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A 4-choosable Graph that is Not (8 : 2)-choosable

Zdeněk Dvořák* Xiaolan Hu† Jean-Sébastien Sereni‡

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Abstract: In 1980, Erdős, Rubin and Taylor asked whether for all positive integers a , b , and m , every $(a : b)$ -choosable graph is also $(am : bm)$ -choosable. We provide a negative answer by exhibiting a 4-choosable graph that is not $(8 : 2)$ -choosable.

Key words and phrases: $(am : bm)$ -conjecture, $(a : b)$ -choosable graph, list coloring.

Coloring the vertices of a graph with sets of colors (that is, each vertex is assigned a fixed-size subset of the colors such that adjacent vertices are assigned disjoint sets) is a fundamental notion, which in particular captures fractional colorings. The fractional chromatic number of a graph G can indeed be defined to be the infimum (which actually is a minimum) of the ratios a/b such that, if every vertex of G is replaced by a clique of order b and every edge of G is replaced by a complete bipartite graph between the relevant cliques, then the chromatic number of the obtained graph is at most a .

In their seminal work on list coloring, Erdős, Rubin and Taylor [2] raised several intriguing questions about the list version of set coloring. Before stating them, let us review the relevant definitions.

Set coloring. A function that assigns a set to each vertex of a graph is a *set coloring* if the sets assigned to adjacent vertices are disjoint. For positive integers a and $b \leq a$, an $(a : b)$ -coloring of a graph G is a set coloring with range $\binom{\{1, \dots, a\}}{b}$, i.e., a set coloring that to each vertex of G assigns a b -element subset of $\{1, \dots, a\}$. The concept of $(a : b)$ -coloring is a generalization of the conventional vertex coloring. In fact, an $(a : 1)$ -coloring is exactly an ordinary proper a -coloring.

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A *list assignment* for a graph G is a function L that to each vertex v of G assigns a set $L(v)$ of colors. A set coloring φ of G is an L -set coloring if $\varphi(v) \subseteq L(v)$ for every $v \in V(G)$. For a positive integer b , we say that φ is an $(L : b)$ -coloring of G if φ is an L -set coloring and $|\varphi(v)| = b$ for every $v \in V(G)$. If such an $(L : b)$ -coloring exists, then G is $(L : b)$ -colorable. For an integer $a \geq b$, we say that G is $(a : b)$ -choosable if G is $(L : b)$ -colorable for every list assignment L such that $|L(v)| = a$ for each $v \in V(G)$. We abbreviate $(L : 1)$ -coloring, $(L : 1)$ -colorable, and $(a : 1)$ -choosable to L -coloring, L -colorable, and a -choosable, respectively.

Questions and results. It is straightforward to see that if a graph is $(a : b)$ -colorable, it is also $(am : bm)$ -colorable for every positive integer m : we can simply replace every color in an $(a : b)$ -coloring by m new colors. However, this argument fails in the list coloring setting, leading Erdős, Rubin and Taylor [2] to ask whether every $(a : b)$ -choosable graph is also $(am : bm)$ -choosable whenever $m \geq 1$. A positive answer to this question is sometimes referred to as “the $(am : bm)$ -conjecture”. Using the characterization of 2-choosable graphs [2], Tuza and Voigt [4] provided a positive answer when $a = 2$ and $b = 1$. In the other direction, Gutner and Tarsi [3] demonstrated that if k and m are positive integers and k is odd, then every $(2mk : mk)$ -choosable graph is also $2m$ -choosable.

Formulated differently, the question is to know whether every $(a : b)$ -choosable graph is also $(c : d)$ -choosable whenever $c/d = a/b$ and $c \geq a$. This formulation raises the same question when $c/d > a/b$, which was also asked by Erdős, Rubin and Taylor [2]. About ten years ago, Gutner and Tarsi [3] answered this last question negatively, by studying the k th choice number of a graph for large values of k . More precisely, the k th choice number of a graph G is $\text{ch}_{\cdot k}(G)$, the least integer a for which G is $(a : k)$ -choosable. Their result reads as follows.

Theorem 1 (Gutner & Tarsi, 2009). *Let G be a graph. For every positive real ε , there exists an integer k_0 such that $\text{ch}_{\cdot k}(G) \leq k(\chi(G) + \varepsilon)$ for every $k \geq k_0$.*

As a direct corollary, one deduces that for all integers $m \geq 3$ and $\ell > m$, there exists a graph that is $(a : b)$ -choosable and not $(\ell : 1)$ -choosable with $\frac{a}{b} = m$. (To see this, one can for example apply Theorem 1 with $\varepsilon = 1$ to the disjoint union of a clique of order $m - 1$ and a complete balanced bipartite graph with choice number $\ell + 1$.)

