

ON THE QUANTUM TEICHMÜLLER INVARIANTS OF FIBRED CUSPED 3-MANIFOLDS

STÉPHANE BASEILHAC¹, RICCARDO BENEDETTI²

¹ Institut Montpellierain Alexander Grothendieck, CNRS, Université de Montpellier
(sbaseilh@univ-montp2.fr)

² Dipartimento di Matematica, Università di Pisa, Largo Bruno Pontecorvo 5, 56127 Pisa, Italy
(benedett@dm.unipi.it)

ABSTRACT. We show that the reduced quantum hyperbolic invariants of pseudo-Anosov diffeomorphisms of punctured surfaces are intertwiners of local representations of the quantum Teichmüller spaces. We characterize them as the only intertwiners that satisfy certain natural cut-and-paste operations of topological quantum field theories and such that their traces define invariants of mapping tori.

CONTENTS

1.	Introduction	1
2.	Topological-combinatorial support	6
3.	Background results	7
3.1.	Classical hyperbolic geometry	7
3.2.	Quantum hyperbolic geometry	9
3.3.	Representations of the quantum Teichmüller spaces	13
3.4.	Intertwiners of local representations	16
4.	Proofs	17
	References	25

1. INTRODUCTION

Let M be a *fibred cusped 3-manifold*, that is, an oriented non compact finite volume complete hyperbolic 3-manifold which fibres over the circle S^1 . In this paper we describe the relationships between the quantum hyperbolic invariants of M (QHI) ([4, 5, 6]), and the intertwiners of finite dimensional representations of the quantum Teichmüller space of S ([7, 8], [11]), where S is a fibre of some fibration of M over S^1 . Recall that M is the interior of a compact manifold \bar{M} with boundary made by tori, and that S is the interior of a compact surface with boundary \bar{S} , properly embedded in \bar{M} , so that $\bar{S} \cap \bar{M}$ is a family of ‘meridian’ curves on $\partial\bar{M}$.

The existence of such relationships has been expected for a long time, and their precise formulation often thought as depending only on the solution of a handful of technical problems. Let us recall the main arguments underlying this opinion. Let q be a primitive N -th root of unity, where N is odd. Then:

- The quantum Teichmüller space \mathfrak{T}_S^q of S consists of the set of Chekhov-Fock algebras \mathfrak{T}_λ^q , considered up to a suitable equivalence relation (see Section 3.3), where λ varies among the ideal triangulations of S . A representation ρ of \mathfrak{T}_S^q is a suitable family of ‘compatible’ representations $\rho = \{\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_\lambda)\}_\lambda$. The irreducible representations of \mathfrak{T}_S^q were classified up to isomorphism in [7], and so-called *local representations* were introduced and classified in [8].

- The local representations are reducible, but more suited to perform the usual cut-and-paste operations of topological quantum field theories. In this framework, it is natural to restrict to a distinguished family of intertwiners, satisfying some functorial properties with respect to the inclusion of triangulated subsurfaces. Theorem 20 of [8] proposed such a family, and argued that for every couple (ρ, ρ') of isomorphic local representations of \mathfrak{T}_S^q and every couple (λ, λ') of ideal triangulations of S , there is a unique projective class of intertwiners in the family which are defined from the representation space V_λ of ρ_λ to the representation space $V_{\lambda'}$ of $\rho_{\lambda'}$.

Let us call the elements of these classes *QT intertwiners*.

- The diffeomorphisms of S act on the local representations of \mathfrak{T}_S^q . Denote by ϕ the monodromy of some fibration of M over S^1 . If a local representation ρ is isomorphic to its pull-back $\phi^*\rho$ under ϕ (an instance is canonically associated to the hyperbolic holonomy of M [7, 8]), then the trace of a suitably normalized QT intertwiner which relates ρ_λ to $\phi^*\rho_{\lambda'}$, $\lambda' := \phi(\lambda)$, should be an invariant of M (possibly depending on ϕ) and the isomorphism class of ρ .
- The QT intertwiners can be constructed by composing elementary intertwiners, associated to the diagonal exchanges in a sequence relating two ideal triangulations of S , and these were identified with the Kashaev $6j$ -symbols in [2].
- The QHI of M can be defined by *QH state sums* over “layered” triangulations of M , that is, roughly, 3-dimensional ideal triangulations that realize sequences of diagonal exchanges relating the surface ideal triangulations λ and $\phi(\lambda)$, in such a way that every diagonal exchange is associated to a tetrahedron of the triangulation. The main building blocks of the QH state sums are tensors called *matrix dilogarithms*, carried by the (suitably decorated) tetrahedra, which were derived in [3] from the Kashaev $6j$ -symbols.

In conclusion, the QHI of M would coincide with the traces of QT intertwiners of local representations of \mathfrak{T}_S^q (at least when M is equipped with the hyperbolic holonomy). Only a suitable choice of normalization of the QT intertwiners and an explicit correspondence between formulas would be missing.

Another fact pointing in the same direction is that both the isomorphism classes of local representations of the quantum Teichmüller spaces and the QHI depend on similar geometric structures. Every local representation ρ of \mathfrak{T}_S^q defines an *augmented character* of $\pi_1(S)$ in $PSL(2, \mathbb{C})$ [8], called its *holonomy*, given by a system of (*exponential*) *shear-bend* coordinates on every ideal triangulation λ of S . Moreover ρ has a *load* h_ρ , defined as a N -th root of the product of the shear-bend coordinates on any ideal triangulation λ of S , and which does not depend on the choice of λ . The holonomy and the load classify ρ up to isomorphism. On another hand, the QHI of any cusped 3-manifold M depend on a choice of augmented character τ of $\pi_1(M)$ in $PSL(2, \mathbb{C})$, lying in the irreducible component of the variety of augmented characters containing the hyperbolic holonomy. If S is a fibre of a fibration of M over S^1 , τ can be recovered from its restriction to $\pi_1(S)$.

However, other facts suggested that the relations between the QT intertwiners and the QHI would be subtler than expected:

- The matrix dilogarithms are obtained from the Kashaev $6j$ -symbols by rewriting them in a non trivial way as tensors depending on geometrically meaningful parameters (which are N -th roots of hyperbolic shape parameters), *but also* by applying a *symmetrization* procedure. This procedure is necessary to get the invariance of the QH state sums over any cusped 3-manifold. It involves several additional choices, and eventually forces to multiply the QH state sums by a normalization factor. It implies furthermore that the QHI must actually depend not only on the choice of an augmented character τ of $\pi_1(M)$ in $PSL(2, \mathbb{C})$, *but also* on some cohomological classes, called *weights*, which cannot be recovered from the holonomy of local representations of the quantum Teichmüller spaces. The weights gauge out the additional choices. Surprisingly, both the normalization factor of the QH state sums and the weights did not appear in quantum Teichmüller theory.

- An issue about the uniqueness of the projective classes of QT intertwiners has been fixed recently. By a careful analysis of the definition of local representations, and adapting the construction of the QT intertwiners from [8], for every couple $(\rho_\lambda, \rho_{\lambda'})$ as above, it is constructed in [11] a unique, minimal set $\mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ of projective classes of QT intertwiners which relate ρ_λ to $\rho_{\lambda'}$, and a free transitive action

$$(1) \quad \psi_{\lambda\lambda'}^{\rho\rho'} : H_1(S; \mathbb{Z}/N\mathbb{Z}) \times \mathcal{L}_{\lambda\lambda'}^{\rho\rho'} \rightarrow \mathcal{L}_{\lambda\lambda'}^{\rho\rho'}.$$

In particular, the set $\mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ is far from being reduced to one point.

The meaning of the normalization factor of the QH state sums has been better understood in [6]. In that paper we showed that for every cusped 3-manifold M , the QH state sums of M *without the normalization factor* define finer invariants $\mathcal{H}_N^{\text{red}}(M, \mathfrak{r}, \kappa; \mathfrak{s})$, called *reduced QH invariants*, well-defined up to multiplication by $4N$ -th roots of unity. They depend on an augmented $PSL(2, \mathbb{C})$ -character \mathfrak{r} of $\pi_1(M)$, a class $\kappa \in H^1(\partial\bar{M}; \mathbb{C}^*)$, $\mathbb{C}^* = \mathbb{C} \setminus \{0\}$ being the multiplicative group, such that κ^N coincides up to sign with the restriction of \mathfrak{r} to $\pi_1(\partial M)$, and an additional structure \mathfrak{s} on M , called a *non ambiguous structure*.

When M fibres over S^1 , a natural choice of \mathfrak{s} is a fibration of M , say with monodromy ϕ . Let us denote by M_ϕ the realization of M as the mapping torus of ϕ , equipped with the corresponding fibration, and the reduced QH invariant by $\mathcal{H}_N^{\text{red}}(M_\phi, \mathfrak{r}, \kappa)$. The latter depends directly on the monodromy ϕ , and hence is apparently more suited to investigate the relationships with the QT intertwiners. Note, however, that the class κ is residual of the cohomological weights involved in the definition of the unreduced QHI, and so the above considerations about the weights still apply.

Denote by S a fibre of M_ϕ . Rather than considering M_ϕ one can consider as well the mapping cylinder C_ϕ of ϕ . In this case, instead of a scalar invariant there is a (*reduced*) *QH-operator*

$$\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w}) \in \text{End}((\mathbb{C}^N)^{\otimes 2m}), \quad m := -\chi(S),$$

which is such that

$$(2) \quad \mathcal{H}_N^{\text{red}}(M_\phi, \mathfrak{r}, \kappa) = \text{Trace} \left(\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w}) \right).$$

More precisely, in the spirit of topological quantum field theories, for every odd $N \geq 3$ the QHI define a contravariant functor from the category of (decorated) $(2+1)$ -cobordisms to the category of vector spaces. In particular, it associates vector spaces U_λ and $U_{\lambda'}$ to the triangulated surfaces (S, λ) and (S, λ') respectively, where $\lambda' := \phi(\lambda)$, and a QH-operator $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w}) : U_{\lambda'} \rightarrow U_\lambda$ to the cylinder C_ϕ . The diffeomorphism ϕ gives an identification between U_λ and $U_{\lambda'}$, so one can consider $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})$ as an element of $\text{End}(U_\lambda)$. Moreover, the construction of the functor provides a natural identification of U_λ with $(\mathbb{C}^N)^{\otimes 2m}$.

The operator $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})$ is defined by means of QH state sums over layered *QH-triangulations* $(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})$ of C_ϕ . Here, by ‘layered QH-triangulation’ we mean that: T_{C_ϕ} is a triangulation of C_ϕ that produces a layered triangulation of M_ϕ under the quotient map $C_\phi \rightarrow M_\phi$; \mathbf{w} is a suitable system of N -th roots of shape parameters, solving the Thurston edge equations associated to the triangulation of M_ϕ induced by T_{C_ϕ} , and which actually encodes both \mathfrak{r} and κ ; finally $\tilde{\mathbf{b}}$ is a way of ordering the elements of \mathbf{w} in each tetrahedron of T_{C_ϕ} , called *weak-branching*. The complete definition of $(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})$ is given in Section 3.2.

Our results describe the relationships between the QH-operators $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})$ and the set $\mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ of QT intertwiners. First note that these are assigned to (λ, λ') in a *covariant* way, as they go from the representation space V_λ of ρ_λ to the representation space $V_{\lambda'}$ of $\rho_{\lambda'}$. For every finite dimensional vector space V , denote by V' its dual space. The natural way to convert a QH operator $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})$ into a covariant one (preserving the value of its trace) is to consider its *dual*, or *transposed operator*

$$\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})^T \in \text{End}(U_{\lambda'}).$$

Using the definitions of U'_λ in the QHI and of V_λ in the theory of local representation of \mathfrak{T}_λ^q one gets natural identifications $U'_\lambda \cong ((\mathbb{C}^N)')^{\otimes 2m} \cong V_\lambda$. Under all these identifications

$$\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})^T \in \text{End} \left(((\mathbb{C}^N)')^{\otimes 2m} \right)$$

is a QT intertwiner. Clearly, the matrix elements of these tensors in the respective canonical bases of $((\mathbb{C}^N)')^{\otimes 2m}$ and $(\mathbb{C}^N)^{\otimes 2m}$ are related by

$$(3) \quad (\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})^T)_{j_1 \dots j_{2m}}^{i_1 \dots i_{2m}} = \mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})_{i_1 \dots i_{2m}}^{j_1 \dots j_{2m}}.$$

We can now state our results precisely. Let S be a punctured surface of negative Euler characteristic, ϕ a pseudo-Anosov diffeomorphism of S , M_ϕ the mapping torus of ϕ equipped with its fibration over S^1 with monodromy ϕ , and consider triples $(M_\phi, \mathfrak{r}, \kappa)$ and $(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})$ as above. Denote by λ, λ' the ideal triangulations of S given by the restriction to $S \times \{0\}$ and $\phi(S) \times \{1\}$ of the (layered) triangulation T_{C_ϕ} .

Theorem 1.1. (1) *The QH-triangulation $(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})$ determines representations ρ_λ and $\rho_{\lambda'}$ of \mathfrak{T}_λ^q and $\mathfrak{T}_{\lambda'}^q$, respectively, belonging to a local representation ρ of \mathfrak{T}_S^q , such that ρ_λ is isomorphic to $\phi^* \rho_{\lambda'}$, and acting on $V_\lambda = V_{\lambda'} = ((\mathbb{C}^N)')^{\otimes 2m}$. Moreover the transposed operator $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})^T$, considered as an element of $\text{Hom}(V_\lambda, V_{\lambda'})$, is a QT intertwiner which intertwins the representations ρ_λ and $\rho_{\lambda'}$.*

(2) *For any other choice of weak branching $\tilde{\mathbf{b}}'$, the operator $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}', \mathbf{w})^T$ intertwins local representations canonically isomorphic to $\rho_\lambda, \rho_{\lambda'}$ respectively.*

We call $\mathcal{H}_{\lambda, \lambda'}^\rho := \mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})^T$ a QHI intertwiner.

Denote by $X_0(M)$ the (unique) irreducible component of the variety of augmented $PSL(2, \mathbb{C})$ -characters of M that contains the discrete faithful holonomy \mathfrak{r}_h . Denote by $X(S)$ the variety of augmented characters of S , and by $i^*: X_0(M) \rightarrow X(S)$ the restriction map. Recall the residual weight $\kappa \in H^1(\partial \bar{M}_\phi; \mathbb{C}^*)$ involved in the definition of the reduced QHI.

Theorem 1.2. (1) *There is a neighborhood of $i^*(\mathfrak{r}_h)$ in $i^*(X_0(M)) \subset X(S)$ such that, for any isomorphism class of local representations of \mathfrak{T}_S^q whose holonomy lies in this neighborhood, there is a representative ρ of the class, representations $\rho_\lambda, \rho_{\lambda'}$ belonging to ρ , and a QHI intertwiner $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{\mathbf{b}}, \mathbf{w})^T$ intertwining ρ_λ and $\rho_{\lambda'}$ as above. The load of ρ is determined by the values of the weight κ at the meridian curves that form $\bar{S} \cap \bar{M}$.*

(2) *The set of QHI intertwiners $\mathcal{H}_{\lambda, \lambda'}^\rho$ is the subset of QT intertwiners which intertwine the representation ρ_λ to $\rho_{\lambda'}$, and whose traces are well defined invariants of the triples $(M_\phi, \mathfrak{r}, \kappa)$ such that the restriction of \mathfrak{r} to $\pi_1(S)$ is the holonomy of ρ .*

Let us make a few comments on Theorem 1.1 and 1.2.

- A key point of Theorem 1.1 (1) is to realize the transposed of the matrix dilogarithms as intertwiners between local representations of the two Chekhov-Fock algebras that one can associate to an ideal square. To this aim we consider an operator theoretic formulation of the matrix dilogarithms. As these are related to the Kashaev $6j$ -symbols (in a non trivial way), one could alternatively develop a proof based on the result of Bai [2] relating the Kashaev $6j$ -symbols with intertwiners as above. However, it is not immediate. Bai's result deals with local representations up to isomorphism, and does not provide explicit relations between actual representatives. Also the relation between the Kashaev $6j$ -symbols and the intertwiners of local representations is expressed in abstract terms, using the cyclic representations of the Weyl algebra, and not in terms of the eventual geometrically relevant parameters, the q -shape parameters (see Section 3.2).

- Theorem 1.2 (1) holds true more generally for any isomorphism class of local representations whose holonomy lies in $i^*(B)$, where B is a determined Zariski open subset of the *eigenvalue* subvariety $E(M)$ of $X_0(M)$, introduced in [12]. Let us note here that $E(M) = X_0(M)$ if M has a single cusp, and in general, $\mathfrak{r}_h \in E(M)$, and $\dim_{\mathbb{C}} E(M)$ equals the number of cusps of M . For simplicity, in this paper we restrict to characters in a neighborhood of $i^*(\mathfrak{r}_h)$ in $i^*(X_0(M))$. The general case is easily deduced from the results of [5].

• We can say that *the invariance property of the QHI selects preferred elements in the set of all traces of QT intertwiners* (which has no a priori basepoints). Note that, the action (1) being transitive, it does not stabilize the set of QHI intertwiners. As well as the reduced QHI of M_ϕ give rise to the QT intertwiners $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T$, with the mild ambiguity by $4N$ -th roots of unity factors, the unreduced QHI of M give rise to the QT intertwiners $\mathcal{H}_N(T_{C_\phi}, \tilde{b}, \mathbf{w})^T$, in the same projective classes as the “reduced” ones. The unreduced QHI of M , hence the associated QT intertwiners, do not depend on the choice of a fibration of M . However they depend on the full set of cohomological weights, which dominates the classes κ . The ratio unreduced/reduced QHI is a simpler invariant called *symmetry defect*, which can be used to study the dependence of the reduced QHI with respect to the fibration of M_ϕ ([6]). Finally, the ambiguity of the invariants up to multiplication by $4N$ -th roots of unity may not be sharp (see [5] for some improvements in the case of the unreduced QHI).

