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# The Markov sequence problem for the Jacobi polynomials and on the simplex \*

D. Bakry †L.Mbarki, ‡

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## Abstract

We provide a simplified proof of the hypergroup property for the Jacobi polynomials on the unit interval, and investigate various generalizations of this property for the family of Dirichlet laws on the simplex.

## 1 Introduction

In this paper, we are interested in the Markov sequence problem. Given a unit orthonormal  $\mathcal{L}^2(\mu)$  basis  $\{f_0 = \mathbf{1}, f_1, \dots, f_n, \dots\}$  on some probability space  $(\mathbb{E}, \mathcal{E}, \mu)$ , this aims at the description of all sequences  $(\lambda_n)$ , such that the linear operator  $K$  defined through  $K(f_n) = \lambda_n f_n$  is a Markov operator, that is positivity preserving and  $K(\mathbf{1}) = \mathbf{1}$ . Since the last property amounts to  $\lambda_0 = 1$ , the problem is reduced to study the positivity preserving property. In what follows, we shall mainly concentrate on the beta measures on the interval and on the Dirichlet laws on the simplex.

This problem arises in many areas, particularly in statistics, special function theory, orthogonal polynomials theory and so on (see [1],[4],[5],[20],[36],[44],.. ) The aim of this paper is to describe these Markov sequences for some families of polynomials in many variables on the simplex  $\{x_i \geq 0; \sum_{i=1}^n x_i \leq 1\} \subset \mathbb{R}^n$ , orthogonal for the Dirichlet measures  $C_{p_1 \dots p_{n+1}} x_1^{\frac{p_1}{2}-1} \dots x_n^{\frac{p_n}{2}-1} (1-x_1-\dots-x_n)^{\frac{p_{n+1}}{2}-1} dx_1 \dots dx_n$ , where  $p_i > 0, i = 1, \dots, n+1$ . (The choice for this parametrization will be explained below).

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These Dirichlet measures again play an important rôle in many areas (statistics, probability, mathematical biology, etc), see ([22],[23],[46]), and are natural generalizations of beta measures on  $(-1, 1)$ . For the beta measure, we shall revisit the fundamental result of Gasper (see [20],[21]), for wich we propoe some extensions to the Dirichlet measures.

The Markov sequence set shares some basic generic properties, whatever the space  $\mathbb{E}$  and the basis  $\mathcal{F}$ .

As we already mentionned, since  $f_0 = \mathbf{1}$ ,  $\lambda_0 = 1$ . Moreover, it is easily seen that for any  $n$ ,  $|\lambda_n| \leq 1$ .

The set of Markov sequences is a convex set (a convex combination of sequences corresponds to the same convex combination of the associated Markov operators), and is closed under pointwise convergence on the sequences. Therefore, through Choquet's representation theorem, the description of all Markov sequences amounts to the description of the extremal ones.

Moreover, it is also stable under pointwise multiplication (which corresponds to the composition of the associated Markov operators).

Let us mention a few classical results concerning the Markov sequence problem.

### 1. Hermite polynomials.

The Hermite polynomials are the orthogonal polynomials for the Gaussian measure on  $\mathbb{R}$ , that is  $\mu(dx) = \frac{1}{\sqrt{2\pi}}e^{-\frac{x^2}{2}}dx$ . Sarmanov and Bratoeva [14] proved that for any Markov sequence, there exists a probability measure  $\nu$  on  $[-1, 1]$  such that  $\lambda_n = \int_{-1}^1 x^n \nu(dx)$ . In other words, the extremal Markov sequences are of the form  $\lambda_n = e^{-nt}$  for some  $t \geq 0$ , or  $(-1)^n e^{-nt}$ , for some  $t \geq 0$ . The sequence  $e^{-nt}$  correspond to a well known family of Markov operators  $K_t$ , namely the heat kernel associated with the Ornstein-Uhlenbeck operator. Indeed,  $K_t = e^{tL}$ , where  $L(f)(x) = f'' - xf'$ . This family of Markov kernels is known as the Ornstein-Uhlenbeck semigroup and there is a large literature devoted to it (see [32], [34]). Moreover, the sequence  $\lambda_n = (-1)^n$  corresponds to the symmetry  $K(f)(x) = f(-x)$ , so that those two operations generate all Markov sequences.

### 2. Ultraspherical polynomials.

The ultraspherical polynomials  $(P_n^\alpha)$  form the family of orthogonal polynomials for the ultraspherical probability measure  $C_\alpha(1 - x^2)^\alpha dx$  on  $(-1, 1)$ , where  $\alpha > -1$  is some real parameter and  $C_\alpha$  the normalizing constant. Then, Bochner's theorem [5] (see also [6, 7] and [12]) asserts that a sequence  $(\lambda_n)$  is a Markov sequence for this basis if and only if there exists a probability measure  $\nu$  on  $(-1, 1)$  such that

$$\lambda_n = \int_{-1}^1 \frac{P_n^\alpha(x)}{P_n^\alpha(1)} \nu(dx).$$

Indeed, at least formally, Sarmanov and Bratoeva's theorem may be deduced from Bochner's one, through a limiting procedure known as Poincaré ansatz,

that is considering the scaling of ultraspherical probability on  $(-\sqrt{a}, \sqrt{a})$  and letting  $a$  go to infinity. But the method followed in [14] is completely different.

### 3. Jacobi polynomials

Gaspar theorem, [9, 10, 11] concerns the beta measures  $C_{a,b}(1-x)^\alpha(1+x)^\beta dx$  on  $(-1, 1)$ , where  $\alpha, \beta > -1$ . As before, the basis is chosen to be the sequence of orthogonal polynomials for this measure, which are the Jacobi polynomials  $P_n^{\alpha,\beta}$ . Then, provided (when  $b \geq a \geq \frac{1}{2}$ , a sequence  $(\lambda_n)$  is a Markov sequence for this family if and only if there exists a probability measure  $\mu$  on  $(-1, 1)$  such that, for any  $n \in \mathbb{N}$ ,

$$\lambda_n = \int_{-1}^1 \frac{P_n^{\alpha,\beta}(x)}{P_n^{\alpha,\beta}(1)} \nu(dx).$$

We shall come back in Section 3 on this result, which is central in our study.

### 4. Eigenvectors of Sturm-Liouville operators

Another remarkable result in this direction is the Achour-Trimèche theorem, which may be stated as follows. Consider the interval  $[-1, 1]$ , and a probability measure  $\mu$  on it, with a smooth density  $\rho$ , that we suppose bounded for simplicity ( $0 < c \leq \rho \leq C < \infty$ ). Then, consider the diffusion operator  $L(f) = f'' + \frac{\rho'}{\rho} f'$ , which is symmetric in  $\mathcal{L}^2(\mu)$ . Then we chose as  $\mathcal{L}^2(\mu)$  basis  $(f_n)$  the one formed by the eigenvector of  $L$  with Neuman boundary condition, such that  $f_0 = 1$ . Then, provided that  $\log \rho$  is concave and symmetric, for any Markov sequence  $(\lambda_n)$  associated with this family  $f_n$ , there exists a probability measure  $\nu$  on  $(-1, 1)$  such that  $\lambda_n = \int_{-1}^1 \frac{f_n(x)}{f_n(1)} \nu(dx)$ .

This situation where the extremal values for the Markov sequence problem are given by the values  $\frac{f_n(x)}{f_n(x_0)}$  for some point  $x_0$  appears in a number of situation. This property is described in [2] where it is called the hypergroup property at the point  $x_0$ . In particular, it is proven there that, in the finite set case, the point  $x_0$  must be of minimal mass for the measure  $\mu$ . The sole exception in the above list is that of Hermite polynomials, which is in fact a degenerate case where the point  $x_0$  is  $+\infty$ .

This property is called the hypergroup property and is developed in Section 2.

Although Gaspar's result looks as a simple generalization of Bochner's one, which itself is a consequence of Achour-Trimèche one and contains as a limiting case the Hermite polynomial sequence, the proof of it is absolutely not straightforward, and it has been considerably simplified by Carlen, Geronimo and Loss [8] by a technique which we shall detail in full generality, and is also used in [17] [4] for the corresponding question for the family of orthogonal polynomials associated to the  $A_2$  root system. We provide here an even simplified proof of the proof of [8]. It relies on the construction of some symmetric diffusion operator having polynomial eigenvectors in some 3 dimensional space.

Moreover, we study this Markov sequence problem for the most direct extension of the beta measures, which are the above mentioned Dirichlet measures on the simplex.

The paper is organized as follows.

In Section 2, we introduce the hypergroup property, which is closely related to the Markov sequence problem. This is a property of some bases of  $\mathcal{L}^2(\mu)$  which provides automatically the answer to the Markov sequence problem. In Section 3, we concentrate on the case of Jacobi polynomials, for which the hypergroup property holds true, thanks to Gasper's theorem. In particular, we present the Carlen-Geronimo-Loss method, which provides in the geometric case a simplified proof of Gasper's theorem. With the help of some basic results on diffusion processes with polynomial eigenvectors, we then provide a simplified proof of Gasper's theorem in the non-geometric situation, following the scheme of Carlen-Geronimo-Loss, which avoids any computation. Finally, in Section 4, we introduce the Dirichlet measure on the simplex, and the natural generalization of the Jacobi polynomials. Although the situation is much more complicated, and despite the fact that the hypergroup property is much harder to investigate, we provide some bases having the hypergroup property, and, for the generalized Jacobi polynomials, we provide a description of Markov sequences, but only for Markov operators which strongly commute with the operator for which these generalized Jacobi polynomials are eigenvectors.

## 2 The hypergroup property : general description

Hypergroups appear in the literature as a natural extension of the notion of locally compact groups, where the convolution of two Dirac masses is a probability measure and no longer a Dirac mass, see for example [8, 28, 38, 39]. For example, they appear naturally when one wants to look at the convolution of class functions in a group.

The hypergroup property (denoted in short HGP in what follows) as described in [7] is just a simplification of this theory, basically valid in the previous situation in the compact setting, and appears as a key tool in many subjects like probability, statistics, statistical mechanics, coding theory and algorithms, reversible Markov chain, etc.

The hypergroup property concerns some properties of a unit  $\mathcal{L}^2(\mu)$  orthonormal basis on a probability space  $(\mathbb{E}, \mathcal{E}, \mu)$ . Consider a probability space  $(\mathbb{E}, \mathcal{E}, \mu)$ , where  $\mathbb{E}$  is a topological space,  $\mathcal{E}$  is the Borel  $\sigma$ -field,  $\mu$  a probability measure. On this space is given an orthonormal basis  $\mathcal{F}$  for  $\mathcal{L}^2(\mu)$   $\mathcal{F} = (f_0, f_1, \dots, f_n, \dots)$ , where we suppose that  $f_0 = 1$ . For everything to make sense, we shall require that the functions  $f_n$  are continuous.

Then, as mentioned earlier, the Markov sequence problem aims at the description of all sequences  $(\lambda_n)$ , with  $\lambda_0 = 0$  such that the (unique) operator such  $K(f_n) = \lambda_n f_n$  is a Markov operator, that is  $K(\mathbf{1}) = \mathbf{1}$  and  $f \geq 0 \implies K(f) \geq 0$ .

We already mentioned that the set of all Markov sequences is a compact (for the pointwise convergence) convex set, and therefore the description of all Markov sequences is therefore reduced to the description of the extremal points in this set.