Another related result that should be mentioned here was obtained by Alon, Tuza and Voigt [1]. They proved that for every graph G ,

$$\inf \left\{ \frac{a}{b} \mid G \text{ is } (a : b)\text{-choosable} \right\} = \inf \left\{ \frac{a}{b} \mid G \text{ is } (a : b)\text{-colorable} \right\}.$$

In other words, the fractional choice number of a graph equals its fractional chromatic number.

The purpose of our work is to provide a negative answer to Erdős, Rubin and Taylor’s question when $a = 4$ and $b = 1$.

Theorem 2. *There exists a graph G that is 4-choosable, but not $(8 : 2)$ -choosable.*

We build such a graph by incrementally combining pieces with certain properties. Each piece is defined, and its relevant properties established, in the forthcoming lemmas.

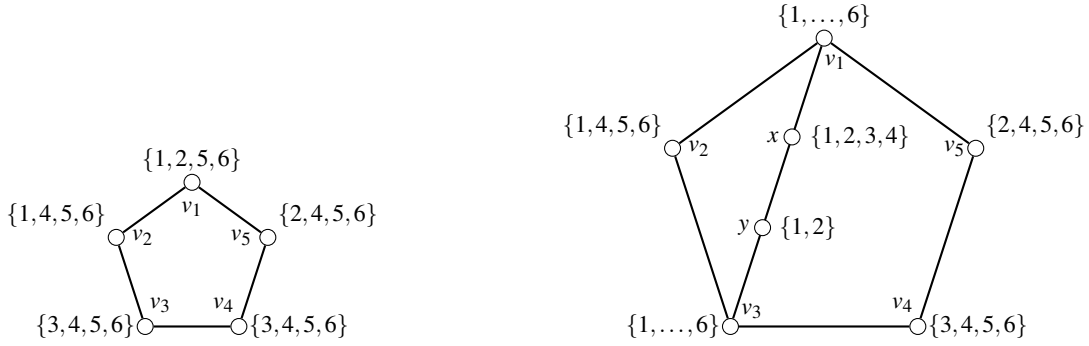


Figure 1: The gadget (C, L_0) of Lemma 3 is depicted on the left; the gadget (G_1, L_1) of Corollary 4 is depicted on the right.

Gadgets and lemmas. A *gadget* is a pair (G, L_0) , where G is a graph and L_0 is an assignment of lists of even size. Given a gadget (G, L_0) , a *half-list assignment* for G is a list assignment L for G such that $|L(v)| = |L_0(v)|/2$ for every $v \in V(G)$. Let us start the construction by a key observation on list colorings of 5-cycles.

Lemma 3. Consider the gadget (C, L_0) , presented in Figure 1, where $C = v_1v_2v_3v_4v_5$ is a 5-cycle, $L_0(v_1) = \{1, 2, 5, 6\}$, $L_0(v_2) = \{1, 4, 5, 6\}$, $L_0(v_3) = L_0(v_4) = \{3, 4, 5, 6\}$ and $L_0(v_5) = \{2, 4, 5, 6\}$. Then C is L -colorable for every half-list assignment L such that $|L(v_1) \cap L(v_3)| \leq 1$, but C is not $(L_0 : 2)$ -colorable.

Proof. The first statement is well known, but let us give the easy proof for completeness: since $|L(v_1) \cap L(v_3)| \leq 1$, we have $|L(v_1) \cup L(v_3)| \geq 3$, and thus $L(v_1)$ or $L(v_3)$ contains a color c_6 not belonging to $L(v_2)$. By symmetry, we can assume that $c_6 \in L(v_1)$. We color v_1 by c_6 and then for $i = 5, 4, 3, 2$ in order, we color v_i by a color $c_i \in L(v_i) \setminus \{c_{i+1}\}$. The resulting coloring is proper—we have $c_2 \neq c_6$, since $c_6 \notin L(v_2)$.

