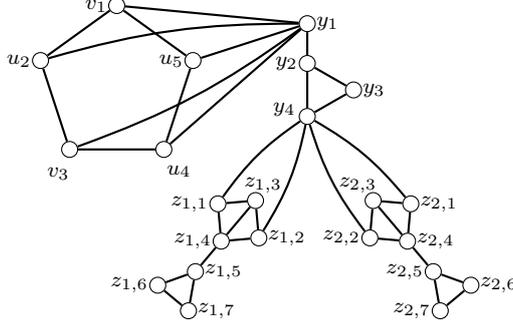


Figure 2: The graph G_3 from Lemma 6

$(v_1, v_3, \{z_{1,7}, z_{2,7}\})$ -relaxed, and $\varphi(z_{1,7}) = \{7, 8\}$ or $\varphi(z_{2,7}) = \{7, 8\}$ for every $(L_3 : 2)$ -coloring φ of G_3 .

Proof. Let L be a half-list assignment for G_3 . The gadget $(G_2, L_3|V(G_2))$ is $(v_1, v_3, \{y_4\})$ -relaxed by lemma 5. Suppose first that (i) holds for the restriction of L to G_2 (with $S = \{y_4\}$), and let ψ_0 be the corresponding L -coloring of $\{v_1, v_3\}$. For $i \in \{1, 2\}$, if $L(z_{i,1}) \cap L(z_{i,2}) \neq \emptyset$, then let c_i be a color in $L(z_{i,1}) \cap L(z_{i,2})$. Otherwise, $|L(z_{i,1}) \cup L(z_{i,2})| = 4 > |L(z_{i,3})|$, and thus we can choose a color $c_i \in (L(z_{i,1}) \cup L(z_{i,2})) \setminus L(z_{i,3})$. Let c be a color in $L(y_4) \setminus \{c_1, c_2\}$. By (i) for G_2 , we know that ψ_0 extends to an L -coloring ψ of G_2 such that $\psi(y_4) = c$. If $L(z_{i,1}) \cap L(z_{i,2}) \neq \emptyset$, then color both $z_{i,1}$ and $z_{i,2}$ by c_i , otherwise color one of them by c_i and the other one by an arbitrary color from its list that is different from c . There are at least two colors in $L(z_{i,4})$ distinct from the colors of $z_{i,1}$ and $z_{i,2}$, choose such a color c'_i so that $L(z_{i,5}) \setminus \{c'_i\} \neq L(z_{i,6})$. Color $z_{i,4}$ by c'_i and extend the coloring to $z_{i,3}$, which is possible by the choice of c_i . Observe that any L -coloring of $z_{i,7}$ extends to an L -coloring of the triangle $z_{i,5}z_{i,6}z_{i,7}$ where the color of $z_{i,5}$ is not c'_i . We conclude that (G_3, L_3) with the half-list assignment L satisfies (i).

Suppose next that (ii) holds for the restriction of L to G_2 (with $S = \{y_4\}$), and let ψ'_0 be the corresponding L -coloring of y_4 . For $i \in \{1, 2\}$, greedily extend ψ'_0 to an L -coloring of $z_{i,1}, \dots, z_{i,7}$ in order, and let ψ_0 be the restriction of the resulting coloring to $\{z_{1,7}, z_{2,7}\}$. Observe that (G_3, L_3) with the half-list assignment L satisfies (ii).

Finally, let φ be an $(L_3 : 2)$ -coloring of G_3 . Lemma 5 implies that $\varphi(y_4) \cap \{7, 8\} \neq \emptyset$. By symmetry, we can assume that $7 \in \varphi(y_4)$. It follows that $\varphi(z_{1,1}) \subset \{1, 2, 3\}$ and $\varphi(z_{1,2}) \subset \{4, 5, 6\}$, and thus $\varphi(z_{1,1}) \cup \varphi(z_{1,2}) \cup \varphi(z_{1,3}) = \{1, \dots, 6\}$. Consequently, $\varphi(z_{1,4}) = \{7, 8\}$, and $\varphi(z_{1,5})$ is a subset of $\{1, 2, 3, 4\}$. This yields that $\varphi(z_{1,5}) \cup \varphi(z_{1,6}) = \{1, 2, 3, 4\}$, and therefore $\varphi(z_{1,7}) = \{7, 8\}$. \square