The theorems above have the following consequences.

Recall that the reduced QH invariants satisfy the identity $\mathcal{H}_N^{\text{red}}(M_\phi, \mathbf{r}, \kappa) = \text{Trace}(\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w}))$, and that M_ϕ is the interior of a compact manifold \bar{M}_ϕ with boundary made by tori. Let us call *longitude* any simple closed curve in $\partial\bar{M}_\phi$ intersecting a fibre of \bar{M}_ϕ in exactly one point. We have:

Corollary 1.3. *The reduced QH invariants $\mathcal{H}_N^{\text{red}}(M_\phi, \mathbf{r}, \kappa)$ do not depend on the values of the weight κ on longitudes.*

We can also use the QH-operators $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})$ to build finer invariants, associated to any irreducible representation of \mathfrak{T}_S^q . Namely, recall that every reducible representation is canonically the direct sum of its *isotypical* components (the maximal direct sums of isomorphic irreducible summands). Every intertwiner fixes globally each isotypical component. Then, we consider the *isotypical intertwiners*

$$L_{\rho_\lambda(\mu)}^\phi := \mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})_{|\rho_\lambda(\mu)}^T$$

obtained by restriction to the isotypical components $\rho_\lambda(\mu)$ of ρ_λ , associated to irreducible representations μ of \mathfrak{T}_λ^q . Any irreducible representation of \mathfrak{T}_λ^q can be embedded as a summand of some local representation ([13]), and so has a corresponding isotypical component.

We have:

Corollary 1.4. (1) *The trace of $L_{\rho_\lambda(\mu)}^\phi$ is an invariant of $(M_\phi, \mathbf{r}, \kappa)$ and μ , well-defined up to multiplication by $4N$ -th roots of unity. It depends on the isotopy class of ϕ and satisfies*

$$\mathcal{H}_N^{\text{red}}(M_\phi, \mathbf{r}, \kappa) = \sum_{\rho_\lambda(\mu) \subset \rho_\lambda} \text{Trace}\left(L_{\rho_\lambda(\mu)}^\phi\right).$$

(2) *The invariants $\text{Trace}(L_{\rho_\lambda(\mu)}^\phi)$ do not depend on the values of the weight κ on longitudes.*

In perspective, another application of Theorem 1.1 would be to study the representations of the Kauffman bracket skein algebras defined by means of the QHI intertwiners, by using the results of [13].

The background material is recalled in Section 2 and 3. The proofs are in Section 4.

Remark about the parameter q . In this paper we denote by q an arbitrary primitive N -th root of unity, where $N \geq 3$ is odd. Our orientation conventions used to define the quantum Teichmüller space, in the relations (14) and (16), imply that q corresponds to q^{-1} in some papers about quantum Teichmüller theory, eg. in [7] or [11]. Our choice of q is motivated by Thurston’s relations between shape parameters in hyperbolic geometry; in the quantum case, this choice gives the most natural form to the tetrahedral and edge relations between q -shape parameters (see Section 3.2).

Acknowledgement. We thank the referee for the several suggestions that have helped us to really improve the text.

2. TOPOLOGICAL-COMBINATORIAL SUPPORT

We fix a compact closed oriented smooth surface S_0 of genus g , and a subset $P = \{p_1, \dots, p_r\} \subset S_0$ of $r \geq 1$ marked points. We denote by S the punctured surface $S_0 \setminus P$. We assume that

$$m := -\chi(S) = (2g - 2) + r > 0.$$

A diffeomorphism $\phi_0 : S_0 \rightarrow S_0$ such that $\phi_0(p_j) = p_j$ for every j induces a diffeomorphism $\phi : S \rightarrow S$. Consider the cylinder $C_0 := S_0 \times [0, 1]$ with the product orientation. Denote by M_0 the mapping torus of ϕ_0 with the induced orientation. That is, $M_0 := C_0 / \sim_{\phi_0}$, where $(x, 0) \sim_{\phi_0} (y, 1)$ if $y = \phi_0(x)$, $x, y \in S_0$. Let \hat{M}_0 be the space obtained by collapsing to one point x_j the image in M_0 of each line $p_j \times [0, 1]$ in C_0 . Then \hat{M}_0 is a pseudo-manifold; $X = \{x_1, \dots, x_r\}$ is the set of singular (ie. non manifold) points of \hat{M}_0 . Every singular point x_j has a conical neighbourhood homeomorphic to the quotient of $T_j^2 \times [0, 1]$ by the equivalence relation identifying $T_j^2 \times \{0\}$ to a point, where T_j^2 is a 2-torus. The point x_j corresponds to the coset of $T_j^2 \times \{0\}$, and $T_j^2 \times \{1\}$ is called the *link* of x_j in \hat{M}_0 . So $M = \hat{M}_0 \setminus X$ is the interior of a compact manifold \hat{M} with boundary formed by r tori. In fact M is the mapping torus $M_\phi = C / \sim_\phi$, where $C = S \times [0, 1]$. From now on, we denote by M_ϕ the manifold M endowed with this fibration over the circle S^1 , and by C_ϕ the mapping cylinder of ϕ .

Layered triangulations of M_ϕ can be constructed as follows (see for instance [10], where one can find also a proof of the well known fact that two ideal triangulations of S are connected by a finite chain of diagonal exchanges). Consider the set of triangulations λ of S_0 whose sets of vertices coincide with P . By removing P , such a triangulation λ is also said an *ideal triangulation* of S ; it has $3m$ edges and $2m$ triangles. Denote by $\lambda' = \phi_0(\lambda)$ the image ideal triangulation of S . Let us fix a chain of diagonal exchanges connecting λ to λ' :

$$\lambda = \lambda_0 \rightarrow \lambda_1 \rightarrow \dots \rightarrow \lambda_k = \lambda'.$$

Possibly by performing some additional diagonal exchanges followed by their inverses, we can assume that this chain is “full”, in the sense that every edge of λ supports some diagonal exchange in the chain. Then the chain induces a 3-dimensional triangulation T of \hat{M}_0 , whose tetrahedra are obtained by “superposing”, for every j , the two squares of λ_j, λ_{j+1} involved in the diagonal exchange $\lambda_j \rightarrow \lambda_{j+1}$ (see Figure 1). The set of vertices of T coincides with X . The ideal triangulation $T \setminus X$ is called a *layered triangulation* of M_ϕ . It lifts to a layered triangulation T_{C_ϕ} of the cylinder C_ϕ .

For each tetrahedron of T , we call *abstract tetrahedron* its underlying simplicial set considered independently of the face pairings in T . Similarly we call abstract edge or face of T any edge or face of an abstract tetrahedra of T . Given any edge e of T , we write $E \rightarrow e$ to mean that an abstract edge E is identified to e in the triangulation T .

From now on we assume that the diffeomorphism ϕ of S is *pseudo-Anosov*, so that M_ϕ is a hyperbolic manifold with r cusps.

By construction every (ideally triangulated) surface (S, λ_j) is embedded in T and T_{C_ϕ} . The union of the surfaces (S, λ_j) forms the 2-skeletons of T and T_{C_ϕ} . Every surface (S, λ_j) divides M_ϕ locally, and the given orientations of S and M_ϕ determines the (local) *positive side* of S in M . In terms of T_{C_ϕ} , the positive side of $(S, \lambda) = (S, \lambda_0)$ lifts to a collar of $S \times \{0\}$, while the *negative side* of $(S, \lambda') = (S, \lambda_k)$ lifts to a collar of $S \times \{1\}$.

The triangulation T is naturally endowed with a *taut pre-branching* ω . Let us recall briefly this notion (see [6], [10]). A taut pre-branching is a choice of transverse co-orientation for each 2-face of T , so that for every tetrahedron, exactly two of its 2-face co-orientations point inward/outward; moreover, it is required that there are exactly two abstract diagonal edges $E \rightarrow e$ for every edge e of T . Here, we call *diagonal edges* the two edges of an abstract tetrahedron whose adjacent 2-face co-orientations point both inward or both outward. The taut pre-branching ω of T is defined by the system of transverse 2-face co-orientations dual to the orientation of the embedded surfaces (S, λ_j) .

We fix also a *weak branching* \tilde{b} compatible with ω (see [5]). Recall that \tilde{b} is a choice of vertex ordering b for each abstract tetrahedron of T , called (*local*) *branching*, satisfying the following constraint. The

abstract 2-faces of the tetrahedron have an orientation induced by b , given by their vertex orderings up to even permutations. Then, it is required that these orientations match, under the 2-face pairings, with the orientations dual to ω (note that we do not require that the abstract vertex orderings match: this defines the stronger notion of *global branching*). Similarly, \tilde{b} gives an orientation to every tetrahedron; without loss of generality, we can assume that it agrees with the orientation of M . A weak branching \tilde{b}_j is also defined on every surface triangulation λ_j , by giving every triangle τ in λ_j the branching induced by the tetrahedron lying on the positive side of τ .

In Figure 1 we show a typical *local* configuration occurring in T or T_{C_ϕ} , that is, a tetrahedron Δ , represented under an orientation preserving embedding in \mathbb{R}^3 , built by gluing squares Q, Q' along the boundary, carrying triangulations λ, λ' related by a diagonal exchange. The orientation of Δ agrees with the standard orientation of \mathbb{R}^3 , and the 2-face co-orientations that define the pre-branching ω are dual to the counter-clockwise orientation of the four faces of Δ . We have chosen one branching b of Δ (any edge being oriented from the lowest to the biggest endpoint) among the two which are compatible with ω and induce the given orientation of Δ . The 2-faces are ordered as the opposite vertices are, accordingly with b .

We advertize that Figure 1 will be used to support all computations. In such a situation we will denote by e_0 the edge $[v_0, v_1]$, by e_1 the edge $[v_1, v_2]$, and by e_2 the edge $[v_0, v_2]$.

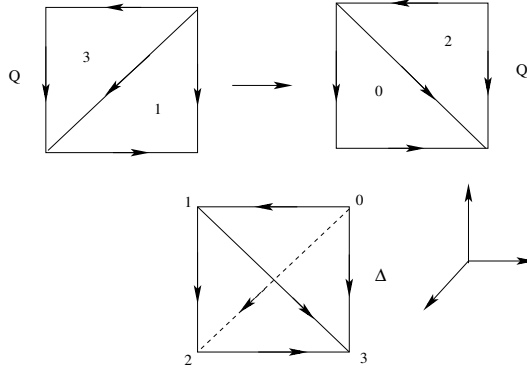


FIGURE 1.

3. BACKGROUND RESULTS

3.1. Classical hyperbolic geometry. Let M be any cusped hyperbolic 3-manifold with r cusps, $r \geq 1$.

We denote by $X(M)$ the variety of *augmented $PSL(2, \mathbb{C})$ -characters* of M . Let us recall the definition. Consider the set of pairs $(\mathbf{r}, \{\xi_\Gamma\}_{\Gamma \in \Pi})$, where $\mathbf{r}: \pi_1(M) \rightarrow PSL(2, \mathbb{C})$ is a group homomorphism, Π is the set of peripheral subgroups of $\pi_1(M)$ (associated to the boundary tori T_i^2 of \bar{M} , for all choices of base points of $\pi_1(T_i^2)$ and $\pi_1(M)$ and paths between them), and for every $\Gamma \in \Pi$, $\xi_\Gamma \in \mathbb{C}P^1$ is a fixed point of $\mathbf{r}(\Gamma)$ such that the assignment $\Gamma \rightarrow \xi_\Gamma$ is equivariant with respect to the action of $\pi_1(M)$ on Π by conjugation and on $\mathbb{C}P^1$ via \mathbf{r} by Moebius transformations. The set $\{(\mathbf{r}, \{\xi_\Gamma\}_{\Gamma \in \Pi})\}$ is a complex affine algebraic set $R(M)$, with an action of $PSL(2, \mathbb{C})$ defined on a pair $(\mathbf{r}, \{\xi_\Gamma\}_\Gamma)$ by conjugation on \mathbf{r} and Moebius transformation on $\{\xi_\Gamma\}_\Gamma$. Then, $X(M)$ is the algebro-geometric quotient of $R(M)$ by $PSL(2, \mathbb{C})$, that is, the set of closed points of the ring of invariant functions $\mathbb{C}[R(M)]^{PSL(2, \mathbb{C})}$.

Similarly we denote by $X(S)$ the variety of augmented characters of a surface S .

In the setting of Section 2, if $M = M_\phi$ and S is a fibre of the fibration, then the inclusion map $i: S = S \times \{0\} \hookrightarrow M_\phi$ induces a regular (restriction) map

$$i^*: X(M_\phi) \rightarrow X(S).$$

Recall that $\pi_1(M_\phi)$ is a HNN-extension of $\pi_1(S)$. That is, given a generating set $\gamma_1, \dots, \gamma_u$ of $\pi_1(S)$ satisfying the relations r_1, \dots, r_v , there is an isomorphism between $\pi_1(M_\phi)$ and the group generated by $\gamma_1, \dots, \gamma_u$ and an element t satisfying the relations r_1, \dots, r_v and $t\alpha t^{-1} = \phi_*(\alpha)$ for all $\alpha \in \pi_1(S)$, where $\phi_*: \pi_1(S) \rightarrow \pi_1(S)$ is the isomorphism induced by ϕ . Therefore, every representation $\mathbf{r}: \pi_1(S) \rightarrow PSL(2, \mathbb{C})$ that may be extended to M_ϕ is such that $\mathbf{r}(\pi_1(S))$ and $\mathbf{r} \circ \phi_*(\pi_1(S))$ are conjugate subgroups of $PSL(2, \mathbb{C})$. Any augmented character of S that may be extended to M_ϕ is thus a fixed point of the map $\Phi_*: X(S) \rightarrow X(S)$ induced by the map $\mathbf{r} \mapsto \mathbf{r} \circ \phi_*$ on representations, and $i^*(X(M_\phi))$ is a subvariety of $\text{Fix}(\Phi_*)$.

Again in the setting of Section 2, let (T, \tilde{b}) be a weakly branched layered triangulation of the fibred cusped manifold M_ϕ . Let us recall a few facts about the *gluing variety* $G(T, \tilde{b})$, that is, the set of solutions of the Thurston gluing equations supported by (T, \tilde{b}) . It is a complex affine algebraic set of dimension greater than or equal to r . The points of $G(T, \tilde{b})$ are classically called *systems of shape parameters*. Their coordinates, the shape parameters, are scalars in $\mathbb{C} \setminus \{0, 1\}$ associated to the abstract edges of T . The shape parameters of opposite edges of a tetrahedron are equal, and the cyclically ordered triple of shape parameters of a tetrahedron encodes an isometry class of hyperbolic ideal tetrahedra.

The set $G(T, \tilde{b})$ is defined by the following two sets of equations. For every branched tetrahedron, set $w_j := w(E_j)$, $j = 0, 1, 2$ (see before Figure 1 for the ordering of the edge E_j). For every edge e of T , define the *total shape parameter* $W(e)$ as the product of the shape parameters $w(E)$, where $E \rightarrow e$. Then we have:

- (*Tetrahedral equation*) For every tetrahedron and $j \in \{0, 1, 2\}$, $w_{j+1} = (1 - w_j)^{-1}$ cyclically; hence $w_0 w_1 w_2 = -1$.
- (*Edge equation*) For every edge e of T , $W(e) = 1$.

A point $w \in G(T, \tilde{b})$ determines a pseudo-developing map $F_w: \tilde{M}_\phi \rightarrow \mathbb{H}^3$, where \tilde{M}_ϕ is the universal cover of M_ϕ , and F_w is well-defined up to post-composition with an orientation-preserving isometry of \mathbb{H}^3 . The map F_w sends homeomorphically the edges of \tilde{T} to complete geodesics, and it satisfies $F_w(g\tilde{x}) = \mathbf{r}_w(g)F_w(\tilde{x})$ for all $\tilde{x} \in \tilde{M}_\phi$, $g \in \pi_1(M_\phi)$, where $\mathbf{r}_w: \pi_1(M_\phi) \rightarrow PSL(2, \mathbb{C})$ is a homomorphism. So w encodes the conjugacy class of \mathbf{r}_w . In fact, F_w determines also some \mathbf{r}_w -equivariant set $\{\xi_\Gamma\}_{\Gamma \in \Pi}$ as above, so that eventually the map $w \mapsto \mathbf{r}_w$ can be lifted to a regular “holonomy” map

$$\text{hol}: G(T, \tilde{b}) \rightarrow X(M_\phi).$$

We have (see Proposition 4.6 of [5] when M_ϕ has a single cusp, and Remark 1.4 of [6] for the general case):

Proposition 3.1. *There is a subvariety A of $G(T, \tilde{b})$ of dimension equal to the number of cusps of M , and a point $w_h \in A$, such that $\text{hol}(w_h)$ is the hyperbolic holonomy of M_ϕ , and $\text{hol}|_A$ is a homeomorphism from a Zariski open subset of A containing w_h onto its image.*

In particular, every point $w \in A$ encodes an augmented character of M_ϕ ; the algebraic closure of $\text{hol}(A)$ is the *eigenvalue* subvariety $E(M_\phi)$ of $X_0(M_\phi)$, the irreducible component of $X(M_\phi)$ containing the discrete faithful holonomy \mathbf{r}_h (see [12]). If M_ϕ has a single cusp, then $E(M_\phi) = X_0(M_\phi)$; in general $E(M_\phi)$ contains \mathbf{r}_h and has complex dimension equal to the number of cusps of M_ϕ .