Suppose now that C has an $(L_0 : 2)$ -coloring, and for $c \in \{1, \dots, 6\}$ let V_c be the set of vertices of C on which the color c is used. Since two colors are used on each vertex of C , we have $\sum_{c=1}^6 |V_c| = 10$. On the other hand, V_c is an independent set of a 5-cycle, and thus $|V_c| \leq 2$ for every color c . Furthermore, color 1 only appears in the lists of v_1 and v_2 , which are adjacent in C . It follows that $|V_1| \leq 1$. The situation is similar for color 2, which appears only in the lists of v_1 and v_5 , and also for color 3, which only appears in the lists of v_3 and v_4 . Consequently, $\sum_{c=1}^6 |V_c| \leq 3 \cdot 2 + 3 \cdot 1 = 9$, which is a contradiction. \square

Corollary 4. Consider the gadget (G_1, L_1) , presented in Figure 1, where G_1 consists of a 5-cycle $C = v_1v_2v_3v_4v_5$ and a path v_1xyv_3 , with $L_1(v_1) = L_1(v_3) = \{1, \dots, 6\}$, $L_1(v_2) = \{1, 4, 5, 6\}$, $L_1(v_4) = \{3, 4, 5, 6\}$, $L_1(v_5) = \{2, 4, 5, 6\}$, $L_1(x) = \{1, 2, 3, 4\}$ and $L_1(y) = \{1, 2\}$. Then G_1 is L -colorable for every half-list assignment L such that $L(v_1) = L(v_3)$, but G_1 is not $(L_1 : 2)$ -colorable.

Proof. Let L be a half-list assignment for G_1 . First L -color y and x by colors $c_y \in L(y)$ and $c_x \in L(x) \setminus \{c_y\}$, respectively. Since $c_x \neq c_y$ and $L(v_1) = L(v_3)$, there exist sets $L'(v_1) \subseteq L(v_1) \setminus \{c_x\}$ and $L'(v_3) \subseteq$

$L(v_3) \setminus \{c_y\}$ of size two such that $L'(v_1) \neq L'(v_3)$. Let $L'(v_i) = L(v_i)$ for $i \in \{2, 4, 5\}$. Lemma 3 implies that C is L' -colorable, which yields an L -coloring of G .

In an $(L_1 : 2)$ -coloring, the vertex y would have to be assigned $\{1, 2\}$ and x would have to be assigned $\{3, 4\}$, and thus the sets of available colors for v_1 and for v_3 would have to be $\{1, 2, 5, 6\}$ and $\{3, 4, 5, 6\}$, respectively. However, no such $(L_1 : 2)$ coloring of C exists according to Lemma 3. \square

Next we construct auxiliary gadgets, which will be combined with the gadget from Corollary 4 to deal with the case where $L(v_1) \neq L(v_3)$. Let G be a graph, let S be a subset of vertices of G and L a list assignment for G . An L -coloring of S is a coloring of the subgraph of G induced by S . Moreover, if S' is a subset of vertices of G that contains S and φ' is an L -coloring of S' , then φ' extends φ if $\varphi'|_S = \varphi$. Let (G, L_0) be a gadget, let v_1 and v_3 be distinct vertices of G , and let S be a set of vertices of G not containing v_1 and v_3 . The gadget is (v_1, v_3, S) -relaxed if every half-list assignment L satisfies at least one of the following conditions.

- (i) There exists an L -coloring ψ_0 of $\{v_1, v_3\}$ such that every L -coloring of $S \cup \{v_1, v_3\}$ extending ψ_0 extends to an L -coloring of G .
- (ii) $L(v_1) = L(v_3)$ and there exists an L -coloring ψ_0 of S such that every L -coloring of $S \cup \{v_1, v_3\}$ extending ψ_0 extends to an L -coloring of G .

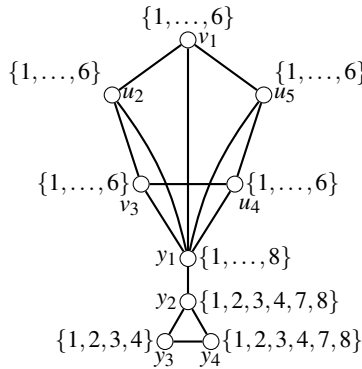


Figure 2: The gadget (G_2, L_2) of Lemma 5

Lemma 5. Consider the gadget (G_2, L_2) presented in Figure 2, where G_2 consists of a 5-cycle $C_2 = v_1u_2v_3u_4u_5$, a vertex y_1 adjacent to all vertices of C_2 , a triangle $y_2y_3y_4$, and an edge y_1y_2 , with $L_2(v) = \{1, \dots, 6\}$ for every $v \in V(C_2)$, $L_2(y_1) = \{1, \dots, 8\}$, $L_2(y_2) = L_2(y_4) = \{1, 2, 3, 4, 7, 8\}$, and $L_2(y_3) = \{1, 2, 3, 4\}$. The gadget is $(v_1, v_3, \{y_4\})$ -relaxed, and $\varphi(y_4) \cap \{7, 8\} \neq \emptyset$ for every $(L_2 : 2)$ -coloring φ of G_2 .

