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Null controllability of a penalized Stokes problem in dimension two with one scalar control.*†

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0 Abstract

In this paper we consider a penalized Stokes equation defined in a regular domain $\Omega \subset \mathbb{R}^2$ and with Dirichlet boundary conditions. We prove that our system is null controllable using a scalar control defined in an open subset inside Ω and whose cost is bounded uniformly with respect to the parameter that converges to 0.

Key words: Carleman inequality, penalized Stokes system, controllability

AMS subject classification: 35K40, 93B05, 93C20

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1 Introduction

1.1 Main results

Let $T > 0$ and $\Omega \subset \mathbb{R}^2$ a regular domain. Throughout this paper we use the term "domain" to refer to a bounded connected non-empty open set. As usual, we denote $Q := (0,T) \times \Omega$ and $\Sigma := (0,T) \times \partial\Omega$. In this paper we work on the Stokes penalized system with Dirichlet boundary conditions. This system is given by the following equations:

$$\begin{cases} v_t^\varepsilon - \Delta v^\varepsilon + \nabla q^\varepsilon = f & \text{in } Q, \\ \varepsilon q^\varepsilon + \nabla \cdot v^\varepsilon = 0 & \text{in } Q, \\ v^\varepsilon = 0 & \text{on } \Sigma, \\ v^\varepsilon(0,\cdot) = v^0 & \text{in } \Omega. \end{cases} \tag{1.1}$$

Here $f : Q \to \mathbb{R}^2$ is a source term, $v^0 : \Omega \to \mathbb{R}^2$ is an initial condition and $\varepsilon > 0$. This system approximates the classical Stokes problem, which is given by the following equations:

$$\begin{cases} v_t - \Delta v + \nabla q = f & \text{in } Q, \\ \nabla \cdot v = 0 & \text{in } Q, \\ v = 0 & \text{on } \Sigma, \\ v(0,\cdot) = v^0 & \text{in } \Omega. \end{cases} \tag{1.2}$$

The main objective of this paper is to prove that system (1.1) is null controllable with a one-dimensional control whose cost is uniformly bounded with respect to ε. We prove it for almost every direction, being these directions different for each Ω. In addition, if Ω is strictly convex we prove it for all the directions.

First of all, we state what hypothesis Ω must satisfy to be controllable by a force parallel to $e_1 := (1,0)$. In order to do so, if $\Omega \subset \mathbb{R}^2$ is a C^2 domain, we use the standard convention to parametrize $\partial\Omega$: we denote the arc-length parametrization of each connected component of $\partial\Omega$ by $\sigma^i = (\sigma_1^i, \sigma_2^i)$ and the signed curvature of $\partial\Omega$ on the point $\sigma^i(\theta)$ by $\kappa^i(\theta)$. In both terms the superscript i is omitted if $\partial\Omega$ is connected. We suppose that each component is parametrized in the standard way; that is, for $U(x,y) := (-y,x)$ (U is the rotation of 90 degrees to the left) and for all $p = \sigma^i(\theta) \in \partial\Omega$, there is $\delta_0(p) > 0$ such that if $\delta \in (0, \delta_0(p))$, then $p + \delta U((\sigma^i)'(\theta)) \in \Omega$.

Remark 1.1. Since the σ^i are arc-length, we have the well-known equalities:

$$\kappa^i = (\sigma_2^i)''(\sigma_1^i)' - (\sigma_1^i)''(\sigma_2^i)' = \frac{(\sigma_2^i)''}{(\sigma_1^i)'} = -\frac{(\sigma_1^i)''}{(\sigma_2^i)'}. \tag{1.3}$$

Hypothesis 1.1. Let $\Omega \subset \mathbb{R}^2$ be a C^2 domain of boundary $\partial\Omega$ parametrized by functions $\sigma^1, \dots, \sigma^k$ as explained in the previous paragraph. Then, for any $i \in \{1, \dots, k\}$ and for any θ such that $(\sigma_1^i)'(\theta) = 0$ or $(\sigma_2^i)'(\theta) = 0$, we have that $\kappa^i(\theta) \neq 0$.

Remark 1.2. Hypothesis 1.1 means that if Ω is a C^2 domain, then on all the points of the boundary of horizontal or vertical tangent line the curvature is not null. We use it to avoid pathologies near those points, since in that case we do not know how to proceed.

Hypothesis 1.1 is not restrictive at all, thanks to the following lemma, which we prove at the beginning of Subsection 4.1:

Lemma 1.3. *Let Ω be a C^2 domain. Then, there is an orthogonal $\mathbb{R}^2$-endomorphism U such that the domain $\widetilde{\Omega} := U(\Omega)$ satisfies Hypothesis 1.1. In fact, if we denote U_ψ the endomorphism characterized by $e_1 := (1,0) \mapsto (\cos(\psi), \sin(\psi))$ and $e_2 := (0,1) \mapsto (-\sin(\psi), \cos(\psi))$, then, for almost every ψ in $[-\pi, \pi]$, $U_\psi(\Omega)$ satisfies Hypothesis 1.1.*

With Lemma 1.3 in mind, we state one of the main results of this paper:

Theorem 1.4. *Let $\Omega \subset \mathbb{R}^2$ be a regular domain that satisfies Hypothesis 1.1 and let $\omega \subset \Omega$ be a non-empty open set. Then, there is $\varepsilon_0 > 0$ such that for all $T > 0$ there is $C > 0$ such that if $\varepsilon \in (0, \varepsilon_0)$ and $y^0 \in \mathbf{L}^2(\Omega)$, there is a scalar-valued function $f^\varepsilon \in L^2((0,T) \times \omega)$ satisfying:*

$$\|f^\varepsilon\|_{L^2((0,T)\times\omega)} \leq C\|y^0\|_{\mathbf{L}^2(\Omega)},$$

and such that the solution of the following system:

$$\begin{cases} y_t^\varepsilon - \Delta y^\varepsilon + \nabla p^\varepsilon = f^\varepsilon 1_\omega e_1 & \text{in } Q, \\ \varepsilon p^\varepsilon + \nabla \cdot y^\varepsilon = 0 & \text{in } Q, \\ y^\varepsilon = 0 & \text{on } \Sigma, \\ y^\varepsilon(0, \cdot) = y^0 & \text{in } \Omega, \end{cases} \tag{1.4}$$

satisfies $y^\varepsilon(T, \cdot) = 0$.

As usual, in Theorem 1.4 and throughout this paper $\mathbf{L}^p$ and $\mathbf{H}^s$ denote respectively the Lebesgue and Sobolev spaces of vector-valued functions.

Remark 1.5. From Lemma 1.3 and since system (1.1) is invariant with respect to rotations, we actually have for almost all directions $e^\theta := (\sin(\theta), \cos(\theta))$ that there is $\varepsilon_0 > 0$ such that for all $T > 0$ there is $C > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$ system (1.1) is null controllable with a force $f^\varepsilon 1_\omega e^\theta$ satisfying:

$$\|f^\varepsilon\|_{L^2((0,T)\times\omega)} \leq C\|y^0\|_{\mathbf{L}^2(\Omega)}.$$

Remark 1.6. A natural question that may arise is the relation between the control problem (1.4) and the control problem:

$$\begin{cases} y_t - \Delta y + \nabla p = f 1_\omega e_1 & \text{in } Q, \\ \nabla \cdot y = 0 & \text{in } Q, \\ y = 0 & \text{on } \Sigma, \\ y(0,\cdot) = y^0 & \text{in } \Omega, \end{cases} \tag{1.5}$$

for $f \in L^2((0,T)\times\omega)$ and for $y^0 \in \mathcal{H}(\Omega)$ (the subspace of $\mathbf{L}^2(\Omega)$ of functions of null divergence and null normal trace). Using weak compactness, we have that there is a sequence f^{ε_k} which converges weakly in $L^2((0,T)\times\omega)$ to some function f^0. Moreover, using the techniques presented in the proof of [31, Theorem I.2], we have that $y^{\varepsilon_k}(t,\cdot)$ converges in the $\mathbf{H}^{-1}(\Omega)$-norm for all $t \in [0,T]$ to $y(t,\cdot)$ (the solution of (1.5)). In particular, since $y^{\varepsilon_k}(T,\cdot) = 0$, we have that $y(T,\cdot) = 0$. Consequently, this provides an alternative way of proving the well-known result that the system (1.5) is null controllable with a one-dimensional control supported in any regular domain (see [12]). In that sense, an interesting problem that remains open is if the control of minimal L^2-norm of (1.4) converges to the control of minimal L^2-norm of (1.5).

In order to prove the null controllability of (1.4), we consider as usual its adjoint system:

$$\begin{cases} -\varphi^\varepsilon_t - \Delta\varphi^\varepsilon + \nabla\pi^\varepsilon = 0 & \text{in } Q, \\ \varepsilon\pi^\varepsilon + \nabla\cdot\varphi^\varepsilon = 0 & \text{in } Q, \\ \varphi^\varepsilon = 0 & \text{on } \Sigma, \\ \varphi^\varepsilon(T,\cdot) = \varphi^T & \text{in } \Omega. \end{cases} \tag{1.6}$$

Indeed, it is a well-known result (see [28, 25]) that the existence in (1.4) of a control f^ε bounded uniformly in ε is equivalent to proving that there is $\varepsilon_0 > 0$ such that for all $T > 0$ there is $C > 0$ such that if $\varepsilon \in (0,\varepsilon_0)$ and $\varphi^T \in \mathbf{L}^2(\Omega)$ we have:

$$\int_\Omega |\varphi^\varepsilon(0,\cdot)|^2 \le C \iint\limits_{(0,T)\times\omega} |\varphi^\varepsilon_1|^2, \tag{1.7}$$

for φ^ε the solution of (1.6). The proof of Theorem 1.4 is thus reduced to the proof of the observability inequality (1.7), and we focus on it from now on.

In order to prove estimate (1.7) we prove a Carleman inequality. Before presenting it, let us define the weights we use throughout the paper as follows:

$$\begin{aligned} &\alpha(t,x) = \frac{e^{2\lambda\|\eta^0\|_\infty} - e^{\lambda\eta^0}}{(t(T-t))^m}, \qquad \xi(t,x) = \frac{e^{\lambda\eta^0}}{(t(T-t))^m}, \\ &\alpha^*(t) = \max_{x\in\overline{\Omega}} \alpha(t,x), \qquad \xi^*(t) = \min_{x\in\overline{\Omega}} \xi(t,x). \end{aligned} \tag{1.8}$$

These weights are classical and widely used in the literature. Here, λ and m are positive real numbers to be fixed later on. Moreover, $\eta^0 \in C^4(\overline{\Omega})$ is a fixed function that satisfies $\eta^0 > 0$ in Ω, $\eta^0 = 0$ on $\partial\Omega$, and $|\nabla\eta^0| > 0$ in $\overline{\Omega \setminus \omega_0}$ for some non-empty domain ω_0 compactly included in ω. We know that such function η^0 exists as long as $\Omega \in C^4$. Indeed, a classical proof of the existence of such function is given in [18], whereas an alternative recent proof is given in [23, Lemma 2.1].

Theorem 1.7. *Let Ω be a regular domain that satisfies Hypothesis 1.1, let $\omega \subset \Omega$ be an open set, and let $m \geq 8$. Then, there is $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have:*

$$s^{15}\lambda^{16} \iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2 \leq Cs^{34}\lambda^{35} \iint_{(0,T)\times\omega} e^{-2s\alpha}\xi^{34}|\varphi_1^\varepsilon|^2, \tag{1.9}$$

for any φ^ε regular solution of (1.6) and for the weights defined in (1.8).

Proving (1.7) from (1.9) is mainly done by an energy estimate on $\chi\varphi^\varepsilon$, for $\chi \geq 0$ a regular cut-off function such that $\chi = 1$ in $[0, T/2]$ and $\chi = 0$ in $[3T/4, T]$. This is a classic and easy procedure, so it is omitted in this paper.