Lemma 7. Consider the gadget (G_4, L_4) obtained from the gadget (G_3, L_3) of Lemma 6 as follows (see Figure 3 for an illustration of G_4). The graph G_4 consists of G_3 ; the three triangles $w_1w_2w_3$ and $w_{i,1}w_{i,2}w_{i,3}$ for $i \in \{1, 2\}$; and the edges $z_{1,7}w_1$, $z_{2,7}w_1$, $w_3w_{1,1}$ and $w_3w_{2,1}$. Let $L_4(v) = L_3(v)$ for $v \in V(G_3)$, and for $i \in \{1, 2\}$, let $L_4(w_1) = L_4(w_3) = L_4(w_{i,1}) = L_4(w_{i,3}) = \{1, 2, 3, 4, 7, 8\}$

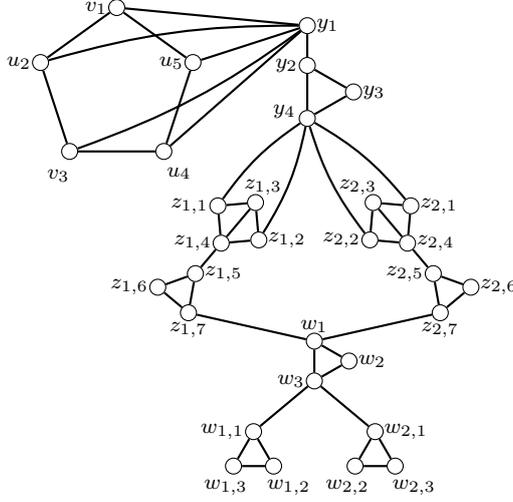


Figure 3: The graph G_4 from Lemma 7.

and $L_4(w_2) = L_4(w_{i,2}) = \{1, 2, 3, 4\}$. The gadget is $(v_1, v_3, \{w_{1,3}, w_{2,3}\})$ -relaxed, and $\varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7, 8\}$ for every $(L_4 : 2)$ -coloring φ of G_4 .

Proof. Let L be a half-list assignment for G_4 . The gadget $(G_3, L_4|V(G_3))$ is $(v_1, v_3, \{z_{1,7}, z_{2,7}\})$ -relaxed by Lemma 5. Suppose first that (i) holds for the restriction of L to G_3 (with $S = \{z_{1,7}, z_{2,7}\}$), and let ψ_0 be the corresponding L -coloring of $\{v_1, v_3\}$. Choose a color $c_1 \in L(z_{1,7})$ so that $L(w_1) \setminus \{c_1\} \neq L(w_2)$. If $c_1 \in L(w_1)$, then choose $c_2 \in L(z_{2,7}) \setminus (L(w_1) \setminus \{c_1\})$, otherwise choose $c_2 \in L(z_{2,7})$ so that $L(w_1) \setminus \{c_2\} \neq L(w_2)$. Choose a color $c_3 \in L(w_3)$ so that $L(w_{i,1}) \setminus \{c_3\} \neq L(w_{i,2})$ for $i \in \{1, 2\}$. By (i) for G_3 , there exists an L -coloring of G_3 extending ψ_0 and assigning c_i to $z_{i,7}$ for each $i \in \{1, 2\}$. Color w_3 by c_3 and observe the L -coloring can be extended to w_1 and w_2 thanks to the choice of c_1 and c_2 . Moreover, the choice of c_3 ensures that for each $i \in \{1, 2\}$, we can color $w_{i,3}$ with any color in $L(w_{i,3})$ and further extend the coloring to $w_{i,1}$ and $w_{i,2}$. We conclude that (G_4, L_4) with the half-list assignment L satisfies (i).

Suppose next that (ii) holds for the restriction of L to G_4 (with $S = \{z_{1,7}, z_{2,7}\}$), and let ψ'_0 be the corresponding L -coloring of $\{z_{1,7}, z_{2,7}\}$. Greedily extend ψ'_0 to an L -coloring of w_1, w_2, w_3 , and $w_{i,1}, w_{i,2}, w_{i,3}$ for $i \in \{1, 2\}$ in order, and let ψ_0 be the restriction of the resulting coloring to $\{w_{1,3}, w_{2,3}\}$. Observe that (G_4, L_4) with the half-list assignment L satisfies (ii).