Proof. Let L be a half-list assignment for G_2 . If not all vertices of C_2 have the same list, then choose a color $c \in L(y_1) \setminus L(y_2)$, and observe there exists an L -coloring of $G_2[V(C_2) \cup \{y_1\}]$ such that y_1 has

color c . Let ψ_0 be the restriction of this coloring to $\{v_1, v_3\}$. Clearly, every L -coloring of $\{v_1, v_3, y_4\}$ extending ψ_0 extends to an L -coloring of G_2 , and thus (i) holds.

If all the vertices of C_2 have the same list (and hence in particular $L(v_1) = L(v_3)$), then let c be a color in $L(y_1) \setminus L(v_1)$. Observe that there exists an L -coloring of $G_2[\{y_1, y_2, y_3, y_4\}]$ such that y_1 has color c . Let ψ_0 be the restriction of this coloring to y_4 . Again, every L -coloring of $\{v_1, v_3, y_4\}$ extending ψ_0 extends to an L -coloring of G_2 , and thus (ii) holds.

It remains to show that if φ is an $(L_2 : 2)$ -coloring of G_2 then $\varphi(y_4) \cap \{7, 8\} \neq \emptyset$. Suppose, on the contrary, that $\varphi(y_4) \cap \{7, 8\} = \emptyset$. It follows that $\varphi(y_4) \cup \varphi(y_3) = \{1, 2, 3, 4\}$, and hence $\varphi(y_2) = \{7, 8\}$. As a result, $\varphi(y_1) \subseteq \{1, \dots, 6\}$ and, by symmetry, we can assume that $\varphi(y_1) = \{5, 6\}$. This implies that $\varphi(v) \subseteq \{1, 2, 3, 4\}$ for each $v \in V(C_2)$. In particular, $\varphi|_{V(C_2)}$ is a $(4 : 2)$ -coloring of C_2 , which is a contradiction since the 5-cycle C_2 has fractional chromatic number $5/2$. \square

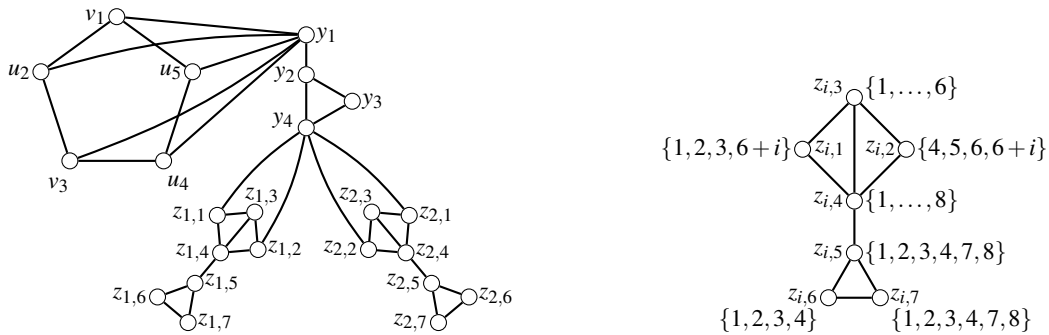


Figure 3: The graph G_3 from Lemma 6 is depicted on the left; the gadget (G_3, L_3) is obtained from (G_2, L_2) by adding to G_3 a disjoint copy of the graph depicted on the right (with the corresponding lists) for each $i \in \{1, 2\}$, and joining y_4 to each of $z_{1,1}, z_{1,2}, z_{2,1}$ and $z_{2,2}$.

Lemma 6. Consider the gadget (G_3, L_3) , obtained from the gadget (G_2, L_2) of Lemma 5 as follows (see Figure 3 for an illustration of G_3). The graph G_3 consists of G_2 and for $i \in \{1, 2\}$, the vertices $z_{i,1}, \dots, z_{i,7}$; the edges $y_4 z_{i,1}$ and $y_4 z_{i,2}$; the edge $z_{i,j} z_{i,k}$ for every j and every k such that $1 \leq j < k \leq 4$ and $(j, k) \neq (1, 2)$; the edges of the triangle $z_{i,5} z_{i,6} z_{i,7}$ and the edge $z_{i,4} z_{i,5}$. Let $L_3(v) = L_2(v)$ for $v \in V(G_2)$, and for $i \in \{1, 2\}$ let $L_3(z_{i,1}) = \{1, 2, 3, 6+i\}$, $L_3(z_{i,2}) = \{4, 5, 6, 6+i\}$, $L_3(z_{i,3}) = \{1, \dots, 6\}$, $L_3(z_{i,4}) = \{1, \dots, 8\}$, $L_3(z_{i,5}) = L_3(z_{i,7}) = \{1, 2, 3, 4, 7, 8\}$ and $L_3(z_{i,6}) = \{1, 2, 3, 4\}$. The gadget (G_3, L_3) is $(v_1, v_3, \{z_{1,7}, z_{2,7}\})$ -relaxed, and $\varphi(z_{1,7}) = \{7, 8\}$ or $\varphi(z_{2,7}) = \{7, 8\}$ for every $(L_3 : 2)$ -coloring φ of G_3 .