Under very generic properties of the probability space, any Markov operator  $K$  may be represented as

$$K(f)(x) = \int f(y)K(x, dy),$$

where  $K(x, dy)$  is a Markov transition kernel, that is for each  $x$  a probability measure on  $E$  such that, for any  $A \in \mathcal{E}$ ,  $x \mapsto K(x, A)$  is measurable. Moreover, as soon as  $\sum_n \lambda_n^2 < \infty$ , then the operator is Hilbert-Schmidt, and the kernel  $K(x, dy)$  has a density with respect to the measure  $\mu$ , that is  $K(x, dy) = k(x, y)\mu(dy)$ , where

$$k(x, y) = \sum_n \lambda_n f_n(x) f_n(y),$$

where it is easily seen that the series converges in  $\mathcal{L}^2(E^2, \mu \otimes \mu)$ .

Then, as soon as  $\lambda_0 = 1$ , the Markov property amounts to check that the function  $k(x, y)$  is non negative. However, since every function  $f_n$  oscillates as soon as  $n \geq 1$ , since it satisfies  $\int_{\mathbb{E}} f_n(x) \mu(dx) = 0$ , it is in general not at all easy to obtain this positivity property from the previous representation.

In [2], the semigroup property is introduced as follows :

**Definition 2.1.** *The family  $\mathcal{F}$  has the hypergroup property at the point  $x_0$  if for any  $x \in E$ , the sequence  $\lambda_n(x) = \frac{f_n(x)}{f_n(x_0)}$  is a Markov sequence.*

The main consequence of [2], is that, when the hypergroup property holds at some point  $x_0$ , then the sequences  $\frac{f_n(x)}{f_n(x_0)}$  form the set of extremal sequences, and therefore, in this situation, for any Markov operator  $K$ , there exists a probability measure  $\nu_K$  on  $\mathbb{E}$  such that

$$\lambda_n = \int_{\mathbb{E}} \frac{f_n(x)}{f_n(x_0)} \nu_K(dx).$$

In the examples described in Section 1, this is the case for ultraspherical polynomials, for the Jacobi polynomials, and for the basis of Neuman eigenvectors of Sturm-Liouville operators, as soon as the reference measure is log-concave and symmetric.

The hypergroup property may be restated (in some more or less formal way however) into the following : for any  $(x, y, z) \in \mathbb{E}^3$ ,

$$(2.1) \quad k(x, y, z) = \sum_i \frac{f_i(x) f_i(y) f_i(z)}{f_i(x_0)} \geq 0.$$

But it may happen that this series is not convergent in  $\mathcal{L}^2(\mathbb{E}^3, \mu \otimes \mu \otimes \mu)$ , and that the formal measure  $k(x, y, z)\mu(dz)$  is not even absolutely continuous with respect to the measure  $\mu$ . Anyhow, one may describe, at least formally, the convolution  $\mu_1 * \mu_2$  of two probability measures  $\mu_1$  and  $\mu_2$  as the measure  $\mu_3$  with density with respect to  $\mu$  equal to  $\int K(x, y, z) d\mu_1(x) d\mu_2(y)$ , and then the measure  $k(x, y, z) d\mu(z)$  appears as the convolutions of the Dirac masses in  $x$  and  $y$ . Then, at least formally, one has

$$\int f_n(x) (\mu_1 * \mu_2)(dx) = \frac{1}{f_n(x_0)} \int f_n d\mu_1 \int f_n d\mu_2.$$

We can extend this convolution to all pairs of measures by bilinearity and to functions by identifying  $f$  to the measure  $f d\mu$ . With this in mind, the link with the usual theory of hypergroups is easily done.

Another aspect of the 3 variable kernel  $K(x, y, z)$  is that it allows some product formulas. Likewise, the probability kernel is nothing else than

$$K(x, y, dz) = \sum_n \frac{f_n(x)f_n(y)f_n(z)}{f_n(x_0)}\mu(dz).$$

In this way, the main result shows that the necessary and sufficient condition for this basis  $\mathcal{F}$  to check the HGP property is there for a probability kernel  $K(x, y, dz)$  for each  $n$ , the functions  $f_n$  satisfy the product formula

$$\frac{f_n(x)f_n(y)}{f_n(x_0)} = \int_{\mathbb{E}} f_n(z)K(x, y, dz).$$

In practise, for all this to make sense, it is useful to have at disposal a family  $\rho_n(t)$  of Markov sequences such that, for any  $t > 0$ ,  $\sum_n \rho_n^2(t) < \infty$ , and which converges (pointwise) as  $t \rightarrow 0$  to 1. Then, one applies all the previous formal computations to the Markov sequences  $\rho_n(t) \frac{f_n(x)}{f_n(x_0)}$ , and let  $t$  go to 0. In general, and in particular in the models studied below, this sequence  $\rho_n(t)$  is provided by some adapted heat kernel.

Another interesting aspect of the hypergroup property is its stability under tensorisation. Namely,

**Proposition 2.2.** *Assume that  $(\mathbb{E}_1, \mathcal{E}_1, \mu_1)$  and  $(\mathbb{E}_2, \mathcal{E}_2, \mu_2)$  are two probability spaces on which there exists two unit orthonormal bases  $(f_0 = \mathbf{1}, f_1, \dots, f_n, \dots)$  and  $(g_0 = \mathbf{1}, g_1, \dots, g_p, \dots)$ , both satisfying the hypergroup property at the respective points  $x_0 \in \mathbb{E}_1$  and  $y_0 \in \mathbb{E}_2$ . Then, on the product space  $(\mathbb{E}_1 \times \mathbb{E}_2, \mathcal{E}_1 \otimes \mathcal{E}_2, \mu_1 \otimes \mu_2)$ , the unit orthonormal basis  $(f_n(x)g_p(y), n, p \geq 0)$  satisfies the hypergroup property at the point  $(x_0, y_0)$ .*

*Proof.* — This is straightforward. If  $K_1^x(x_1, dx_2)$  is a Markov kernel on  $\mathbb{E}_1$  with eigenvectors  $f_n$  associated with the eigenvalue  $\frac{f_n(x)}{f_n(x_0)}$ , and  $K_2^y(y_1, dy_2)$  is a Markov kernel on  $\mathbb{E}_2$  with eigenvectors  $g_p$  associated with the eigenvalue  $\frac{g_p(y)}{g_p(y_0)}$ , then the product kernel  $K_1^x \otimes K_2^y$  has eigenvectors  $f_n(x_1)g_p(y_1)$  with associated eigenvalue  $\frac{f_n(x)}{f_n(x_0)} \frac{g_p(y)}{g_p(y_0)}$ . ■

let us finally mention that this HGP property may be seen as the dual of the GKS property, named after by Griffith-Kelly and Sherman who described the so called GKS inequality in statistical mechanics, and asserts the that the product of two elements of the  $\mathcal{L}^2(\mu)$  basis may be expressed as a linear combination of the elements of the basis with non negative coefficients.

## 3 Gasper's theorem

### 3.1 Jacobi Polynomials

As mentioned earlier, Gasper's theorem is the statement that the hypergroup property is valid for the family of Jacobi polynomials. One may find many proofs of it

in the literature see [1],[20],[21], [36]. It plays an important rôle in many areas, as in example in the proof of Bieberbach conjecture, see [29].

As described in the introduction (and with a small change in the notations that will be justified later), the beta measure  $\beta_{p,q}(dx)$  on  $(-1, 1)$  is defined as

$$\beta_{p,q}(dx) = C_{p,q}(1-x)^{\frac{p}{2}-1}(1+x)^{\frac{q}{2}-1}dx,$$

where  $p$  and  $q$  are positive and  $C_{p,q}$  is the normalizing constant which makes  $\beta_{p,q}$  is a probability measure. In what follows, we find convenient to move everything on  $(0, 1)$  through  $x \mapsto \frac{1+x}{2}$ , so that the beta measure is now, with another normalizing constant,

$$\beta_{p,q}(dx) = C_{p,q}x^{p/2-1}(1-x)^{q/2-1}dx.$$

The Jacobi polynomials are then as the unique family of orthogonal polynomials associated with  $\beta_{p,q}$  and positive dominant coefficient. We shall denote by  $P_n^{p,q}(x)$  the Jacobi polynomial of degree  $n$ .

The Jacobi polynomials are also the eigenvectors of the Jacobi operator on  $(0, 1)$

$$(3.2) \quad J_{p,q} = x(1-x)\frac{d^2}{dx^2} + \left[\frac{q}{2} - \left(\frac{q+p}{2}\right)x\right]\frac{d}{dx}$$

with eigenvalue equal to  $\lambda_n = -n(n + \frac{p+q}{2} - 1)$ . The specificity of these polynomials is that represent the unique family of orthogonal polynomials in dimension 1 (together with their limiting cases the Laguerre and Hermite polynomials) which are simultaneously the eigenvectors of diffusion operators, that is elliptic second order differential operators with no zero order terms.

Through a simple change of variables,  $P_n^{p,q}(\cos^2(t))$  are the eigenvector of the Sturm-Liouville operator  $\frac{d^2}{dt^2} + ((q-1)\cot(t) - (p-1)\tan(t))\frac{d}{dt}$  on  $[0, \pi]$ , with Neuman boundary condition, and is symmetric with respect of the measure  $\sin^{q-1}(t)\cos^{p-1}(t)dt$ .

Under this form, one may check that the density of the measure is log-concave as soon as  $p, q \geq 1$ , and is symmetric whenever  $p = q$ . So that, after a translation of  $-\pi/2$ , the latter case enter in the scope of Achour-Trimèche theorem, which is not the case when  $q \neq p$ .

For this family, we have

**Theorem 3.1.** *Let  $p, q \geq 0$ . Then, the hypergroup property holds for the family of Jacobi polynomials at the point  $x_0 = 1$  if and only if  $q \geq p \geq 1$ .*

As already mentioned in the introduction, Gasper's theorem is indeed an extension of a previous theorem due to Bochner [5], which deals with the symmetric case  $p = q$ , that is the case of ultraspherical (or Gegenbauer) polynomials. However, although the arguments for the symmetric case are quite easy to follow, the proofs of Gasper's theorem remained quite complicated, up to the paper [8], which provided an illuminating argument that we shall briefly recall below in Section 3.2.

Moreover, in the case  $p = q$ , letting  $p$  go to  $\infty$ , scaling  $x$  onto  $\frac{x}{\sqrt{p}}$  then the measure  $\mu_{p,p}$  converges to the Gaussian measure, the Jacobi polynomials converge to Hermite ones and  $\frac{2}{p}\mathcal{L}_{p,p}$  converge to the Hermite operator. With this in mind, Sarmanov-Bratoeva's result may be seen again as a limiting case of Bochner's theorem.



In the Jacobi polynomials case, it is worth to observe that the set of parameters for which the hypergroup property is valid is closed. Later on, the next Lemma 3.2 will allow us to restrict to cases where the auxiliary measures used in the proof have smooth densities.

**Lemma 3.2.** *If the hypergroup property for the Jacobi polynomials  $(P_n^{p_n, q_n})$  holds true for a sequence  $(p_n, q_n)$  converging to  $(p, q)$ , then it holds for  $(p, q)$ .*

*Proof.* — The family of orthogonal polynomials  $P_n^{p, q}$  is obviously continuous in the parameters  $(p, q)$ . The hypergroup property may be stated as the fact that the operator  $K(x)$  with eigenvalues  $\frac{P_n^{p, q}(x)}{P_n^{p, q}(1)}$  is positivity preserving. But this may be checked on polynomials, since any positive function may be approximated by positive polynomials, and any positive polynomial is a sum of squared polynomials. Therefore, it is enough to check that for any polynomial  $Q$  with degree  $k$ , one has  $K(Q^2) \geq 0$ .

But this translates into

$$K(Q^2)(y) = \int Q^2(z) \sum_{r=1}^{2k} \frac{P_r^{p, q}(x)}{P_r^{p, q}(1)} P_r^{p, q}(y) P_r^{p, q}(z) \mu_{p, q}(dz),$$

since  $Q^2$  is orthogonal to  $P_r^{p, q}$  for any  $r > 2k$ .