We remark that in (1.6) we can simplify the pressure and get the following equations for φ^ε:

$$\begin{cases} -\frac{\varepsilon}{1+\varepsilon}\partial_t\varphi_1^\varepsilon - \partial_{xx}\varphi_1^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{yy}\varphi_1^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_2^\varepsilon, \\ -\frac{\varepsilon}{1+\varepsilon}\partial_t\varphi_2^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{xx}\varphi_2^\varepsilon - \partial_{yy}\varphi_2^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_1^\varepsilon. \end{cases} \tag{1.10}$$

Using standard methods for coupled parabolic equations we know that the important term is $\partial_{xy}\varphi_2^\varepsilon$, which appears in the equation satisfied by φ_1^ε. The main difficulty on proving (1.9) is that $\|\partial_{xy}\cdot\|_{L^2(\Omega)}$ is not necessarily a norm (see Remark 3.2 and Proposition 3.3 below). However, if we add some information on the first and second order derivatives of $\partial_{xy}\varphi^\varepsilon$ and on the boundary, we do have a norm. In that sense, we consider the following operator:

$$L_a u = -a\partial_{xx}u - \partial_{yy}u. \tag{1.11}$$

Theorem 1.8. *Let Ω be a C^4 domain that satisfies Hypothesis 1.1. Then, for $a_0 > 0$ small enough, there is $C > 0$ such that for any function $u \in H^4(\Omega) \cap H_0^1(\Omega)$ and for any $a \in (0, a_0]$ we have that:*

$$\|\partial_x u\|_{C^0(\overline{\Omega})} \leq C(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \tag{1.12}$$

Remark 1.9. By continuity, (1.12) remains true for $a = 0$.

Remark 1.10. Thanks to Poincaré inequality, there is $C > 0$ and $a_0 > 0$ such that for all $a \in [0, a_0]$ and for all $u \in H^4(\Omega) \cap H_0^1(\Omega)$, we have:

$$\|u\|_{C^0(\overline{\Omega})} \leq C(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \tag{1.13}$$

Remark 1.11. By symmetry, we get an analogous estimate for $\partial_y u$ if instead of L_a we have the operator:

$$\tilde{L}_a u := -\partial_{xx} u - a\partial_{yy} u. \tag{1.14}$$

Indeed, in that case, there is $C > 0$ and $a_0 > 0$ such that for all $a \in [0, a_0]$ and for any function $u \in H^4(\Omega) \cap H_0^1(\Omega)$, we have:

$$\|\partial_y u\|_{C^0(\overline{\Omega})} \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|\tilde{L}_a u\|_{H^1(\partial\Omega)}).$$

In particular, in this case we also have an estimate of $\|u\|_{C^0(\overline{\Omega})}$ similar to (1.13).

The reason why (1.12) is useful to prove (1.9) is the following one:

Remark 1.12. Let us consider φ^ε a solution of (1.6). Using (1.10) and the Dirichlet boundary conditions, we have that, on $\partial\Omega$, for all $t \in [0, T)$:

$$\begin{cases} -\partial_{xx}\varphi_1^\varepsilon - \frac{\varepsilon}{1+\varepsilon}\partial_{yy}\varphi_1^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_2^\varepsilon, \\ -\frac{\varepsilon}{1+\varepsilon}\partial_{xx}\varphi_2^\varepsilon - \partial_{yy}\varphi_2^\varepsilon = \frac{1}{1+\varepsilon}\partial_{xy}\varphi_1^\varepsilon. \end{cases}$$

Thus, by Remark 1.10 and Remark 1.11, we get that there is $C > 0$ and $\varepsilon_0 > 0$ such that for all $t \in [0, T)$ and $\varepsilon \in (0, \varepsilon_0]$:

$$\|\varphi^\varepsilon(t, \cdot)\|_{\mathbf{L}^2(\Omega)} \leq C\|\partial_{xy}\varphi^\varepsilon(t, \cdot)\|_{\mathbf{H}^2(\Omega)}. \tag{1.15}$$

Finally, let us make some remarks about possible extensions of the work:

Remark 1.13. The case of Theorem 1.4 and Theorem 1.7 for $\Omega \subset \mathbb{R}^3$ is left for future research. The main complication that arises is to prove an analogous result to Theorem 1.8 because there is a larger variety of domains in $\mathbb{R}^3$ than in $\mathbb{R}^2$. Indeed, in $\mathbb{R}^3$ there is one curvature for each direction.

Remark 1.14. The construction provided in Section 2.1 for Lipschitz domains in which (1.4) is not null-controllable just gives one problematic ε for each Ω. Thus, it is an open problem to know if for all Lipschitz domain $\Omega \subset \mathbb{R}^2$ there is ε_0 small enough such that if $\varepsilon \in (0, \varepsilon_0)$ system (1.4) is null controllable.

1.2 Historical background

Getting an approximation of the Stokes and the Navier-Stokes systems by approximating the incompressibility condition by a term involving the pressure was made for the first time in [31], where the author considered the almost incompressible Navier-Stokes system. Many other ways of approximating the Navier-Stokes equations have been presented throughout the years. In the survey [29] the author presents different ways of approximating the Navier-Stokes system through

the incompressibility condition and compares them. Moreover, there are physical systems which satisfy in some ways the property of being almost incompressible, as shown in [30] and the references therein.

The interior null controllability of system (1.1) was first proved in [23, Section 4] with a control bounded uniformly with respect to ε, for ε small enough. Then, in [4], this same property is proved with an additional first order term. Moreover, in [4] the author also proves the local controllability to trajectories of the penalized Navier-Stokes system uniformly on ε for ε small enough.

There is an extensive literature on controllability of partial differential equations uniformly with respect to a vanishing parameter. For a transport equation with a small diffusion term, see [11] (see also [19] and [27]). The case of the KdV equation is treated in [20], [7] and [8], while a chemotaxis system is presented in [10].

As for the restriction of having controls with a reduced number of components, it is not new in the Navier-Stokes mathematical context. This same property has already been proved for the Stokes problem (1.2) in [12]. Consequently, in this paper we prove that a system which approximates the Stokes system conserves that property after choosing a valid reference system. Moreover, controllability results with controls having one null component have been proved for other systems: for instance, the local null controllability of the Navier-Stokes system (see [6]), the local controllability to the trajectories of the Navier-Stokes and the Boussineq system when the domain "touches" the boundary (see [17]), or the existence of insensitizing controls (see [22, 9]). Similarly, the approximate controllability of the Stokes system in a cylindrical domain with a control having two null components is proved in [26]. Finally, the local null controllability of the Navier-Stokes system in dimension three with one scalar control is proved in [13].

Outside the Navier-Stokes context, there is a huge literature on controllability results with controls having a reduced number of components. For instance, the null controllability in the context of linear thermoelasticity (see [24]), the existence of insensitizing controls for the heat equation (see [14]), the controllability to trajectories in phase-field models (see [2]), the controllability in cascade-like systems (see [21]) and the controllability in reaction-diffusion systems (see [1]). For more results on the controllability of parabolic systems with a reduced number of control, see the survey [3] and the references therein.

Remark 1.15. The main difference of the problem we consider in this paper with respect to the above cited papers is the coupling. Indeed, in all the papers cited above (and in the literature as far as we know) the coupling is constituted by a zero, first or second order term which induces

a norm in the subset of $H^2(\Omega)$ which satisfies their respective boundary conditions. However, in our situation this is clearly not the case since the differential operator ∂_{xy} with Dirichlet boundary conditions does not induce a norm for some Ω (see Remark 3.2 below).

The rest of the paper is organized as follows: in Section 2 we present some analytical results; in Section 3 we prove Theorem 1.8 when Ω is strictly convex; in Section 4 we prove Lemma 1.3 and Theorem 1.8; and in Section 5 we end the proof of Theorem 1.7. Finally, in the Appendix we prove some technical results stated in Section 2.

2 Some previous and intermediary results

In this section we present some results that are either interesting for understanding the problem or needed later. The section is split in three parts: first, in Subsection 2.1 we prove that there is a domain Ω which is not C^2 and where we do not even have approximate null controllability; then, in Subsection 2.2 we present several results on Cauchy problems and a classical result on linear ordinary differential equations; finally, in Subsection 2.3 we present some Carleman estimates.

2.1 A negative controllability result

In this subsection we provide a counterexample on null controllability with one component of (1.4) when Ω is not C^2, even if $\omega = \Omega$. With that purpose, we show that we do not even have the approximate null controllability.

Definition 2.1. We recall that system (1.4) is approximately null controllable if for all $y^0 \in \mathbf{L}^2(\Omega)$ and for all $\eta > 0$ there is a function $f^\varepsilon \in L^2\big((0,T)\times\omega\big)$ such that the solution y^ε of (1.4) satisfies $\|y^\varepsilon(T,\cdot)\|_{\mathbf{L}^2(\Omega)} \leq \eta$.

Proposition 2.2. *Let $\varepsilon > 0$. Then, there is $\Omega \subset \mathbb{R}^2$ a domain that is just Lipschitz such that, even for $\omega = \Omega$, (1.4) is not approximately null controllable.*

In order to prove Proposition 2.2 we use the technique presented in [26, Section 3].

Proof. It is a classical result that system (1.4) is not approximately null controllable if there is $\varphi^\varepsilon \neq 0$ solution of (1.6) such that $\varphi_1^\varepsilon = 0$ in $(0,T)\times\omega$. Suppose that the following scalar-valued system has a nonzero solution u^ε:

$$\begin{cases} -\partial_{xx}u^\varepsilon - \frac{1+\varepsilon}{\varepsilon}\partial_{yy}u^\varepsilon = \lambda u^\varepsilon & \text{in } \Omega, \\ u^\varepsilon = 0 & \text{on } \partial\Omega, \\ \partial_{xy}u^\varepsilon = 0 & \text{in } \Omega. \end{cases} \tag{2.1}$$

Then, $\varphi^\varepsilon\big(t,(x,y)\big) := (0, e^{\lambda t}u^\varepsilon(x,y))$ is a non-trivial solution of (1.6) which satisfies $\varphi_1^\varepsilon = 0$. Consequently, it suffices to find a domain Ω with a nonzero solution of (2.1).

The third equation of (2.1) is satisfied if $u^\varepsilon(x,y) = f(x)+g(y)$ for any $f,g \in C^2(\mathbb{R};\mathbb{R})$. Moreover, u^ε satisfies the first equation of (2.1) if:

$$-f''(x) - \lambda f(x) = \frac{1+\varepsilon}{\varepsilon} g''(y) + \lambda g(y). \tag{2.2}$$

Since both sides of equation (2.2) depend on independent variables, they must be constant. So we have to solve an ordinary differential equation with constant coefficients. We can suppose that they are equal to 0. Otherwise, if they are equal to some other value α, we have that $\overline{f} := f + \frac{\alpha}{\lambda}$ and $\overline{g} := g - \frac{\alpha}{\lambda}$ are solutions for the case $\alpha = 0$ such that $u(x,y) = \overline{f}(x) + \overline{g}(y)$.

The solutions (f,g) of the system:

$$\begin{cases} f''(x) + \lambda f(x) = 0, \\ \frac{1+\varepsilon}{\varepsilon} g''(y) + \lambda g(y) = 0, \end{cases}$$

are exponential, affine or trigonometric functions, depending on the value of λ. Since we need u^ε to be null on a bounded boundary, then necessarily they must be trigonometric; that is, $\lambda > 0$. In fact, the first equation of (2.1) is satisfied by:

$$u^\varepsilon(x,y) = \sin\left(\sqrt{\lambda}x\right) - \sin\left(\sqrt{\frac{\varepsilon\lambda}{1+\varepsilon}}y\right). \tag{2.3}$$

Finally, we have to consider that the function u^ε given in (2.3) is null on the lines:

$$\begin{cases} x = \sqrt{\frac{\varepsilon}{1+\varepsilon}}y, \\ x = \sqrt{\frac{\varepsilon}{1+\varepsilon}}y + \frac{2\pi}{\sqrt{\lambda}}, \\ x = -\sqrt{\frac{\varepsilon}{1+\varepsilon}}y + \frac{\pi}{\sqrt{\lambda}}, \\ x = -\sqrt{\frac{\varepsilon}{1+\varepsilon}}y - \frac{\pi}{\sqrt{\lambda}}. \end{cases} \tag{2.4}$$

Consequently, the function u^ε given in (2.3) is a solution of (2.1) in $\mathcal{R}_{\varepsilon,\lambda}$, for $\mathcal{R}_{\varepsilon,\lambda}$ the domain limited by (2.4), which is a rhombus. □

Remark 2.3. With this method we can find for $\mathcal{R}_{\varepsilon,\lambda}$ (for ε and λ fixed) a sequence of eigenfunctions which satisfy (2.1). The sequence is given by:

$$u_n^\varepsilon(x,y) = \sin\left((2n+1)\sqrt{\lambda}x\right) - \sin\left((2n+1)\sqrt{\frac{\varepsilon\lambda}{1+\varepsilon}}y\right), \quad n \in \mathbb{N},$$

and their respective eigenvalues are $\lambda_n = (2n+1)^2\lambda$.