Proof. Let L be a half-list assignment for G_3 . The gadget $(G_2, L_3|_{V(G_2)})$ is $(v_1, v_3, \{y_4\})$ -relaxed by Lemma 5. Suppose first that (i) holds for the restriction of L to G_2 (with $S = \{y_4\}$), and let ψ_0 be the corresponding L -coloring of $\{v_1, v_3\}$. For $i \in \{1, 2\}$, if $L(z_{i,1}) \cap L(z_{i,2}) \neq \emptyset$, then let c_i be a color in $L(z_{i,1}) \cap L(z_{i,2})$. Otherwise, $|L(z_{i,1}) \cup L(z_{i,2})| = 4 > |L(z_{i,3})|$, and thus we can choose a color $c_i \in (L(z_{i,1}) \cup L(z_{i,2})) \setminus L(z_{i,3})$. Let c be a color in $L(y_4) \setminus \{c_1, c_2\}$. By (i) for G_2 , we know that ψ_0 extends to an L -coloring ψ of G_2 such that $\psi(y_4) = c$. If $L(z_{i,1}) \cap L(z_{i,2}) \neq \emptyset$, then color both $z_{i,1}$ and $z_{i,2}$ by c_i , otherwise color one of them by c_i and the other one by an arbitrary color from its list that is different

from c . There are at least two colors in $L(z_{i,4})$ distinct from the colors of $z_{i,1}$ and $z_{i,2}$, choose such a color c'_i so that $L(z_{i,5}) \setminus \{c'_i\} \neq L(z_{i,6})$. Color $z_{i,4}$ by c'_i and extend the coloring to $z_{i,3}$, which is possible by the choice of c_i . Observe that any L -coloring of $z_{i,7}$ extends to an L -coloring of the triangle $z_{i,5}z_{i,6}z_{i,7}$ where the color of $z_{i,5}$ is not c'_i . We conclude that (G_3, L_3) with the half-list assignment L satisfies (i).

Suppose next that (ii) holds for the restriction of L to G_2 (with $S = \{y_4\}$), and let ψ'_0 be the corresponding L -coloring of y_4 . For $i \in \{1, 2\}$, greedily extend ψ'_0 to an L -coloring of $z_{i,1}, \dots, z_{i,7}$ in order, and let ψ_0 be the restriction of the resulting coloring to $\{z_{1,7}, z_{2,7}\}$. Observe that (G_3, L_3) with the half-list assignment L satisfies (ii).

Finally, let φ be an $(L_3 : 2)$ -coloring of G_3 . Lemma 5 implies that $\varphi(y_4) \cap \{7, 8\} \neq \emptyset$. By symmetry, we can assume that $7 \in \varphi(y_4)$. It follows that $\varphi(z_{1,1}) \subset \{1, 2, 3\}$ and $\varphi(z_{1,2}) \subset \{4, 5, 6\}$, and thus $\varphi(z_{1,1}) \cup \varphi(z_{1,2}) \cup \varphi(z_{1,3}) = \{1, \dots, 6\}$. Consequently, $\varphi(z_{1,4}) = \{7, 8\}$, and $\varphi(z_{1,5})$ is a subset of $\{1, 2, 3, 4\}$. This yields that $\varphi(z_{1,5}) \cup \varphi(z_{1,6}) = \{1, 2, 3, 4\}$, and therefore $\varphi(z_{1,7}) = \{7, 8\}$. \square