The polynomial  $Q$  being fixed, this property is obviously satisfied in the limit  $(p, q)$  as soon as it holds for a sequence  $(p_n, q_n)$ . ■

An important feature of the Jacobi operator is that, when  $p$  and  $q$  are integers, there is a natural interpretation of it through the unit sphere in dimension  $p + q - 1$ , and the Jacobi operator (3.2) is an image of the spherical Laplace operator.

Indeed, if one considers the unit sphere  $\mathbb{S}^{p+q-1} \subset \mathbb{R}^{p+q}$ , there is a diffusion operator on it, namely the spherical Laplace operator  $\Delta^{\mathbb{S}^{p+q-1}}$ , which commutes to rotations and is unique up to scaling. If one considers the function  $\mathbf{x} = (x_1, \dots, x_{p+q}) \in \mathbb{R}^{p+q} \mapsto y = \sum_{i=1}^p x_i^2 \in (0, 1)$ , one has, for any smooth function  $f : (-1, 1) \mapsto \mathbb{R}$

$$(3.3) \quad \Delta^{\mathbb{S}^{p+q-1}}(f(y)) = 4J_{p, q}(f)(y).$$

As such, the Jacobi operator  $J_{p, q}$  may be seen as an image of the spherical Laplace operator, and this remark is the key tool in the Carlen-Geronimo-Loss method to obtain the hypergroup property in this geometric case.

### 3.2 The Carlen-Geronimo-Loss method

The Carlen-Geronimo-Loss scheme appears to be a quite general method to obtain the hypergroup property in various contexts (see for example [4]).

Recall that we consider probability some space  $(\mathbb{E}, \mathcal{E}, \mu)$  on which we have a  $\mathcal{L}^2(\mu)$  orthonormal basis  $\mathcal{F} = (f_0 = 1, f_1, \dots, f_n, \dots)$ . As before, in order for everything to make sense, we shall assume that  $\mathbb{E}$  is a topological space, that  $\mathcal{E}$  is the Borel sigma-algebra, and that all the functions  $f_i$  are continuous.

We assume that we have some dense linear subspace  $\mathcal{A}$  in  $\mathcal{L}^2(\mu)$ , containing all the functions  $(f_n)$  of the basis  $\mathcal{F}$ , and a symmetric operator  $L : \mathcal{A} \mapsto \mathcal{A}$ . The basis  $\mathcal{F}$  is formed of eigenvectors of  $L$ , that is  $\mathcal{L}(f_n) = \rho_n f_n$ , for some real sequence  $(\rho_n)$ . In our example,  $\mathcal{A}$  will be the space of polynomials.

We assume that there is an auxiliary topological space  $(\mathbb{E}_1, \mathcal{E}_1, \mu_1)$ , endowed also with a dense subspace  $\mathcal{A}_1 \subset \mathcal{L}^2(\mu_1)$ , and another symmetric operator  $L_1 : \mathcal{A}_1 \mapsto \mathcal{A}_1$ . Moreover, there exists a continuous map  $\pi : \mathbb{E}_1 \mapsto \mathbb{E}$ , and another continuous map  $\phi : \mathbb{E}_1 \mapsto \mathbb{E}_1$ , with properties described in Proposition 3.3. We assume that the image of  $\mu_1$  under  $\pi$  is  $\mu$ . For a function  $f : \mathbb{E} \mapsto \mathbb{R}$ , we denote by  $\pi(f) : \mathbb{E}_1 \mapsto \mathbb{R}$  the function  $\pi(f)(y) = f(\pi(y))$ . Similarly, for a function  $g : \mathbb{E}_1 \mapsto \mathbb{R}$ , we denote  $\phi(g)(y) = g(\phi(y))$ . We also assume that  $f \in \mathcal{A} \implies \pi(f) \in \mathcal{A}_1$  and similarly  $g \in \mathcal{A}_1 \implies \phi(g) \in \mathcal{A}_1$ .

**Proposition 3.3.** *Assume the following*

1. For each  $n$ , the eigenspace of  $L$  associated with the eigenvalue  $\rho_n$  is one dimensional.
2.  $\pi L = L_1 \pi$ .
3.  $\phi L_1 = L_1 \phi$
4. For two points  $x_0$  and  $x$  in  $\mathbb{E}$ , if  $Y$  is a random variable with values in  $\mathbb{E}_1$  with law  $\mu_1$ , then the conditional law of  $\pi(\phi(Y))$  given that  $\pi(Y) = x_0$  is a Dirac mass at  $x$ .

Then, the sequence  $\frac{f_n(x)}{f_n(x_0)}$  is a Markov sequence for the basis  $(f_n)$ . (If  $f_n(x_0) = 0$ , then the conclusion is that we also have  $f_n(x) = 0$ ).

*Proof.* — Although the proof of this proposition is more or less implicit in [19], and fully developed in [17], we provide a sketch of it for completeness.

We denote  $\langle f, g \rangle$  the scalar product in  $\mathcal{L}^2(\mu)$  and  $\langle f, g \rangle_1$  the scalar product in  $\mathcal{L}^2(\mu_1)$ .

We consider the correlation operator  $K$  defined on bounded Borel functions  $f : \mathbb{E} \mapsto \mathbb{R}$  as follows

$$K(f)(x) = \mathbb{E}(\phi(\pi(f))(Y) / \pi(Y) = x),$$

where  $Y$  is a random variable with law  $\mu_1$ . It is clearly a Markov operator. We shall see that  $K(f_n) = \mu_n f_n$ , where  $\mu_n = \frac{f_n(x)}{f_n(x_0)}$ .

The main remark is that the hypothesis imply that  $K$  commutes with  $L$ . Indeed, the operator  $K$  is entirely determined by the following property, which is just a rephrasing of what a conditional expectation means

$$(3.4) \quad \forall f, g \in \mathcal{A}, \langle K(f), g \rangle = \langle \phi \pi(f), \pi g \rangle_1.$$

Then, for any pair  $(f, g) \in \mathcal{A}$ , we have

$$\begin{aligned} \langle LK(f), g \rangle &= \langle K(f), Lg \rangle = \langle \phi \pi(f), \pi L(g) \rangle_1 = \langle \phi \pi(f), L_1 \pi(g) \rangle_1 \\ &= \langle L_1 \phi \pi(f), \pi(g) \rangle_1 = \langle \phi L_1 \pi(f), \pi(g) \rangle_1 = \langle \phi \pi L(f), \pi(g) \rangle_1 \\ &= \langle KL(f), g \rangle, \end{aligned}$$

which proves the commutation property between  $K$  and  $L$ .

Therefore, if  $f_n$  is an eigenvector of  $L$ , with eigenvalue  $\rho_n$ , then  $K(f)$  is again an eigenvector of  $L$  with the same eigenvalue. Since the eigenspaces are one dimensional, then  $K(f_n) = \mu_n f_n$ , for some sequence  $\mu_n$ , which is therefore a Markov sequence.

Looking at the values at the point  $x_0$ , we get

$$f_n(x) = \mu_n f_n(x_0),$$

from which the conclusion follows. ■

**Remark 3.4.** *If there exist many such functions  $\phi$  such that the corresponding points  $x_\phi$  of the hypothesis 4 in Proposition 3.3 cover all the space  $\mathbb{E}$ , then  $f_n(x_0) \neq 0$ , since unless the function  $f_n$  would vanish identically.*

To obtain the property, one requires however many different maps  $\phi$  so that the associated points  $x = x_\phi$  cover all the space  $\mathbb{E}$ .

With this in mind, Gasper's theorem in the geometric case follows easily. Of course, in this context, the auxiliary space  $\mathbb{E}_1$  is  $\mathbb{S}^{p+q-1}$ ,  $\mathcal{L}_1$  is the spherical Laplace operator, and the map  $\pi$  is the map  $\mathbf{x} \mapsto y = \sum_{i=1}^p x_i^2$  described in Section 3.1.

The maps  $\phi$  are as follows : since  $p \leq q$ , for some point  $\mathbf{x} = (x_1, \dots, x_{p+q}) \in \mathbb{R}^{p+q}$ , we extract  $\mathbf{x}_1 = (x_1, \dots, x_p)$ ,  $\mathbf{x}_2 = (x_{p+1}, \dots, x_{2p})$  and  $\mathbf{x}_3 = (x_{2p+1}, \dots, x_{p+q})$  (the last one may be empty). Then, for  $\theta \in [0, 2\pi]$ ,  $\phi_\theta(\mathbf{x}) = (\mathbf{y}_1, \mathbf{y}_2, \mathbf{x}_3)$ , where

$$(3.5) \quad \mathbf{y}_1 = \cos(\theta)\mathbf{x}_1 + \sin(\theta)\mathbf{x}_2, \mathbf{y}_2 = -\sin(\theta)\mathbf{x}_1 + \cos(\theta)\mathbf{x}_2.$$

Then,  $\mathbf{x} \mapsto \phi_\theta(\mathbf{x})$  is a rotation in  $\mathbb{R}^{p+q}$ , and as such commutes with the spherical Laplace operator.

Then, it remains to observe that whenever  $\pi(\mathbf{x}) = 1$ , then  $\mathbf{x}_2 = \mathbf{x}_3 = 0$ , so that  $\pi(\phi_\theta(\mathbf{x})) = \cos^2(\theta)$ . Then, the conditional law property is satisfied (with  $x = \cos^2(\theta)$  and  $x_0 = 1$ ), and therefore we obtain the hypergroup property in this case.

To extend this proof to the general case, we shall require a few concepts from the general diffusion theory.

### 3.3 Symmetric diffusions and orthogonal polynomials

Most of the material presented here is borrowed from [1] for the general situation and from [3] for the particular case where orthogonal polynomials come into play.

A diffusion operator in an open set  $\Omega \subset \mathbb{R}^d$  is a second order semi-elliptic differential operator with no zero order terms. As such, it may be written in a given system of coordinates as

$$(3.6) \quad L(f)(x) = \sum_{ij} g^{ij}(x) \partial_{ij}^2 f + \sum_i b^i(x) \partial_i f,$$

where here and in what follows the coefficients  $g^{ij}(x)$  and  $b^i(x)$  are assumed to be smooth (indeed, for our purpose, they always will be polynomials in the variables  $(x_i)$  which are the coordinates of the point  $x$ ). The matrix  $g = (g^{ij})$  is always

symmetric and, in this paper, positive definite in  $\Omega$  (that is our operator  $L$  is indeed elliptic).

We are interested in the case where these operators are symmetric with respect to some measure  $\mu(dx)$  which has a smooth positive density  $\rho(x)$  with respect to the Lebesgue measure, that is, for any pair  $(f, g)$  of smooth functions  $\Omega \mapsto \mathbb{R}$ , compactly supported in  $\Omega$ , we require

$$(3.7) \quad \int_{\Omega} L(f)(x)g(x)\rho(x)dx = \int_{\Omega} f(x)L(g)(x)\rho(x)dx.$$

For this to happen, a necessary and sufficient condition is that

$$(3.8) \quad \forall i = 1, \dots, d, \quad b^i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}(x)\partial_j \log(\rho)(x),$$

since, by integration by parts

$$(3.9) \quad \int_{\Omega} L(f)(x)g(x)\rho(x)dx = - \int_{\Omega} g^{ij}\partial_i f\partial_j g\rho dx + \int_{\Omega} g\partial_i f[b_i - r_i]\rho dx,$$

where  $r_i(x) = \sum_j \partial_j g^{ij}(x) + \sum_j g^{ij}(x)\partial_j \log(\rho)(x)$ .

Such a measure is often called a reversible measure. It is unique in general, up to a multiplicative constant.

We then see that the coefficients  $b^i$  are then entirely determined by the second order terms  $g^{ij}$  and by the density  $\rho(x)$ .