Remark 2.4. The "reason" why unique continuation fails is that $\partial\mathcal{R}_{\varepsilon,\lambda}$ contains lines of a specific slope. Indeed, if for ε fixed we try to replicate the proof of Theorem 1.8 for the rhombus $\mathcal{R}_{\varepsilon,\lambda}$ we find out that the information that (1.6) provides on the boundary is equivalent to the information provided by the Dirichlet condition. More precisely, for the case of $\Omega = \mathcal{R}_{\varepsilon,\lambda}$ we are blocked in (3.7) below, since we have $\kappa = 0$ and $(\sigma_1')^2 - \frac{\varepsilon}{1+\varepsilon}(\sigma_2')^2 = 0$. This problem somehow persists when we try to generalize the proof to regular domains and the only solution we have found is to exclude a few directions (a subset of $[0, 2\pi)$ of null measure).

2.2 Results on Cauchy problems

In this subsection we present some results about the Stokes penalized problem: first with Dirichlet boundary conditions and then with Neumann boundary conditions. We also present a classical estimate about a linear differential equation. But before, we recall the definition of the interpolation spaces, for $p, q \geq 0$:

$$H^{p,q}(Q) := H^p\big(0,T;L^2(\Omega)\big) \cap L^2\big(0,T;H^q(\Omega)\big),$$
$$H^{p,q}(\Sigma) := H^p\big(0,T;L^2(\partial\Omega)\big) \cap L^2\big(0,T;H^q(\partial\Omega)\big).$$

Lemma 2.5. *Let $i \in \mathbb{N}$, $\Omega \in C^{2i}$. Then, there is $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $v^0 = 0$ and $f \in \mathbf{H}^{i-1,2i-2}(Q)$ satisfying $\partial_t^m f(t,\cdot) = 0$ for all $m \in \mathbb{N} \cap [0, i-2]$, we have that the solution v^ε of (1.1) satisfies $v^\varepsilon \in \mathbf{H}^{i,2i}(Q)$ with the estimate:*

$$\|v^\varepsilon\|_{\mathbf{H}^{i,2i}(Q)} + \varepsilon^{-1}\|\nabla\cdot v^\varepsilon\|_{H^{i-1,2i-1}(Q)} \leq C\|f\|_{\mathbf{H}^{i-1,2i-2}(Q)}. \tag{2.5}$$

The proof of Lemma 2.5 is mainly by induction. The base case ($i = 0$) can be proved by Galerkin method (we just have to replicate the method in [15, Chapter 7.1] and see that the constants are independent of ε). As for the inductive case, we get the regularity in time by considering that v_t^ε is a solution of (1.1) with $(f, 0)$ replaced by $(f_t, 0)$ and using again the Galerkin method. Moreover, we get the regularity in space by using the estimate for the steady Stokes problem given in [32, Proposition I.2.2].

Let us now state the Stokes penalized system with non-homogeneous Neumann boundary conditions:

$$\begin{cases} v_t^\varepsilon - \Delta v^\varepsilon + \nabla q^\varepsilon = f & \text{in } Q, \\ \varepsilon q^\varepsilon + \nabla\cdot v^\varepsilon = 0 & \text{in } Q, \\ \partial_n v^\varepsilon - q^\varepsilon n = h & \text{on } \Sigma, \\ v^\varepsilon(0,\cdot) = v^0 & \text{in } \Omega. \end{cases} \tag{2.6}$$

We have the following regularity and existence results, which are proved in Annex A:

Lemma 2.6. *Let $\Omega \in C^2$. Then, there is $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $v^0 \in \mathbf{H}^1(\Omega)$, $f \in \mathbf{L}^2(Q)$ and $h \in \mathbf{H}^{1,1/2}(\Sigma)$, system (2.6) has a unique solution:*

$$(v^\varepsilon, q^\varepsilon) \in \mathbf{H}^{1,2}(Q) \times H^{0,1}(Q).$$

In addition, that solution satisfies the estimate:

$$\begin{aligned}\|v^\varepsilon\|_{\mathbf{H}^{1,2}(Q)} + \|q^\varepsilon\|_{H^{0,1}(Q)} &\leq C\sqrt{1+T}\big(\|f\|_{\mathbf{L}^2(Q)} + \|h\|_{\mathbf{H}^{1,1/2}(\Sigma)}\big) \\ &+ C\left(\|v^0\|_{\mathbf{H}^1(\Omega)} + \left\|\frac{\nabla \cdot v^0}{\varepsilon}\right\|_{\mathbf{L}^2(\Omega)} + \|h(0,\cdot)\|_{\mathbf{L}^2(\Sigma)} + \|h(T,\cdot)\|_{\mathbf{L}^2(\Sigma)}\right). \quad (2.7)\end{aligned}$$

Remark 2.7. It is not necessary to assume that ε is small enough if we just want to prove existence and uniqueness of the energy solution of (2.6). Indeed, we prove collaterally that for all $\varepsilon \in \mathbb{R}^+$, (2.6) has a solution in $\mathbf{H}^{1,1}(Q)$ and that the norm $\|v^\varepsilon\|_{\mathbf{H}^{1,1}(Q)}$ can be estimated by the right-hand side of (2.7) for a constant C independent of ε (see (A.4) below).

Lemma 2.8. *Let $\Omega \in C^4$. Then, there is $\varepsilon_0 > 0$ and $C > 0$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $v^0 = 0$, $f \in \mathbf{H}^{1,2}(Q)$ satisfies $f(0,\cdot) = 0$ and $h \in \mathbf{H}^{2,5/2}(\Sigma)$ satisfies:*

$$h(0,\cdot) = 0,\ h(T,\cdot) = 0,\ \partial_t h(0,\cdot) = 0 \text{ and } \partial_t h(T,\cdot) = 0,$$

we have that the solution v^ε of (2.6) belongs to $\mathbf{H}^{2,4}(Q)$ with the estimate:

$$\|v^\varepsilon\|_{\mathbf{H}^{2,4}(Q)} \leq C\sqrt{1+T}\left(\|f\|_{\mathbf{H}^{1,2}(Q)} + \|h\|_{\mathbf{H}^{2,5/2}(\Sigma)}\right). \quad (2.8)$$

Remark 2.9. These results are not optimal in terms of the regularity imposed on h, but they are enough for our purpose.

Finally, we recall the following classical estimate for a linear ordinary differential equation:

Lemma 2.10. *Let $T > 0$ and let x be the solution in $C^0([0,T])$ of the following ordinary differential equation:*

$$\begin{cases} a(t)x(t) + x'(t) = g(t) & t \in (0,T), \\ x(0) = x_0, \end{cases}$$

for $x_0 \in \mathbb{R}$, $a \in L^1(0,T)$ and $g \in L^1(0,T)$. Then, we have the estimate:

$$\|x\|_{C^0([0,T])} \leq (|x_0| + \|g\|_{L^1(0,T)})e^{\|a\|_{L^1(0,T)}} \quad (2.9)$$

2.3 Results about Carleman estimates

In this subsection we present some Carleman estimates that are needed later. We first state a Carleman estimate which concerns a parabolic equation with non-homogeneous Neumann boundary conditions. More precisely, we consider the following system:

$$\begin{cases} -\delta\varphi_t - \Delta\varphi = f & \text{in } Q, \\ \partial_n\varphi = h & \text{on } \Sigma, \\ \varphi(T,\cdot) = \varphi^T & \text{in } \Omega, \end{cases} \tag{2.10}$$

for $\varphi^T \in L^2(\Omega)$, $f \in L^2(Q)$ and $\delta \in (0,1]$.

Lemma 2.11. *Let Ω be a C^4 domain, let $\widehat{\omega}$ be an open set included in Ω such that $\overline{\omega_0} \subset \widehat{\omega}$, let $m \geq 1$ and let $r \in \mathbb{R}$. Then, there is $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\delta \in (0,1]$, $\varphi^T \in L^2(\Omega)$, $f \in L^2(Q)$, $h \in L^2(\Sigma)$, $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$ we have:*

$$\begin{aligned} & s^{3+r}\lambda^{4+r} \iint_Q e^{-2s\alpha}\xi^{3+r}|\varphi|^2 + s^{1+r}\lambda^{2+r} \iint_Q e^{-2s\alpha}\xi^{1+r}|\nabla\varphi|^2 \\ & \leq C\left(s^r\lambda^r \iint_Q e^{-2s\alpha}\xi^r|f|^2 + s^{3+r}\lambda^{4+r} \iint_{(0,T)\times\widehat{\omega}} e^{-2s\alpha}\xi^{3+r}|\varphi|^2 + s^{1+r}\lambda^{1+r} \iint_\Sigma e^{-2s\alpha}\xi^{1+r}|h|^2 \right), \end{aligned} \tag{2.11}$$

for φ is the solution of (2.10).

The case $m = 1$, $\delta = 1$ and $r = 0$ of Lemma 2.11 is proved in [16, Theorem 1]. We get the case for $m \geq 1$ and $r \in \mathbb{R}$ repeating all the steps in [16] and we get uniformity on δ following the steps of, for instance, [23].

Next, we also need the following elliptic inequality, whose proof can be found in [12, Lemma 3]:

Lemma 2.12. *Let Ω be a C^4 domain, let $m \geq 1$ and let $r \in \mathbb{R}$. Then, there is $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\lambda \geq \lambda_0$, $s \geq CT^{2m}$ and $u \in L^2\big(0,T;H^1(\Omega)\big)$, we have:*

$$s^{2+r}\lambda^{3+r} \iint_Q e^{-2s\alpha}\xi^{2+r}|u|^2 \leq C\left(s^r\lambda^{1+r} \iint_Q e^{-2s\alpha}\xi^r|\nabla u|^2 + s^{2+r}\lambda^{3+r} \iint_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{2+r}|u|^2 \right). \tag{2.12}$$

Finally, we need a Carleman inequality for the backwards solution of (2.6) (see (2.16) below). For a simpler statement of the Carleman inequality, we define the weights:

$$\eta(t) := (s\xi^*(t))^{1/4+1/m}e^{-s\alpha^*(t)}, \tag{2.13}$$

and

$$\tilde{\eta}(t) := (s\xi^*(t))^{-3/4} e^{-s\alpha^*(t)}. \tag{2.14}$$

Proposition 2.13. *Let Ω be a C^4 domain, let $\tilde{\omega}$ be an open subset Ω such that $\overline{\omega_0} \subset \widetilde{\omega}$ and let $m \geq 8$. Then, there is $\varepsilon_0 > 0$, $C > 0$ and $\lambda_0 \geq 1$ such that if $T > 0$, $\varepsilon \in (0, \varepsilon_0)$, $\varphi^T \in \mathbf{L}^2(\Omega)$, $h \in \mathbf{H}^{2,5/2}(\Sigma)$, $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$, we have:*

$$\begin{aligned} s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2 + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi^\varepsilon|^2 \\ \leq C\left(s^4\lambda^5 \iint\limits_{(0,T)\times\widetilde{\omega}} e^{-2s\alpha}\xi^4|\varphi^\varepsilon|^2 + (1+T)\left(\|\eta h\|^2_{\mathbf{H}^{1,1/2}(\Sigma)} + \|\tilde{\eta} h\|^2_{\mathbf{H}^{2,5/2}(\Sigma)}\right)\right), \end{aligned} \tag{2.15}$$

for φ^ε the solution of the following system:

$$\begin{cases} -\varphi^\varepsilon_t - \Delta\varphi^\varepsilon + \nabla\pi^\varepsilon = 0 & \text{in } Q, \\ \varepsilon\pi^\varepsilon + \nabla\cdot\varphi^\varepsilon = 0 & \text{in } Q, \\ \partial_n\varphi^\varepsilon - \pi^\varepsilon n = h & \text{on } \Sigma, \\ \varphi^\varepsilon(T,\cdot) = \varphi^T & \text{in } \Omega. \end{cases} \tag{2.16}$$

The proof of this Carleman estimate is presented in Annex B.