It follows from Proposition 3.1 that a point of $i^*(X_0(M_\phi)) \subset X(S)$ close enough to $i^*(\mathbf{r}_h)$ is encoded by a point $w \in A$.

From now on, we consider only systems of shape parameters w lying in the subvariety A of $G(T, \tilde{b})$. Denote by (T, \tilde{b}, w) the layered triangulation T of M_ϕ endowed with the weak branching \tilde{b} and the labelling of the abstract edges of T by a system of shape parameters w . For every $j = 0, \dots, k$, consider the ideally triangulated surfaces (S, λ_j) embedded into (T, \tilde{b}, w) as in Section 2. For every edge e of λ_j , define the *lateral shape parameter* $W_j^+(e)$ as the product of the shape parameters of the abstract edges $E \rightarrow e$ carried by the tetrahedra lying on the *positive* side of (S, λ_j) .

Lemma 3.2. *For any edge e of λ_j , the (exponential) shear-bend coordinate of e defined by any pleated surface $F_{w|\tilde{S}}$ (ie. any lift $(\tilde{S}, \tilde{\lambda}_j)$ of (S, λ_j) to \bar{M}_ϕ , and any pseudo-developing map $F_{w|\tilde{S}}: \tilde{S} \rightarrow \mathbb{H}^3$), coincides with the parameter $x^j(e) := -W_j^+(e)$.*

The proof follows from the definitions (see eg. [7]). Note that $x^0(e) = x^k(\phi_0(e))$ for every edge e of $\lambda = \lambda_0$. Also, the shear-bend coordinate of any edge \tilde{e} of $\tilde{\lambda}_j$ is eventually ‘attached’ to the corresponding edge e of λ_j , because for different choices of \tilde{S} or F_w the images of $F_{w|\tilde{S}}$ differ only by an hyperbolic isometry.

Remarks 3.3. The parameter $x^j(e)$ is the *opposite* of $W_j^+(e)$ because the (oriented) bending angle along an edge of λ_j is traditionally measured by the external dihedral angle $\pi - \theta$ (see eg. [7, 8]), whereas the shape parameters use the internal dihedral angle θ . So $\theta = 0$ when two adjacent ideal triangles $F(\tau)$ and $F(\tau')$ coincide, for triangles τ, τ' of $(\tilde{S}, \tilde{\lambda}_j)$.

3.2. Quantum hyperbolic geometry. We keep the setting of the previous section.

Recall that $N \geq 3$ is an odd integer, and q a primitive N -th root of unity.

We begin with a few qualitative, structural features of the *reduced quantum hyperbolic state sum* $\mathcal{H}_N^{\text{red}}(T, \tilde{b}, \mathbf{w})$ defined in [5, 6]. It is a regular rational function defined on a covering of the gluing variety $G(T, \tilde{b})$. The points of this covering over a point $w \in G(T, \tilde{b})$ are certain systems of N -th roots $\mathbf{w}(E)$ of the shape parameters $w(E)$, which label the abstract edges E of T and are called *quantum shape parameters*. Alike the ‘classical’ ones, opposite edges of an abstract tetrahedron are given the same quantum shape parameter. Moreover, the quantum shape parameters verify the following relations, which are ‘quantum’ counterparts of the defining equations of the gluing variety. For every branched tetrahedron of (T, \tilde{b}) put $\mathbf{w}_j := \mathbf{w}(E_j)$ (with the usual edge ordering fixed before Figure 1). For every edge e of $(T, \tilde{b}, \mathbf{w})$, define the *total quantum shape parameter* $\mathbf{W}(e)$ as the product of the quantum shape parameters of the abstract edges $E \rightarrow e$. Then we have:

- (Tetrahedral relation) For every tetrahedron, $\mathbf{w}_0 \mathbf{w}_1 \mathbf{w}_2 = -q$.
- (Edge relation) For every edge e of T , $\mathbf{W}(e) = q^2$.

For simplicity we will work with systems of quantum shape parameters \mathbf{w} such that the corresponding systems of ‘classical’ shape parameters w belong to a simply connected open neighborhood $A_0 \subset A \subset G(T, \tilde{b})$ of w_h (the general case of systems \mathbf{w} with arbitrary $w \in A$ is described in [5]). Note that A_0 is chosen so that $\mathbf{w}(E)$ varies continuously with $w \in A_0$.

Remarks 3.4. (1) For every $z \in \mathbb{C} \setminus \{0\}$, denote by $\log(z)$ the determination of the logarithm which has the imaginary part in $] -\pi, \pi]$. There is a \mathbb{Z} -labelling d of the abstract edges of T such that, for every system \mathbf{w} of quantum shape parameters over a point $w \in A_0$, and every abstract edge E , we have

$$(4) \quad \mathbf{w}(E) = \exp \left(\frac{1}{N} (\log(w(E)) + \pi i(N+1)d(E)) \right).$$

The above relations verified by the quantum shape parameters can be equivalently rephrased in terms of the \mathbb{Z} -labelling d . For any point $w \in A_0$, a \mathbb{Z} -labelling d satisfying these relations defines a system of quantum shape parameters over w by the formula (4).

(2) In [4, 5, 6] we solved (mainly in terms of the \mathbb{Z} -labellings d) the existence problem of triples $(T, \tilde{b}, \mathbf{w})$ for any cusped hyperbolic 3-manifold in the special case $q = -\exp(-i\pi/N)$. The same method works for an arbitrary q , up to minor changes.

The following result summarizes in a qualitative way the invariance properties of $\mathcal{H}_N^{\text{red}}(T, \tilde{b}, \mathbf{w})$. Denote by $A_{0,N}$ the set of systems of quantum shape parameters over A_0 . Recall from Section 2 that M_ϕ can be considered as the interior of a compact manifold \bar{M}_ϕ bounded by tori. Fix a basis $(l_1, m_1), \dots, (l_r, m_r)$ of $\pi_1(\partial \bar{M}_\phi)$, and use it to identify $H^1(\partial \bar{M}_\phi; \mathbb{C}^*)$ with $(\mathbb{C}^*)^{2r}$. Any augmented character $[(\mathbf{r}, \{\xi_\Gamma\}_\Gamma)] \in X(M_\phi)$ determines the square of one of the two (reciprocally inverse) eigenvalues of $\mathbf{r}(\gamma)$, for any non trivial simple closed curve γ on $\partial \bar{M}_\phi$ and representative \mathbf{r} of the character. Namely, by taking \mathbf{r} in its conjugacy class so that $\mathbf{r}(\pi_1(\partial \bar{M}_\phi))$ fixes the point ∞ on $\mathbb{C}P^1$, $\mathbf{r}([\gamma])$ acts

on \mathbb{C} as $w \mapsto \gamma_{\mathbf{r}}w + b$, where $\gamma_{\mathbf{r}} \in \mathbb{C}^*$ and $b \in \mathbb{C}$. The coefficient $\gamma_{\mathbf{r}}$ is that squared eigenvalue selected by $[(\mathbf{r}, \{\xi_{\Gamma}\}_{\Gamma})]$.

We have (see [5] and [6]):

Theorem 3.5. (1) *There exists a determined regular rational map $\kappa_N: A_{0,N} \rightarrow (\mathbb{C}^*)^{2r}$ such that the image $\kappa_N(A_{0,N})$ is the open subset of $(\mathbb{C}^*)^{2r}$ made of all the classes $\kappa \in H^1(\partial\bar{M}_{\phi}; \mathbb{C}^*)$ such that $\kappa(l_j)^N$ and $\kappa(m_j)^N$ are equal up to a sign respectively to the squared eigenvalues selected by $[(\mathbf{r}, \{\xi_{\Gamma}\}_{\Gamma})]$ at the curves l_j and m_j , where $\mathbf{r} := [(\mathbf{r}, \{\xi_{\Gamma}\}_{\Gamma})] = \text{hol}(w)$ for some $w \in A_0$.*

(2) *Given $\mathbf{r} \in \text{hol}(A_0)$ and $\kappa \in H^1(\partial\bar{M}_{\phi}; \mathbb{C}^*)$ as above, the value of $\mathcal{H}_N^{\text{red}}(T, \tilde{b}, \mathbf{w})$ is independent, up to multiplication by $4N$ -th roots of unity, of the choice of $(T, \tilde{b}, \mathbf{w})$ among all weakly branched layered triangulations of M_{ϕ} endowed with a system of quantum shape parameters \mathbf{w} such that $\text{hol}(w) = \mathbf{r}$ and $\kappa_N(\mathbf{w}) = \kappa$.*

We denote the resulting invariant by $\mathcal{H}_N^{\text{red}}(M_{\phi}, \mathbf{r}, \kappa) := \mathcal{H}_N^{\text{red}}(T, \tilde{b}, \mathbf{w})$. By the results of [6], it actually depends on the fibration of M_{ϕ} .

The map κ_N in Theorem 3.5 is a lift to $A_{0,N}$ of $j^* \circ \text{hol}|_{A_0}$, where j^* is the restriction map $X(M_{\phi}) \rightarrow X(\partial M_{\phi})$. Here is a concrete way to compute $\kappa_N(\mathbf{w})(\alpha)$, for $\alpha \in H_1(\partial\bar{M}_{\phi}; \mathbb{Z})$. It is well known that the *truncated tetrahedra* of an ideal triangulation T of M_{ϕ} provide a cell decomposition of \bar{M}_{ϕ} which restricts to a triangulation ∂T of $\partial\bar{M}_{\phi}$. Every abstract vertex of T corresponds to an abstract triangle of ∂T (at which a tetrahedron has been “truncated”); every vertex of such a triangle is contained in one abstract edge of T . Represent the class α by an oriented simple closed curve a on $\partial\bar{M}_{\phi}$, transverse to ∂T . The intersection of a with a triangle F of ∂T is a collection of oriented arcs. We can assume that none of these arcs enters and exits F by a same edge. Then, each one turns around a vertex of F . If Δ is a tetrahedron of T and F corresponds to a vertex of Δ , for every vertex v of F we denote by E_v the edge of Δ containing v , and write $a \rightarrow E_v$ to mean that some subarcs of a turn around v . We count them algebraically, by using the orientation of a : if there are s_+ (resp. s_-) such subarcs whose orientation is compatible with (respectively, opposite to) the orientation of $\partial\bar{M}_{\phi}$ as viewed from v , then we set $\text{ind}(a, v) := s_+ - s_-$. Then

$$(5) \quad \kappa_N(\mathbf{w})(\alpha) = \prod_{a \rightarrow E_v} \mathbf{w}(E_v)^{\text{ind}(a, v)}.$$

In the case where α is the class of a positively oriented meridian curve m_i of $\partial\bar{M}_{\phi}$, this gives

$$(6) \quad \kappa_N(\mathbf{w})(\alpha) = \prod_{m_i \rightarrow E_v} \mathbf{w}(E_v).$$

Let us give now more details about the definition of the reduced QH state sum $\mathcal{H}_N^{\text{red}}(T, \tilde{b}, \mathbf{w})$. We are going to do it in terms of the QH-operator associated to the cylinder C_{ϕ} , already mentioned in the Introduction.

By cutting $(T, \tilde{b}, \mathbf{w})$ along one of the surfaces (S, λ_j) (that is, a triangulated fibre of M_{ϕ}), we get a *QH-triangulation* $(T_{C_{\phi}}, \tilde{b}, \mathbf{w})$ of the cylinder C_{ϕ} having as “source” boundary component (S, λ) , and as “target” boundary component (S, λ') , where $\lambda = \lambda_j$ and $\lambda' = \phi(\lambda_j)$. It carries in a *contravariant way* the QH-operator $\mathcal{H}_N^{\text{red}}(T_{C_{\phi}}, \tilde{b}, \mathbf{w}) \in \text{End}((\mathbb{C}^N)^{\otimes 2m})$, which is such that

$$\text{Trace} \left(\mathcal{H}_N^{\text{red}}(T_{C_{\phi}}, \tilde{b}, \mathbf{w}) \right) = \mathcal{H}_N^{\text{red}}(T, \tilde{b}, \mathbf{w}).$$

We define the QH-operator $\mathcal{H}_N^{\text{red}}(T_{C_{\phi}}, \tilde{b}, \mathbf{w})$ by means of elementary tetrahedral and 2-face operators, as follows.

We can regard the QH triangulation $(T, \tilde{b}, \mathbf{w})$ of M_{ϕ} as a *network of abstract QH-tetrahedra* (Δ, b, \mathbf{w}) , with gluing data along the 2-faces. To each QH-tetrahedron (Δ, b, \mathbf{w}) we associate a linear isomorphism called *basic matrix dilogarithm* (“basic” refers to the fact that we have dropped a symmetrization factor from the matrix dilogarithms that enter the definition of the unreduced QHI),

$$\mathcal{L}_N(\Delta, b, \mathbf{w}): V_2 \otimes V_0 \rightarrow V_3 \otimes V_1$$

where V_j is a copy of \mathbb{C}^N associated to the 2-face of (Δ, b) opposite to the vertex v_j . The basic matrix dilogarithms will be defined explicitly below. For the moment, note that V_1, V_3 correspond to the 2-faces with pre-branching co-orientation pointing inside Δ . Concerning the gluing data, every 2-face F of T is obtained by gluing a pair of abstract 2-faces. Denote by F_s and F_t the “source” and “target” 2-face of the pair, with respect to the transverse co-orientation defined by the pre-branching $\omega_{\tilde{b}}$. Denote by V_{F_s} and V_{F_t} the copies of \mathbb{C}^N associated to F_s and F_t . The identification $F_s \rightarrow F_t$ is given by an even permutation on three elements, which encodes the image of the vertices of F_s in F_t . So it can be encoded by an element $r(F) \in \mathbb{Z}/3\mathbb{Z}$. The triangulation T_{C_ϕ} of the cylinder C_ϕ has $2m$ free 2-faces at both the source and target boundary components, (S, λ) and (S, λ') . Note that at every 2-face of (S, λ) the pre-branching orientation points inside C_ϕ , while it points outside at the 2-faces of (S, λ') . For every gluing $F_s \rightarrow F_t$ occurring at an internal 2-face F of T_{C_ϕ} , as well as to every 2-face F of (S, λ') , we associate an endomorphism (again in contravariant way)

$$(7) \quad Q_N^{r(F)} : V_{F_t} \rightarrow V_{F_s}.$$

By definition, the QH-operator $\mathcal{H}_N^{red}(T_{C_\phi}, \tilde{b}, \mathbf{w})$ is the total contraction of the network of tensors $\{\mathcal{L}_N(\Delta, b, \mathbf{w})\}_\Delta$ and $\{Q_N^{r(F)}\}_F$. In formulas:

$$(8) \quad \mathcal{H}_N^{red}(T_{C_\phi}, \tilde{b}, \mathbf{w}) = \sum_s \prod_{\Delta} \mathcal{L}_N(\Delta, b, \mathbf{w})_s \prod_F Q_N^{r(F)}$$

where the sum ranges over all maps $s: \{\text{abstract 2-faces of } T\} \cup \{2\text{-faces of } (S, \lambda')\} \rightarrow \{0, \dots, N-1\}$ (the *states* of $(T_{C_\phi}, \tilde{b}, \mathbf{w})$), and $\mathcal{L}_N(\Delta, b, \mathbf{w})_s$ and $Q_{N,s}$ denote the entries of the tensors $\mathcal{L}_N(\Delta, b, \mathbf{w})$ and Q_N selected by s , when the tensors are written in the canonical basis $\{e_j\}$ of \mathbb{C}^N . Note that the domain of s contains two copies of each 2-face F in the target boundary component of C_ϕ . They correspond to the source and target spaces V_{F_s}, V_{F_t} of $Q_N^{r(F)}$.

Finally, we provide an operator theoretic definition of the basic matrix dilogarithms (it is a straightforward rewriting of formula (32) in [4]), as well as their entries and those of the endomorphisms Q_N . Since it is the transposed tensor $\mathcal{H}_N^{red}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T$ mentioned in the Introduction (see formula (3)) that occurs in Theorem 1.1, we consider the transposed endomorphisms of Q_N and of the basic matrix dilogarithm instead:

$$Q_N^T : V'_{F_s} \rightarrow V'_{F_t}, \quad \mathcal{L}_N^T(\Delta, b, \mathbf{w}) : V'_3 \otimes V'_1 \rightarrow V'_2 \otimes V'_0$$

where $V'_{F_s} = V'_{F_t} = V'_j = (\mathbb{C}^N)'$, the dual space of \mathbb{C}^N , for every j . The matrix elements will be given with respect to the canonical basis of $((\mathbb{C}^N)')^{\otimes 2}$.