Finally, let φ be an $(L_4 : 2)$ -coloring of G_4 . By Lemma 6 and by symmetry, we can assume that $\varphi(z_{1,7}) = \{7, 8\}$. Consequently, $\varphi(w_1) \subset \{1, 2, 3, 4\}$, and thus $\varphi(w_1) \cup \varphi(w_2) = \{1, 2, 3, 4\}$, which yields that $\varphi(w_3) = \{7, 8\}$. We conclude analogously that $\varphi(w_{1,3}) = \{7, 8\} = \varphi(w_{2,3})$. \square

We can now combine (G_1, L_1) with (G_4, L_4) to obtain a gadget (G_5, L_5) that is L -colorable from each half-list assignment L , but not $(L_5 : 2)$ -colorable.

Lemma 8. Consider the gadget (G_5, L_5) obtained from the gadgets (G_1, L_1) of Corollary 4 and (G_4, L_4) of Lemma 7 as follows (see Figure 4 for an illustration

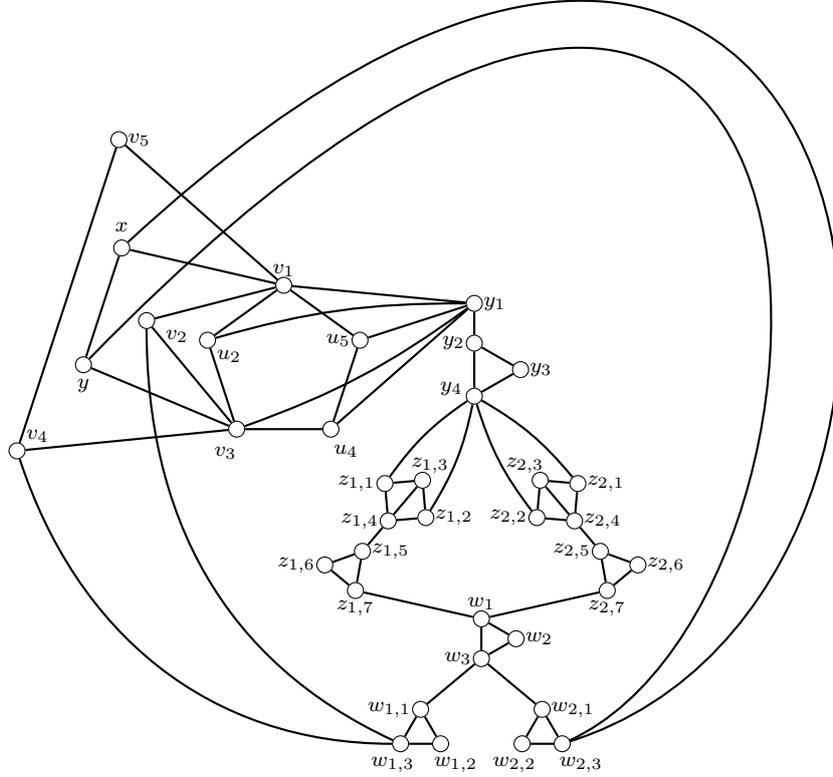


Figure 4: The graph G_5 from Lemma 8.

of G_5). The graph G_5 is obtained from the union of the graphs G_1 and G_4 (intersecting in $\{v_1, v_3\}$) by adding the edges $w_{1,3}v_2$, $w_{1,3}v_4$, $w_{2,3}x$ and $w_{2,3}y$. Let $L_5(v) = L_4(v)$ for $v \in V(G_4)$, $L_5(v) = L_1(v)$ for $v \in V(G_1) \setminus \{v_2, v_4, x, y\}$, and $L_5(v) = L_1(v) \cup \{7, 8\}$ for $v \in \{v_2, v_4, x, y\}$. Then G_5 is L -colorable for every half-list assignment L , but not $(L_5 : 2)$ -colorable.

Proof. Let L be a half-list assignment for G_5 . The gadget $(G_4, L_5|V(G_4))$ is $(v_1, v_3, \{w_{1,3}, w_{2,3}\})$ -relaxed by Lemma 7. Suppose first that (i) holds for the restriction of L to G_4 (with $S = \{w_{1,3}, w_{2,3}\}$), and let ψ_0 be the corresponding L -coloring of $\{v_1, v_3\}$. Greedily extend ψ_0 to an L -coloring ψ of G_1 . Choose $c_1 \in L(w_{1,3}) \setminus \{\psi(v_2), \psi(v_4)\}$ and $c_2 \in L(w_{2,3}) \setminus \{\psi(x), \psi(y)\}$. By (i) for G_4 , there exists an L -coloring of G_4 that extends ψ_0 and assigns to $w_{i,3}$ the color c_i for each $i \in \{1, 2\}$. This yields, together with ψ , an L -coloring of G_5 .