Moreover, let us introduce the carré du champ  $\Gamma(f, g) = \frac{1}{2}(L(fg) - fL(g) - gL(f))$ . We have

$$\Gamma(f, g) = \sum_{ij} g^{ij}(x)\partial_i f\partial_j g,$$

and this bilinear operator characterizes the second order terms ( $g^{ij}$ ) of the operator  $L$ . We have  $g^{ij}(x) = \Gamma(x_i, x_j)$ , and, when the operator  $L$  is symmetric, for any pair of smooth compactly supported functions  $(f, g)$ , we have

$$(3.10) \quad \int_{\Omega} L(f)g\rho(x)dx = - \int_{\Omega} \Gamma(f, g)\rho(x)dx.$$

This is the integration by parts formula.

Moreover, the operator  $\Gamma$  allows to describe the so-called "change of variable formula", which is a way to describe in a general setting second order differential operators with no zero order terms. More precisely, when  $f_1, \dots, f_q$  are smooth functions  $\Omega \mapsto \mathbb{R}$ , then, for any smooth function  $\Phi : \mathbb{R}^q \mapsto \mathbb{R}$ , one has

$$(3.11) \quad L(\Phi(f_1, \dots, f_q)) = \sum_{ij} \Gamma(f_i, f_j)\partial_{ij}^2 \Phi(f_1, \dots, f_q) + \sum_i L(f_i)\partial_i \Phi(f_1, \dots, f_q).$$

It is also worth to observe that  $\Gamma$  is a bilinear operator which is first order in each of its variables, which translates into

$$(3.12) \quad \Gamma(\Phi_1(f_1, \dots, f_q), \Phi_2(f_1, \dots, f_q)) = \sum_{ij} \Gamma(f_i, f_j)\partial_i \Phi_1(f_1, \dots, f_q)\partial_j \Phi_2(f_1, \dots, f_q).$$

From this, one sees that in order to describe locally a symmetric diffusion operator, it is enough to describe in some coordinate basis  $(x_1, \dots, x_d)$  the quantities  $\Gamma(x_i, x_j)$  and either  $\rho$ , either  $L(x_i) = b^i(x)$ , provided they satisfy equation (3.8) for some  $\rho$ .

It is not necessary to restrict diffusion operators to open sets in  $\mathbb{R}^d$ . One may as well consider operators defined on smooth manifolds (and quite often compact manifolds such as spheres), or closed sets with boundaries. Then, the operator may be described through equation (3.6) in any local system of coordinates, and formula (3.11) allows to change coordinates to obtain a coherent system. However, when considering such operators on manifolds with boundaries, one has in general to describe to which functions one may apply the integration by parts formula (3.10). This is done in general through the prescription of the so called "boundary conditions" (such as Neuman or Dirichlet). In what follows, we shall require the possibility to apply this formula to any polynomial (or even any restriction to  $\Omega$  of any smooth function defined in a neighborhood of  $\Omega$ ), and this requires some extra conditions concerning the behavior of the matrix  $(g^{ij})$  at the boundary. Indeed, the fundamental property for that (assuming that the boundary is piecewise smooth) is that, for any regular point  $x_0$  of the boundary, the normal unit vector belongs to the kernel of the matrix  $(g^{ij})$ : in this situation, the extra term in integration by parts formula (3.10), coming from the boundary term in Stokes formula, vanishes (see [3], for example). It is easily seen that this condition is also sufficient.

This is what is hidden indeed in the boundary equation (3.13) below, which is the translation of this property when the boundary is described through some algebraic equation (see [3]).

A key feature is the notion of image of a diffusion operator  $L_1$  on some set  $\mathbb{E}_1$ . This is the basic tool to construct new diffusion operators  $L$  on a set  $\mathbb{E}$  and maps  $\pi : \mathbb{E}_1 \mapsto \mathbb{E}$  such that  $\pi L = L_1 \pi$ , as in Proposition 3.3.

Let  $\mathbb{E}_1$  be some space on which we have a diffusion operator  $L_1$  and  $d$  applications  $y_1, \dots, y_d : \mathbb{E}_1 \mapsto \mathbb{R}$ . Consider the map  $\pi : \mathbb{E}_1 \mapsto \mathbb{E} \subset \mathbb{R}^d$   $\pi(x) = (y_1, \dots, y_d)$ . Then, assume that for any  $i$ ,  $L(y_i) = B^i(\pi)$ , and for any  $i, j = 1, \dots, d$ , one has  $\Gamma(y_i, y_j) = G^{ij}(\pi)$ , for some functions  $B^i$  and  $G^{ij} : \mathbb{E} \mapsto \mathbb{R}$ . We say in this situation that we have a closed system. Then, the operator

$$L = \sum_{ij} G^{ij} \partial_{ij}^2 + \sum_i B^i \partial_i$$

defined on  $\mathbb{E}$  is such that  $L_1 \pi = \pi L$  (this is just the translation of equation (3.11)). Moreover,  $L$  is a diffusion operator which is symmetric as soon as  $L_1$  is, with reversible measure which is the image through  $\mathbf{y}$  of the reversible measure  $\mu_1$  of  $L_1$ . In this situation, we say that  $L$  is the image of  $L_1$  through  $\pi$ , or that  $L_1$  projects onto  $L$  through  $\pi$ . An example of this is the case of the spherical Laplace operator  $\Delta^{\mathbb{S}^{p+q-1}}$  which projects (up to the factor 4) onto the Jacobi operator through the map  $\mathbf{x} \mapsto \sum_i x_i^2$  as described in equation (3.3), so that the beta measure  $\beta_{p,q}$  is the image measure of the uniform measure on the sphere through this projection.

As mentioned above, the symmetry identity (3.7) is not enough for our purpose. We shall require it to be valid for pair of polynomials. In what follows, we shall be concerned with symmetric diffusion operators which may be diagonalized in a basis

of orthogonal polynomials. That is, for every  $n \geq 0$ , there exists a basis of the space of polynomials in  $d$  variables with degree less than  $n$  and which are at the same time eigenvectors for  $L$ . When this happens, we say that  $(\Omega, \Gamma, \rho)$  is a polynomial model, and  $\Omega$  is a polynomial domain.

When the set  $\Omega$  is bounded with a piecewise  $\mathcal{C}^1$  boundary, this requires the boundary of  $\Omega$  to be an algebraic set and requires some extra algebraic condition relating the boundary and the coefficients  $g^{ij}$ , called the boundary equation, see [3].

More precisely, the boundary  $\partial\Omega$  is included in an algebraic set  $\{P_1 \cdots P_k = 0\}$ , where  $P_i$  are real polynomials, which are irreducible in the complex field. Here, we assume that  $P_1 \cdots P_k = 0$  is the reduced equation of the boundary, that is

1. For each regular point  $x \in \partial\Omega$ , there exists a neighborhood  $\mathcal{V}(x)$  which contains  $x$  and a unique  $i$  such that  $\mathcal{V}(x) \cap \partial\Omega = \mathcal{V}(x) \cap \{P_i = 0\}$ .
2. For  $i = 1 \cdots k$ , there exist a regular point  $x \in \partial\Omega$  such that  $P_i(x) = 0$ .

Then, following [3], bounded polynomial models are characterized by the following

1. For any  $i, j = 1, \dots, d$ ,  $g^{ij}(x)$  is a polynomial with degree at most 2.
2. For any  $i = 1, \dots, d$ ,  $b^i(x)$  is a polynomial with degree at most 1.
3. For any  $i = 1, \dots, d$  and any  $q = 1, \dots, k$ , there exists a polynomial  $L_{i,q}$  with degree at most 1 such that

$$(3.13) \quad \sum_j g^{ij} \partial_j \log P_q = L_{i,q}.$$

(This is called the boundary equation).

As a consequence of the previous, each polynomial  $P_q$  is a factor of the polynomial (of degree at most  $2d$ ) of  $\det(g^{ij})$ . Moreover, every polynomial  $P_q$  has a constant sign on the open set  $\Omega$  and we may decide that they are all positive on it. Beyond this, provided  $(g^{ij})$  satisfies the boundary equation (3.13), for any choice of parameters  $a_1, \dots, a_k$  such that  $P_1^{a_1} \cdots P_k^{a_k}$  is integrable on  $\Omega$ , the density measure

$$(3.14) \quad \rho(x) = C_{a_1 \dots a_k} P_1^{a_1} \cdots P_k^{a_k},$$

where  $C_{a_1 \dots a_k}$  is the normalizing constant, is such that  $(\Omega, \Gamma, \rho)$  is a polynomial model.

Indeed, for the integration by parts formula to be true for a pair of polynomial functions, and thanks to the boundary equation (3.13), one may allow the parameters  $a_i$  in equation (3.14) to be negative, as long as  $a_i > -1$ , which is anyway a necessary condition for the measure  $\rho(x)dx$  to be finite on  $\Omega$ .

Sometimes one needs to extend those polynomial models using weighted degrees, that is deciding that the degree of a monomial  $x_1^{p_1} \cdots x_d^{p_d}$  is  $\sum_i n_i p_i$ , where  $n_1, \dots, n_d$  are some positive integers. All the picture remains valid, except that  $g^{ij}$  must have degree  $n_i + n_j$  and  $b^i$  must have degree  $n_i$ . We call the sequence  $(n_1, \dots, n_d)$  the weights of the polynomial model.

It is worth to observe that whenever  $(\Omega, \Gamma, \rho)$  is a polynomial model, and when we have a closed system  $(y_1, \dots, y_q)$  where the functions  $y_i$  are polynomials,

then the image model is again a polynomial system. But the degree may change. For example, if one starts from a polynomial model with the usual degree (that is  $n_i = 1$  for any  $i$ ), and if the degree of  $y_i$  is  $n_i$ , then we get a polynomial degree with weights  $n_1, \dots, n_d$ . Of course, one may always reduce to the case where the degrees have no common factor.

### 3.4 A proof of Gasper's theorem in the general case

In this section, we extend the proof of Gasper's theorem provided in Section 3.2 which was valid only in the geometric case (that is when  $p$  and  $q$  are integers) to the general case. For this, we need to construct a model  $(\mathbb{E}_1, L_1, \mu_1)$ , with the adapted functions  $\pi : \mathbb{E}_1 \mapsto \mathbb{E}$  and  $\phi_\theta : \mathbb{E}_1 \mapsto \mathbb{E}_1$  with the properties required in Proposition 3.3. The key observation is that, in the geometric picture, one just requires the knowledge of  $\|\mathbf{x}_1\|^2$ ,  $\|\mathbf{x}_2\|^2$  and the scalar product  $\mathbf{x}_1 \cdot \mathbf{x}_2$  to describe the action of the rotations  $\phi_\theta$  on  $\|\mathbf{x}_1\|^2$ .

For this, we first observe the action of the spherical Laplace operator on those variables. Following [1], the spherical Laplace operator in dimension  $d$  may be described through its action on the coordinates (that is considering the restrictions of the various coordinates  $x_1, \dots, x_{d+1}$  to the spheres as functions  $\mathbb{S}^d \mapsto \mathbb{R}$ ). Then, we get

$$(3.15) \quad \Gamma^{\mathbb{S}}(x_i, x_j) = \delta_{ij} - x_i x_j, \quad \Delta^{\mathbb{S}^d}(x_i) = -d x_i.$$

It is worth to observe that  $\Gamma$  does not depend on the dimension  $d$ . The image through  $\Delta^{\mathbb{S}^d}$  of polynomial in the variables  $x_i$  with degree less than  $n$  is again a polynomial in the variables  $x_i$  with degree less than  $n$ . From this, it is easily seen whenever we have a closed system made of polynomials, then the image of  $\Delta^{\mathbb{S}^d}$  through this system is a polynomial model.