Remark 3.6. If $\{e_j\}$ is the canonical basis of \mathbb{C}^N , and $\{e^j\}$ the dual basis of $(\mathbb{C}^N)'$, the identification map $\iota : \mathbb{C}^N \rightarrow (\mathbb{C}^N)'$, $\iota(e_j) = e^j$, extends to a canonical identification, also denoted by ι , between $(\mathbb{C}^N)^{\otimes 2}$ and $((\mathbb{C}^N)')^{\otimes 2}$. Below we will rather deal with the endomorphisms

$$\iota^{-1} \circ Q_N^T \circ \iota \in \text{End}(\mathbb{C}^N), \quad \iota^{-1} \circ \mathcal{L}_N^T(\Delta, b, \mathbf{w}) \circ \iota \in \text{End}((\mathbb{C}^N)^{\otimes 2}).$$

Obviously, they have the same matrix elements as Q_N^T and $\mathcal{L}_N^T(\Delta, b, \mathbf{w})$ with respect to the canonical basis of \mathbb{C}^N and $(\mathbb{C}^N)'$, and $(\mathbb{C}^N)^{\otimes 2}$ and $((\mathbb{C}^N)')^{\otimes 2}$, respectively. Similarly we will consider

$$\iota^{-1} \circ \mathcal{H}_N^{red}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T \circ \iota \in \text{End}((\mathbb{C}^N)^{\otimes 2m}).$$

For simplicity we will systematically omit the maps ι from the notations.

The endomorphism $Q_N^T : V_{F_s} \rightarrow V_{F_t}$ is defined in the standard basis $\{e_j\}$ of (the copies V_{F_s}, V_{F_t} of) \mathbb{C}^N by

$$(9) \quad Q_N^T(e_k) = \frac{1}{\sqrt{N}} \sum_{j=0}^{N-1} q^{2kj+j^2} e_j.$$

Define the endomorphisms $A_0, A_1, A_2 \in \text{End}(\mathbb{C}^N)$ by

$$(10) \quad A_0 e_k = q^{2k} e_k, \quad A_1 e_k = e_{k+1}, \quad A_2 e_k = q^{1-2k} e_{k-1}.$$

Note that $A_0 A_1 A_2 = q \text{Id}_{\mathbb{C}^N}$. We stress that these operators arise in the representation of the *triangle algebra* (see Section 3.3.2 below); their occurrence in the definition of the basic matrix dilogarithm is a key point to perform the computations at the end of the paper that will establish the bridge between the QHI and the QT intertwiners.

Put

$$g(x) = \prod_{j=1}^{N-1} (1 - xq^{2j})^{j/N}, h(x) = \frac{g(x)}{g(1)}$$

$$\Upsilon = \frac{1}{N} \sum_{i,j=0}^{N-1} q^{-2ij-j} A_0^i \otimes (A_1 A_0)^{-j}.$$

For any triple of quantum shape parameters $\mathbf{w} = (\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2)$ and any endomorphism U such that $U^N = -\text{Id}$, set

$$\Psi_{\mathbf{w}}(U) = h(\mathbf{w}_0) \sum_{i=0}^{N-1} U^i \prod_{j=1}^i \frac{-q^{-1}\mathbf{w}_2}{1 - \mathbf{w}_0^{-1}q^{2j}}$$

where by convention we set the product equal to 1 when $i = 0$. Then

$$(11) \quad \mathcal{L}_N^T(\Delta, b, \mathbf{w}) = \Psi_{\mathbf{w}}(-A_0 A_1 \otimes A_1^{-1}) \circ \Upsilon.$$

Later we will need the following elementary but crucial fact, which belongs (in a different form) to Faddeev-Kashaev [9].

Lemma 3.7. *Let $\mathbf{w}_0, \mathbf{w}_2$ be complex numbers such that $\mathbf{w}_2^N = 1 - \mathbf{w}_0^{-N}$, and U an endomorphism such that $U^N = -\text{Id}$. Then $\Psi_{\mathbf{w}}(U)$ defined as above is the unique endomorphism up to scalar multiplication which is a solution of the functional relation*

$$(12) \quad \Psi_{\mathbf{w}}(q^2 U) = (\mathbf{w}_0 + q^{-1}\mathbf{w}_0\mathbf{w}_2 U) \Psi_{\mathbf{w}}(U).$$

Proof. The hypothesis on $\mathbf{w}_0, \mathbf{w}_2$ implies that the summands of $\Psi_{\mathbf{w}}(U)$ are periodic in i , with period N . Hence we get the same sum if i ranges from 1 to N . With this observation, we have

$$\begin{aligned} q^{-1}\mathbf{w}_0\mathbf{w}_2 U \Psi_{\mathbf{w}}(U) &= h(\mathbf{w}_0) \sum_{i=0}^{N-1} U^{i+1} q^{-1}\mathbf{w}_0\mathbf{w}_2 \prod_{j=1}^i \frac{-q^{-1}\mathbf{w}_2}{1 - \mathbf{w}_0^{-1}q^{2j}} \\ &= h(\mathbf{w}_0) \sum_{i=0}^{N-1} U^{i+1} q^{-1}\mathbf{w}_0\mathbf{w}_2 \left(\frac{1 - \mathbf{w}_0^{-1}q^{2i+2}}{-q^{-1}\mathbf{w}_2} \right) \prod_{j=1}^{i+1} \frac{-q^{-1}\mathbf{w}_2}{1 - \mathbf{w}_0^{-1}q^{2j}} \\ &= h(\mathbf{w}_0) \left(-\mathbf{w}_0 \sum_{i=0}^{N-1} U^{i+1} \prod_{j=1}^{i+1} \frac{-q^{-1}\mathbf{w}_2}{1 - \mathbf{w}_0^{-1}q^{2j}} + \sum_{i=0}^{N-1} (q^2 U)^{i+1} \prod_{j=1}^{i+1} \frac{-q^{-1}\mathbf{w}_2}{1 - \mathbf{w}_0^{-1}q^{2j}} \right) \\ &= -\mathbf{w}_0 \Psi_{\mathbf{w}}(U) + \Psi_{\mathbf{w}}(q^2 U). \end{aligned}$$

Hence $\Psi_{\mathbf{w}}(U)$ solves the equation (12). Conversely, $U^N = -\text{Id}$ implies that U is diagonalizable with eigenvalues in the set $\{-q^{2i}, i = 0, \dots, N-1\}$. Hence $\Psi_{\mathbf{w}}(U)$ is a polynomial in U completely determined by its eigenvalues, which are of the form $\Psi_{\mathbf{w}}(-q^{2i})$. By (12) they are given for $i \geq 1$ by $\Psi_{\mathbf{w}}(-q^{2i}) = \prod_{j=1}^i (\mathbf{w}_0 - q^{2j-3}\mathbf{w}_0\mathbf{w}_2) \Psi_{\mathbf{w}}(-1)$, and hence uniquely determined up to the choice of $\Psi_{\mathbf{w}}(-1)$. Therefore (12) has a unique solution $\Psi_{\mathbf{w}}(U)$ up to scalar multiplication. \square

Remark 3.8. The normalization factor $h(\mathbf{w}_0)$ is chosen so that $\det(\Psi_{\mathbf{w}}(-A_0 A_1 \otimes A_1^{-1})) = 1$.

In particular, when $(\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2)$ is a triple of quantum shape parameters we have

$$(13) \quad \Psi_{\mathbf{w}}(q^2 U) = (\mathbf{w}_0 - \mathbf{w}_1^{-1} U) \Psi_{\mathbf{w}}(U).$$

Finally the entries of $\mathcal{L}_N^T(\Delta, b, \mathbf{w})$ in the basis $\{e_k \otimes e_l\}_{k,l}$ of $(\mathbb{C}^N)^{\otimes 2}$ are as follows. Identify $\mathbb{Z}/N\mathbb{Z}$ with $\{0, 1, \dots, N-1\}$. For every $a \in \mathbb{Z}$ set $\delta(a) = 1$ if $a \equiv 0 \pmod{N}$, and $\delta(a) = 0$ otherwise. For

every $n \in \mathbb{Z}/N\mathbb{Z}$, consider the function $\omega(x, y|n)$ defined on the curve $\{(x, y \in \mathbb{C}^2 \mid x^N + y^N = 1)\}$ by

$$\omega(x, y|0) = 1, \omega(x, y|n) = \prod_{j=1}^n \frac{y}{1 - xq^{-2j}}, \quad n \geq 1.$$

Then, for every $i, j, k, l \in \mathbb{Z}/N\mathbb{Z}$ we have

$$\mathcal{L}_N^T(\Delta, b, \mathbf{w})_{i,j}^{k,l} = h(\mathbf{w}_0)q^{-2kj-k^2}\omega(\mathbf{w}_0, \mathbf{w}_1^{-1}|i-k)\delta(i+j-l).$$

3.3. Representations of the quantum Teichmüller spaces. Unless stated differently, the results recalled in this section are proved in [7, 8] or [11].

Let λ be an ideal triangulation of S , and q as above. Recall that $m = -\chi(S)$. Put $n := 3m$ and fix an ordering e_1, \dots, e_n of the edges λ . For all distinct i, j set $\sigma_{ij} := a_{ij} - a_{ji} \in \{0, \pm 1, \pm 2\}$, where a_{ij} is the number of times e_i is on the *right* of e_j in a triangle of λ , using the orientation of S . The *Chekhov-Fock algebra* \mathfrak{T}_λ^q is the algebra over \mathbb{C} with generators $X_i^{\pm 1}$ associated to the edges e_i , and relations

$$(14) \quad X_i X_j = q^{2\sigma_{ij}} X_j X_i.$$

When $q = 1$, \mathfrak{T}_λ^1 is just the algebra of Laurent polynomials $\mathbb{C}[X_1^{\pm 1}, \dots, X_n^{\pm 1}]$, which is the ring of functions on the *classical* (enhanced) Teichmüller space generated by the exponential shear-bend coordinates on λ .

The algebra \mathfrak{T}_λ^q has a well-defined fraction algebra $\hat{\mathfrak{T}}_\lambda^q$, and any diagonal exchange $\lambda \rightarrow \lambda'$ induces an isomorphism of algebras

$$\varphi_{\lambda\lambda'}^q : \hat{\mathfrak{T}}_{\lambda'}^q \rightarrow \hat{\mathfrak{T}}_\lambda^q.$$

The above definition of \mathfrak{T}_λ^q works as well for any ideally triangulated punctured compact oriented surface S , possibly with boundary, where each boundary component is a union of edges. In particular, take S the ideal triangulated squares Q, Q' in Figure 1. Number the edges of their triangulations λ, λ' as in Figure 2. In such a situation $\varphi_{\lambda\lambda'}^q$ has the form

$$(15) \quad \begin{aligned} \varphi_{\lambda\lambda'}^q(X'_5) &= X_5^{-1} \\ \varphi_{\lambda\lambda'}^q(X'_1) &= (1 + qX_5)X_1 & \varphi_{\lambda\lambda'}^q(X'_3) &= (1 + qX_5)X_3 \\ \varphi_{\lambda\lambda'}^q(X'_2) &= (1 + qX_5^{-1})^{-1}X_2 & \varphi_{\lambda\lambda'}^q(X'_4) &= (1 + qX_5^{-1})^{-1}X_4. \end{aligned}$$

It is proved in [1] (see also Theorem 1.22 in [11]) that this case of the ideal square determines, for *any* punctured compact oriented surface S , a unique family $\{\varphi_{\lambda\lambda'}^q\}_{\lambda\lambda'}$ of algebra isomorphisms $\varphi_{\lambda\lambda'}^q : \hat{\mathfrak{T}}_{\lambda'}^q \rightarrow \hat{\mathfrak{T}}_\lambda^q$ defined for all ideal triangulations λ and λ' of S , if the family satisfies certain natural properties with respect to composition, diffeomorphisms of S , and decomposition into ideally triangulated subsurfaces.

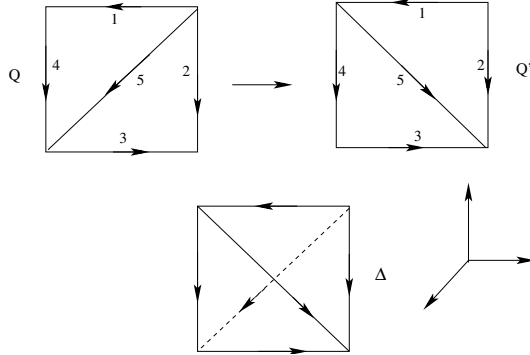


FIGURE 2.

Given any ideal triangulations λ, λ' of S and finite dimensional representation $\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_\lambda)$, one says that $\rho_\lambda \circ \varphi_{\lambda\lambda'}^q$ *makes sense* if for every generator $X'_i \in \mathfrak{T}_{\lambda'}^q$, the element $\varphi_{\lambda\lambda'}^q(X'_i)$ can be

written as $P_i Q_i^{-1} \in \widehat{\mathfrak{T}}_\lambda^q$, where $P_i, Q_i \in \mathfrak{T}_\lambda^q$ and $\rho_\lambda(P_i), \rho_\lambda(Q_i)$ are invertible endomorphisms of V_λ . In such a case, $\rho_\lambda \circ \varphi_{\lambda\lambda'}^q(X'_i) = \rho_\lambda(P_i)\rho_\lambda(Q_i)^{-1}$ is well-defined (ie. independent of the choice of the pair (P_i, Q_i)).

By definition, the *quantum Teichmüller space* of S is the quotient set $\mathfrak{T}_S^q := (\coprod_\lambda \widehat{\mathfrak{T}}_\lambda^q) / \sim$, where \sim identifies the algebras $\widehat{\mathfrak{T}}_\lambda^q$ and $\widehat{\mathfrak{T}}_{\lambda'}^q$ by $\varphi_{\lambda\lambda'}^q$, for all ideal triangulations λ, λ' of S . A *representation* of \mathfrak{T}_S^q is a family of representations

$$\rho := \{\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_\lambda)\}_\lambda$$

indexed by the set of ideal triangulations of S , such that $\rho_\lambda \circ \varphi_{\lambda\lambda'}^q$ makes sense for every λ, λ' and is isomorphic to $\rho_{\lambda'}$. In fact, it is enough to check the isomorphisms $\rho_{\lambda'} \cong \rho_\lambda \circ \varphi_{\lambda\lambda'}^q$ whenever λ and λ' differ by a flip.

3.3.1. Local representations. The irreducible representations of \mathfrak{T}_S^q were classified up to isomorphism in [7]. The *local* representations of \mathfrak{T}_S^q are special *reducible* representations defined as follows (see [8, 11]). Define the *triangle algebra* \mathcal{T} as the algebra over \mathbb{C} with generators $Y_0^{\pm 1}, Y_1^{\pm 1}, Y_2^{\pm 1}$ and relations

$$(16) \quad Y_0 Y_1 = q^2 Y_1 Y_0, \quad Y_1 Y_2 = q^2 Y_2 Y_1, \quad Y_2 Y_0 = q^2 Y_0 Y_2.$$

Fix an ordering τ_1, \dots, τ_{2m} of the triangles of λ . Order the abstract edges e_0^j, e_1^j, e_2^j of τ_j so that the induced cyclic ordering is counter-clockwise with respect to the orientation of S . Associate to τ_j a copy \mathcal{T}_j of the algebra \mathcal{T} , with generators denoted by Y_0^j, Y_1^j, Y_2^j , so that Y_i^j is associated to the edge e_i^j . There is an algebra embedding

$$(17) \quad \mathfrak{i}_\lambda : \mathfrak{T}_\lambda^q \rightarrow \mathcal{T}_1 \otimes \dots \otimes \mathcal{T}_{2m}$$

defined on generators by (we denote a monomial $\otimes_j A_j$ in $\otimes_j \mathcal{T}_j$ by omitting the terms $A_j = 1$):

- If e_i is an edge of two distinct triangles τ_{i_i} and τ_{r_i} , and $e_{a_i}^l, e_{b_i}^{r_i}$ are the edges of τ_{i_i}, τ_{r_i} respectively identified to e_i , then $\mathfrak{i}_\lambda(X_i) := Y_{a_i}^{l_i} \otimes Y_{b_i}^{r_i}$.
- If e_i is an edge of a single triangle τ_{k_i} , and $e_{a_i}^{k_i}, e_{b_i}^{k_i}$ are the edges of τ_{k_i} identified to e_i , with $e_{a_i}^{k_i}$ on the right of $e_{b_i}^{k_i}$, then $\mathfrak{i}_\lambda(X_i) := q^{-1} Y_{a_i}^{k_i} Y_{b_i}^{k_i} = q Y_{b_i}^{k_i} Y_{a_i}^{k_i}$.

A representation ρ_λ of \mathfrak{T}_λ^q is *local* if

$$\rho_\lambda = (\rho_1 \otimes \dots \otimes \rho_{2m}) \circ \mathfrak{i}_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_1 \otimes \dots \otimes V_{2m})$$

for some irreducible representations $\rho_j : \mathcal{T} \rightarrow \text{End}(V_j)$ of \mathcal{T} , $j \in \{1, \dots, 2m\}$. So, a local representation is an equivalence class of tuples $(\rho_1, \dots, \rho_{2m})$, where two tuples are equivalent if their restrictions to the subalgebra $\mathfrak{i}_\lambda(\mathfrak{T}_\lambda^q)$ of $\mathcal{T}_1 \otimes \dots \otimes \mathcal{T}_{2m}$ define the same representation. Two local representations $\rho_\lambda = (\rho_1 \otimes \dots \otimes \rho_{2m}) \circ \mathfrak{i}_\lambda$ and $\rho'_\lambda = (\rho'_1 \otimes \dots \otimes \rho'_{2m}) \circ \mathfrak{i}_\lambda$ of \mathfrak{T}_λ^q are isomorphic if there are linear isomorphisms $L_j : V_j \rightarrow V'_j$ such that for every $j = 1, \dots, 2m$ and $Y \in \mathcal{T}_j$ we have

$$L_j \circ \rho_j(Y) \circ (L_j)^{-1} = \rho'_j(Y).$$

It is straightforward to check that this definition is independent of the choice of tuples $(\rho_1, \dots, \rho_{2m}), (\rho'_1, \dots, \rho'_{2m})$. By definition a local representation ρ of \mathfrak{T}_S^q is a representation formed by local representations ρ_λ .