Remark 2.14. Proposition 2.13 with $h = 0$ implies that there is $\varepsilon_0 > 0$ and $C > 0$ such that for all $T > 0$, $\varepsilon \in (0, \varepsilon_0]$ and $y^0 \in \mathbf{L}^2(\Omega)$, there is $f^\varepsilon \in \mathbf{L}^2\big((0,T)\times\widetilde{\omega}\big)$ such that

$$\|f^\varepsilon\|_{\mathbf{L}^2((0,T)\times\widetilde{\omega})} \leq C\|y^0\|_{\mathbf{L}^2(\Omega)};$$

and such that the solution of:

$$\begin{cases} y^\varepsilon_t - \Delta y^\varepsilon + \nabla p^\varepsilon = f^\varepsilon 1_{\widetilde{\omega}} & \text{in } Q, \\ \varepsilon p^\varepsilon + \nabla\cdot y^\varepsilon = 0 & \text{in } Q, \\ \partial_n y^\varepsilon - p^\varepsilon n = 0 & \text{on } \Sigma, \\ y^\varepsilon(0,\cdot) = y^0 & \text{in } \Omega, \end{cases}$$

satisfies $y^\varepsilon(T,\cdot) = 0$.

Up to our knowledge the result presented in Remark 2.14 is new.

3 Proof and optimality of Theorem 1.8 when Ω is strictly convex

In Section 3 we first give some remarks about how much Theorem 1.8 can be improved and we then prove Theorem 1.8 when Ω is strictly convex. The proof is simpler, clearer and more explicit

than when we are in a general domain. We recall that Ω strictly convex means that its boundary consists of one connected component and that:

$$\min_{\theta\in[0,|\partial\Omega|]} \kappa(\theta) > 0. \tag{3.1}$$

Moreover, we remark that a strictly convex domain always satisfies Hypothesis 1.1.

When Ω is a strictly convex convex domain, we do not need a_0 to be small. We state this in the following proposition:

Proposition 3.1. *Let Ω be a strictly convex C^4 domain. Then, there is $C > 0$ such that for all $a \in (0, 1]$ and for any real valued function $u \in H^4(\Omega) \cap H^1_0(\Omega)$, we have:*

$$\|\partial_x u\|_{C^0(\overline{\Omega})} \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \tag{3.2}$$

Throughout this section we prove Proposition 3.1, which automatically implies Theorem 1.8 if Ω is strictly convex.

Remark 3.2. Estimate (3.2) is false if we remove the term $\|L_a u\|_{H^1(\partial\Omega)}$. Indeed, we just have to consider the domain $\Omega = \{(x, y) : x^2 + y^2 \leq 1\}$ and the function $u(x, y) = 1 - x^2 - y^2$.

Although it might be possible that the spaces we give on the right of (3.2) are not optimal, the statement is false if we replace $\|\partial_{xy} u\|_{H^2(\Omega)}$ by $\|\partial_{xy} u\|_{L^2(\Omega)}$ in the right-hand side of (3.2), even if we replace $\|\partial_x u\|_{C^0(\overline{\Omega})}$ by $\|u\|_{L^2(\Omega)}$ in the left-hand side of (3.2).

Proposition 3.3. *Let Ω a C^4 domain with a boundary characterized by an equation of the following kind:*

$$g(x) + h(y) = 0,$$

for $g, h \in C^4(\mathbb{R})$. Then, there is no $C > 0$ such that for any $u \in H^4(\Omega) \cap H^1_0(\Omega)$, we have:

$$\|u\|_{L^2(\Omega)} \leq C \left(\|\partial_{xy} u\|_{L^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}\right). \tag{3.3}$$

Remark 3.4. The hypothesis of Proposition 3.3 includes circles, ellipses and p-norm spheres (for $p \geq 2$) among others.

Proof of Proposition 3.3. We prove this assertion by contradiction. Let us suppose that there is $C > 0$ such that, for any $u \in H^4(\Omega) \cap H^1_0(\Omega)$, we have (3.3). Let us consider the function $w(x, y) := g(x) + h(y)$, for g and h the functions stated in the hypothesis of Proposition 3.3. Then, $w \in H^2(\Omega) \cap H^1_0(\Omega)$ and $\partial_{xy} w = 0$.

We consider $f_n \in \mathcal{D}(\Omega)$ a sequence such that $f_n \to L_a w$ in $L^2(\Omega)$ and we define u_n as the solution of the equations:

$$\begin{cases} L_a u_n = f_n & \text{in } \Omega, \\ u_n = 0 & \text{on } \partial\Omega. \end{cases}$$

By usual theorems on elliptic regularity we have $u_n \in H^4(\Omega) \cap H_0^1(\Omega)$. Thus, applying (3.3) and since f_n vanishes on the boundary, we get that:

$$\|u_n\|_{L^2(\Omega)} \leq C\|\partial_{xy} u_n\|_{L^2(\Omega)}. \tag{3.4}$$

Moreover, by continuity with respect to the force in the elliptic problem, we have that $u_n \to u$ in $H^2(\Omega)$, where u is the solution of

$$\begin{cases} L_a u = L_a w & \text{in } \Omega, \\ u = 0 & \text{on } \partial\Omega. \end{cases}$$

Consequently, $u_n \to w$ in $H^2(\Omega)$.

Thus, if we take limits in (3.4) we get:

$$\|w\|_{L^2(\Omega)} \leq C\|\partial_{xy} w\|_{L^2(\Omega)} = 0,$$

which is absurd. □

Proof of Proposition 3.1. In order to make the proof more understandable we split it in three steps: first, we obtain a differential equation on the boundary in terms of $\partial_{xy} u$ and $L_a u$; then, we define an auxiliary function and perform estimates on it; finally, we estimate $\partial_x u$ in terms of the auxiliary function.

Step 1: Getting an equation on the boundary.
In order to get a differential equation on the boundary, we consider that because of the Dirichlet boundary condition u satisfies the equation:

$$u\big(\sigma_1(\theta), \sigma_2(\theta)\big) = 0, \qquad \forall \theta \in [0, |\partial\Omega|].$$

If we differentiate this, we have:

$$\sigma_1' \partial_x u + \sigma_2' \partial_y u = 0, \qquad \forall \theta \in [0, |\partial\Omega|]. \tag{3.5}$$

Moreover, if we differentiate (3.5), we get:

$$\sigma_1'' \partial_x u + \sigma_2'' \partial_y u + (\sigma_1')^2 \partial_{xx} u + (\sigma_2')^2 \partial_{yy} u + 2\sigma_1' \sigma_2' \partial_{xy} u = 0, \qquad \forall \theta \in [0, |\partial\Omega|]. \tag{3.6}$$

The idea is to get an equality from (3.6) in which we only have $\partial_x u$, $\partial_{xx} u$, $\partial_{xy} u$ and $L_a u$. In order to get it, we multiply (3.6) by σ_2' and use (3.5) and (1.11). We get that:

$$(\sigma_1''\sigma_2' - \sigma_2''\sigma_1')\partial_x u + \sigma_2' \left((\sigma_1')^2 - a(\sigma_2')^2\right)\partial_{xx} u = -2\sigma_1'(\sigma_2')^2\partial_{xy} u + (\sigma_2')^3 L_a u, \quad \forall \theta \in [0, |\partial\Omega|].$$

Recalling (1.3) we can rewrite the previous equation as follows:

$$-\kappa\partial_x u + \sigma_2' \left((\sigma_1')^2 - a(\sigma_2')^2\right)\partial_{xx} u = -2\sigma_1'(\sigma_2')^2\partial_{xy} u + (\sigma_2')^3 L_a u, \quad \forall \theta \in [0, |\partial\Omega|]. \tag{3.7}$$

Thanks to (3.1), we can divide (3.7) by κ:

$$-\partial_x u + \frac{\sigma_2'}{\kappa}\left((\sigma_1')^2 - a(\sigma_2')^2\right)\partial_{xx} u = -\frac{2\sigma_1'(\sigma_2')^2}{\kappa}\partial_{xy} u + \frac{(\sigma_2')^3}{\kappa} L_a u, \quad \forall \theta \in [0, |\partial\Omega|]. \tag{3.8}$$

In order to shorten this expression, we introduce the following notation:

$$A(\theta) := \frac{\sigma_2'(\theta)}{\kappa(\theta)}\left((\sigma_1'(\theta))^2 - a(\sigma_2'(\theta))^2\right) = \frac{\sigma_2'(\theta)}{\kappa(\theta)}\left(1 - (a+1)(\sigma_2'(\theta))^2\right). \tag{3.9}$$

Thus, (3.8) turns into:

$$-\partial_x u + A\partial_{xx} u = -\frac{2\sigma_1'(\sigma_2')^2}{\kappa}\partial_{xy} u + \frac{(\sigma_2')^3}{\kappa} L_a u \quad \forall \theta \in [0, |\partial\Omega|]. \tag{3.10}$$

Step 2: Defining an auxiliary function.
We now consider the lower part of the boundary:

$$\Gamma := \{(\sigma_1(\theta), \sigma_2(\theta)) : \sigma_1'(\theta) \geq 0\}.$$

We can extend the functions κ, σ_i' and σ_{ii}' $(i = 1, 2)$ to Ω. In order to do so, we define $\Theta_h(x)$ as the only value $\theta \in [0, |\partial\Omega|]$ such that $\sigma_1(\theta) = x$ and $\sigma(\theta) \in \Gamma$. We consider the following auxiliary function in $\overline{\Omega}$:

$$g(x, y) := -\partial_x u(x, y) + A(\Theta_h(x))\partial_{xx} u(x, y). \tag{3.11}$$

Besides, for any set $S \subset \mathbb{R}^2$, we define:

$$\mathcal{O}(S) := \left(S + \mathbb{R}e_2\right) \cap \Omega,$$
$$P_h(S) := \sigma(\Theta_h(S)).$$

We estimate g and $\partial_x g$ on horizontal segments; that is, on segments of the type:

$$l := [x_l, x_r] \times \{y\} \subset \overline{\Omega}. \tag{3.12}$$

First, we estimate the L^1-norm of $\partial_x g$ on any horizontal segment $l \subset \overline{\Omega}$. We consider the following equality:

$$\partial_x g(x,y) = \partial_x g(\sigma(\Theta_h(x))) + \int_{\sigma_2(\Theta_h(x))}^{y} \partial_{xy} g(x,z)dz, \qquad \forall (x,y) \in \overline{\Omega}. \tag{3.13}$$

The second term in this equality is clearly estimated in $L^1(l,dx)$-norm by $\|\partial_{xy}u\|_{H^2(\Omega)}$ (see (3.11)). In order to estimate the first term in the previous equality, we differentiate (3.10) in the direction θ:

$$\sigma_1' \partial_x g = -\sigma_2' \partial_y g + \partial_\theta \left(-\frac{2\sigma_1'(\sigma_2')^2}{\kappa} \partial_{xy} u + \frac{(\sigma_2')^3}{\kappa} L_a u \right), \qquad \theta \in [\tilde{\theta}, \hat{\theta}].$$