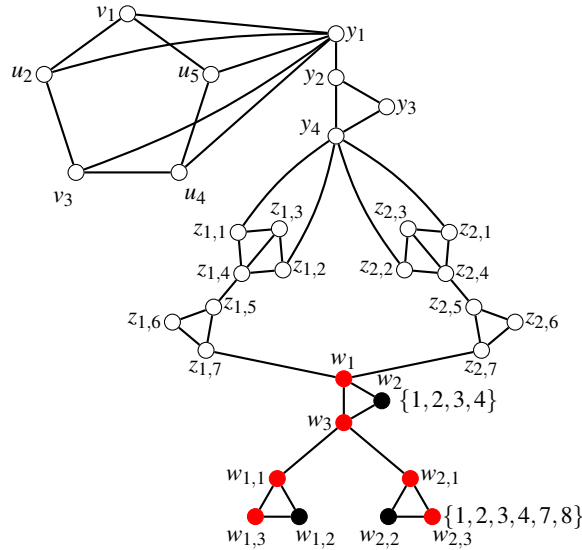


Figure 4: The graph G_4 from Lemma 7: the filled vertices are those added to G_3 to form G_4 ; the red ones have list $\{1, 2, 3, 4, 7, 8\}$ while the black ones have list $\{1, 2, 3, 4\}$.

Lemma 7. Consider the gadget (G_4, L_4) obtained from the gadget (G_3, L_3) of Lemma 6 as follows (see Figure 4 for an illustration of G_4). The graph G_4 consists of G_3 ; the three triangles $w_1w_2w_3$ and $w_{i,1}w_{i,2}w_{i,3}$ for $i \in \{1, 2\}$; and the edges $z_{1,7}w_1$, $z_{2,7}w_1$, $w_3w_{1,1}$ and $w_3w_{2,1}$. Let $L_4(v) = L_3(v)$ for $v \in V(G_3)$, and for $i \in \{1, 2\}$, let $L_4(w_1) = L_4(w_3) = L_4(w_{i,1}) = L_4(w_{i,3}) = \{1, 2, 3, 4, 7, 8\}$ and $L_4(w_2) = L_4(w_{i,2}) = \{1, 2, 3, 4\}$. The gadget is $(v_1, v_3, \{w_{1,3}, w_{2,3}\})$ -relaxed, and $\varphi(w_{1,3}) = \varphi(w_{2,3}) = \{7, 8\}$ for every $(L_4 : 2)$ -coloring φ of G_4 .

Proof. Let L be a half-list assignment for G_4 . The gadget $(G_3, L_4|V(G_3))$ is $(v_1, v_3, \{z_{1,7}, z_{2,7}\})$ -relaxed by Lemma 6. Suppose first that (i) holds for the restriction of L to G_3 (with $S = \{z_{1,7}, z_{2,7}\}$), and

let ψ_0 be the corresponding L -coloring of $\{v_1, v_3\}$. Choose a color $c_1 \in L(z_{1,7})$ so that $L(w_1) \setminus \{c_1\} \neq L(w_2)$. If $c_1 \in L(w_1)$, then choose $c_2 \in L(z_{2,7}) \setminus (L(w_1) \setminus \{c_1\})$, otherwise choose $c_2 \in L(z_{2,7})$ so that $L(w_1) \setminus \{c_2\} \neq L(w_2)$. Choose a color $c_3 \in L(w_3)$ so that $L(w_{i,1}) \setminus \{c_3\} \neq L(w_{i,2})$ for $i \in \{1, 2\}$. By (i) for G_3 , there exists an L -coloring of G_3 extending ψ_0 and assigning c_i to $z_{i,7}$ for each $i \in \{1, 2\}$. Color w_3 by c_3 and observe that the L -coloring can be extended to w_1 and w_2 thanks to the choice of c_1 and c_2 . Moreover, the choice of c_3 ensures that for each $i \in \{1, 2\}$, we can color $w_{i,3}$ with any color in $L(w_{i,3})$ and further extend the coloring to $w_{i,1}$ and $w_{i,2}$. We conclude that (G_4, L_4) with the half-list assignment L satisfies (i).

Suppose next that (ii) holds for the restriction of L to G_4 (with $S = \{z_{1,7}, z_{2,7}\}$), and let ψ'_0 be the corresponding L -coloring of $\{z_{1,7}, z_{2,7}\}$. Greedily extend ψ'_0 to an L -coloring of w_1, w_2, w_3 , and $w_{i,1}, w_{i,2}, w_{i,3}$ for $i \in \{1, 2\}$ in order, and let ψ_0 be the restriction of the resulting coloring to $\{w_{1,3}, w_{2,3}\}$. Observe that (G_4, L_4) with the half-list assignment L satisfies (ii).