3.3.2. Isomorphism classes and standard local representations. The isomorphism classes of irreducible representations of the triangle algebra \mathcal{T} are parametrized by tuples of non zero scalars $(y_0, y_1, y_2, h) \in (\mathbb{C}^\bullet)^4$ such that $h^N = y_0 y_1 y_2$. The parameter h is called the *load* of the class. The isomorphism class with parameters (y_0, y_1, y_2, h) can be represented by *standard* representations $\rho : \mathcal{T} \rightarrow \text{End}(\mathbb{C}^N)$, which have the form

$$(18) \quad \rho(Y_i) = \mathbf{y}_i A_i, \quad i \in \{0, 1, 2\}$$

where $\mathbf{y}_0, \mathbf{y}_1, \mathbf{y}_2 \in \mathbb{C}^*$ satisfy

$$\mathbf{y}_i^N = y_i, \quad i \in \{0, 1, 2\}, \quad h = \mathbf{y}_0 \mathbf{y}_1 \mathbf{y}_2$$

and A_0, A_1, A_2 are the endomorphisms of \mathbb{C}^N defined in (10).

Any local representation of \mathfrak{T}_λ^q has dimension N^{2m} . The isomorphism class of a local representation ρ_λ of \mathfrak{T}_λ^q is determined by:

- a non zero complex *weight* x_i associated to each edge of λ ;
- a N -th root h of $x_1 \dots x_n$, called the *load*.

The weights x_i and the load h are such that

$$(19) \quad \rho_\lambda(X_i^N) = x_i \text{Id}_{V_1 \otimes \dots \otimes V_{2m}}, \quad \rho_\lambda(H) = h \text{Id}_{V_1 \otimes \dots \otimes V_{2m}}$$

where H is a central element of \mathfrak{T}_λ^q , called *the principal central element*, given by

$$(20) \quad H = q^{-\sum_{l < l'} \sigma_{ll'}} X_1 \dots X_n.$$

Note that $h^N = x_1 \dots x_n$. It is straightforward to check that two local representations are isomorphic if and only if they are isomorphic as local representations. We call $(\{x_1, \dots, x_n\}, h)$ the *parameters* of ρ_λ . We say that a local representation $\rho_\lambda = (\rho_1 \otimes \dots \otimes \rho_{2m}) \circ \mathbf{i}_\lambda$ is standard if every ρ_i is. This notion naturally extends to representations of \mathfrak{T}_S^q . Every point of $(\mathbb{C}^*)^n$ can be realized as the n -tuple of parameters x_i , $i = 1, \dots, n$, of a standard local representation of \mathfrak{T}_λ^q . There is a one-to-one correspondence between isomorphism classes of local representations $\{\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_1 \otimes \dots \otimes V_{2m})\}_\lambda$ of \mathfrak{T}_S^q and families of parameters $\{(\{x_1, \dots, x_n\}_\lambda, h_\lambda)\}_\lambda$ such that

$$(21) \quad h_\lambda = h_{\lambda'} \text{ and } x_i = \varphi_{\lambda\lambda'}^1(x'_i)$$

for every $i = 1, \dots, n$ and any two ideal triangulations λ, λ' of S with edge weights $\{x_1, \dots, x_n\}, \{x'_1, \dots, x'_n\}$ and loads $h_\lambda, h_{\lambda'}$ respectively. The edge weights x_i of a local representation $\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_\lambda)$ define its “classical shadow”

$$sh(\rho_\lambda) : \mathfrak{T}_\lambda^1 \rightarrow \text{End}(\mathbb{C})$$

by taking the restriction of ρ_λ to the subalgebra $\mathbb{C}[X_1^{\pm N}, \dots, X_n^{\pm N}] \cong \mathfrak{T}_\lambda^1$ of the center of \mathfrak{T}_λ^q . This notion extends immediately to local representations $\rho = \{\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_\lambda)\}_\lambda$ of \mathfrak{T}_S^q ; the classical shadow

$$sh(\rho) = \{sh(\rho_\lambda) : \mathfrak{T}_\lambda^1 \rightarrow \text{End}(\mathbb{C})\}_\lambda$$

is a representation of the coordinate ring \mathfrak{T}_S^1 of the Teichmüller space (considered as a rational manifold with transition functions the maps $\varphi_{\lambda\lambda'}^1$). Every representation of \mathfrak{T}_S^1 is the shadow of N local representations of \mathfrak{T}_S^q . The shadows of local representations encode the Zariski open subset of $X(S)$ made of the so-called *peripherally generic characters*. These include all the augmented characters of injective representations.

The load h has the following geometric interpretation. Let $[(\mathbf{r}, \{\xi_\Gamma\}_{\Gamma \in \Pi})]$ be the augmented character of S determined by $sh(\rho)$. Each $\Gamma \in \Pi$ is a group generated by the class in $\pi_1(S)$ of a small loop m_j in S going once and counter-clockwise around the j -th puncture, for some $j \in \{1, \dots, r\}$ and some choice of basepoints. Since $\pi_1(S)$ is a free group, $\mathbf{r} : \pi_1(S) \rightarrow PSL(2, \mathbb{C})$ can be lifted to a homomorphism $\hat{\mathbf{r}} : \pi_1(S) \rightarrow SL(2, \mathbb{C})$; the fixed point $\xi_\Gamma \in \mathbb{P}^1$ is then contained in an eigenspace of $\hat{\mathbf{r}}(m_j)$, corresponding to an eigenvalue $a_j \in \mathbb{C}$. Then

$$(22) \quad h^N = (-1)^r a_1^{-1} \dots a_r^{-1}.$$

Finally, the decomposition into irreducible summands of a local representation ρ_λ of \mathfrak{T}_λ^q is ([13])

$$(23) \quad \rho_\lambda = \bigoplus_{\rho_\lambda(\mu) \subset \rho_\lambda} \rho_\lambda(\mu)$$

where the sum ranges over the set of all spaces $\rho_\lambda(\mu)$ formed by intersecting one eigenspace for each of the so-called (central) *puncture elements* of \mathfrak{T}_λ^q . Each space $\rho_\lambda(\mu)$ is also the *isotypical* component of an irreducible representation μ of \mathfrak{T}_λ^q , that is, the direct sum of all irreducible summands of ρ_λ isomorphic to μ . Every irreducible representation of \mathfrak{T}_λ^q has an isotypical summand appearing in some local representation, and has multiplicity N^g in it.

3.4. Intertwiners of local representations. Let (S, λ) and q be as before. Any surface R obtained by splitting S along some (maybe all) edges of λ inherits an orientation from S , and an ideal triangulation μ from λ . By gluing along edges backwards, one says that (S, λ) is obtained *by fusion* from (R, μ) . In such a case, every local representation $\eta_\mu = (\eta_1 \otimes \dots \otimes \eta_{2m}) \circ \mathbf{i}_\mu$ of \mathfrak{T}_μ^q determines a local representation ρ_λ of \mathfrak{T}_λ^q by setting $\rho_\lambda = (\eta_1 \otimes \dots \otimes \eta_{2m}) \circ \mathbf{i}_\lambda$. One says that η_μ *represents* ρ_λ .

Given local representations $\eta = \{\eta_\mu : \mathfrak{T}_\mu^q \rightarrow \text{End}(W_\mu)\}_\mu$, $\rho = \{\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_\lambda)\}_\lambda$ of \mathfrak{T}_R^q and \mathfrak{T}_S^q respectively, one says that ρ *is obtained by fusion from* η if η_μ represents ρ_λ for all ideal triangulations μ of R , where λ is the ideal triangulation of S obtained by fusion from μ .

In [11], the following result is proved. The intertwiners $L_{\lambda\lambda'}^{\rho\rho'} \in \mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ in the statement are called *QT intertwiners*.

Theorem 3.9. *There exists a collection $\{(\mathcal{L}_{\lambda\lambda'}^{\rho\rho'}, \psi_{\lambda\lambda'}^{\rho\rho'})\}$, indexed by the couples of isomorphic local representations $\rho = \{\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_\lambda)\}_\lambda$, $\rho' = \{\rho'_{\lambda'} : \mathfrak{T}_{\lambda'}^q \rightarrow \text{End}(V_{\lambda'})\}_{\lambda'}$ of \mathfrak{T}_S^q and by the couples of ideal triangulations λ, λ' of S , such that:*

(1) $\mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ is a set of projective classes of linear isomorphisms $L_{\lambda\lambda'}^{\rho\rho'} : V_\lambda \rightarrow V_{\lambda'}$ such that for every $X' \in \mathfrak{T}_{\lambda'}^q$, we have

$$L_{\lambda\lambda'}^{\rho\rho'} \circ (\rho_\lambda \circ \varphi_{\lambda\lambda'}^q)(X') \circ (L_{\lambda\lambda'}^{\rho\rho'})^{-1} = \rho'_{\lambda'}(X').$$

(2) $\psi_{\lambda\lambda'}^{\rho\rho'} : H_1(S; \mathbb{Z}/N\mathbb{Z}) \times \mathcal{L}_{\lambda\lambda'}^{\rho\rho'} \rightarrow \mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ is a free transitive action.

(3) Let R be a surface such that S is obtained by fusion from R . Let $\eta = \{\eta_\mu : \mathfrak{T}_\mu^q \rightarrow \text{End}(W_\mu)\}_\mu$, $\eta' = \{\eta'_\mu : \mathfrak{T}_\mu^q \rightarrow \text{End}(W'_\mu)\}_\mu$ be two local representations of \mathfrak{T}_R^q such that ρ, ρ' are obtained respectively by fusion from η, η' . Then, for every ideal triangulations μ, μ' of R , if λ, λ' are the corresponding ideal triangulations of S , there exists an inclusion map $j : \mathcal{L}_{\mu\mu'}^{\eta\eta'} \rightarrow \mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ such that for every $L \in \mathcal{L}_{\mu\mu'}^{\eta\eta'}$ and every $c \in H_1(R; \mathbb{Z}/N\mathbb{Z})$ the following holds:

$$(j \circ \psi_{\mu\mu'}^{\eta\eta'})(c, L) = \psi_{\lambda\lambda'}^{\rho\rho'}(\pi_*(c), j(L))$$

where $\pi : R \rightarrow S$ is the projection map.

(4) For every isomorphic local representations ρ, ρ', ρ'' of \mathfrak{T}_S^q and for every ideal triangulations $\lambda, \lambda', \lambda''$ of S the composition map

$$\begin{aligned} \mathcal{L}_{\lambda\lambda'}^{\rho\rho'} \times \mathcal{L}_{\lambda'\lambda''}^{\rho'\rho''} &\longrightarrow \mathcal{L}_{\lambda\lambda''}^{\rho\rho''} \\ (L_1, L_2) &\longmapsto L_2 \circ L_1 \end{aligned}$$

is well-defined, and for all $c, d \in H_1(S; \mathbb{Z}/N\mathbb{Z})$ it satisfies

$$\psi_{\lambda'\lambda''}^{\rho'\rho''}(d, L_2) \circ \psi_{\lambda\lambda'}^{\rho\rho'}(c, L_1) = \psi_{\lambda\lambda''}^{\rho\rho''}(c + d, L_2 \circ L_1).$$

It is also proved in [11] that any collection of intertwiners satisfying a weak form of conditions (3) and (4) (not involving the actions $\psi_{\lambda\lambda'}^{\rho\rho'}$) contains the collection $\{\mathcal{L}_{\lambda\lambda'}^{\rho\rho'}\}$. So the latter is minimal with respect to these conditions.

Note that property (3) describes the behaviour of the intertwiners in $\mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ under cut-and-paste of subsurfaces along edges of λ, λ' . In particular, it implies that they can be decomposed into elementary intertwiners as follows. Any two ideal triangulations λ, λ' of S can be connected by a sequence $\lambda = \lambda_0 \rightarrow \dots \rightarrow \lambda_{k+1} = \lambda'$ consisting of k diagonal exchanges followed by an edge reindexing $\lambda_k \rightarrow \lambda_{k+1}$ (with same underlying triangulation). Then, any projective class of intertwiners $[L_{\lambda\lambda'}^{\rho\rho'}] \in \mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ can be decomposed as

$$(24) \quad L_{\lambda\lambda'}^{\rho\rho'} = L_{\lambda'\lambda'}^{\rho\rho'} \circ L_{\lambda_k\lambda_{k+1}}^{\rho\rho} \circ \dots \circ L_{\lambda_0\lambda_1}^{\rho\rho}$$

where $L_{\lambda_i\lambda_{i+1}}^{\rho\rho} \in \mathcal{L}_{\lambda_i\lambda_{i+1}}^{\rho\rho}$ intertwins ρ_{λ_i} and $\rho_{\lambda_{i+1}} \circ (\varphi_{\lambda_i\lambda_{i+1}}^q)^{-1}$, related by the i -th diagonal exchange, for every $i \in \{0, \dots, k-1\}$, and $L_{\lambda'\lambda'}^{\rho\rho'}$ intertwins $\rho_{\lambda'}$ and $\rho'_{\lambda'}$ on the triangulation λ' . In general the intertwiner $L_{\lambda\lambda'}^{\rho\rho'}$ depends on the choice of sequence $\lambda = \lambda_0 \rightarrow \dots \rightarrow \lambda_{k+1} = \lambda'$, but the set $\mathcal{L}_{\lambda\lambda'}^{\rho\rho'}$ and the action $\psi_{\lambda\lambda'}^{\rho\rho'}$ do not.

4. PROOFS

We are ready to prove our main Theorems 1.1, 1.2, and Corollary 1.4. Let us reformulate them by using the background material recalled in the previous sections.

Let M_ϕ be a fibred cusped hyperbolic 3-manifold realized as the mapping torus of a pseudo Anosov diffeomorphism ϕ of a punctured surface S . Put $m = -\chi(S) > 0$. Let C_ϕ be the associated cylinder. Let T be a layered triangulation of M_ϕ , and T_{C_ϕ} the induced layered triangulation of C_ϕ , with source boundary component the ideally triangulated surface (S, λ) , and target boundary component (S, λ') , where $\lambda' = \phi(\lambda)$ (as in Section 2).

For every odd $N \geq 3$ and every primitive N -th root of unity q , let $(T, \tilde{b}, \mathbf{w})$ be a QH layered triangulation of M_ϕ , where \mathbf{w} is a system of quantum shape parameters over $w \in A$, where A is the subvariety of the gluing variety $G(T, \tilde{b})$ as in Proposition 3.1. Let $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ be the induced QH triangulation of C_ϕ , and $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w}) \in \text{End}((\mathbb{C}^N)^{\otimes 2m})$ the QH operator defined by means of the QH state sum carried by $(T_{C_\phi}, \tilde{b}, \mathbf{w})$. Associated to it we have the transposed operator $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T \in \text{End}((\mathbb{C}^N)'^{\otimes 2m})$ and $\iota^{-1} \circ \mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T \circ \iota \in \text{End}((\mathbb{C}^N)^{\otimes 2m})$ as in Section 3.2, still denoted by $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T$ (see Remark 3.6).

Now we re-state and prove our first main theorem, Theorem 1.1.

First Main Theorem. (1) Every layered QH-triangulation $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ determines representations ρ_λ and $\rho_{\lambda'}$ of \mathfrak{S}_λ^q and $\mathfrak{S}_{\lambda'}^q$, respectively, belonging to a local representation ρ of \mathfrak{S}_S^q and such that $V_\lambda = V_{\lambda'} = (\mathbb{C}^N)^{\otimes 2m}$, and ρ_λ is isomorphic to $\phi^* \rho_{\lambda'}$. Moreover, the operator $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T$, considered as an element of $\text{Hom}(V_\lambda, V_{\lambda'})$, is a QT intertwiner which intertwins the representations ρ_λ and $\rho_{\lambda'}$.

(2) For any other choice of weak branching \tilde{b}' , $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}', \mathbf{w})^T$ intertwins local representations canonically isomorphic to $\rho_\lambda, \rho_{\lambda'}$ respectively.

Proof. We organize the proof in several steps:

Step 1. Every transposed basic matrix dilogarithm $\mathcal{L}_N^T(\Delta, b, \mathbf{w}) \in \text{End}((\mathbb{C}^N)^{\otimes 2})$ (see Section 3.2) intertwins standard local representations of $\mathfrak{S}_\lambda^q, \mathfrak{S}_{\lambda'}^q$ (see Section 3.3.2), where λ, λ' are the ideal triangulations of the squares Q, Q' in Figure 3.

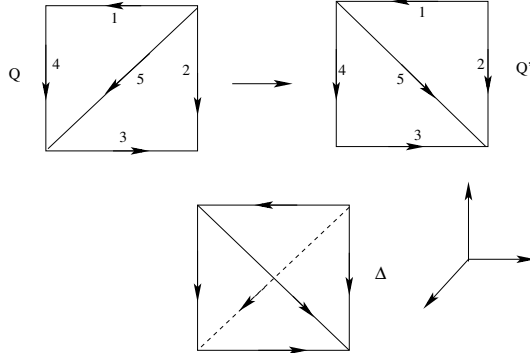


FIGURE 3.