Now fix  $d$  large enough and, for  $p \leq [d/2]$ , consider the 3 variables  $\mathbb{S}^d \mapsto \mathbb{R}$  defined as

$$X = \sum_{i=1}^p x_i^2, \quad Y = \sum_{i=p+1}^{2p} x_i^2, \quad U = \sum_{i=1}^p x_i x_{i+p}.$$

With the help of the change of variables formulas (3.11) and (3.12), we get

$$\begin{cases} \Gamma(X, X) = 4X(1 - X), \Gamma(Y, Y) = 4Y(1 - Y), \Gamma(U, U) = X + Y - 4U^2 \\ \Gamma(X, Y) = -4XY, \Gamma(X, U) = -4XU + 2U, \Gamma(Y, U) = -4YU + 2U \\ \Delta^{\mathbb{S}^d}(X) = -2(d+1)X + 2p, \Delta^{\mathbb{S}^d}(Y) = -2(d+1)Y + 2p, \Delta^{\mathbb{S}^d}(U) = -2(d+1)U, \end{cases}$$

which shows that the triple  $(X, Y, U)$  form a closed system for the spherical Laplace operator.

It is worth to observe that  $X$  itself is a sub-closed system of this closed system (and the image of the spherical Laplace operator is nothing else than the Jacobi operator, up to some affine transformation on the variable and scaling). Such is  $\{X, Y\}$ , but neither  $\{U\}$  or  $\{X, U\}$ , for example.

Let us consider the image of the sphere under  $x \mapsto (X, Y, U)$ . It is a polynomial domain in  $\mathbb{R}^3$  with boundary equation  $\{(1 - X - Y)(XY - U^2) = 0\}$ .

The image of  $\mathbb{S}^{d-1}$  through the map  $(X, Y, U)$  is therefore a polynomial model, with domain  $\mathbb{E}_1$  being the bounded set which is the connected component of  $\mathbb{R}^3$  which is the complement of the set  $\{(1 - X - Y)(XY - U^2) = 0\}$ , and which contains for example the point  $(1/4, 1/4, 1/8)$ . It is worth to observe that the boundary equation (3.13) is automatically satisfied for this model. Indeed, since the spherical operator may be diagonalized in a basis of orthogonal polynomials in the variable  $(x_i)$  (the eigenvectors are the restrictions to the sphere of the harmonic homogeneous polynomials in dimension  $d$ ), and one sees that the eigenvectors of this operator are nothing else than those polynomial eigenvectors which depend only on the variables  $X, Y, U$ .

With the  $\Gamma$  operator given in these coordinates as

$$G = (G^{ij}) := \begin{pmatrix} 4X(1 - X) & -4XY & -4XU + 2U \\ -4XY & 4Y(1 - Y) & -4YU + 2U \\ -4XU + 2U & -4YU + 2U & X + Y - 4U^2 \end{pmatrix},$$

one may check (but, as already mentioned, this is automatic) that the two polynomials  $1 - X - Y$  and  $XY - U^2$  satisfy the boundary equation. The reversible measure has density (up to a normalizing constant)  $(1 - X - Y)^a (XY - U^2)^b$ , where the coefficients  $a$  and  $b$  may be computed through equation (3.8). Then, we get

$$a = \frac{d-1}{2} - p, \quad b = \frac{p-3}{2},$$

Now, this diffusion operator again projects, up to a factor 4, on the Jacobi operator  $J_{p,q}$  through the map  $(X, Y, U) \mapsto X$ , whenever  $d = p + q - 1$ .

We may now consider this polynomial model  $(\mathbb{E}_1, \Gamma)$  with a new measure with density  $\rho(X, Y, U) = C(1 - X - Y)^a (XY - U^2)^b$ , where now  $a$  and  $b$  are real numbers..

It is easily seen that this measure is integrable on the domain  $\mathbb{E}_1$  as soon as  $a > -1$  and  $b > -1$ . Setting  $a = (q - p)/2 - 1$  and  $b = \frac{p-3}{2}$ , this requires  $q > p > 1$ , where now  $p$  and  $q$  are no longer integers but again real numbers.

We get in such a way a model  $(\mathbb{E}_1, \Gamma_1, \mu_1)$  which projects through the map  $\pi : (X, Y, U) \mapsto X$  on  $4J_{pq}$ , where  $J_{pq}$  is the Jacobi operator defined in equation (3.2).

To complete the picture, it remains to describe the operators  $\Phi_\theta : \mathbb{E}_1 \mapsto \mathbb{E}_1$  which commute with  $L_1$ . From the geometric picture, when  $p$  and  $q$  are integers, one may describe the action of the rotations  $\Phi_\theta$  defined in equation (3.5). We get  $\Phi_\theta(X) = A(X, Y, U)$ , where  $A$  is the linear operator with matrix

$$(3.16) \quad \begin{pmatrix} \cos^2(\theta) & \sin^2(\theta) & 2 \sin(\theta) \cos(\theta) \\ \sin^2(\theta) & \cos^2(\theta) & -2 \sin(\theta) \cos(\theta) \\ -\sin(\theta) \cos(\theta) & \cos(\theta) \sin(\theta) & \cos^2(\theta) - \sin^2(\theta) \end{pmatrix}.$$

To check that it commutes with  $L_1$ , and following Section 3.3, it is enough to check its action on the variables  $X, Y, U$  for  $L_1$  and  $\Gamma$ . The property for  $\Gamma$  comes from the geometric picture (only the action of  $L_1$  changes in the general case). As for the action of  $L_1$ , it may be checked directly.

As before, the point  $x_0$  is 1. Whenever  $\pi(X, Y, U) = 1$ , then  $(X, Y, U) = (1, 0, 0)$  and  $\pi\Phi_\theta(1, 0, 0) = \cos^2(\theta)$ .

This completes the proof of Gasper's theorem in the case  $q > p > 1$ . The general case  $q \geq p \geq 1$  comes from Lemma 3.2.



**Remark 3.5.** *If one considers the kernel  $K_\theta(f)(\xi) = \mathbb{E}\{f(\pi(R_\theta Z)) / \pi(Z) = \xi\}$ , the previous representation allows to compute it explicitly through some integral expression. However, the result is quite complicated, but it is easy to see that the kernel  $K_\theta(\xi, dy)$  has support  $[0, (\sqrt{\xi} \cos \theta + \sqrt{1 - \xi} \sin \theta)^2]$ .*

## 4 Dirichlet laws and diffusion processes on the simplex

### 4.1 Dirichlet laws, and a first basis with the HGP property

The  $d$ -dimensional simplex  $\mathbb{D}_d$  is the set of points  $(x_1, \dots, x_d) \in \mathbb{R}^d$  such that,  $\forall i = 1, \dots, d$ ,  $x_i \geq 0$  and that  $\sum_{i=1}^d x_i \leq 1$ . In what follows, it will be convenient to set  $x_{d+1} = 1 - \sum_{i=1}^d x_i$ , so that  $x_{d+1} \geq 0$  and  $\sum_{i=1}^{d+1} x_i = 1$ .

The Dirichlet laws  $\mu_{d,\mathbf{p}}$  depend on a multi-index real parameter  $\mathbf{p} = \{p_1, \dots, p_{d+1}\}$ ,  $p_i > 0, i = 1, \dots, d+1$ , are probability measures on  $\mathbb{D}_d$  with densities with respect to the Lebesgue measure  $dx_1 \dots dx_d$  of the form

$$C_{d,\mathbf{p}} x_1^{a_1} x_2^{a_2} \dots x_d^{a_d} x_{d+1}^{a_{d+1}},$$

where for  $i = 1, \dots, d+1$ ,  $a_i = \frac{p_i}{2} - 1$ .  $C_{d,\mathbf{p}}$  is the normalizing constant  $\frac{\Gamma(\sum_{i=1}^{d+1} a_i)}{\prod_{i=1}^{d+1} \Gamma(a_i)}$ , where  $\Gamma$  is the Euler function, which insures  $\mu_{d,\mathbf{p}}$  to be a probability. The choice of the parameters  $p_i$  instead of  $a_i = \frac{p_i}{2} - 1$ , similar to the choice made for beta measures, comes from geometric considerations which will be described below.

Dirichlet measures appear as extensions the beta measures on the interval. It turns out that the simplex is a polynomial domain as described in Section 3.3, so that the Dirichlet laws are the natural measures associated to it, the boundary of the domain having reduced equation  $x_1 \dots x_d (1 - x_1 - \dots - x_d) = 0$ .

When the parameters  $p_i$  are integers, then this Dirichlet law is the image measure of the uniform measure on the unit sphere in  $\mathbb{R}^n$ , with  $n = \sum_{i=1}^{d+1} p_i$ . Indeed, if we consider some partition of  $\{1, \dots, n\}$  in sets  $I_1, \dots, I_{d+1}$  with respective size  $p_1, \dots, p_{d+1}$ , and, for  $(y_1, \dots, y_n) \in \mathbb{S}^{n-1} \subset \mathbb{R}^n$ , consider the variables  $x_i = \sum_{j \in I_i} y_j^2$ , then  $\{x_1, \dots, x_d\} \in \mathbb{D}_d$ , and the image measure of the uniform measure on the sphere through the map  $y \mapsto (x_1, \dots, x_d)$  is  $\mu_{d,\mathbf{p}}$ . This will be obvious later on when we shall identify some diffusion operator on  $\mathbb{D}_d$  with reversible measure  $\mu_{d,\mathbf{p}}$  as the image of the spherical Laplace operator, as are the beta measures on  $[0, 1]$ .

It is worth to observe that the change of variables  $x_i \mapsto 1 - x_{d+1}$  allows to exchange the parameters  $p_i$  and  $p_{d+1}$ , so that one may order the parameters  $p_i, i = 1, \dots, d+1$ , in whichever order we like.

The change of variables  $x_i = y_i(1 - x_1)$ , for  $i = 2, \dots, d$  transforms the measure  $\mu_{d,\mathbf{p}}$  into a product measure  $\beta_{p_1, n-p_1}(dx_1) \otimes \mu_{d-1,\mathbf{q}}(dy_2 \dots dy_d)$ , where  $n = \sum_{i=1}^{d+1} p_i$ , and  $\mathbf{q} = \{p_2, \dots, p_{d+1}\}$ . Iterating the procedure, one may transform the Dirichlet

measure into a product of beta measure on  $[0, 1]^d$

$$\beta_{p_1, n-p_1} \otimes \beta_{p_2, n-p_1-p_2} \otimes \cdots \otimes \beta_{p_d, n-n_1-\cdots-p_d}.$$

We may now chose a basis for  $\mathcal{L}^2(\mathbb{D}_d, \mu_{d, \mathbf{p}})$  made of products of Jacobi polynomials associated to each of the factors (to be more precise, the image of these products under the inverse change of variables which maps  $[0, 1]^d$  to  $\mathbb{D}_d$ ). Now, provided that, for  $i = 1, d+1$ ,  $p_i \geq 1$ , one may apply Gasper's theorem and the tensorisation procedure of Proposition (2.2), and we therefore get the hypergroup property for this basis.

It is worth to observe that this procedure depends on the choice of the ordering in the parameters  $p_1, \dots, p_{d+1}$ , so that one may construct in this way many different bases. But this bases are not most natural direct extensions of the Jacobi polynomial bases on the simplex. In particular, in the coordinates  $(x_1, \dots, x_d)$ , they do not appear as polynomials, but as rational functions. That is why we explore some new polynomial bases in the next paragraph.

## 4.2 Diffusion operators on the simplex having polynomial eigenvectors

To describe the diffusion processes which may be diagonalized in a system of orthogonal polynomials on the simplex, we have just to describe their carré du champ  $\Gamma$ , since the measure is given. It is a special feature of the simplex that there are many such  $\Gamma$  structures which answer the question, beyond the mere scaling factor, and this situation is very peculiar (in the dimension 2 classification of [3], only the simplex, the circle, and a particular case of the double parabola have this property).