Suppose next that (ii) holds for the restriction of L to G_4 (with $S = \{w_{1,3}, w_{2,3}\}$), and let ψ_0 be the corresponding L -coloring of $\{w_{1,3}, w_{2,3}\}$. Note that $L(v_1) = L(v_3)$ in this case. Corollary 4 implies that G_1 has an L -coloring ψ such that $\psi(v_2) \in L(v_2) \setminus \{\psi_0(w_{1,3})\}$, $\psi(v_4) \in L(v_4) \setminus \{\psi_0(w_{1,3})\}$, $\psi(x) \in L(x) \setminus \{\psi_0(w_{2,3})\}$, and $\psi(y) \in L(y) \setminus \{\psi_0(w_{2,3})\}$. By (ii), the restriction of $\psi \cup \psi_0$ to $\{v_1, v_3, w_{1,3}, w_{2,3}\}$ extends to an L -coloring of G_4 , which together with ψ gives an L -coloring of G_5 .

It remains to show that G_5 is not $(L_5 : 2)$ -colorable. If φ were an $(L_5 : 2)$ -coloring of G_5 , then by Lemma 7 we would have $\varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7, 8\}$, and thus the restriction of φ to G_1 would be an $(L_1 : 2)$ -coloring, thereby contradicting Corollary 4. \square

The final graph. We are now in a position to prove Theorem 2 by simply using a standard construction to ensure uniform lengths of lists.

Proof of Theorem 2. Let G be a graph and L' an assignment of lists of size 8 obtained as follows. Let K be a clique with vertices r_1, \dots, r_4 , and let $L'(r_1) = \dots = L'(r_4) = \{9, \dots, 16\}$. For every $(L' : 2)$ -coloring ψ of K , let G_ψ be a copy of the graph G_5 from the gadget (G_5, L_5) of Lemma 8, and for each vertex $v \in V(G_\psi)$ such that $|L_5(v)| = 2k$ with $k \in \{2, 3\}$, we add the edges vr_1, \dots, vr_{4-k} and let $L'(v) = L_5(v) \cup \bigcup_{i=1}^{4-k} \psi(r_i)$. If G had an $(L' : 2)$ -coloring φ , then letting ψ be the restriction of φ to K , the restriction of φ to G_ψ would be an $(L_5 : 2)$ -coloring of G_5 , thereby contradicting Lemma 8.

Consider now a list assignment L for G such that $|L(v)| = 4$ for every $v \in V(G)$. Let φ be any L -coloring of K . For each $(L' : 2)$ -coloring ψ of K , let L_ψ be the list assignment for G_ψ obtained by, for each $v \in V(G_\psi)$, removing the colors of neighbors in K according to φ , and possibly removing further colors to ensure that $|L_\psi(v)| = |L_5(v)|/2$. By Lemma 8, the graph G_ψ has an L_ψ -coloring. The union of these colorings and φ yields an L -coloring of G . \square

Concluding remarks. It follows from Theorem 2 that for each integer $a \geq 4$, there exists a graph G_a that is a -choosable but not $(2a : 2)$ -choosable—if we have such a graph G_a , taking the disjoint union of $\binom{2(a+1)}{2}$ copies of G_a and adding a vertex adjacent to all other vertices yields G_{a+1} , by an argument analogous to the list uniformization procedure used for the proof of Theorem 2. It is natural to ask whether there exists a graph that is 3-choosable but not $(6 : 2)$ -choosable. We believe this to be the case; in particular, Corollary 4 only requires lists of size at most 6. However, it does not seem easy to construct a gadget that satisfies the properties stated in Lemma 5 without using a vertex with a list of size 8.

References

- [1] N. ALON, Z. TUZA, AND M. VOIGT, *Choosability and fractional chromatic numbers*, Discrete Mathematics, 165–166 (1997), pp. 31 – 38. Graphs and Combinatorics.
- [2] P. ERDŐS, A. L. RUBIN, AND H. TAYLOR, *Choosability in graphs*, Congr. Numer., 26 (1980), pp. 125–157.
- [3] S. GUTNER AND M. TARSI, *Some results on $(a : b)$ -choosability*, Discrete Math., 309 (2009), pp. 2260–2270.
- [4] Z. TUZA AND M. VOIGT, *Every 2-choosable graph is $(2m, m)$ -choosable*, J. Graph Theory, 22 (1996), pp. 245–252.