Since $\sigma_1(\Theta_h(x)) = x$, we have that:

$$\Theta_h'(x) = \frac{1}{\sigma_1'(\Theta_h(x))}.$$

So, if we combine this with the fact that $|\sigma'| = 1$, with the fact that $\Theta_h(\sigma_1(\theta)) = \theta$, and recalling the notation presented in (3.12), we have that:

$$\begin{aligned}
&\int_l \left(\frac{1}{|\sigma_1'|} \left| -\sigma_2'((\partial_y g) \circ \sigma) + \partial_\theta \left(-\frac{2\sigma_1'(\sigma_2')^2}{\kappa}((\partial_{xy}u) \circ \sigma) + \frac{(\sigma_2')^3}{\kappa}((L_a u) \circ \sigma) \right) \right| \right) (\Theta_h(x))dx \\
&\quad = \int_l \left| \frac{d\big(\sigma(\Theta_h(x))\big)}{dx} \right| \Bigg| -\sigma_2' \partial_y g - \partial_\theta \left(\frac{2\sigma_1'(\sigma_2')^2}{\kappa} \right) \partial_{xy} u - \frac{2\sigma_1'(\sigma_2')^2}{\kappa}(\sigma_1' \partial_x + \sigma_2' \partial_y)\partial_{xy} u \\
&\qquad\qquad + \partial_\theta \left(\frac{(\sigma_2')^3}{\kappa} \right) L_a u + \frac{(\sigma_2')^3}{\kappa}(\sigma_1' \partial_x + \sigma_2' \partial_y) L_a u \Bigg| (\sigma(\Theta_h(x)))dx \\
&\quad = \int_{P_h(l)} \Bigg| -\sigma_2' \partial_y g - \partial_\theta \left(\frac{2\sigma_1'(\sigma_2')^2}{\kappa} \right) \partial_{xy} u - \frac{2\sigma_1'(\sigma_2')^2}{\kappa}(\sigma_1' \partial_x + \sigma_2' \partial_y)\partial_{xy} u \\
&\qquad\qquad\qquad + \partial_\theta \left(\frac{(\sigma_2')^3}{\kappa} \right) L_a u + \frac{(\sigma_2')^3}{\kappa}(\sigma_1' \partial_x + \sigma_2' \partial_y) L_a u \Bigg|,
\end{aligned}$$

for the geometric functions in the second and third integral above evaluated in $\Theta_h(x)$, when we are in a point $p = (x,y) \in \Gamma$. Thus, recalling that $\partial_y g = -\partial_{xy} u + A \partial_{xxy} u$ and recalling (3.13) we get that:

$$\|\partial_x g\|_{L^1(l,dx)} \leq C \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right). \tag{3.14}$$

Next, we estimate the C^0-norm of g on any horizontal segment $l \subset \overline{\Omega}$. Indeed, if we use (3.8), (3.11) and the formula:

$$g(x,y) = g(\sigma(\Theta_h(x))) + \int_{\sigma_2(\Theta_h(x))}^{y} \partial_y g(x,z)dz, \qquad \forall (x,y) \in \overline{\Omega},$$

we get, for some $p \in l$, an estimate of $g(p)$ in terms of $\|\partial_{xy}u\|_{H^2(\Omega)}$ and $\|L_a u\|_{H^1(\partial\Omega)}$ with a constant depending on l. So, if we also consider (3.14), we get that:

$$\|g\|_{C^0(l)} \leq C(l)(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \tag{3.15}$$

Step 3: Getting the information from the ordinary differential equation (3.11).
We split Ω in different subsets depending on the sign of $A(\Theta_h(x))$. With that purpose, we denote θ_0, $\theta_a^\pm$ and $\theta_1^\pm$ the values such that:

$$\begin{cases} \sigma'(\theta_0) = e_1, \\ \sigma'(\theta_a^\pm) = \left(\sqrt{\frac{a}{a+1}}, \pm\sqrt{\frac{1}{a+1}}\right), \\ \sigma'(\theta_1^\pm) = \pm e_2. \end{cases} \tag{3.16}$$

Since the domain is strictly convex, all θ_i are uniquely determined for a fixed. We set:

$$x_0 := \sigma_1(\theta_0);\ x_a^\pm := \sigma_1(\theta_a^\pm) \text{ and } x_1^\pm := \sigma_1(\theta_1^\pm).$$

Because of (3.9) we have that $A(\theta_0) = 0$ and $A(\theta_a^\pm) = 0$. By (3.11) and (3.15), this implies that we can estimate $\partial_x u$ in the vertical segments given by $\{x = x_0\}$ and $\{x = x_a^\pm\}$. More generally, if we have an estimate of $|\partial_x u(x^*, y^*)|$, we have an estimate of $|\partial_x u(x^*, y)|$ for all y such that $(x^*, y) \in \Omega$ because $\|\partial_{xy} u\|_{C^0(\overline{\Omega})}$ can be estimated by $\|\partial_{xy} u\|_{H^2(\Omega)}$. Consequently, we just have to transmit horizontally the punctual estimates of $|\partial_x u|$. In order to do so, in the rest of the proof we fix some appropriate horizontal segments and regard equality (3.11) as an ordinary differential equation.

We first prove estimate (3.2) in $\Omega \cap \{x \geq x_0\}$. We consider a sequence of values $s_0 < \cdots < s_n$ such that $s_0 := x_0$, $s_n := x_1^+$ and such that for any $i \in \{1, \ldots, n\}$ there exists an horizontal segment $l_i \subset \overline{\Omega}$ such that the abscissa of its left endpoint (respectively its right endpoint) is s_{i-1} (respectively s_i). Because we are in a regular bounded convex domain, all this can be done (see Figure 1 for one such example). We denote by y_i the second coordinate of any point of l_i. Since the segments $l_1, \ldots, l_n$ do not depend on a, we have, due to (3.15), the following estimate in those segments with a constant that only depends on Ω:

$$\|g\|_{C^0(\cup_{i=1}^n l_i)} \leq C(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \tag{3.17}$$

Let us set $j_a \in \{1, \ldots, n\}$ such that $\theta_a^+ \in \Theta_h(l_{j_a})$. If there are two such segments, we choose the one on the right. We first get an estimate in $\mathcal{O}(\{x \geq x_a^+\})$, where we have:

$$A(\Theta_h(x)) \leq 0 \tag{3.18}$$

(see (3.9)). For that purpose, we define $l_a^r := l_{j_a} \cap \mathcal{O}(\{x \geq x_a^+\})$.

First, considering that $\partial_x u(x_a^+, y) = -g(x_a^+, y)$ and (3.17), we obtain the following estimate:

$$|\partial_x u(x_a^+, y_{j_a})| \leq C \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right). \tag{3.19}$$

Next, we have that, for any $z \in [x_a^+, s_{j_a}]$:

$$\begin{aligned} -\int_{x_a^+}^{z} (x - z - 2)(\partial_x u \partial_{xx} u)(x, y_{j_a}) dx &= \frac{1}{2} \int_{x_a^+}^{z} |\partial_x u(x, y_{j_a})|^2 dx \\ &\quad + \frac{(x_a^+ - z - 2)}{2} |\partial_x u(x_a^+, y_{j_a})|^2 - (-1)|\partial_x u(z, y_{j_a})|^2. \end{aligned} \tag{3.20}$$

Thus, considering (3.11), (3.19) and (3.20), we have that for a constant C independent of z:

$$\begin{aligned} &\frac{1}{2} \int_{x_a^+}^{z} |\partial_x u(x, y_{j_a})|^2 dx + \int_{x_a^+}^{z} (x - z - 2) A\left(\Theta_h(x)\right) |\partial_{xx} u(x, y_{j_a})|^2 dx + |\partial_x u(z, y_{j_a})|^2 \\ &\qquad \leq \int_{x_a^+}^{z} (x - z - 2)(g \partial_{xx} u)(x, y_{j_a}) dx + C \left(\|\partial_{xy} u\|_{H^2(\Omega)}^2 + \|L_a u\|_{H^1(\partial\Omega)}^2 \right). \end{aligned} \tag{3.21}$$

So, the last term to be estimated in (3.21) is the one with g. In order to do that, we integrate by parts:

$$\begin{aligned} \int_{x_a^+}^{z} (x - z - 2)(g \partial_{xx} u)(x, y_{j_a}) dx &= -\int_{x_a^+}^{z} (x - z - 2)(\partial_x g \partial_x u)(x, y_{j_a}) dx \\ &\quad - \int_{x_a^+}^{z} (g \partial_x u)(x, y_{j_a}) dx - (x_a^+ - z + 2) g(x_a^+, y_{j_a}) \partial_x u(x_a^+, y_{j_a}) - 2g(z, y_{j_a}) \partial_x u(z, y_{j_a}) \\ &\qquad \leq C \left(\|\partial_x g\|_{L^1(l_a^r, dx)} + \|g\|_{C^0(l_a^r)} \right) \|\partial_x u\|_{C^0(l_a^r)}. \end{aligned} \tag{3.22}$$

Using that $ab \leq \frac{1}{4\eta} a^2 + \eta b^2$ and estimating the norms of g by (3.14) and (3.17), we obtain the following from (3.21):

$$\int_{x_a^+}^{z} |\partial_x u(x, y_{j_a})|^2 dx + |\partial_x u(z, y_{j_a})|^2 \leq C_\eta \left(\|\partial_{xy} u\|_{H^2(\Omega)}^2 + \|L_a u\|_{H^1(\partial\Omega)}^2 \right) + \eta \|\partial_x u\|_{C^0(l_a^r)}^2.$$

Since z is arbitrary, we deduce that:

$$\|\partial_x u\|_{C^0(l_a^r)} \leq C_\eta \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right) + \eta \|\partial_x u\|_{C^0(l_a^r)}.$$

Thus, taking η small enough, we can absorb the last term on the right-hand side. Moreover, as $\|\partial_{xy} u\|_{C^0(\overline{\Omega})}$ is estimated in terms of $\|\partial_{xy} u\|_{H^2(\Omega)}$, we get:

$$\|\partial_x u\|_{C^0(\overline{\mathcal{O}(l_a^r)})} \leq C \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right). \tag{3.23}$$

This method also works for $l_{j_a+1}, \ldots, l_n$ because the trace of $\partial_x u$ on the left of these segments is obtained first by (3.23) and then inductively. Indeed, in l_i it suffices to multiply at both sides

of the identity (3.11) by $(x - s_i - 2)\partial_{xx}u$ and integrate by parts as above. Therefore, we can get inductively the estimate:

$$\|\partial_x u\|_{C^0(\overline{\mathcal{O}(l_a^r)})} + \sum_{i=j_a+1}^{n} \|\partial_x u\|_{C^0(\overline{\mathcal{O}(l_i)})} \leq C \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right),$$

which implies that:

$$\|\partial_x u\|_{C^0(\overline{\mathcal{O}(\{x \geq x_a^+)\})}} \leq C \left(\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)} \right).$$

As for getting the estimate in $\mathcal{O}(\{x \in [x_0, x_a^+]\})$, we can obtain it analogously as before: first in $l_a^l := l_{j_a} \cap O(\{x \leq x_a^+\})$, then inductively in $l_{j_a-1}, \ldots, l_1$. The only difference is that this time we multiply by $x + x_a^+ + 2$ and we spread the estimates to the left. The reason for these changes is that for $x \in [x_0, x_a^+]$, $A(\Theta_h(x))$ is positive instead of negative (see (3.9)).

Finally, it is quite clear that we can get the estimate in a similar way for $\Omega \cap \{x < x_0\}$. Thus, (3.2) is true. □

4 Proof of Theorem 1.8 and some geometrical results

In this section we present the proof of Theorem 1.8 as well as some strongly related results. First, in Subsection 4.1 we prove Lemma 1.3. Second, in Subsections 4.2 and 4.3 we state and prove some geometrical consequences of Hypothesis 1.1. Finally, in Subsection 4.4 we prove Theorem 1.8, using, among others, Section 3.

We recall that some of the notation used in this section has been introduced above Hypothesis 1.1.

4.1 Proof of Lemma 1.3

Lemma 1.3 is a consequence of Sard's Theorem:

Theorem 4.1 (Sard's Theorem)**.** *Let $f : \mathbb{R} \to \mathbb{R}$ a C^1 function. Let*

$$X := \{x \in \mathbb{R} : f'(x) = 0\}.$$

Then, $f(X)$ has zero measure.