Step 2. Every transposed 2-face operator $(Q_N^T)^{r(F)} : V_s \rightarrow V_t$ (see (9)) intertwins representations of the triangle algebra $\rho_s : \mathcal{T} \rightarrow \text{End}(\mathbb{C}^N), \rho_t : \mathcal{T} \rightarrow \text{End}(\mathbb{C}^N)$, associated to the abstract 2-faces F_s, F_t so that for each one the generator Y_0 labels the edge joining the lowest vertex to the middle one (with respect to the vertex ordering induced by the branching), and Y_1 labels the edge joining the lowest vertex to the biggest one.

Step 3. Every layered QH-triangulation $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ of the cylinder C_ϕ , associated to a sequence of diagonal exchanges $\lambda = \lambda_0 \rightarrow \lambda_1 \rightarrow \dots \rightarrow \lambda_k = \lambda'$, determines standard local representations

$\rho_j: \mathfrak{T}_{\lambda_j}^q \rightarrow \text{End}((\mathbb{C}^N)^{\otimes 2m})$ associated to the surfaces (S, λ_j) , for every $j \in \{0, \dots, k\}$. These representations belong to a local representation ρ of the quantum Teichmüller space \mathfrak{T}_S^q .

By Step 1, Step 2, and its actual definition by means of a QH state sum, $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T$ intertwines the local representations ρ_0 and ρ_k .

Step (2) proves the claim (2) in the First Main Theorem; the three steps together prove (1). The details follow.

4.0.1. *Proof of Step 1.* Consider the squares Q, Q' and the branched tetrahedron (Δ, b) in Figure 3. Use the branching to order as e_0, e_1, e_2 the edges of any triangle of the squares Q, Q' , so that e_0 goes from the lowest vertex to the middle one, and e_1 goes from the lowest vertex to the biggest one. In such a situation there are two embeddings (17), of the form $i_\lambda: \mathfrak{T}_\lambda^q \rightarrow \mathcal{T}_3 \otimes \mathcal{T}_1, i_{\lambda'}: \mathfrak{T}_{\lambda'}^q \rightarrow \mathcal{T}_2 \otimes \mathcal{T}_0$, where \mathcal{T}_i is the copy of the triangle algebra associated to the i -th 2-face (ie. opposite to the i -th vertex) of (Δ, b) , and

$$\begin{aligned} i_\lambda(X_1) &= Y_0^3 \otimes 1, i_\lambda(X_2) = 1 \otimes Y_1^1, i_\lambda(X_3) = 1 \otimes Y_2^1, i_\lambda(X_4) = Y_2^3 \otimes 1, i_\lambda(X_5) = Y_1^3 \otimes Y_0^1. \\ i_{\lambda'}(X'_1) &= Y_0^2 \otimes 1, i_{\lambda'}(X'_2) = Y_1^2 \otimes 1, i_{\lambda'}(X'_3) = 1 \otimes Y_2^0, i_{\lambda'}(X'_4) = 1 \otimes Y_0^0, i_{\lambda'}(X'_5) = Y_2^2 \otimes Y_1^0. \end{aligned}$$

Note that Y_j^i is associated to the edge e_j of the i -th 2-face of (Δ, b) , that we have specified above. Recall the notion of standard local representation in Section 3.3.2.

Proposition 4.1. *Let $\rho_\lambda = (\rho_3 \otimes \rho_1) \circ i_\lambda: \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_3 \otimes V_1)$ be a standard local representation, and \mathbf{y}_j^i be such that $\rho_i(Y_j^i) = \mathbf{y}_j^i A_j$, for $i \in \{1, 3\}, j \in \{0, 1, 2\}$. Let \mathbf{w} be a triple of quantum shape parameters of the branched tetrahedron (Δ, b) such that $\mathbf{w}_2 = -q\mathbf{y}_1^3\mathbf{y}_0^1$. Then, for all $X \in \mathfrak{T}_\lambda^q$ we have*

$$(25) \quad \mathcal{L}_N^T(\Delta, b, \mathbf{w}) \circ \rho_\lambda(X) = \rho_{\lambda'}((\varphi_{\lambda\lambda'}^q)^{-1}(X)) \circ \mathcal{L}_N^T(\Delta, b, \mathbf{w})$$

where $\rho_{\lambda'} = (\rho_2 \otimes \rho_0) \circ i_{\lambda'}: \mathfrak{T}_{\lambda'}^q \rightarrow \text{End}(V_2 \otimes V_0)$ is the standard representation given by

$$(26) \quad \mathbf{y}_0^2 = \mathbf{w}_0^{-1}\mathbf{y}_0^3, \quad \mathbf{y}_1^2 = \mathbf{w}_1^{-1}\mathbf{y}_1^1, \quad \mathbf{y}_0^0 = \mathbf{w}_1^{-1}\mathbf{y}_2^3, \quad \mathbf{y}_2^0 = \mathbf{w}_0^{-1}\mathbf{y}_2^1, \quad \mathbf{y}_2^2\mathbf{y}_1^0 = -q\mathbf{w}_2^{-1}.$$

Remark 4.2. Note that \mathbf{w}_2 and $w_1 = \mathbf{w}_1^N = (1 - \mathbf{w}_2^N)^{-1}$ and $w_0 = 1 - w_1^{-1}$ are determined by ρ_λ , but the choice of N -th root \mathbf{w}_1 (or \mathbf{w}_0) is free; this choice determines \mathbf{w} , and hence $\rho_{\lambda'}$, completely by the relation $\mathbf{w}_0\mathbf{w}_1\mathbf{w}_2 = -q$. Also, we have $\rho_\lambda(X_5) = \mathbf{y}_1^3\mathbf{y}_0^1 A_1 \otimes A_0 = -q^{-1}\mathbf{w}_2 A_1 \otimes A_0$, $\rho_{\lambda'}(X'_5) = -q\mathbf{w}_2^{-1} A_2 \otimes A_1$, so that

$$\rho_\lambda(X_5^N) = -w_2 I_N \otimes I_N, \rho_{\lambda'}((X'_5)^N) = -w_2^{-1} I_N \otimes I_N.$$

Proof. It is enough to check (25) for $X \in \{X_1, \dots, X_5\}$, using the relations (15). Let us do the cases $X = X_5$ and X_4 , the other cases being respectively similar to this last. For $X = X_5$ recall that $\varphi_{\lambda\lambda'}^q(X'_5) = X_5^{-1}$; in this case the identity (25) reads

$$\mathcal{L}_N^T(\Delta, b, \mathbf{w}) \circ (\mathbf{y}_0^1\mathbf{y}_1^3 A_1 \otimes A_0) = (\mathbf{y}_1^0\mathbf{y}_2^2 A_2 \otimes A_1)^{-1} \circ \mathcal{L}_N^T(\Delta, b, \mathbf{w}).$$

Consider the factorization formula (11). We have $(A_2 \otimes A_1)^{-1} = qA_1 A_0 \otimes A_1^{-1}$, so it commutes with $\Psi_{\mathbf{w}}(-A_0 A_1 \otimes A_1^{-1})$ and we get

$$\begin{aligned} & (\mathbf{y}_1^0\mathbf{y}_2^2 A_2 \otimes A_1)^{-1} \circ \mathcal{L}_N^T(\Delta, b, \mathbf{w}) = \\ & = \Psi_{\mathbf{w}}(-A_0 A_1 \otimes A_1^{-1}) \circ (q(\mathbf{y}_1^0\mathbf{y}_2^2)^{-1} A_1 A_0 \otimes A_1^{-1}) \circ \left(\frac{1}{N} \sum_{i,j=0}^{N-1} q^{-2ij-j} A_0^i \otimes (A_1 A_0)^{-j} \right) \\ & = \Psi_{\mathbf{w}}(-A_0 A_1 \otimes A_1^{-1}) \circ \left(\frac{1}{N} \sum_{i,j=0}^{N-1} q^{-2(i+1)(j+1)-(j+1)+3} A_0^i \otimes (A_1 A_0)^{-j} \right) \circ \\ & \quad \circ (q(\mathbf{y}_1^0\mathbf{y}_2^2)^{-1} A_1 A_0 \otimes A_1^{-1}) \\ & = \mathcal{L}_N^T(\Delta, b, \mathbf{w}) \circ (q^3 A_0^{-1} \otimes A_1 A_0) \circ (q(\mathbf{y}_1^0\mathbf{y}_2^2)^{-1} A_1 A_0 \otimes A_1^{-1}) \\ & = \mathcal{L}_N^T(\Delta, b, \mathbf{w}) \circ ((\mathbf{y}_1^0\mathbf{y}_2^2)^{-1} A_1 \otimes A_0). \end{aligned}$$

Hence the identity (25) holds true for $X = X_5$ whenever

$$(27) \quad (\mathbf{y}_1^0\mathbf{y}_2^2)^{-1} = \mathbf{y}_0^1\mathbf{y}_1^3 = -q^{-1}\mathbf{w}_2.$$

For $X = X_4$ recall that $\varphi_{\lambda\lambda'}^q(X'_4) = (1 + qX_5^{-1})^{-1}X_4$; in this case the identity (25) reads

$$(28) \quad \mathcal{L}_N^T(\Delta, b, \mathbf{w}) \circ (\mathbf{y}_2^3 A_2 \otimes I_N) = (1 + q\mathbf{y}_2^2 \mathbf{y}_1^0 A_2 \otimes A_1) \circ (I_N \otimes \mathbf{y}_0^0 A_0) \circ \mathcal{L}_N^T(\Delta, b, \mathbf{w}).$$

Now we have:

$$\begin{aligned} & \mathcal{L}_N^T(\Delta, b, \mathbf{w}) \circ (\mathbf{y}_2^3 A_2 \otimes I_N) \\ &= \Psi_{\mathbf{w}}(-A_0 A_1 \otimes A_1^{-1}) \circ (\mathbf{y}_2^3 A_2 \otimes I_N) \circ \left(\frac{1}{N} \sum_{i,j=0}^{N-1} q^{-2i(j+1)-j} A_0^i \otimes (A_1 A_0)^{-j} \right) \\ &= \Psi_{\mathbf{w}}(-A_0 A_1 \otimes A_1^{-1}) \circ (q\mathbf{y}_2^3 A_2 \otimes A_1 A_0) \circ \Upsilon \\ &= (q\mathbf{y}_2^3 A_2 \otimes A_1 A_0) \circ \Psi_{\mathbf{w}}(q^2(-A_0 A_1 \otimes A_1^{-1})) \circ \Upsilon \\ &= (q\mathbf{y}_2^3 A_2 \otimes A_1 A_0) \circ (\mathbf{w}_0 + q^2 \mathbf{w}_1^{-1} A_1 A_0 \otimes A_1^{-1}) \circ \Psi_{\mathbf{w}}(-A_0 A_1 \otimes A_1^{-1}) \circ \Upsilon \\ &= (1 + \mathbf{w}_0 \mathbf{w}_1 (A_1 A_0)^{-1} \otimes A_1) \circ (\mathbf{w}_1^{-1} A_1 A_0 \otimes A_1^{-1}) \circ \\ & \quad \circ (q\mathbf{y}_2^3 A_2 \otimes A_1 A_0) \circ \mathcal{L}_N^T(\Delta, b, \mathbf{w}) \\ &= (1 - q^2 \mathbf{w}_2^{-1} A_2 \otimes A_1) \circ (I_N \otimes \mathbf{w}_1^{-1} \mathbf{y}_2^3 A_0) \circ \mathcal{L}_N^T(\Delta, b, \mathbf{w}). \end{aligned}$$

Note that we used the relation (13) in the fourth equality. Hence, using (27) we see that (25) holds true for $X = X_4$ whenever $\mathbf{y}_0^0 = \mathbf{w}_1^{-1} \mathbf{y}_2^3$. The other cases $X = X_1, X_2$ or X_3 are similar. \square

4.0.2. *Proof of Step 2.* Consider QH-tetrahedra (Δ, b, \mathbf{w}) , $(\Delta', b', \mathbf{w}')$ glued along a 2-face F . Denote as usual by F_s, F_t the abstract 2-faces of Δ, Δ' corresponding to F . Recall that the vertices v_0, v_1, v_2 of F_s and F_t are ordered by the branchings b and b' respectively; the gluing $F_s \rightarrow F_t$ is encoded by an even permutation $p := \sigma^{r(F)}$ of the vertices, where σ is the 2-cycle $(v_0 v_1 v_2)$ and $r(F) \in \mathbb{Z}/3\mathbb{Z}$.

Assume that a standard representation of the triangle algebra $\rho_s: \mathcal{T} \rightarrow \text{End}(\mathbb{C}^N)$ is given on F_s , where the generators Y_0 labels the edge $[v_0, v_1]$, and Y_1 labels the edge $[v_0, v_2]$. Using the gluing, ρ_s is the pull-back of a standard representation $\rho_t: \mathcal{T} \rightarrow \text{End}(\mathbb{C}^N)$ on F_t , where now Y_0 labels the edge $[v_{p(0)}, v_{p(1)}]$, and Y_1 labels the edge $[v_{p(0)}, v_{p(2)}]$. So Y_i on F_s corresponds to $Y_{p^{-1}(i)}$ on F_t .

We have:

Lemma 4.3. *The endomorphism $(Q_N^T)^{r(F)}$ intertwines ρ_s and ρ_t . Namely, for all X in \mathcal{T} we have $(Q_N^T)^{r(F)} \circ \rho_s(X) = \rho_t(X) \circ (Q_N^T)^{r(F)}$.*

Proof. It is enough to check this on generators, where it follows from $Q_N^T A_i (Q_N^T)^{-1} = A_{i-1}$ (indices mod(3)). This is straightforward to check. \square

Note that Q_N has order 3 up to multiplication by 4-th roots of 1 (see [5], Lemma 7.3: if $N = 2n+1$, we have $Q_N^3 = \phi_N^{-1} I_N$, where $\phi_N = (\frac{n+1}{N})$ if $N \equiv 1 \pmod{4}$, and $\phi_N = (\frac{n+1}{N}) i$ if $N \equiv 3 \pmod{4}$).

4.0.3. *Proof of Step 3.* Let $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ be a layered QH-triangulation of the cylinder C_ϕ as usual. Recall that for every $j \in \{0, \dots, k\}$, the triangulated surface (S, λ_j) inherits a weak branching \tilde{b}_j from its positive side in $(T, \tilde{b}, \mathbf{w})$. Order as $\tau_1^j, \dots, \tau_{2m}^j$ the branched abstract triangles of (λ_j, \tilde{b}_j) . For every $i \in \{1, \dots, 2m\}$, order as e_0^i, e_1^i, e_2^i the edges of the triangle τ_i^j , as described before Proposition 4.1 and Lemma 4.3. Label e_k^i with the generator Y_k^i of a copy of the triangle algebra \mathcal{T} .

Assume that for some $j \in \{0, \dots, k\}$ we are given a standard local representation

$$\rho_j = (\rho_1^j \otimes \dots \otimes \rho_{2m}^j) \circ i_{\lambda_j}: \mathfrak{T}_{\lambda_j}^q \rightarrow \text{End}((\mathbb{C}^N)^{\otimes 2m})$$

where the standard representations $\rho_i^j: \mathcal{T} \rightarrow \text{End}(\mathbb{C}^N)$ are associated to the triangles τ_i^j . As in (18), put $\rho_i^j(Y_k) := \mathbf{y}_{i,k}^j A_k$. By the very definition of i_{λ_j} (see (17)), for any edge e_s of λ_j , with edge generator $X_s \in \mathfrak{T}_{\lambda_j}^q$, we have (as usual we denote a monomial $\otimes_j A_j$ in $\mathcal{T}^{\otimes 2m}$ by omitting the terms $A_j = 1$):

- (a) $\rho_j(X_s) = \mathbf{y}_{l_i, a_i}^j \mathbf{y}_{r_i, b_i}^j A_{a_i}^{l_i} \otimes A_{b_i}^{r_i}$, if e_s is an edge of two distinct triangles $\tau_{l_i}^j$ and $\tau_{r_i}^j$, and $e_{a_i}^{l_i}, e_{b_i}^{r_i}$ are the edges of $\tau_{l_i}^j, \tau_{r_i}^j$ respectively identified to e_s .

- (b) $\rho_j(X_s) = q^{-1} \mathbf{y}_{k_i, a_i}^j \mathbf{y}_{k_i, b_i}^j A_{a_i}^{k_i} A_{b_i}^{k_i}$, if e_s is an edge of a single triangle $\tau_{k_i}^j$, and $e_{a_i}^{k_i}, e_{b_i}^{k_i}$ are the edges of $\tau_{k_i}^j$ identified to e_s , with $e_{a_i}^{k_i}$ on the right of $e_{b_i}^{k_i}$.