Finally, let φ be an $(L_4 : 2)$ -coloring of G_4 . By Lemma 6 and by symmetry, we can assume that $\varphi(z_{1,7}) = \{7, 8\}$. Consequently, $\varphi(w_1) \subset \{1, 2, 3, 4\}$, and thus $\varphi(w_1) \cup \varphi(w_2) = \{1, 2, 3, 4\}$, which yields that $\varphi(w_3) = \{7, 8\}$. We conclude analogously that $\varphi(w_{1,3}) = \{7, 8\} = \varphi(w_{2,3})$. \square

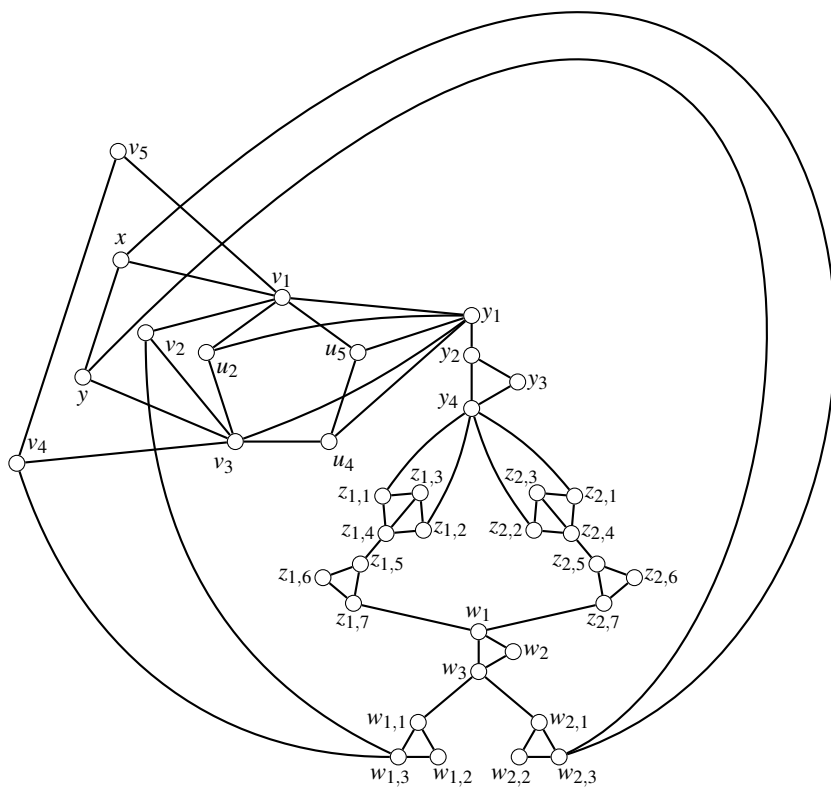


Figure 5: The graph G_5 from Lemma 8.

We can now combine (G_1, L_1) with (G_4, L_4) to obtain a gadget (G_5, L_5) that is L -colorable from each half-list assignment L , but not $(L_5 : 2)$ -colorable.

Lemma 8. *Consider the gadget (G_5, L_5) obtained from the gadgets (G_1, L_1) of Corollary 4 and (G_4, L_4) of Lemma 7 as follows (see Figure 5 for an illustration of G_5). The graph G_5 is obtained from the union of the graphs G_1 and G_4 (intersecting in $\{v_1, v_3\}$) by adding the edges $w_{1,3}v_2$, $w_{1,3}v_4$, $w_{2,3}x$ and $w_{2,3}y$. Let $L_5(v) = L_4(v)$ for $v \in V(G_4)$, $L_5(v) = L_1(v)$ for $v \in V(G_1) \setminus \{v_2, v_4, x, y\}$, and $L_5(v) = L_1(v) \cup \{7, 8\}$ for $v \in \{v_2, v_4, x, y\}$. Then G_5 is L -colorable for every half-list assignment L , but not $(L_5 : 2)$ -colorable.*

Proof. Let L be a half-list assignment for G_5 . The gadget $(G_4, L_5|_{V(G_4)})$ is $(v_1, v_3, \{w_{1,3}, w_{2,3}\})$ -relaxed by Lemma 7. Suppose first that (i) holds for the restriction of L to G_4 (with $S = \{w_{1,3}, w_{2,3}\}$), and let ψ_0 be the corresponding L -coloring of $\{v_1, v_3\}$. Greedily extend ψ_0 to an L -coloring ψ of G_1 . Choose $c_1 \in L(w_{1,3}) \setminus \{\psi(v_2), \psi(v_4)\}$ and $c_2 \in L(w_{2,3}) \setminus \{\psi(x), \psi(y)\}$. By (i) for G_4 , there exists an L -coloring of G_4 that extends ψ_0 and assigns to $w_{i,3}$ the color c_i for each $i \in \{1, 2\}$. This yields, together with ψ , an L -coloring of G_5 .