Proof of Lemma 1.3. In order to apply Sard's Theorem, we consider the functions:

$$(\sigma_1^1)', \dots, (\sigma_1^k)', (\sigma_2^1)', \dots, (\sigma_2^k)'.$$

Let us denote $Z \subset [-1,1]$ the set of values such that if $(\sigma_1^i)'(\theta) = z$ or if $(\sigma_1^i)'(\theta) = -z$ or if $(\sigma_2^i)'(\theta) = z$ or if $(\sigma_2^i)'(\theta) = -z$; then, $\kappa^i(\theta) \neq 0$. Since $\kappa^i(\theta) = 0$ implies that $(\sigma_1^i)''(\theta) = 0$ and $(\sigma_2^i)''(\theta) = 0$ (see (1.3)), thanks to Sard's Theorem (and to the fact that a finite union of null-measure sets and translations of those sets are still of null measure) we get that the measure of $[-1,1] \setminus Z$ is 0.

Let us consider $\psi \in \sin^{-1}(Z)$. Then, we have that $\partial\left(U_\psi(\Omega)\right) = U_\psi(\partial\Omega)$ and it can be parametrized by $\widetilde{\sigma}^i := U_\psi(\sigma^i)$. Moreover, we have $(U_\psi(\sigma^i))' = U_\psi((\sigma^i)')$. Consequently, $(\widetilde{\sigma}^i)' = \pm e_2$ if and only if $(\sigma^i)'(\theta) = U(-\psi)(0, \pm 1)$; that is, if and only if $(\sigma^i)'(\theta) = \pm(\sin(\psi), \cos(\psi))$. Similarly, $(\widetilde{\sigma}^i)' = \pm e_1$ if and only if $(\sigma^i)'(\theta) = \pm(\cos(\psi), -\sin(\psi))$. Furthermore, if $\widetilde{\kappa}^i(\theta)$ is by definition the curvature in $\widetilde{\sigma}^i(\theta)$, we have, by the non-variation of the curvature by rotations, $\widetilde{\kappa}^i(\theta) = \kappa^i(\theta)$. Thus, by the definition of Z, taking $z = \sin\psi$, we have that all the points of $\partial\left(U_\psi(\Omega)\right)$ with horizontal or vertical tangent vector have a non zero curvature.

Finally, the measure of $\mathbb{R} \setminus \sin^{-1}(Z)$ is null because, for all $k \in \mathbb{Z}$, the sinus is a diffeomorphism from $(\pi(k-1/2), \pi(k+1/2))$ to $(-1,1)$ (and because a countable union of sets of null measure has null measure). □

4.2 Geometrical consequences of Hypothesis 1.1

In order to prove Theorem 1.8, we need to define equivalent notions to the ones presented in the convex case (see Section 3).

Definition 4.2. We define Γ as the subset of $\partial\Omega$ such that $p = \sigma^i(\theta) \in \Gamma$ if and only if at least one of the following properties is satisfied:

- $\exists \delta_0(p) > 0 : \forall \delta \in (0, \delta_0(p)),\ p + \delta e_2 \in \overline{\Omega}$,
- $(\sigma^i)'(\theta) = \pm e_2$.

When Ω is convex we have that Γ is the bottom of Ω. Moreover, for an illustration on what Γ may look like in a non-convex domain, we can regard Figure 2 below.

Remark 4.3. The relative boundary of Γ is given by points of tangent vectors $\pm e_2$. Indeed, the components of $\partial\Omega$ are closed curves; thus, by regularity, having a vertical tangent vector is the only possibility.

Definition 4.4. Let $(x, y) \in \overline{\Omega}$. We define:

$$\mathbb{P}_h(x, y) := (x, y) - \lambda e_2 \text{ such that } \lambda := \min\{\lambda \in \mathbb{R}^+ : (x, y) - \lambda e_2 \in \Gamma\}.$$

We remark that when Ω is convex $\mathbb{P}_h$ represents the vertical projection on Γ, is continuous and does not depend on y. In the general situation, though, there is a dependence on y and $\mathbb{P}_h$ is not continuous when Γ is not connected (see B^1 in Figure 2 below). Yet, we can define an application P_h that coincides with the one given in Section 3 when Ω is convex:

Definition 4.5. Let $l_i = [x_l^i, x_r^i] \times \{y^i\} \subset \overline{\Omega}$ a segment. Then,

$$P_h(l_i) := \overline{\mathbb{P}_h((x_l^i, x_r^i) \times \{y^i\})}.$$

Moreover, we see that Hypothesis 1.1 implies the existence of segments like in the case of a convex domain:

Lemma 4.6. *Let Ω be a domain that satisfies Hypothesis 1.1. Then, there is a subset $S \subset \overline{\Omega}$ such that:*

- *S is a finite union of horizontal segments $l_i := [x_l^i, x_r^i] \times \{y^i\}$.*
- *$\mathbb{P}_h(S) = \Gamma$.*
- *$\mathbb{P}_h$ is continuous in the relative interior of each segment l_i.*

Example 4.7. In Figure 2 S is given by the segments: $[A^1, A^2]$, $[A^2, A^5]$, $[C^1, C^5]$, $[D^1, D^6]$ and $[E^1, E^2]$.

The proof of Lemma 4.6 is postponed to Section 4.3. We first prove some geometrical results:

Lemma 4.8. *Let Ω be a domain that satisfies Hypothesis 1.1. We have:*

1. *If $(\sigma_1^i)'(\theta) = 0$ or if $(\sigma_2^i)'(\theta) = 0$, then, for some $\delta(\theta) > 0$, κ^i does not change of sign in $(\theta - \delta(\theta), \theta + \delta(\theta))$.*
2. *The number of points in $\partial\Omega$ with tangent vectors $\pm e_1$ or $\pm e_2$ is finite.*
3. *Given any $c \in \mathbb{R}$, the number of points in $\partial\Omega \cap \{x = c\}$ or in $\partial\Omega \cap \{y = c\}$ is finite.*
4. *Given any $c \in \mathbb{R}$, there is $\delta(c) > 0$ such that:*
   - *We have*
   $$([c - \delta(c), c + \delta(c)] \times \mathbb{R}) \cap \partial\Omega = \bigcup_{p = \sigma^{i_p}(\theta_p) \in \partial\Omega \cap \{x = c\}} \sigma^{i_p}(I_p),$$
   for $I_p = (\theta_p^1, \theta_p^2)$, for some $\theta_p^1 < \theta_p < \theta_p^2$.

- *In the set*

$$\big(([c-\delta(c), c+\delta(c)] \setminus \{c\}) \times \mathbb{R}\big) \cap \partial\Omega,$$

we do not have $p = \sigma^i(\theta)$ *with* $(\sigma^i)'(\theta) = \pm e_2$.

5. *There is some* $\eta > 0$ *such that for all points* $p = \sigma^i(\theta_p) \in \partial\Omega$ *with* $(\sigma^i)'(\theta_p) = \pm e_1$, *there exists a neighbourhood* $V_p = \sigma^i(I_p) \subset \partial\Omega$ *(*$I_p = (\theta_p^1, \theta_p^2)$, *for some* $\theta_p^1 < \theta_p < \theta_p^2$*) such that* $\sigma_2^i(\theta_p^1) = \sigma_2^i(\theta_p^2)$ *and such that* $|\kappa^i| > \eta$.

6. *There exists* $a_0 > 0$ *small enough such that, for all* $a \in (0, a_0)$, *for each point* $p = \sigma^i(\theta) \in \partial\Omega$ *with* $(\sigma^i(\theta))' = \pm e_2$ *there is a neighbourhood* $U_p \subset \partial\Omega$ *which has exactly a point of tangent vector* $\pm\left(\sqrt{\frac{a}{1+a}}, \sqrt{\frac{1}{1+a}}\right)$ *and exactly another one of tangent vector* $\pm\left(\sqrt{\frac{a}{1+a}}, -\sqrt{\frac{1}{1+a}}\right)$. *Reciprocally, if* $p_a = \sigma^i(\theta^a) \in \partial\Omega$ *satisfies* $(\sigma^i)'(\theta^a) = \left(\pm\sqrt{\frac{a}{1+a}}, \pm\sqrt{\frac{1}{1+a}}\right)$, *then* $p_a \in U_p$, *for* U_p *one of the above defined neighbourhoods. Finally, we can suppose that for some* $\eta > 0$, $|\kappa^i| > \eta$ *on those neighbourhoods.*

Proof of Lemma 4.8. Firstly, implication 1 is an easy consequence of Ω being at least C^2.

Secondly, we prove implication 2 for points of tangent vector $\pm e_2$ by contradiction. If they are not finite, by (pre-)compactness and regularity of Ω, there is a point $p = \sigma^i(\theta)$ and $\theta_n \to \theta$ such that $(\sigma_1^i)'(\theta_n) = 0$. Obviously $(\sigma_1^i)'(\theta) = 0$. But, because of the regularity of Ω, we also have $(\sigma_1^i)''(\theta) = 0$. Indeed,

$$(\sigma_1^i)''(\theta) = \lim_{s\to\theta} \frac{(\sigma_1^i)'(s) - (\sigma_1^i)'(\theta)}{s-\theta} = \lim_{n\to\infty} \frac{(\sigma_1^i)'(\theta_n) - (\sigma_1^i)'(\theta)}{\theta_n - \theta} = 0. \tag{4.1}$$

Thus, we get by (1.3) that $\kappa^i(\theta) = 0$, which contradicts Hypothesis 1.1. The proof for points with tangent vector $\pm e_1$ is analogous.

Thirdly, given any line $x = c$, we prove by contradiction that there is a finite number of points in $\partial\Omega \cap \{x = c\}$. Indeed, if we have an infinite number of points, by regularity and compactness we can write a sequence of distinct elements as $\sigma^i(\theta_n)$ with $\theta_n \to \theta$. Since $\sigma_1^i(\theta_n) = c$, by an equality similar to (4.1), $(\sigma_1^i)'(\theta) = 0$. Since $\kappa^i(\theta) \neq 0$ by Hypothesis 1.1, $\sigma^i(\theta_n)$ cannot be in $\{x = c\}$ for n large enough, contradicting the choice of θ_n. The proof for $y = c$ is analogous.

Fourthly, statement 4 is a consequence of assertion 2. Indeed, the only possibility is that there is an infinite number of curves of $\partial\Omega$ that approach the line $x = c$ and then move away (like a parabola). But this implies that there is an infinite number of points of tangent vector $\pm e_2$, which contradicts statement 2.

Fifthly, assertion 5 is an easy consequence of statements 1 and 2 and of picking the neighbourhoods small enough.

Finally, statement 6 is a consequence of assertion 2. Indeed, we consider U_{p_j} some neighbourhoods of p_j, for p_j the points of tangent vector $\pm e_2$. We have the bound:

$$\inf_i \inf_{\theta:\sigma^i(\theta)\notin U_{p_j}\forall j} |(\sigma_1^i)'(\theta)| > 0.$$

Moreover, by making them smaller if necessary, we have due to Hypothesis 1.1 and the continuity of the κ^i:

$$\inf_i \inf_{\theta:\exists j:\sigma^i(\theta)\in U_{p_j}} |\kappa^i(\theta)| > 0.$$

Consequently, we can fix some smaller neighbourhoods U_{p_j} and the parameters a_0 and η. □

4.3 Proof of Lemma 4.6 and some remarks

In this subsection we first present the proof of Lemma 4.6; and then, we state some direct consequences.

First of all, we define some useful notation:

Definition 4.9. Let $\Omega \subset \mathbb{R}^2$ be a domain and $x \in \mathbb{R}$. We define:

$$\Omega_x := \Omega \cap \big((-\infty, x) \times \mathbb{R}\big).$$

Now we are ready to present the proof:

Proof of Lemma 4.6. Without loss of generality we can suppose that:

$$0 = \min\{x : \exists y \text{ with } (x, y) \in \overline{\Omega}\}.$$

We consider:

$$I = \{x \in \mathbb{R}^+ : \forall s \in [0, x]\ \exists S_s \subset \overline{\Omega}_s : \\ S_s \text{ satisfies the conclusion of Lemma 4.6 with } \Gamma \text{ replaced by } \Gamma \cap \overline{\Omega_s}\}.$$

First, we remark that $0 \in I$ because $\Omega_0 = \emptyset$. Thus, in order to prove the result it suffices to show that if $[0, c) \subset I$, we have $c \in I$ and that if $c \in I$, we have $(c, c + \delta(c)) \subset I$ for $\delta(c) > 0$ sufficiently small.