The various  $\Gamma$  operators on the simplex such that  $(\mathbb{D}_d, \Gamma, \mu_{d, \mathbf{p}})$  are a polynomial model have been described for example in [13]. They depend on a symmetric parameter matrix  $\mathbf{A}$  with entries  $A_{rs}$  as follows

$$(4.17) \quad g^{rs} := \Gamma_{\mathbf{A}}(x_r, x_s) = -A_{rs}x_r x_s + \delta_{rs}x_r \sum_{k=1}^{d+1} A_{rk}x_k, \quad 1 \leq r \leq s \leq d$$

where  $A_{rs} = A_{sr}$ ,  $1 \leq r \leq s \leq d+1$  are non negative real parameters. The operator is elliptic on the simplex as soon as, for every  $r \neq s$ ,  $A_{rs} \neq 0$ . One should check that the value of  $A_{ii}$  plays no rôle in the definition of  $\Gamma_{\mathbf{A}}$ , and we shall set  $A_{ii} = 0$ .

For this operator, and for the Dirichlet measure  $\mu_{d, \mathbf{p}}$ , one has

$$L_{\mathbf{A}, \mathbf{p}}(x_i) = \frac{1}{2} \sum_{k=1}^{d+1} A_{ik}(x_k p_i - x_i p_k).$$

One may check the validity of the boundary equation (3.13), that is the fact that  $\sum_{i=1}^d g^{ij} \partial_j \log P_p$  is an affine function for every boundary polynomial  $P_p = x_1, \dots, x_{d+1}$ .

Indeed, for  $k = 1, \dots, d+1$ , one has

$$\sum_{j=1}^d g^{ij} \partial_j \log x_k = -A_{ik}x_i + \sum_{q=1}^{d+1} A_{iq}x_q.$$

It is worth to write  $L_{\mathbf{A}, \mathbf{p}}$  as

$$L_{\mathbf{A}, \mathbf{p}} = \sum_{i < j} A_{ij} L_{ij, \mathbf{p}},$$

where  $L_{ij, \mathbf{p}}$  has a carré du champ  $\Gamma_{ij}$  with

$$(4.18) \quad \Gamma_{ij}(x_r, x_s) = x_i x_j [\delta_{rs}(\delta_{ri} + \delta_{rj}) - (\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si})]$$

and

$$(4.19) \quad L_{ij, \mathbf{p}}(x_r) = \frac{1}{2}(\delta_{ri} - \delta_{rj})(x_j p_i - x_i p_j).$$

In the case where all the  $A_{pq}$  are set to 1 (let us denote this matrix  $\mathbf{1}$ ), and when the parameters  $p_i$  are integers, there is a natural interpretation for this operator coming from the spherical Laplace operator in dimension  $n = \sum_{i=1}^{d+1} p_i - 1$ , that is for the sphere imbedded in  $\mathbb{R}^n$ .

Indeed, let  $n$  be an integer and, as in the previous subsection 4.1, consider the  $n - 1$  dimensional spherical Laplace operator acting on the unit sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ , defined through the equation  $\{\sum_1^n y_i^2 = 1\}$ . Let us look at a partition of the index set  $\{1, \dots, n\}$  into  $d+1$  disjoint sets  $I_1, \dots, I_{d+1}$  with respective size  $p_1, \dots, p_{d+1}$ , and as before the variables  $x_j = \sum_{i \in I_j} y_i^2$ . As already observed, the map  $y \in \mathbb{S}^{n-1} \mapsto \{x_1, \dots, x_d\}$  maps the sphere onto the simplex  $\mathbb{D}_d$ .

Moreover, following equation (3.15), we see that

$$(4.20) \quad \Gamma^{\mathbb{S}}(x_i, x_j) = 4(\delta_{ij}x_i - x_i x_j), \Delta^{\mathbb{S}^{n-1}}(x_i) = 2(p_i - nx_i).$$

The variables  $(x_1, \dots, x_d)$  form a closed system, and we see that those formulas are the one obtained for  $4L_{\mathbf{1}, \mathbf{p}}$ . One may therefore follow the same path as before to obtain the hypergroup property for the family of orthogonal polynomials which are the eigenvectors of this operator. Unfortunately, it turns out that the eigenspaces for  $L_{\mathbf{1}, \mathbf{p}}$  are not one dimensional.

Indeed, looking at a polynomial eigenvector of degree  $k$  and look at the action of  $L_{\mathbf{1}, \mathbf{p}}$  on its highest degree term  $x_{\mathbf{k}} := x_1^{k_1} \dots x_d^{k_d}$ , where  $k = \sum_1^d k_i$ . The highest degree term of  $L_{\mathbf{1}, \mathbf{a}}(x_{\mathbf{k}})$  is

$$-k(k + \frac{n-2}{2})x_{\mathbf{k}}$$

so that the corresponding eigenvalue is  $\nu_{\mathbf{k}} = -k(k + \frac{n-2}{2})$  depends only on  $k = \sum_1^d k_i$ . The eigenspaces have then dimension  $\binom{k+d-1}{k}$ . However, for this operator, one may follow the scheme of [8] and construct a new space  $\mathbb{E}_1$ , (the sphere in the geometric case), with a symmetric diffusion operator  $L_1$  on it, together with maps  $\pi = \mathbb{E}_1 \mapsto \mathbb{E}$  and  $\phi : \mathbb{E}_1 \mapsto \mathbb{E}_1$  with the properties that  $\pi L = L_1 \pi$ ,  $\Phi L_1 = L_1 \phi$ , together with the conditional law property at the point  $(1, 0, \dots, 0)$ . But the fundamental property that the eigenspaces of  $L$  are one dimensional is missing, and the analysis of Markov sequences is therefore much more delicate.

Indeed, following the scheme of the proof of Gasper's theorem, one may first concentrate on the geometric case. To understand the difficulty, let us also concentrate

on the case  $d = 2$ . In this situation, one has 3 integer parameters  $p_1 \leq p_2 \leq p_3$ , and, setting  $n = p_1 + p_2 + p_3$ , we look at the sphere  $\mathbb{S}^{n-1} \subset \mathbb{R}^n$ . Then, one considers three subsets  $I_1, I_2, I_3$  of  $\{1, \dots, n\}$ , with respective size  $p_1, p_2, p_3$  and three vectors  $\mathbf{x}_1 = (y_i, i \in I_1)$ ,  $\mathbf{z}_2 = (y_i, i \in I_2)$  and  $\mathbf{z}_3 = (y_i, i \in I_3)$ . Moreover, we split  $I_2$  and  $I_3$  into disjoint sets  $I_2 = J_2 \cup K_2$ ,  $I_3 = J_3 \cup K_3$ , with  $|J_2| = |J_3| = p_1$ . Then, we consider the vectors  $\mathbf{x}_2 = (y_i, i \in J_2)$ ,  $\mathbf{y}_3 = (y_i, i \in J_3)$ ,  $\mathbf{y}_2 = (y_i, i \in K_2)$  and  $\mathbf{y}_3 = (y_i, i \in K_3)$ .

We consider now the variables  $x_i = \|\mathbf{x}_i\|^2$ ,  $i = 1, 2, 3$ , and  $y_i = \|\mathbf{z}_i\|^2$ ,  $i = 2, 3$ . Moreover, we look at the variables  $u_{ij} = \mathbf{x}_i \cdot \mathbf{x}_j$ ,  $1 \leq i < j \leq 3$ . For simplicity, we stick to the case where  $p_1 < p_2 \leq p_3$ , and, observing that  $y_3 = 1 - x_1 - x_2 - x_3 - y_2$ , we are left to the 7 variables

$$(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}).$$

It happens that these 7 variables form a closed system for the spherical Laplace operator, and we obtain some operator  $L_7$  on some bounded polynomial domain  $\Omega_7 \subset \mathbb{R}^7$ . Moreover, the operator  $L_{1,\mathbf{p}}$  is the image of  $L_7$  under the map  $\pi_1 : \Omega_7 \mapsto \mathbb{D}_2$ ,  $\pi_1(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}) = (x_1, x_2 + y_2)$ . Let us denote by  $\pi_2$  the projection from the sphere onto  $\Omega_7$ , and  $\pi : \mathbb{S}^{n-1} \mapsto \mathbb{D}_2 = \pi_2 \pi_1$ .

The introduction of those variables allows us to look, for  $1 \leq i < j \leq 3$  at the horizontal rotations  $R_\theta^{ij}$ , which leave invariant the vectors  $\mathbf{y}_2, \mathbf{y}_3$  and  $\mathbf{x}_k$ ,  $k \neq \{i, j\}$ , and are defined through

$$R_\theta^{ij}(\mathbf{x}_i, \mathbf{x}_j) = (\cos \theta \mathbf{x}_i + \sin \theta \mathbf{x}_j, -\sin \theta \mathbf{x}_i + \cos \theta \mathbf{x}_j).$$

One may immediately see the action of these rotations on the variables

$$(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}),$$

as we did in dimension 1.

Observe that if we require to have enough horizontal rotations  $R$  with associated points  $x_R$  satisfying that the law of  $\pi R Y$  given  $\pi(Y) = (1, 0, \dots, 0)$  cover all the simplex  $\mathbb{D}_2$ , this system of 7 variables seem to be the minimal one.

Observe also that instead of considering those horizontal rotations  $R_\theta^{ij}$ , one could have considered the full  $SO(3)$  group acting in this horizontal way. It turns out that this is not necessary for our purpose.

In order to apply the one dimensional scheme, one may expect to find a common orthonormal base in the eigenspaces of  $L_{1,\mathbf{p}}$  in which the correlation operators  $K_\theta^{ij}(f)(x) = \mathbb{E}(R_\theta^{ij} \pi f(Y) / \pi(Y) = x)$ , where  $Y$  is uniformly distributed on the sphere, are jointly diagonalizable. We shall see that it is impossible. Indeed, if such was the case, they would commute to each other. But this is not the case, as shows the next Proposition 4.1.

**Proposition 4.1.** *The operators  $K_\theta^{12}$  and  $K_\theta^{13}$  do not commute to each other.*

*Proof.* — The operators  $K_\theta^{ij}$  are not easy to describe. But we may look at the easier operators  $S_{ij} = \partial_\theta K_{|\theta=0}^{ij}$ . But we shall see that those operators vanish identically. We may therefore compute  $R_{ij} = \partial_\theta^2 K_{|\theta=0}^{ij}$ .

To compute these operators  $S_{ij}$  and  $K_{ij}$  on the simplex, for the pairs  $(1, 2), (1, 3), (2, 3)$ , we observe that for two bounded polynomial functions  $f(x, y)$  and  $g(x, y)$  on  $\mathbb{D}_d$ , up to a constant 2, we have

$$\langle S_{12}(f), g \rangle = 2 \int_{\mathbb{S}^{D-1}} u_{12}(\partial_1 f - \partial_2 f)(\pi(\mathbf{y}))g(\pi\mathbf{y})d\mathbf{y},$$

where  $\pi(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23}) = (x_1, x_2 + y_2)$ .