Suppose next that (ii) holds for the restriction of L to G_4 (with $S = \{w_{1,3}, w_{2,3}\}$), and let ψ_0 be the corresponding L -coloring of $\{w_{1,3}, w_{2,3}\}$. Note that $L(v_1) = L(v_3)$ in this case. Corollary 4 implies that G_1 has an L -coloring ψ such that $\psi(v_2) \in L(v_2) \setminus \{\psi_0(w_{1,3})\}$, $\psi(v_4) \in L(v_4) \setminus \{\psi_0(w_{1,3})\}$, $\psi(x) \in L(x) \setminus \{\psi_0(w_{2,3})\}$, and $\psi(y) \in L(y) \setminus \{\psi_0(w_{2,3})\}$. By (ii), the restriction of $\psi \cup \psi_0$ to $\{v_1, v_3, w_{1,3}, w_{2,3}\}$ extends to an L -coloring of G_4 , which together with ψ gives an L -coloring of G_5 .

It remains to show that G_5 is not $(L_5 : 2)$ -colorable. If ϕ were an $(L_5 : 2)$ -coloring of G_5 , then by Lemma 7 we would have $\phi(w_{1,3}) = \phi(w_{2,3}) = \{7, 8\}$, and thus the restriction of ϕ to G_1 would be an $(L_1 : 2)$ -coloring, thereby contradicting Corollary 4. \square

The final graph. We are now in a position to prove Theorem 2 by simply using a standard construction to ensure uniform lengths of lists.

Proof of Theorem 2. Let G be a graph and L' an assignment of lists of size 8 obtained as follows. Let K be a clique with vertices r_1, \dots, r_4 , and let $L'(r_1) = \dots = L'(r_4) = \{9, \dots, 16\}$. For every $(L' : 2)$ -coloring ψ of K , let G_ψ be a copy of the graph G_5 from the gadget (G_5, L_5) of Lemma 8, and for each vertex $v \in V(G_\psi)$ such that $|L_5(v)| = 2k$ with $k \in \{2, 3\}$, we add the edges vr_1, \dots, vr_{4-k} and let $L'(v) = L_5(v) \cup \bigcup_{i=1}^{4-k} \psi(r_i)$. If G had an $(L' : 2)$ -coloring ϕ , then letting ψ be the restriction of ϕ to K , the restriction of ϕ to G_ψ would be an $(L_5 : 2)$ -coloring of G_5 , thereby contradicting Lemma 8.

Consider now a list assignment L for G such that $|L(v)| = 4$ for every $v \in V(G)$. Let ϕ be any L -coloring of K . For each $(L' : 2)$ -coloring ψ of K , let L_ψ be the list assignment for G_ψ obtained by, for each $v \in V(G_\psi)$, removing the colors of neighbors in K according to ϕ , and possibly removing further colors to ensure that $|L_\psi(v)| = |L_5(v)|/2$. By Lemma 8, the graph G_ψ has an L_ψ -coloring. The union of these colorings and ϕ yields an L -coloring of G . \square

Concluding remarks. It follows from Theorem 2 that for each integer $a \geq 4$, there exists a graph G_a that is a -choosable but not $(2a : 2)$ -choosable—if we have such a graph G_a , taking the disjoint union of $\binom{2(a+1)}{2}$ copies of G_a and adding a vertex adjacent to all other vertices yields G_{a+1} , by an argument analogous to the list uniformization procedure used for the proof of Theorem 2. It is natural to ask

whether there exists a graph that is 3-choosable but not $(6 : 2)$ -choosable. We believe this to be the case; in particular, Corollary 4 only requires lists of size at most 6. However, it does not seem easy to construct a gadget that satisfies the properties stated in Lemma 5 without using a vertex with a list of size 8.

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AUTHORS

Zdeněk Dvořák
 Associate Professor
 Computer Science Institute (CSI) of Charles University
 Prague, Czech Republic
rakdver@iuuk.mff.cuni.cz
<https://iuuk.mff.cuni.cz/~rakdver/>

Xiaolan Hu
 Assistant Professor
 School of Mathematics and Statistics & Hubei Key Laboratory of Mathematical Sciences
 Central China Normal University, Wuhan, PR China.
xlhu@mail.ccnu.edu.cn
<http://maths.ccnu.edu.cn/info/1042/15221.htm>

Jean-Sébastien Sereni
 Directeur de recherche
 Centre National de la Recherche Scientifique
 CSTB (ICube), Strasbourg, France.
sereni@kam.mff.cuni.cz
<http://lbgi.fr/~sereni/index.html>