Next, let us show that $[0, c) \subset I$ implies $c \in I$. We consider $\delta(c)$ given by statement 4 in Lemma 4.8. In $\overline{\Omega_{c-\delta(c)}}$ we already have the segments $S_{c-\delta(c)}$. Thus, it suffices to define in $\overline{\Omega_c \setminus \Omega_{c-\delta(c)}}$ a finite number of segments slightly above the curves of Γ, which can be done by regularity and pre-compactness. If there is some point $p = \sigma^i(\theta) \in \Gamma$ with $x = c$ such that $(\sigma^i)'(\theta) = \pm e_2$ we might have to start a segment in p and spread it to the left (see the segment $[E^2, E^1]$ in Figure 2). So we have $c \in I$.

Finally, let us show that $c \in I$ implies $c + \delta(c) \in I$, for $\delta(c)$ sufficiently small. Again, we pick $\delta(c)$ small enough to satisfy the conclusion of statement 4 in Lemma 4. We already have S_c, so now we consider the possible situation for the points in $\Gamma \cap \{x = c\}$ (which are finite by statement 3 in Lemma 4.8):

- For the points in $\{x = c\} \cap \Gamma$ which do not have as tangent vector $\pm e_2$, we just have to extend the segment of S_c with right endpoint in $x = c$ to the right of $\overline{\Omega}_c$, unless some segment has a right endpoint which belongs to $\partial\Omega$. In that case, we just have to start a new segment whose left endpoint can be joint by a vertical segment inside $\overline{\Omega}$ with the right endpoint of the former one (see the dashed segments near C^5 in Figure 2).
- As for the points in $\{x = c\} \cap \Gamma$ of tangent vector $\pm e_2$, we might not be allowed to extend the segment because $\mathbb{P}_h$ becomes discontinuous and we need to start two new segments (for instance, in A^2 in Figure 2 we must stop and start two new segments: one with left endpoint A^2, the other one with left endpoint C^1). It might also be the case that we need to start a new segment above the point (see near D^1 in Figure 2).

Summing up, since all this happens for a finite number of situations, for $\delta(c)$ sufficiently small we have $(c, c + \delta(c)) \subset I$. □

Remark 4.10. Because of the conclusion of Lemma 4.6, the left endpoint of each segment l_i is either a point $p = \sigma^i(\theta) \in \Gamma$ with $(\sigma^i)'(\theta) = \pm e_2$ and $\kappa^i(\theta) > 0$ (the case of A^1 in Figure 2) or it can be joined by a vertical segment (including degenerated segments) inside Ω with some other segment l_j such that $x_l^j < x_l^i \leq x_r^j$ (the case of A^2, C^1, D^1 and E^1 in Figure 2).

Remark 4.11. Another easy consequence of Lemma 4.6 is that, if Ω satisfies Hypothesis 1.1, since $\mathbb{P}_h(S) = \Gamma$, for all $p \in \overline{\Omega}$ there is $\lambda \in \mathbb{R}$ such that $[p, p + \lambda e_2] \subset \overline{\Omega}$ and such that $p + \lambda e_2 \in S$. Consequently, since $\|\partial_{xy} u\|_{C^0(\overline{\Omega})}$ is estimated by $\|\partial_{xy} u\|_{H^2(\Omega)}$, if suffices to get an estimate of $\|\partial_x u\|_{C^0(S)}$ to prove Theorem 1.8.

Remark 4.12. Given any segment l_i as defined in Lemma 4.6, because of its third property, it makes sense to define $\Theta_h^i(x)$, $A_i(x)$ and $g_i(x, y)$ as in the convex case (see Section 3) by looking at $P_h(l_i)$.

Indeed, they make sense in the domain limited superiorly by l_i and inferiorly by $P_h(l_i)$. Moreover, we recall that:

$$g_i(x,y) := -\partial_x u(x,y) + A_i(\Theta_h^i(x))\partial_{xx}u(x,y). \tag{4.2}$$

We also remark that if $\min_{l_i} |\kappa(\Theta_h^i(x))| > 0$, we can prove the following estimate as in the convex case (see (3.14) and (3.15)) for a constant C depending only on S:

$$\|g_i\|_{C^0(l_i)} + \|\partial_x g_i\|_{L^1(l_i,dx)} \leq C(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \tag{4.3}$$

4.4 Proof of Theorem 1.8

In order to prove Theorem 1.8 we get an estimate in each segment l_i given by Lemma 4.6. Indeed, we prove inductively that:

$$\|\partial_x u\|_{C^0(l_i)} \leq C(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}). \tag{4.4}$$

By inductively, we mean that (4.4) is proved for any other segment l_j such that $x_l^j < x_l^i$ (see Lemma 4.6 for the notation).

First, for getting a pointwise estimate on x_l^i, we consider the two situations given in Remark 4.10. In the first case, because of statement 6 on Lemma 4.8, we can get an estimate on x_l^i by $\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}$ as in the convex case (see Section 3). In the second case, by the induction hypothesis and the fact that $\|\partial_{xy}u\|_{C^0(\overline{\Omega})}$ can be estimated by $\|\partial_{xy}u\|_{H^2(\Omega)}$, we get the estimate on x_l^i by $\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}$. So, in both cases we have:

$$|\partial_x u(x_l^i, y^i)| \leq C\left(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}\right). \tag{4.5}$$

So, once we have (4.5), we have to propagate the estimate in l_i. Indeed, we can split, extend and move the segments l_i (see the points $A^3, A^4, C^3, C^4, D^2, D^3, D^4$ in Figure 2, which allow us to split their respective segment into smaller ones) so that we only have one of the four following possibilities for $P_h(l_i)$:

1. $P_h(l_i)$ is the intersection of Γ with one of the neighbourhoods $\overline{U_p}$ (see statement 6 in Lemma 4.8).
2. $P_h(l_i)$ has null intersection with all the neighbourhoods U_p and V_p.
3. $P_h(l_i)$ is one of the neighbourhoods $\overline{V_p}$ (see statement 5 in Lemma 4.8) which has a positive curvature.
4. $P_h(l_i)$ is one of the neighbourhoods $\overline{V_p}$ (see statement 5 in Lemma 4.8) which has a negative curvature.

- Let us first deal with the case 1. We recall that by Remark 4.12, functions like g_i or Θ_h^i or A_i make sense in l_i. We start in the subcase in which $(\sigma_1^i)'(\Theta_h^i(x_r^i)) = 0$. In Figure 3, Ω_1 illustrates an example in which the surface is locally convex and Ω_2 an example in which it is locally concave. We remark that, in the concave case, the ratio $\frac{(\sigma_2^i)'}{\kappa^i}$ has the same sign as in the convex case (by the criteria fixed above Hypothesis 1.1, $(\sigma_2^i)' < 0$, $(\sigma_1^i)' \geq 0$ and $\kappa^i < 0$ in that part of Γ). Analogously to the convex case (see Section 3), we define $p_a \in V_q$ as the point such that $A_i(\Theta_h(p_a)) = 0$. Moreover, q_a is the point in l_i such that $\mathbb{P}_h(q_a) = p_a$. Thus, in both cases, we can multiply (4.2) by $x + C$ for C large enough in the segment $[q_1, q_a]$ and by $x - C$ for C large enough in $[q_a, q_2]$ and then follow the procedure of the convex case. The case in which $(\sigma_1^i)'(\Theta_h^i(x_l^i)) = 0$ is analogous. Consequently, we have (4.4) in the case 1.

- If $P_h(l_i)$ belongs neither to U_p nor to V_p (situation 2), we have, by statement 6 in Lemma 4.8, that there is $\eta > 0$ such that:
$$|A_i(\Theta_h^i(l_i))| > \eta$$
(see Remark 4.12 or (3.9) for definition of A_i). Consequently, we can divide the equation (4.2) by A_i and hope to have uniform estimates. With that purpose, we define:
$$\widetilde{g_i}(x, y) := -\frac{1}{A_i(\Theta_h^i(x))}\partial_x u(x, y) + \partial_{xx} u(x, y). \tag{4.6}$$
In that sense, we can get the same estimate for $\widetilde{g_i}$ as the one for g_i in (4.3) by following similar steps. Thus, seeing (4.6) as an ordinary linear differential equation in l_i whose initial value is in (x_l^i, y^i), we get (4.4) by Lemma 2.10.

- As for the case 3 (see Figure 4 for the notation), we mainly replicate the method of the previous section. In this paragraph and in the following one, we do all the estimates by $\|\partial_{xy} u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}$ even if we do not explicitely write it. We already have by (4.5) a pointwise estimate in q_{-1}, so we can prove (4.4) in $[q_{-1}, q]$ following the method of the convex case. As for $[q, q_1]$, to replicate the method of the convex case, we need an estimate of $|\partial_x u(q_1)|$. In order to get it, first we use the estimate of $|\partial_x u(q_{-1})|$ in (4.5) to get an estimate of $|\partial_x u(p_{-1})|$. Then, using the Dirichlet boundary condition, we get an estimate of $|\partial_y u(p_{-1})|$. Next, with the estimate of $\|\partial_{xy} u\|_{C^0([p_{-1}, p_1])}$, we have an estimate of $|\partial_y u(p_1)|$. To continue with, again by Dirichlet conditions, we have an estimate of $|\partial_x u(p_1)|$. Finally, by estimating $\|\partial_{xy} u\|_{C^0([p_1, q_1])}$, we have an estimate of $|\partial_x u(q_1)|$, so we can replicate the method of the previous section to prove (4.4) in $[q, q_1]$.

- The last case is that $P_h(l_i)$ is locally concave (see Figure 5 for the notation). The technique is the same one as in the case 3 (recall that the ratio $\frac{(\sigma_2^i)'}{\kappa^i}$ and consequently A^i have the

same sign as in the convex case), but this time we face the extra difficulty of showing that having the estimate of $|\partial_y u(p_{-1})|$ implies having the estimate of $|\partial_y u(p_1)|$. In order to do it, we consider that in each component of $\partial\Omega$ we have the following equality:

$$a\kappa^i \partial_y u - (\sigma_1^i)'(1-(a+1)((\sigma_2^i)')^2)\partial_{yy}u = ((\sigma_1^i)')^3 L_a u - 2a((\sigma_1^i)')^2(\sigma_2^i)'\partial_{xy}u.$$

This equality can be obtained similarly to (3.7). So, similarly as in Remark 4.12, we define locally at the left of $\sigma^i((\theta_p^1,\theta_p))$ (see statement 5 in Lemma 4.8 for the notation) a function $\Theta_v^i(y)$ and a function:

$$\widehat{g}(x,y) := a\kappa^i(\Theta_v^i(y))\partial_y u(x,y) - \left((\sigma_1^i)'(1-(a+1)((\sigma_2^i)')^2)\right)(\Theta_v^i(y))\partial_{yy}u(x,y). \tag{4.7}$$

Moreover, we can obtain as (4.3) that (see Figure 5 for the notation):

$$\|\widehat{g}\|_{L^1([q_{-1},k_{-1}],dy)} \leq C(\|\partial_{xy}u\|_{H^2(\Omega)} + \|L_a u\|_{H^1(\partial\Omega)}).$$

So, by seeing (4.7) as an ordinary differential equation whose initial data is p_{-1}, because there is $\eta > 0$ such that $(\sigma_1^i)'(1-(a+1)((\sigma_2^i)')^2) \geq \eta$ in V_p, we can use Lemma 2.10 to get an estimate of $|\partial_y u(k_{-1})|$. Then, because $\|\partial_{xy}u\|_{C^0([k_{-1},k_1])}$ can be estimated, we have an estimate of $|\partial_y u(k_1)|$. Finally, we can propagate the pointwise estimate of $|\partial_y u|$ in the segment $[k_1,p_1]$ in an analogous way as in the segment $[p_{-1},k_{-1}]$. Consequently, we have the estimate of $|\partial_y u(p_1)|$ and we can replicate the method of the case 3 to obtain (4.4).

Finally, by Remark 4.6, since we have (4.4) for all segments in S, we have (1.12).