So that  $S_{12}(f) = 2s_{12}(x)(\partial_i f - \partial_j f)$ , where

$$s_{12}(x) = \mathbb{E}(u_{12}(\mathbf{y})/\pi(\mathbf{y}) = (x_1, x_2 + y_2)),$$

which is 0 by symmetry, and

$$\langle K_{12}f, g \rangle = \int_{\mathbb{S}^{D-1}} [2(x_2 - x_1)(\partial_1 f - \partial_2 f) + 4u_{12}^2(\partial_1 - \partial_2)^2 f]\pi(\mathbf{y})g(\pi\mathbf{y})d\mathbf{y}.$$

So that

$$K_{12}(f) = 2k_{12}(\partial_1 - \partial_2)f + 4t_{12}(\partial_1 - \partial_2)^2 f,$$

where

$$k_{12}(x, y) = \mathbb{E}(x_1 - x_2/(x_1, x_2 + y_2) = (x, y)), \quad t_{12}(x, y) = \mathbb{E}(u_{12}^2(Y)/\pi(Y) = (x_1, x_2 + y_2) = (x, y)).$$

For the operators  $S_{13}$  and  $K_{13}$ , we may perform a similar computation, and obtain a similar computation,

$$K_{13}(f) = 2k_{13}(\partial_1 f - \partial_2 f) + 4t_{13}(\partial_1 - \partial_2)^2 f,$$

with

$$k_{13}(x, y) = \mathbb{E}(x_1 - x_3/(x_1, x_2 + y_2) = (x, y)), \quad t_{13}(x, y) = \mathbb{E}(u_{13}^2(Y)/(x_1, x_2 + y_2) = (x, y)),$$

and for  $K_{23}$ , we obtain

$$K_{23}(f) = 2k_{23}\partial_2 f + 4t_{23}\partial_2^2 f,$$

with

$$k_{23}(x, y) = \mathbb{E}(x_2 - x_3/(x_1, x_2 + y_2) = (x, y)), \quad t_{23}(x, y) = \mathbb{E}(u_{23}^2/(x_1, x_2 + y_2) = (x, y)),$$

It remains to compute those conditional laws.

Following the computations of Section 3.4, we may compute the law of the set of variables  $(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23})$  under the uniform measure on the sphere through the action of the spherical Laplace operator  $\Delta_{\mathbb{S}^{n-1}}$  on these variables. The Gamma operator acts on the variables as

$$\begin{cases} \Gamma(x_p, x_q) = 4x_p(\delta_{pq} - x_q), \\ \Gamma(x_i, y_2) = -4x_i y_2, \\ \Gamma(y_2, y_2) = 4y_2(1 - y_2) \end{cases}$$

while

$$\begin{cases} \Gamma(y_2, u_{ij}) = -4y_2u_{ij}, \\ \Gamma(x_i, u_{lk}) = -4x_iu_{lk} + 2\delta_{il}u_{ik} + 2\delta_{ik}u_{il}, \\ \Gamma(u_{ij}, u_{kl}) = -4u_{ij}u_{kl} + \delta_{ik}u_{jl} + \delta_{il}u_{jk} + \delta_{jk}u_{il} + \delta_{jl}u_{ik} \end{cases}.$$

where, in the last formulas,  $u_{ii}$  stands for  $x_i$ . Moreover, we have, with  $n = p_1 + p_2 + p_3$ ,

$$\begin{cases} \Delta_{\mathbb{S}^{n-1}}(x_i) = -2nx_i + 2p_1, \\ \Delta_{\mathbb{S}^{n-1}}(y_2) = -2ny_2 + 2(p_2 - p_1), \\ \Delta_{\mathbb{S}^{D-1}}(u_{ij}) = -2nu_{ij} \end{cases}$$

Then, the image measure of the sphere is the reversible measure for this operator, that we compute through equation (3.8). Up to some normalizing constant, we may compute the density through formula (3.8). In order to compute density with respect to the product measure  $dx_1 dx_2 dx_3 dy_1 du_{12} du_{13} du_{23}$ , we introduce

$$\begin{cases} F_1 = x_1 x_2 x_3 + 2u_{12} u_{13} u_{23} - x_1 u_{23}^2 - x_2 u_{13}^2 - x_3 u_{12}^2, \\ F_2 = 1 - x_1 - x_2 - x_3 - y_2 \end{cases}$$

Observe that  $F_1$  the determinant of the Gram matrix associated with the vectors  $\mathbf{x}_1, \mathbf{x}_2, \mathbf{x}_3$ .

Rewriting these variables  $(x_1, x_2, x_3, y_2, u_{12}, u_{13}, u_{23})$  as  $(x_1, x_2, x_3, x_4, x_5, x_6, x_7)$  in his order, (to have a more compact presentation of what follows), we get, with  $G_{ij} = \frac{1}{4}\Gamma(x_i, x_j)$ ,

$$\begin{cases} \sum_j \partial_j G_{ij} = 2 - 8x_i, \quad i = 1, 2, 3 \\ \sum_j \partial_j G_{4j} = 1 - 8x_4, \\ \sum_j \partial_j G_{ij} = -8x_i, \quad i = 5, 6, 7; \\ \sum_j G_{ij} \partial_j \log F_1 = 1 - 3x_i, \quad i = 1, 2, 3 \\ \sum_j G_{ij} \partial_j \log F_1 = -3x_i, \quad i = 4, 5, 6, 7, \\ \sum_i G_{ij} \partial_j \log F_2 = -x_i, \quad i = 1, \dots, 7 \\ \sum_i G_{ij} \partial_j \log x_4 = -x_i + \delta_{i4}, \quad i = 1, \dots, 7 \end{cases}$$

In the end, through formula (3.8), we are able to compute the density of the measure, which is, up to some normalizing constant

$$\rho = F_1^\alpha F_2^\beta y_2^\gamma,$$

with

$$\alpha = \frac{p_1}{2} - 2, \quad \beta = \frac{n - p_2}{2} - p_1 - 1, \quad \gamma = \frac{p_2 - p_1}{2} - 1.$$

It is worth to observe that the equation  $F_1 F_2 y_2 = 0$  is indeed the reduced equation of the set  $\Omega_7$ .

To compute the conditional law, is is worth to change variables in order to transform the measure  $\rho(x)dx$  into a product measure. For this, we set

$$u_{ij} = \sqrt{x_i x_j} \sigma_{ij}, \quad y_2 = z - x_2, \quad x_2 = uz, \quad x_3 = v(1 - x_1 - z),$$

so that the measure becomes a product measure, of the form

$$\mu(dx_1, dz)\beta_1(du)\beta_2(dv)\gamma(d\sigma_{12}, d\sigma_{23}, d\sigma_{13}),$$

where  $\mu$  is as expected the Dirichlet law in dimension 2  $\mu_{2,(p_1,p_2)}$ .

With this in mind, it is easy to check that we have

$$\begin{cases} k_{12} = 2(x - a_1y), & t_{12} = b_1xy, \\ k_{13} = 2(x - a_2(1 - x - y)), & t_{13} = b_2x(1 - x - y), \\ k_{23} = 2(a_3y - a_4(1 - x - y)), & t_{23} = b_3y(1 - x - y), \end{cases}$$

for some constants  $a_i, b_j$  that we are not going to identify directly, but where we may assert that  $b_i > 0$  for example.

(Indeed, knowing that those differential operators  $K_{ij}$  must commute with  $L_{2,\mathbf{p}}$  allows to compute them up to some constant.)

Now, if one wants to see that those operators do not commute, we may look at  $[\frac{1}{b_1}K_{12}, \frac{1}{b_3}K_{13}]$  for example. This is a third order operator whose leading term is  $2(1 - x - y)(x - y)(\partial_1 - \partial_2)^3$ , which clearly does not vanish. ■

We now concentrate on the operators  $L_{\mathbf{A},\mathbf{p}}$ . We shall show that in the generic case (that is for some dense set for the parameters  $A_{ij}$  and  $p_i$ ), their eigenspaces are one dimensional.

There is still a geometric interpretation for them, in the geometric case  $p_i \in \mathbb{N}$ , as we shall see below. And this geometric interpretation allows to use the same space  $\mathbb{E}_1$  with the projection  $\pi : \mathbb{E}_1 \mapsto \mathbb{E}$ , which may be extended to the general case  $p_i \notin \mathbb{N}$  as we did in section 3.4. But the problem now is that the horizontal rotations do not commute with the lift of  $L_{\mathbf{A},\mathbf{p}}$  to the geometric model. We may therefore not apply the Carlen-Geronimo-Loss scheme to them.

The geometric interpretation of  $L_{\mathbf{A},\mathbf{p}}$  that we present now is inspired from [13], where a similar interpretation is carried out for the matrix simplex. In  $\mathbb{R}^n$ , consider the infinitesimal rotations in the coordinate plane  $(i, j)$   $D_{ij} = y_i\partial_j - y_j\partial_i$ . Consider now as before a partition  $\{I_1, \dots, I_{d+1}\}$  of the set  $\{1, \dots, n\}$ , where  $|I_i| = p_i$ . For  $i \neq j$  consider the following second order diffusion operator on the sphere  $\mathbb{S}^{n-1}$

$$\Delta_{ij} = \sum_{p \in I_i, q \in I_j} D_{pq}^2.$$

The action of  $\Delta_{ij}$ , and its associated carré du champ  $\Gamma_{ij}$  on the variables  $x_r = \sum_{p \in I_r} y_p^2$  and  $x_s = \sum_{p \in I_s} y_p^2$  is as follows.

**Proposition 4.2.**

$$\Gamma_{ij}(x_r, x_s) = 4[\delta_{rs}x_ix_j(\delta_{ri} + \delta_{rj}) - (\delta_{ri}\delta_{sj} + \delta_{rj}\delta_{si})x_rx_s].$$

$$\Delta_{ij}(x_r) = 2(\delta_{ir} - \delta_{jr})(x_jp_i - x_ip_j).$$

*Proof.* — We start by the computation of this action on the variables  $y_p, y_q; \mathbb{S}^{D-1} \mapsto \mathbb{R}$ .

$$(4.21) \quad \begin{cases} \Delta_{i,j}(y_p) = -y_p(\mathbf{1}_{p \in I_j} p_i + \mathbf{1}_{p \in I_i} p_j) \\ \Gamma_{i,j}(y_p, y_q) = \delta_{pq}(\mathbf{1}_{p \in I_i} x_j + \mathbf{1}_{p \in I_j} x_i) - y_p y_q (\mathbf{1}_{p \in I_i} \mathbf{1}_{q \in I_j} + \mathbf{1}_{p \in I_j} \mathbf{1}_{q \in I_i}). \end{cases}$$

From this, using the change of variable formula (3.12), we get

$$\Gamma_{i,j}(x_p, x_q) = 4x_i x_j [\delta_{pq}(\delta_{pi} + \delta_{pj}) - (\delta_{pi} \delta_{qj} + \delta_{pj} \delta_{qi})].$$

In the same way, we obtain the formula for  $\Delta_{ij}(x_r)$  using formula (3.11). ■

As a corollary, and comparing with formulae (4.18) and (4.19), we get

**Corollary 4.3.** *The operator  $4L_{\mathbf{A}, \mathbf{p}}$  is the image of the operator  $\sum_{i < j} A_{ij} \Delta_{ij}$  through the map  $y \mapsto (x_1, \dots, x_d)$  which maps  $\mathbb{S}^{n-1}$  onto  $\mathbb{D}_d$ , where  $n = \sum_{i=1}^{d+1} p_i$ .*

**Remark 4.4.** *In view of equation (4.20), it is worth to observe that the spherical Laplace operator may be written as  $\sum_{i < j} \Delta_{ij}$ . Therefore, comparing with Proposition 4.3, we see that what are missing is the operator  $\sum_i \Delta_{ii}$ , where  $\Delta_{ii} = \sum_{p < q, p \in I_i, q \in I_i} D_{pq}^2$ . But it is easily seen that  $\Delta_{ii}$  has not action on the variables  $x_p$  :  $\Gamma_{ii}(x_p, x_q) = \Delta_{ii}(x_p) = 0$ .*

*It is also worth to observe that one may split some subset  $I_i$  into two subsets  $I_{i_1}$  and  $I_{i_2}$ . More precisely, suppose that we have a partition  $\{I_1, \dots, I_{d+1}\}$  of  $\{1, \dots, D\}$  and that we split say  $I_1$  into two disjoint sets  $I_{1a} \cup I_{1b}$ . Then we may consider a new operator on  $\mathbb{D}_{d+1}$   $L_{\mathbf{A}_1, \mathbf{a}_1}$ , for some matrix  $\mathbf{A}_1$  and some vector  $\mathbf{a}_1$ . Then, provided provided that for any  $j > 1$ ,  $A_{1a,j} = A_{1b,j} = A_{1j}$ , the image of  $L_{\mathbf{A}_1, \mathbf{a}_1}$  on  $\mathbb{D}_d$  under the map  $(x_{1a}, x_{1b}, x_2, \dots, x_d) \mapsto (x_{1a} + x_{1b}, x_2, \dots, x_d)$  is  $L_{\mathbf{A}, \mathbf{a}}$ , where  $\mathbf{a} = (a_{1a} + a_{1b}, a_2, \dots, a_d)$ .*

*Of course, the same reasoning applies for any any parameter  $i$  instead of 1.*

For the sake of completeness, we show below that the eigenspaces of  $L_{\mathbf{A}, \mathbf{p}}$  have dimension 1 in the generic case.