Consider the diagonal exchange $\lambda_j \rightarrow \lambda_{j+1}$, and the corresponding triangulated (abstract) squares Q and Q' , as in Figure 3. The standard representations $\rho_i^j : \mathcal{T} \rightarrow \text{End}(\mathbb{C}^N)$ associated to the triangles of Q define a standard local representation ρ_Q of $\mathfrak{T}_{\lambda_j|Q}^q$. Denote by $\rho_{Q'}$ the standard local representation of $\mathfrak{T}_{\lambda_{j+1}|Q'}$ defined from ρ_Q by the relations (26), where (Δ, b, \mathbf{w}) is the QH-tetrahedron of $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ bounded by Q and Q' ; we describe below the relations between \mathbf{w} and ρ_j (and hence ρ_Q). The representation $\rho_{Q'}$ extends (by fusion) to a standard local representation

$$\rho_{j+1} = (\rho_1^{j+1} \otimes \dots \otimes \rho_{2m}^{j+1}) \circ i_{\lambda_j} : \mathfrak{T}_{\lambda_{j+1}}^q \rightarrow \text{End}((\mathbb{C}^N)^{\otimes 2m})$$

by stipulating that $\rho_i^{j+1} = \rho_i^j$ on the common triangles of (S, λ_j) and (S, λ_{j+1}) . Since (S, λ_{j+1}) is given the weak-branching \tilde{b}_{j+1} induced by its positive side in $(T_{C_\phi}, \tilde{b}, \mathbf{w})$, for every triangle τ_i^{j+1} of (S, λ_{j+1}) where \tilde{b}_{j+1} differs from the weak-branching given by (S, λ_j) or Q' , replace the corresponding representation ρ_i^{j+1} by $(Q_N^T)^{r(F)} \circ \rho_i^{j+1}(X) \circ (Q_N^T)^{-r(F)}$, as in Lemma 4.3. For simplicity, we keep however the same notation for ρ_{j+1} . Define

$$L_{\lambda_j \lambda_{j+1}}^{\rho_j \rho_{j+1}} : \text{End}((\mathbb{C}^N)^{\otimes 2m}) \rightarrow \text{End}((\mathbb{C}^N)^{\otimes 2m})$$

as the operator acting as the identity on all factors except on the representation space $\mathbb{C}^N \otimes \mathbb{C}^N \hookrightarrow (\mathbb{C}^N)^{\otimes 2m}$ of ρ_Q , where it acts by $\left((Q_N^T)^{r(F_2)} \otimes (Q_N^T)^{r(F_0)} \right) \circ \mathcal{L}_N^T(\Delta, b, \mathbf{w})$. Here, as usual we denote by F_2 and F_0 the 2-faces supporting the target space of $\mathcal{L}_N^T(\Delta, b, \mathbf{w})$. By construction, for every $X \in \mathfrak{T}_{\lambda_{j+1}}^q$ we have

$$(29) \quad L_{\lambda_j \lambda_{j+1}}^{\rho_j \rho_{j+1}} \circ \rho_j \circ \varphi_{\lambda_j \lambda_{j+1}}^q(X') \circ (L_{\lambda_j \lambda_{j+1}}^{\rho_j \rho_{j+1}})^{-1} = \rho_{j+1}(X).$$

Finally, assuming that a standard local representation ρ_j is given for $j = 0$, by working as above we can define inductively a sequence of standard local representation ρ_j , $j = 0, \dots, k$, and set

$$L_{\lambda_0, \lambda_k}^{\rho_0, \rho_k} := L_{\lambda_{k-1} \lambda_k}^{\rho_{k-1} \rho_k} \circ \dots \circ L_{\lambda_0 \lambda_1}^{\rho_0 \rho_1}.$$

Comparing with (8) we see that

$$L_{\lambda_0, \lambda_k}^{\rho_0, \rho_k} = \mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T.$$

Now we have to specialize the choice of ρ_0 , structurally related to the triangulation $(T_{C_\phi}, \tilde{b}, \mathbf{w})$. Similarly to the ‘‘classical’’ lateral shape parameters $W_j^+(e)$ (Section 3.1), put:

Definition 4.4. For every edge e of (S, λ_j) , the *lateral quantum shape parameter* $\mathbf{W}_j^+(e)$ is the product of the quantum shape parameters $\mathbf{w}(E)$ of the abstract edges $E \rightarrow e$ carried by tetrahedra lying on the positive side of (S, λ_j) . The *quantum shear-bend coordinates* of (S, λ_j) are the scalars

$$\mathbf{x}^j(e) := -q^{-1} \mathbf{W}_j^+(e).$$

Denote by $\lambda_j^{(1)}$ the set of edges of the ideal triangulation λ_j .

By Proposition 3.2, the quantum shear-bend coordinates $\mathbf{x}^j(e)$, $e \in \lambda_j^{(1)}$, form a system of N -th roots of the shear-bend coordinates $x^j(e)$ of λ_j . By fixing w and varying \mathbf{w} over w in the tuple $(T_{C_\phi}, \tilde{b}, \mathbf{w})$, one obtains a family of systems $\{\mathbf{x}^j(e)\}_{e \in \lambda_j^{(1)}}$ such that $\mathbf{x}^0(e) = \mathbf{x}^k(\phi(e))$ for every edge e of $\lambda = \lambda_0$.

The QH-triangulation $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ determines $\mathbf{x}^j(e_s)$ for all $j \in \{0, \dots, k\}$ and every edge e_s of λ_j . Consider the case $j = 0$, and the regular map

$$\pi_0 : \{\mathbf{y}^0 = (\mathbf{y}_{i,k}^0)_{i,k}\} \rightarrow \{\mathbf{x}^0 = (\mathbf{x}^0(e_s))_{e_s \in \lambda_0^{(1)}}\}$$

defined by

$$(30) \quad \mathbf{x}^0(e_s) = \mathbf{y}_{i, a_i}^0 \mathbf{y}_{r_i, b_i}^0 \quad \text{or} \quad \mathbf{y}_{k_i, a_i}^0 \mathbf{y}_{k_i, b_i}^0$$

according to the cases (a) or (b) described at the beginning of Step 3. A simple dimensional count shows that π_0 is surjective, with generic fibre isomorphic to $(\mathbb{C}^*)^{3m}$. Then, take a point $\mathbf{y}^0 \in \pi_0^{-1}(\mathbf{x}^0)$. We get

$$(31) \quad \rho_0(X_s) = \mathbf{x}^0(e_s)A_{a_i}^{l_i} \otimes A_{b_i}^{r_i} \quad \text{or} \quad \rho_0(X_s) = q^{-1}\mathbf{x}^0(e_s)A_{a_i}^{k_i}A_{b_i}^{k_i}$$

again according to the cases (a) or (b) above. We have to check that the sequence of local representations ρ_j constructed by starting from ρ_0 is consistent, that is, compatible with \mathbf{w} under the diagonal exchanges. Using the notations of Figure 3, by the tetrahedral and edge relations satisfied by the quantum shape parameters, for all $j \in \{0, \dots, k\}$ we have

$$(32) \quad \mathbf{x}^j(e_5) = -q^{-1}\mathbf{w}_2, \mathbf{x}^{j+1}(e_5) = -q\mathbf{w}_2^{-1}$$

and

$$(33) \quad \begin{aligned} \mathbf{x}^{j+1}(e_1) &= \mathbf{x}^j(e_1)\mathbf{w}_0^{-1} & , & \quad \mathbf{x}^{j+1}(e_2) = \mathbf{x}^j(e_2)\mathbf{w}_1^{-1} \\ \mathbf{x}^{j+1}(e_3) &= \mathbf{x}^j(e_3)\mathbf{w}_0^{-1} & , & \quad \mathbf{x}^{j+1}(e_4) = \mathbf{x}^j(e_4)\mathbf{w}_1^{-1}. \end{aligned}$$

In particular, (32) for $j = 0$ is consistent with the relation $\mathbf{w}_2 = -q\mathbf{y}_1^3\mathbf{y}_0^1$ required by Proposition 4.1 (where $\mathbf{y}_1^3 = \mathbf{y}_{r_i, b_i}^0$ and $\mathbf{y}_0^1 = \mathbf{y}_{l_i, a_i}^0$). Then, comparing (33) and (26) we see that for every $j \in \{0, \dots, k\}$, ρ_j is indeed defined by identities like (31), with \mathbf{x}^0 replaced by \mathbf{x}^j .

By (29), the representations ρ_0, \dots, ρ_k belong to a local representation $\rho = \{\rho_\lambda : \mathfrak{T}_\lambda^q \rightarrow \text{End}(V_\lambda)\}_\lambda$ of \mathfrak{T}_S^q . By (19), (31) and the similar identities for ρ_j , we see that the parameters of ρ_j are

$$(\{x^j(e)\}_{e \in \lambda_j^{(1)}}, h)$$

where $x^j(e) = (\mathbf{x}^j(e))^N$ and

$$(34) \quad h = \prod_{e \in \lambda_j^{(1)}} \mathbf{x}^j(e).$$

For this identity, first note that the right hand side, say h^j , does not depend on j . Indeed, by (32), (33) and the tetrahedral and edge relations for $\mathbf{w}_0, \mathbf{w}_1$ and \mathbf{w}_2 , we see that

$$\prod_{i=1}^5 \mathbf{x}^j(e_i) = \prod_{i=1}^5 \mathbf{x}^{j+1}(e_i)$$

under a diagonal exchange. Since the quantum shear-bend coordinates of edges not touched by a diagonal exchange are not altered, we deduce that $h^j = h^{j+1}$ for every $j = 0, \dots, k-1$. Now we can check (34) as follows. Consider a triangle τ_s^j of λ_j , with edges X_i, X_j, X_k , where $i < j < k$. The weak branching \tilde{b} provides a bijection $p: \{i, j, k\} \mapsto \{0, 1, 2\}$. Then $i_{\lambda_j}(H)$ is a tensor product of monomials associated to the triangles of λ_j , and the monomial for τ_s^j is $q^{-\sigma_{|\tau_s^j}} Y_{p(i)}^s Y_{p(j)}^s Y_{p(k)}^s$ where $-\sigma_{|\tau_s^j}$ is the contribution to the summand $-\sigma_{ij} - \sigma_{ik} - \sigma_{jk}$ of $-\sum_{l < l'} \sigma_{ll'}$ coming from the triangle τ_s^j . Noting that the product $Y_0^s Y_1^s Y_2^s$ is invariant under cyclic permutations, one checks immediately that $q^{-\sigma_{|\tau_s^j}} Y_{p(i)}^s Y_{p(j)}^s Y_{p(k)}^s = q^{-1} Y_0^s Y_1^s Y_2^s$. So

$$i_{\lambda_j}(H) = \otimes_{s=1}^{2m} (q^{-1} Y_0^s Y_1^s Y_2^s)$$

and hence

$$\rho_j(H) = \otimes_{s=1}^{2m} (\mathbf{y}_0^s \mathbf{y}_1^s \mathbf{y}_2^s q^{-1} A_0^s A_1^s A_2^s) = \mathbf{x}^j(e_1) \dots \mathbf{x}^j(e_n) \text{Id}_{(\mathbb{C}^N)^{\otimes 2m}}.$$

Summarizing our discussion, we have proved the following result. It concludes the proof of Step 3 of our First Main Theorem, and hence of Theorem 1.1 in the introduction.

Proposition 4.5. *The QH triangulation $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ determines a sequence of standard local representations $\rho_j : \mathfrak{T}_{\lambda_j}^q \rightarrow \text{End}((\mathbb{C}^N)^{\otimes 2m})$ associated to the triangulated surfaces (S, λ_j) , with parameters*

$$(\{x^j(e)\}_{e \in \lambda_j^{(1)}}, h), \quad h = \prod_{e \in \lambda_j^{(1)}} \mathbf{x}^j(e),$$

and belonging to a local representation ρ of \mathfrak{T}_S^q . Moreover, $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T$ intertwines the representations ρ_0 and ρ_k .

Let us consider now Theorem 1.2. For the convenience of the reader we re-state it. We keep the usual setting. Recall that M_ϕ is the interior of a compact manifold \bar{M}_ϕ with boundary made by tori.

Second Main Theorem. *There is a neighborhood of $i^*(\mathfrak{r}_h)$ in $i^*(X_0(M_\phi)) \subset X(S)$ such that:*

(1) *For any isomorphism class of local representations of \mathfrak{S}_S^q whose holonomy lies in this neighborhood, there are:*

- *A representative ρ of the class, and representations $\rho_\lambda, \rho_{\lambda'}$ belonging to ρ and acting on $(\mathbb{C}^N)^{\otimes 2m}$,*
- *a layered QH triangulation $(T, \tilde{b}, \mathbf{w})$ of M_ϕ such that $\mathcal{H}_N^{\text{red}}(T, \tilde{b}, \mathbf{w}) = \mathcal{H}_N^{\text{red}}(M_\phi, \mathfrak{r}, \kappa)$, for some weight $\kappa \in H^1(\partial\bar{M}_\phi, \mathbb{C}^*)$ and augmented character \mathfrak{r} of $\pi_1(M_\phi)$ such that the restriction of \mathfrak{r} to $\pi_1(S)$ is the holonomy of ρ ,*

such that the operator $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T$ is a QHI intertwiner which intertwins ρ_λ and $\rho_{\lambda'}$ as in Proposition 4.5. The load of ρ is determined by the values of the weight κ at the meridian curves that form $\bar{S} \cap \bar{M}_\phi$.

(2) *The family of QHI intertwiners $\{\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})^T\}$ consists of the QT intertwiners which intertwine ρ_λ to $\rho_{\lambda'}$ and whose traces are well defined invariants of triples $(M_\phi, \mathfrak{r}, \kappa)$ such that the restriction of \mathfrak{r} to $\pi_1(S)$ is the holonomy of ρ .*

Proof. (1) Let $\rho: \mathfrak{S}_\lambda^q \rightarrow \text{End}((\mathbb{C}^N)^{\otimes 2m})$ be a local representation with parameters $(\{x(e)\}_{e \in \lambda^{(1)}}, h)$. Recall Proposition 3.1 and the discussion that follows. Assume that the augmented character $[(\mathbf{r}, \{\xi_\Gamma\}_\Gamma)]$ associated to ρ lies in $i^*(X_0(M_\phi))$, and can be encoded by a system of shape parameters $w \in A_0 \subset A$, the simply connected open neighborhood of w_h in $G(T, \tilde{b})$ chosen before Remark 3.4. Then we can lift w to a system of quantum shape parameters $\mathbf{w} \in A_{0,N}$, and using the QH-triangulation $(T_{C_\phi}, \tilde{b}, \mathbf{w})$, we can define a local representation $\rho': \mathfrak{S}_\lambda^q \rightarrow \text{End}((\mathbb{C}^N)^{\otimes 2m})$ in the same way as ρ_0 in the proof of Proposition 4.5 (see (31)). It has quantum shear-bend coordinates $\mathbf{x}(e)$, $e \in \lambda^{(1)}$, and parameters $(\{x(e)\}_{e \in \lambda^{(1)}}, h')$, where $h' = \prod_{e \in \lambda^{(1)}} \mathbf{x}(e)$. The system $\{x(e)\}_{e \in \lambda^{(1)}}$ is the same for ρ and ρ' , so in order that $\rho' \cong \rho$, it remains to show that we can choose \mathbf{w} so that $h' = h$. But $(h')^N = \prod_{e \in \lambda^{(1)}} x(e) = h^N$, so $h' = \zeta h$ for some N -th root of unity ζ . On the other hand, by Theorem 3.5 (1), at all the meridian curves m_j of $\partial\bar{M}_\phi$ we have

$$\kappa_N(\mathbf{w})(m_j)^N = \pm a_j^{-2}$$

where a_j is defined before the identity (22), and we note that a_j^{-2} is the dilation factor of the similarity transformation associated to the conjugate of $\mathbf{r}(m_j)$ fixing ∞ , whence the squared eigenvalue at m_j selected by $[(\mathbf{r}, \{\xi_\Gamma\}_\Gamma)]$. Hence

$$(h')^{2N} = a_1^{-2} \dots a_r^{-2} = \pm \kappa_N(\mathbf{w})(m_1)^N \dots \kappa_N(\mathbf{w})(m_r)^N.$$

Moreover, there is $\mathbf{w}' \in A_{0,N}$ so that $\kappa_N(\mathbf{w}')(m_1) \dots \kappa_N(\mathbf{w}')(m_r)$ is any given N -th root of $a_1^{-2} \dots a_r^{-2}$. Let ζ' be the N -th root of unity such that $(h')^2 = \zeta' \kappa_N(\mathbf{w})(m_1) \dots \kappa_N(\mathbf{w})(m_r)$. By using the formula (6) it is easy to check that

$$(35) \quad \kappa_N(\mathbf{w})(m_j) = (-q^{-1})^{t_j} \prod_{e_1 p_j \in \partial e} \mathbf{W}^+(e)$$

where $\mathbf{W}^+(e)$ is the lateral quantum shape parameter of (S, λ) at the edge e of λ , t_j is the number of spikes of the triangles of λ at the j -th puncture p_j of S , and the product is over all the edges e , counted with multiplicity, having p_j as an endpoint. Recalling that there are $3m = -3\chi(S)$ edges in λ , we deduce

$$(36) \quad \kappa_N(\mathbf{w})(m_1) \dots \kappa_N(\mathbf{w})(m_r) = (-q^{-1})^{3.2m} \prod_{e \in \lambda^{(1)}} \mathbf{W}^+(e)^2$$

$$(37) \quad = (-q^{-1})^{6m} (-q)^{6m} \prod_{e \in \lambda^{(1)}} \mathbf{x}(e)^2 = (h')^2.$$

Hence $\zeta' = 1$. Then take $\mathbf{w}' \in A_{0,N}$ so that $\kappa_N(\mathbf{w}')(m_1) \dots \kappa_N(\mathbf{w}')(m_r) = (h')^2 \zeta^{-2}$. The load h'' of the representation ρ'' associated to \mathbf{w}' satisfies $(h'')^2 = (h')^2 \zeta^{-2} = h^2$. Since N is odd, and again $(h'')^N = h^N$, eventually $h = h''$. This achieves the proof of the claim (1) of the theorem.

(2) We need to describe the general form of the QT intertwiners associated to sequences of diagonal exchanges $\lambda \rightarrow \dots \rightarrow \lambda'$, for arbitrary standard local representations $\rho_\lambda, \rho_{\lambda'}$ of $\mathfrak{T}_\lambda^q, \mathfrak{T}_{\lambda'}^q$. This is done in Lemma 4.8. For that, we reconsider the arguments of the proof of the claim (1) in the First Main Theorem. First, we look at the case of a diagonal exchange $\lambda \rightarrow \lambda'$, occurring in squares Q, Q' as in Figure 3. We fix standard local representations $\rho_\lambda, \rho_{\lambda'}$ of $\mathfrak{T}_\lambda^q, \mathfrak{T}_{\lambda'}^q$, such that $\rho_{\lambda'} \circ (\varphi_{\lambda\lambda'}^q)^{-1}$ is isomorphic to ρ_λ , and differs from it only on Q, Q' , where $\rho_\lambda, \rho_{\lambda'}$ are represented by irreducible representations ρ_j of the triangle algebra with parameters \mathbf{y}_k^j , where $j \in \{0, 1, 2, 3\}$ and $k \in \{0, 1, 2\}$.

Lemma 4.6. *With the above notations we have*

$$(38) \quad \mathbf{y}_0^3/\mathbf{y}_0^2 = \mathbf{y}_2^1/\mathbf{y}_2^0, \quad \mathbf{y}_1^1/\mathbf{y}_1^2 = \mathbf{y}_2^3/\mathbf{y}_2^0, \quad \mathbf{y}_1^3\mathbf{y}_0^1 = (\mathbf{y}_2^2\mathbf{y}_1^0)^{-1}.$$

Moreover, denoting these scalars by $\tilde{\mathbf{w}}_0, \tilde{\mathbf{w}}_1$ and $\tilde{\mathbf{w}}_2$ respectively, $(w_0, w_1, w_2) := (\tilde{\mathbf{w}}_0^N, \tilde{\mathbf{w}}_1^N, \tilde{\mathbf{w}}_2^N)$ is a triple of shape parameters on (Δ, b) , that is $w_{i+1} = (1 - w_i)^{-1}$ for $i = 0, 1$.

We say that $(\tilde{\mathbf{w}}_0, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2)$ is a triple of *rough* q -shape parameters on (Δ, b) . Each of the scalars $\tilde{\mathbf{w}}_j$ corresponds to a pair of opposite edges of Δ . Note that, in general, the product $\tilde{\mathbf{w}}_0\tilde{\mathbf{w}}_1\tilde{\mathbf{w}}_2$ can be any N -th root of -1 , the special case of $-q$ being achieved by the q -shape parameters.

Proof. We prove first the second claim, that is, the N -th powers of the scalars set equal in (38) are indeed equal, and form a triple of shape parameters (w_0, w_1, w_2) . By the relations (15) we have

$$\begin{aligned} \varphi_{\lambda\lambda'}^q((X_5')^N) &= \varphi_{\lambda\lambda'}^q((X_5')^N) = X_5^{-N} \\ \varphi_{\lambda\lambda'}^q((X_1')^N) &= \varphi_{\lambda\lambda'}^q((X_1')^N) = (X_1 + qX_5X_1)^N = X_1^N + X_5^N X_1^N \\ \varphi_{\lambda\lambda'}^q((X_2')^N) &= \varphi_{\lambda\lambda'}^q((X_2')^N) = (X_2^{-1} + qX_2^{-1}X_5^{-1})^{-N} = (X_2^{-N} + X_2^{-N}X_5^{-N})^{-1} \end{aligned}$$

and a relation like the second for X_3', X_3 , and like the fourth for X_4', X_4 . Note that we use the q -binomial formula and the fact that q^2 is a primitive N -th root of 1, N odd, to deduce the results. Hence the shear-bend parameters x_i, x_i' at the edges $e_i, i \in \{1, \dots, 5\}$, satisfy

$$x_5' = x_5^{-1}, \quad x_1' = x_1(1 + x_5), \quad x_3' = x_3(1 + x_5), \quad x_2' = x_2(1 + x_5^{-1})^{-1}, \quad x_4' = x_4(1 + x_5^{-1})^{-1}.$$

Set $x_5 = -w_2$. By a mere rewriting of formulas, it follows easily from this, (30) and $\mathbf{x}(e)^N = x(e)$ for all edges e of λ , that

$$(39) \quad \begin{aligned} -w_2 &= (\mathbf{y}_1^3\mathbf{y}_0^1)^N = (\mathbf{y}_2^2\mathbf{y}_1^0)^{-N} \\ w_0 &= x_1/x_1' = x_3/x_3' = (\mathbf{y}_0^3/\mathbf{y}_0^2)^N = (\mathbf{y}_2^1/\mathbf{y}_2^0)^N \\ w_1 &= x_2/x_2' = x_4/x_4' = (\mathbf{y}_1^1/\mathbf{y}_1^2)^N = (\mathbf{y}_2^3/\mathbf{y}_2^0)^N. \end{aligned}$$

and $w_{i+1} = (1 - w_i)^{-1}$ for $i = 0, 1$. Now we prove (38). Let us reconsider Proposition 4.1 in the present situation. Instead of ρ_λ and the system $(\mathbf{w}_0, \mathbf{w}_1, \mathbf{w}_2)$ of quantum shape parameters on (Δ, b) , it is the pair of local representations $\rho_\lambda, \rho_{\lambda'}$ that are given on Q, Q' , and we are looking for $(\tilde{\mathbf{w}}_0, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2)$ such that the identity (25) holds true. Again (25) yields the first equality of (27), which is the last relation of (38), without implying any constraints on $(\tilde{\mathbf{w}}_0, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2)$. Then, use it to define $\tilde{\mathbf{w}}_2$ by the same formula. Replacing all uses of the relation (13) by (12) in the other computations defines $\tilde{\mathbf{w}}_0 = \mathbf{y}_0^3/\mathbf{y}_0^2$ and $\tilde{\mathbf{w}}_1 = \mathbf{y}_1^1/\mathbf{y}_1^2$, and eventually proves the first two identities in (38). For instance, in the case of $X = X_4$ the result becomes

$$\mathcal{L}_N^T(\Delta, b, \tilde{\mathbf{w}}) \circ (\mathbf{y}_2^3 A_2 \otimes I_N) = (1 - q^2 \tilde{\mathbf{w}}_2^{-1} A_2 \otimes A_1) \circ (I_N \otimes (-q^{-1} \tilde{\mathbf{w}}_0 \tilde{\mathbf{w}}_2 \mathbf{y}_2^3 A_0)) \circ \mathcal{L}_N^T(\Delta, b, \tilde{\mathbf{w}})$$

and hence (28) holds true whenever $\tilde{\mathbf{w}}_0 = -q\mathbf{y}_0^0/\mathbf{y}_2^3\tilde{\mathbf{w}}_2$. Comparing with the result of the similar computation for $X = X_2$ we find the second identity in (38). The other cases are similar. \square

Now we have:

Proposition 4.7. (See [7, 11]) *Assume that S is an ideal polygon with $p \geq 3$ vertices. Then for every ideal triangulation λ of S , every local representation of \mathfrak{T}_λ^q is irreducible.*

In the situation of the lemma we have the ideal squares Q, Q' , and so the proposition implies that $\mathcal{L}_N^T(\Delta, b, \tilde{\mathbf{w}})$ is a representative of the unique projective class of intertwiners from ρ_λ to $\rho_{\lambda'} \circ (\varphi_{\lambda\lambda'}^q)^{-1}$. More generally, let λ, λ' be ideal triangulations of S connected by a sequence of diagonal exchanges, and let ρ, ρ' be isomorphic standard local representations of \mathfrak{T}_S^q . Then every QT intertwiner $L_{\lambda\lambda'}^{\rho\rho'}$

between ρ_λ and $\rho_{\lambda'}$, has a representative of the form (24), where $L_{\lambda_i \lambda_{i+1}}^{\rho \rho'} = \mathcal{L}_N^T(\Delta, b, \tilde{\mathbf{w}})$ for some system $\tilde{\mathbf{w}} := (\tilde{\mathbf{w}}_0, \tilde{\mathbf{w}}_1, \tilde{\mathbf{w}}_2)$ of rough q -shape parameters (as in Lemma 4.6) on the tetrahedron (Δ, b) associated to the i -th diagonal exchange.

Hence, proceeding as in the proof of Proposition 4.5 we deduce that:

Lemma 4.8. *Every projective class of QT intertwiner in $\mathcal{L}_{\lambda \lambda'}^{\rho \rho'}$ is represented by a QH state sum $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \tilde{\mathbf{w}})^T$, where $\tilde{\mathbf{w}}$ is a system of rough q -shape parameters on T_{C_ϕ} .*

The proof of the claim (2) of the Second Main Theorem then follows by analysing the invariance properties of the QH state sums $\text{Trace} \left(\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \tilde{\mathbf{w}}) \right)$. Such an analysis has been done in [4] and [5] in the case of the unreduced QH state sums on arbitrary closed pseudo-manifold triangulations. It applies verbatim to the reduced QH state sums by combining the results of Section 5 of [4] with Proposition 6.3 (1) and 8.3 of [6].

The result is that $\text{Trace} \left(\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \tilde{\mathbf{w}}) \right)$ is an invariant of the triple $(M_\phi, \mathbf{r}, \kappa)$ if and only if it is invariant under any layered ‘2 \leftrightarrow 3 transit’ or ‘lune transit’ of $(T_{C_\phi}, \tilde{b}, \tilde{\mathbf{w}})$; such transits enhance to \tilde{b} and $\tilde{\mathbf{w}}$ the usual 2 \leftrightarrow 3 and lune moves between layered triangulations (which correspond to the pentagon and square relations for diagonal exchanges between surface triangulations). This invariance property happens if and only if $\tilde{\mathbf{w}}$ satisfies the tetrahedral and edge relations along every interior edge of T_{C_ϕ} , that is, when $\tilde{\mathbf{w}}$ is a genuine system of quantum shape parameters. In that situation $\text{Trace} \left(\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \tilde{\mathbf{w}}) \right)$ a QHI intertwiner.

For instance, the relation (27) implies that, modifying T_{C_ϕ} by introducing in the sequence $\lambda \rightarrow \dots \rightarrow \lambda'$ two consecutive diagonal exchanges along an edge e , $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \tilde{\mathbf{w}})$ is unchanged if and only if the total quantum shape parameter at e satisfies $\mathbf{W}(e) = q^2$.

This concludes the proof of our Second Main Theorem. \square

Finally, let us consider Corollary 1.3 and 1.4. For the reader’s convenience we re-state them below as Corollary 1 and 2. We keep the usual setting. Let us fix a QH-triangulation $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ of the cylinder C_ϕ . Recall that $\mathcal{H}_N^{\text{red}}(M_\phi, \mathbf{r}, \kappa) = \text{Trace} \left(\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w}) \right)$, and that M_ϕ is the interior of a compact manifold \bar{M}_ϕ with boundary made by tori. Call *longitude* any simple closed curve in $\partial \bar{M}_\phi$ intersecting a fibre of \bar{M}_ϕ in exactly one point.

Corollary 1. The reduced QH invariants $\mathcal{H}_N^{\text{red}}(M_\phi, \mathbf{r}, \kappa)$ do not depend on the values of the weight κ on the longitudes.

Proof. Denote by ρ_λ the local representation of \mathfrak{T}_λ^q associated to $(T_{C_\phi}, \tilde{b}, \mathbf{w})$ as in Proposition 4.5, where $\lambda = \lambda_0$. Since $\kappa = \kappa_N(\mathbf{w})$ is given by its values on the longitudes and the meridian curves m_j of $\partial \bar{M}_\phi$, it is enough to show that \mathbf{r} and the scalars $\kappa(m_j)$ determine the parameters $(\{x_1, \dots, x_n\}, h)$ of ρ_λ . We already know that x_1, \dots, x_n are determined by \mathbf{r} . Now, by the identities (36) and (37) we have $h^2 = \kappa(m_1) \dots \kappa(m_r)$. Among the two square roots of h^2 , the load h is the only one which is an N -th root of $(-1)^r a_1^{-1} \dots a_r^{-1}$ (using the notations of (22)). The eigenvalues a_i being determined by the augmented character \mathbf{r} , the conclusion follows. \square

Recall that given a local representation ρ_λ and an irreducible representation μ of \mathfrak{T}_λ^q , we denote by $\rho_\lambda(\mu)$ the isotypical component associated to μ in the direct sum decomposition of ρ_λ into irreducible factors. Consider the *isotypical intertwiners* $L_{\rho_\lambda(\mu)}^\phi := \mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}', \mathbf{w})_{|\rho_\lambda(\mu)}^T$.

Corollary 2. (1) The trace of $L_{\rho_\lambda(\mu)}^\phi$ is an invariant of $(M_\phi, \mathbf{r}, \kappa)$ and μ , well-defined up to multiplication by $4N$ -th roots of unity. It depends on the isotopy class of ϕ and satisfies

$$\mathcal{H}_N^{\text{red}}(M_\phi, \mathbf{r}, \kappa) = \sum_{\rho_\lambda(\mu) \subset \rho_\lambda} \text{Trace} \left(L_{\rho_\lambda(\mu)}^\phi \right).$$

(2) The invariants $\text{Trace} \left(L_{\rho_\lambda(\mu)}^\phi \right)$ do not depend on the values of the weight κ on longitudes.

Proof. (1) We take back the notations used for Corollary 1. Once again, $\mathcal{H}_N^{\text{red}}(M_\phi, \mathfrak{r}, \kappa)$ is determined by ϕ and the isomorphism class of the local representation ρ_λ , because it is the trace of $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})$. Each isotypical component of ρ_λ is the intersection of one eigenspace of each of the so called *puncture elements* of \mathfrak{T}_λ^q (see [7] and [13]). Hence it is determined by the parameters $(\{x_1, \dots, x_n\}, h)$ of ρ_λ , together with a system of *puncture weights* $p_j \in \mathbb{C}^*$, the eigenvalues of the puncture elements. These can be any N -th root of $\kappa(m_j)^N$, where $\kappa(m_j)$ is computed in (35). Hence \mathfrak{r} and κ determine one isotypical component of ρ_λ . The others correspond to all other systems of puncture weights obtained by multiplying each p_j with some N -th root of unity. Since $\mathcal{H}_N^{\text{red}}(T_{C_\phi}, \tilde{b}, \mathbf{w})$ intertwines ρ_λ and $\rho_{\lambda'}$, it intertwines their isotypical components too. The usual invariance proof of the reduced QHI still apply, and we get that the traces of the isotypical intertwiners are singly invariant.

By the results of [5, 6], the invariants $\mathcal{H}_N^{\text{red}}(M_\phi, \mathfrak{r}, \kappa)$ actually depend on the choice of mapping torus realization M_ϕ of M . Therefore the invariants $\text{Trace}(L_{\rho_\lambda(\mu)}^\phi)$ do as well. By the same arguments as for $\mathcal{H}_N^{\text{red}}(M_\phi, \mathfrak{r}, \kappa)$, they are also defined for any augmented character \mathfrak{r} in a Zariski open subset of the eigenvalue subvariety of $X_0(M)$ (see the comments on Theorem 1.2 in the Introduction).

The proof of (2) is the same as the one of the previous corollary. \square

REFERENCES

- [1] H. Bai, *A uniqueness property for the quantization of Teichmüller Spaces*, *Geom. Dedicata* 128:1 (2007) 1–16
- [2] H. Bai, *Quantum Teichmüller space and Kashaev’s 6j-symbols*, *Alg. Geom. Topol.* 7 (2007) 1541–1560
- [3] S. Baseilhac, R. Benedetti, *Quantum Hyperbolic invariants of 3-manifolds with $PSL(2, \mathbb{C})$ -characters*, *Topology* 43 (6) (2004) 1373–1423
- [4] S. Baseilhac, R. Benedetti, *Classical and quantum dilogarithmic invariants of flat $PSL(2, \mathbb{C})$ -bundles over 3-manifolds*, *Geom. Topol.* 9 (2005) 493–570
- [5] S. Baseilhac, R. Benedetti, *Analytic families of quantum hyperbolic invariants*, *Alg. Geom. Topol.* 15 (2015) 1983–2063
- [6] S. Baseilhac, R. Benedetti, *Non ambiguous structures on 3-manifolds and quantum symmetry defects*, to appear in *Quantum Topology*, ArXiv:1506.01174v2 (2016)
- [7] H. F. Bonahon, X. Liu, *Representations of the quantum Teichmüller space and invariants of surface diffeomorphisms*, *Geom. Topol.* 11 (2007) 889–937
- [8] H. Bai, F. Bonahon, X. Liu, *Local representations of the quantum Teichmüller space*, ArXiv:0707.2151 (2007)
- [9] L.D. Faddeev, R.M. Kashaev, *Quantum dilogarithm*, *Mod. Phys. Lett. A* 9 (5) (1994) 427–434
- [10] M. Lackenby, *Taut ideal triangulations of 3-manifolds*, *Geom. Topol.* 4 (2000) 369–395
- [11] F. Mazzoli, *Intertwining operators of the quantum Teichmüller space*, ArXiv:1610.06056 (2016)
- [12] B. Klaff, S. Tillmann, *A birationality result for character varieties*, *Math. Res. Lett.* (4) 23 (2016), 1099–1110
- [13] J. Toulisse, *Irreducible decomposition for local representations of quantum Teichmüller space*, Arxiv:1404.4938v2 (2016)