**Proposition 4.5.** *For a dense set for the parameters  $A_{ij}$  and  $p_i$ , the eigenspaces of the operator  $L_{\mathbf{A}, \mathbf{p}}$  are one dimensional.*

*Proof.* — Since the space  $\mathcal{P}_n$  of polynomials with total degree  $n$  is preserved by  $L_{\mathbf{A}, \mathbf{p}}$ , one may concentrate on its action on  $\mathcal{P}_n$ . To understand the eigenvalues of this restriction, which do not come from the restriction to  $\mathcal{H}_{n-1}$ , it is enough to look at the restriction of  $L_{\mathbf{A}, \mathbf{p}}$  to homogeneous polynomial of degree  $n$ , and consider for such polynomial  $P$ , the degree  $n$  homogeneous part of  $L_{\mathbf{A}, \mathbf{p}}(P)$ .

Then, the eigenvalues of  $L_{\mathbf{A}, \mathbf{p}}$  are the eigenvalues of this linear operator, represented by some matrix  $M_{n, \mathbf{A}, \mathbf{p}}$  in the natural basis of these homogeneous polynomials  $e_{k_1, \dots, k_d} = \{x_1^{k_1} \dots x_d^{k_d}, \sum_i k_i = n\}$ . We shall see that for each  $n$ , there exists a dense subset  $\Omega_n$  of parameters (even with a complementary with Lebesgue measure 0) such that the eigenvalues of  $M_{n, \mathbf{A}, \mathbf{p}}$  are all distinct for this parameters. Then, on  $\cap_n \Omega_n$ , all the eigenvalues of  $L_{\mathbf{A}, \mathbf{p}}$  are distinct.



To assert that the eigenvalues  $M_{n,\mathbf{A},\mathbf{p}}$  are distinct, it is enough to check that the characteristic polynomial has distinct roots, or in other words that its discriminant does not vanish. But the discriminant is a polynomial in the coefficients of the characteristic polynomial, which themselves are polynomials in the entries of the matrix, which themselves are polynomials in the variables  $A_{ij}$  and  $p_i$ . Therefore, there exists some polynomial  $Q$  in the variables  $A_{ij}, p_i$ , depending on the degree  $n$ , such that, if  $Q \neq 0$ , all the eigenvalues of  $M_{n,\mathbf{A},\mathbf{p}}$  are distinct.

It remains to show that  $Q$  does not vanish identically, that is that there exists some choice for the parameters  $A_{ij}$  and  $p_i$  for which the eigenvalues are distinct.

Let us choose the matrix  $A_{ij}$  such that  $A_{ij} = A_{i(d+1)}$  for  $j > i$ . Then, if we order the elements of the basis  $\{e_{k_1, \dots, k_d}, \sum_1^d k_i = n\}$  according to their inverse lexicographic order of  $(k_1, \dots, k_{d-1})$  (so that  $(n, \dots, 0, 0)$  is the lowest term), then one may check that all the elements of  $M_{n,\mathbf{A},\mathbf{p}}$  which are above the diagonal vanish. Then, the eigenvalues of  $M_{n,\mathbf{A},\mathbf{p}}$  are the diagonal elements. On the diagonal, the coefficient corresponding to  $e_{k_1, \dots, k_d}$  is

$$-\sum_{i \neq j} k_i k_j A_{ij} - \sum_i k_i (k_i - 1) A_{i,d+1} + \frac{1}{2} \sum_i k_i (A_{i,d+1} p_i - \sum_{k=1}^{d+1} A_{ik} p_k).$$

With the choice that we made, for  $i \neq j$ ,  $A_{i,j} = a_{\min(i,j)}$  for some sequence  $a_i, i = 1, \dots, d$ . Then, it is not hard to see that there exists a choice for the sequences  $a_i, i = 1, \dots, d$  and  $p_i, i = 1, \dots, d+1$  for which all these terms are different, for all the sequences of integers  $(k_1, \dots, k_d)$  such that  $\sum_1^d k_i = n$ . ■

**Remark 4.6.** For any horizontal rotation  $R$ , the associated kernel

$$K_R(f)(x) = \mathbb{E}(f(\pi(R\mathbf{x})) / \pi(\mathbf{x}) = x)$$

leaves invariant all the eigenspaces of  $L_{\mathbf{1},\mathbf{p}}$ . But the question of their action on this space remains completely open.

### 4.3 Representations of Markov sequences

In what follows, we restrict ourselves to the case where all the coefficients  $A_{ij}, i \neq j$  are set to 1. Since the eigenspaces  $E_n$  are not one dimensional, we also restrict our attention to the study of Markov operators which have constant eigenvalues on the space  $E_n$ . That is, instead of looking at Markov operators which commute with  $L_{\mathbf{1},\mathbf{p}}$ , we look at Markov operators which are functions of  $L_{\mathbf{1},\mathbf{p}}$ . We say that such a Markov operator strongly commutes with  $L_{\mathbf{1},\mathbf{p}}$

It is worth to observe first that, for any choice of a strict subset  $I \subset \{1, \dots, d+1\}$ , the projection  $\pi : \mathbb{D}_d \mapsto [0, 1]$   $\pi(x) = \sum_{i \in I} x_i$  maps the Dirichlet law  $\mu_{d,\mathbf{p}}$  on the beta measure  $\beta_{q,n-q}$ , where  $q = \sum_{i \in I} p_i$  and  $n = \sum_1^{d+1} p_i$ . (We recall that by convention,  $x_{d+1} = 1 - \sum_1^d x_i$ ). As usual, for any function  $f : [0, 1] \mapsto \mathbb{R}$ , we denote  $\pi f = \mathbb{D}_d \mapsto \mathbb{R}$  the function  $\pi f(y) = f(\pi(y))$ . Then, with the Jacobi operator  $L_{1,q,n-q}$ , one has

$$\pi L_{1,q,n-q} = L_{\mathbf{1},\mathbf{p}} \pi,$$

as may be checked directly and easily, computing  $L_{\mathbf{1},\mathbf{p}}\pi(x)$  and  $\Gamma_{\mathbf{1},\mathbf{p}}(\pi(x), \pi(x))$ .

Now, the eigenvalues of  $J_{p,n-q}$  and  $L_{\mathbf{1},\mathbf{p}}$  are the same (namely  $-k(k + \frac{n-2}{2})$ ), acting on polynomials of degree  $k$ . In other words, any eigenspace for  $L_{\mathbf{1},\mathbf{p}}$  contains an eigenvector of the form  $P(\pi(x))$ .

Now,  $K$  being a Markov operator on  $\mathbb{D}_d$  which strongly commutes with  $L_{\mathbf{1},\mathbf{p}}$ , consider the new Markov operator  $K_1$  on  $[0, 1]$  defined as  $K_1(f) = K(\pi(f))$ . It is clear that  $K_1$  commutes with  $J_{q,n-q}$ .

If  $\mu_n$  is the eigenvalue of  $K$  on the eigenspace  $E_n$  of  $L_{\mathbf{1},\mathbf{p}}$ , then, for any Jacobi polynomial with degree  $k$ ,  $K_1(P) = \mu_n P$ . One may now apply Gasper's theorem and we have obtained

**Proposition 4.7.** *Let  $K$  a Markov operator on  $\mathbb{D}_d$  which strongly commutes with  $L_{\mathbf{1},\mathbf{p}}$ , and let  $(\mu_k)$  be the sequence of its eigenvalues on the eigenspace  $E_k$  of  $L_{\mathbf{1},\mathbf{p}}$ . Choose  $I \subset \{1, \dots, d+1\}$ ,  $I \neq \{1, \dots, d+1\}$ , and let  $q = \sum_{i \in I} p_i$ , and  $n = \sum_{i=1}^{d+1} p_i$ . Then, there exists a Probability measure  $\nu$  on  $[0, 1]$  such that, for any  $k \in \mathbb{N}$*

$$\mu_k = \int_0^1 \frac{P_k^{q,n-q}(x)}{P_n^{q,n-q}(x_0)} \nu(dx),$$

where  $P_k^{p,n-q}$  is the Jacobi polynomial with degree  $k$  for the measure  $\beta_{q,n-q}$ , and  $x_0 = 1$  or  $x_0 = 0$  according to  $p \leq n - q$  or not.

**Remarks 4.8.**

Contrary to the one dimensional case, it is not true in general that for any probability measure  $\nu$  on  $[0, 1]$ , the associated sequence  $\mu_n$  may be the sequence of eigenvalues of a Markov operator. Indeed, if such was the case, then for some value of  $q = \sum_{i \in I} p_i$ , one would have that the sequence  $\frac{P_k^{q,n-q}(x)}{P_k^{q,n-q}(1)}$  is such a strong Markov sequence. Choosing another value of  $q$ , say  $q_1$ , associated to another subset  $I_1$  of  $\{1, \dots, d+1\}$ , one would therefore get some measure  $\nu(x, dy)$  on  $[0, 1]$  such that

$$\frac{P_k^{q,n-q}(x)}{P_k^{q,n-q}(1)} = \int_0^1 \frac{P_k^{q_1,n-q_1}(y)}{P_k^{q_1,n-q_1}(1)} \nu(x, dy).$$

Repeating the operation with  $\frac{P_k^{q_1,n-q_1}(y)}{P_k^{q_1,n-q_1}(1)}$  and another measure  $\nu_1(y, dz)$ , one would get

$$\frac{P_k^{q,n-q}(x)}{P_k^{q,n-q}(1)} = \int \frac{P_k^{q,n-q}(z)}{P_k^{q,n-q}(1)} \nu_2(x, dz),$$

where  $\nu_2(x, dz) = \int \nu(x, dy) \nu_2(y, dz)$ .

Then,  $\nu_2(x, dz)$  is the Dirac mass in  $x$ . As a consequence, for  $\nu(x, dy)$  almost every  $y$ ,  $\nu_1(y, dz)$  is a Dirac mass in some point  $h(y)$ , and moreover this point is constant. This is clearly wrong, since the Jacobi polynomials for different values of the parameters do not coincide.

In view of Proposition 3.3, in order to obtain the true hypergroup representation, that is the set of extremal points for Markov which strongly commutes to  $L_{\mathbf{1},\mathbf{p}}$ , it would be enough to produce the associated space  $\mathbb{E}_1$  and the corresponding operations  $\pi$  and

$\phi$  such that the associated correlation operator  $K(f) = E(f(\phi\pi f(Y))/\pi(Y) = x)$  strongly commutes with  $L_{\mathbf{1},\mathbf{p}}$ . Even in the geometric case, when the parameters  $p_i$  are integers, it does not seem to be the case for the horizontal rotations described in (3.5).

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