5 Proof of Theorem 1.7

For the proof of this theorem we define a subdomain $\widetilde{\omega}$ compactly contained in ω such that $\overline{\omega_0} \subset \widetilde{\omega}$. Moreover, we consider a cut-off function $\chi \geq 0$ satisfying $\text{supp}(\chi) \subset \omega$ and $\chi = 1$ in $\widetilde{\omega}$. In addition, we define D^i as the tensor which contains all the derivatives of order i with respect to the x and y variable. In order to clarify the proof we divide it in several steps.

Step 1: Estimates of the crossed derivative.
First of all, we consider estimate (1.15) squared, multiplied by $(s\xi^*)^{15}\lambda^{16}e^{-2s\alpha^*}$ and integrated in time. If we also bound the weights (see (1.8)), we get that:

$$s^{15}\lambda^{16}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2 \leq Cs^{15}\lambda^{16}\sum_{i=0}^{2}\iint_Q e^{-2s\alpha}\xi^{15}|D^i\partial_{xy}\varphi^\varepsilon|^2. \tag{5.1}$$

Next, we apply the elliptic estimate (2.12) to $D^i\partial_{xy}\varphi^\varepsilon$ for $i=0,\dots,7$ (we take $D^0\partial_{xy}\varphi^\varepsilon := \partial_{xy}\varphi^\varepsilon$). We get that:

$$\sum_{i=0}^{7} s^{19-2i}\lambda^{20-2i}\iint_Q e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2 \le C\Bigg(s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|D^8\partial_{xy}\varphi^\varepsilon|^2 + \sum_{i=0}^{7} s^{19-2i}\lambda^{20-2i}\iint\limits_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2\Bigg). \quad (5.2)$$

Moreover, since under our hypothesis $1\le Cs\xi$, we combine that fact with (5.2) and (5.1), and we get that:

$$s^{15}\lambda^{16}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2 + \sum_{i=0}^{7} s^{19-2i}\lambda^{20-2i}\iint_Q e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2 \le C\left(s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|D^8\partial_{xy}\varphi^\varepsilon|^2 + \sum_{i=0}^{7} s^{19-2i}\lambda^{20-2i}\iint\limits_{(0,T)\times\omega_0} e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2\right). \quad (5.3)$$

To continue with, we deal with each term of $D^8\partial_{xy}\varphi^\varepsilon$. In order to do so, we use Proposition 2.13 (on each term of $D^8\partial_{xy}\varphi^\varepsilon$) and get that:

$$s\lambda^2\iint_Q e^{-2s\alpha}\xi|D^9\partial_{xy}\varphi^\varepsilon|^2 + s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|D^8\partial_{xy}\varphi^\varepsilon|^2 \le C\left(s^4\lambda^5\iint\limits_{(0,T)\times\widetilde{\omega}} e^{-2s\alpha}\xi^4|D^8\partial_{xy}\varphi^\varepsilon|^2 + (1+T)\left(\|\eta h\|^2_{\mathbf{H}^{1,1/2}(\Sigma)} + \|\tilde{\eta}h\|^2_{\mathbf{H}^{2,5/2}(\Sigma)}\right)\right), \quad (5.4)$$

for $h := \partial_n D^8\varphi^\varepsilon + \varepsilon^{-1}\nabla\cdot D^8\varphi^\varepsilon$.

Remark 5.1. It is well-known since [12] that by taking enough derivatives we can absorb the trace. Indeed, each time we use (2.12), the weight is, up to a constant, divided by $s^2\lambda^2\xi^2$. Moreover, with the weights α^* and ξ^* (the weights on the boundary), we can formally "remove a derivative" by multiplying the weight by $C(s\xi^*)^{1+1/m}$ (this is rigorously done in the next step). Consequently, it is clear that we can absorb the trace using Proposition 2.13 by taking enough derivatives. In our case, if we take less derivatives, what happens when we remove the derivatives of the trace is that we get something with a weight larger than $s^{15}\lambda^{16}e^{-2s\alpha^*}(\xi^*)^{15}$, which cannot be absorbed by the left-hand side of (5.1).

Step 2: Absorbing the trace terms.

Let us start absorbing $(1+T)\|\eta h\|^2_{\mathbf{H}^{1,1/2}(\Sigma)}$. We consider the continuous injections:

$$H^{6,12}(Q) \subset H^1(0,T;H^{10}(\Omega)) \text{ and } H^{5,11}(Q) \subset H^1(0,T;H^{8+4/5}(\Omega)).$$

Consequently, we have that:

$$\begin{aligned}\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)} &\leq C\big(\|\eta\varphi^\varepsilon\|_{H^1(0,T;\mathbf{H}^{10}(\Omega))} + \varepsilon^{-1}\|\nabla\cdot(\eta\varphi^\varepsilon)\|_{\mathbf{H}^{0,9}(Q)} \\ &\quad + \varepsilon^{-1}\|\nabla\cdot(\eta\varphi^\varepsilon)\|_{H^1(0,T;\mathbf{H}^{8+4/5}(\Omega))}\big) \leq C\left(\|\eta\varphi^\varepsilon\|_{\mathbf{H}^{6,12}(Q)} + \varepsilon^{-1}\|\nabla\cdot(\eta\varphi^\varepsilon)\|_{\mathbf{H}^{5,11}(Q)}\right). \quad (5.5)\end{aligned}$$

We recall that:

$$(t,x) \mapsto \eta(T-t)\varphi^\varepsilon(T-t,x)$$

is a solution of (1.1) with null initial value and force $-\eta'(T-t)\varphi^\varepsilon(T-t,x)$. Thus, applying estimate (2.5) with $i=6$ to (5.5), we have that:

$$\|\eta\varphi^\varepsilon\|_{\mathbf{H}^{6,12}(Q)} + \varepsilon^{-1}\|\nabla\cdot(\eta\varphi^\varepsilon)\|_{\mathbf{H}^{5,11}(Q)} \leq C\|\eta'\varphi^\varepsilon\|_{\mathbf{H}^{5,10}(Q)}.$$

If we repeat this reasoning five times, we get that:

$$\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)} \leq C\|\eta^{vi)}\varphi^\varepsilon\|_{\mathbf{L}^2(Q)}.$$

We have that, if $s \geq e^{C\lambda}(T^m + T^{2m})$ and $m \geq 8$ (see (2.13)):

$$(1+T^{1/2})|\eta^{vi)}| \leq C(s\xi^*)^{6+1/4+7/m}e^{-s\alpha^*} \leq C(s\xi^*)^{15/2}e^{-s\alpha^*}.$$

Consequently, if we also have $\lambda \geq \lambda_0$, we can absorb the term $(1+T)\|\eta h\|_{\mathbf{H}^{1,1/2}(\Sigma)}$ by the left-hand side of (5.3).

Finally, we have to absorb $(1+T^{1/2})\|\tilde{\eta}h\|_{\mathbf{H}^{2,5/2}(\Sigma)}$. In order to do so, we recall that:

$$H^{7,14}(Q) \subset H^2(0,T;H^{10}(\Omega)) \text{ and } H^{6,13}(Q) \subset H^2(0,T;H^{8+2/3}(\Omega)).$$

Thus, we have that:

$$\begin{aligned}\|\tilde{\eta}h\|_{\mathbf{H}^{2,5/2}(\Sigma)} &\leq C\big(\|\tilde{\eta}\varphi^\varepsilon\|_{\mathbf{H}^{0,12}(Q)} + \|\tilde{\eta}\varphi^\varepsilon\|_{H^2(0,T;\mathbf{H}^{10}(\Omega))} + \varepsilon^{-1}\|\nabla\cdot(\tilde{\eta}\varphi^\varepsilon)\|_{\mathbf{H}^{0,11}(Q)} \\ &\quad + \varepsilon^{-1}\|\nabla\cdot(\tilde{\eta}\varphi^\varepsilon)\|_{H^2(0,T;\mathbf{H}^{8+2/3}(\Omega))}\big) \leq C\left(\|\tilde{\eta}\varphi^\varepsilon\|_{\mathbf{H}^{7,14}(Q)} + \varepsilon^{-1}\|\nabla\cdot(\tilde{\eta}\varphi^\varepsilon)\|_{\mathbf{H}^{6,13}(Q)}\right).\end{aligned}$$

Consequently, using estimate (2.5) seven times, we get that:

$$\|\tilde{\eta}h\|_{\mathbf{H}^{2,5/2}(\Sigma)} \leq \|\tilde{\eta}^{vii)}\varphi^\varepsilon\|_{\mathbf{L}^2(Q)};$$

which is a term that can be absorbed by the left-hand side of (5.3) if $\lambda \geq \lambda_0$, $s \geq e^{C\lambda}(T^m + T^{2m})$ and $m \geq 8$, because under those hypothesis (see (2.14)):

$$(1+T^{1/2})|\tilde{\eta}^{vii)}| \leq C(s\xi^*)^{6+1/4+7/m}e^{-s\alpha^*} \leq C(s\xi^*)^{15/2}e^{-s\alpha^*}. \quad (5.6)$$

Summing up, if we combine (5.3) and (5.4), and then do the corresponding absorptions, we have the estimate:

$$s^{15}\lambda^{16}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2+\sum_{i=0}^{9}s^{19-2i}\lambda^{20-2i}\iint_Q e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2$$
$$\leq C\left(\sum_{i=0}^{7}s^{19-2i}\lambda^{20-2i}\iint\limits_{(0,T)\times\omega_0}e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2+s^4\lambda^5\iint\limits_{(0,T)\times\widetilde{\omega}}e^{-2s\alpha}\xi^4|D^8\partial_{xy}\varphi^\varepsilon|^2\right). \quad (5.7)$$

Step 3: Bounding the local terms.
In order to bound the local terms, we start estimating everything by a local term of $\partial_{xy}\varphi^\varepsilon$. We do it with the usual technique: we bound each $1_{\widetilde{\omega}}$ by χ (which is defined at the beginning of this section) to a sufficiently high power, we integrate by parts and we use properly weighted Cauchy-Schwarz inequalities. After all this process, we get from (5.7) that:

$$s^{15}\lambda^{16}\iint_Q e^{-2s\alpha^*}(\xi^*)^{15}|\varphi^\varepsilon|^2+\sum_{i=0}^{9}s^{19-2i}\lambda^{20-2i}\iint_Q e^{-2s\alpha}\xi^{19-2i}|D^i\partial_{xy}\varphi^\varepsilon|^2$$
$$+\sum_{i=1}^{8}s^{28-3i}\lambda^{29-3i}\iint\limits_{(0,T)\times\omega}\chi^{4+2i}e^{-2s\alpha}\xi^{28-3i}|D^i\partial_{xy}\varphi^\varepsilon|^2\leq Cs^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\chi^4e^{-2s\alpha}\xi^{28}|\partial_{xy}\varphi^\varepsilon|^2. \quad (5.8)$$

Indeed, when $i=1,\dots,8$, we have that:

$$s^{28-3i}\lambda^{29-3i}\iint\limits_{(0,T)\times\omega}\chi^{4+2i}e^{-2s\alpha}\xi^{28-3i}|D^i\partial_{xy}\varphi^\varepsilon|^2$$
$$=s^{28-3i}\lambda^{29-3i}\iint\limits_{(0,T)\times\omega}D\left(\chi^{4+2i}e^{-2s\alpha}\xi^{28-3i}D^i\partial_{xy}\varphi^\varepsilon\right)\cdot D^{i-1}\partial_{xy}\varphi^\varepsilon$$
$$\leq C_\delta s^{28-3(i-1)}\lambda^{29-3(i-1)}\iint\limits_{(0,T)\times\omega}\chi^{4+2(i-1)}e^{-2s\alpha}\xi^{28-3(i-1)}|D^{i-1}\partial_{xy}\varphi^\varepsilon|^2$$
$$+\delta\Bigg(s^{28-3i}\lambda^{29-3i}\iint\limits_{(0,T)\times\omega}\chi^{4+2i}e^{-2s\alpha}\xi^{28-3i}|D^i\partial_{xy}\varphi^\varepsilon|^2$$
$$+s^{28-3(i+1)}\lambda^{29-3(i+1)}\iint\limits_{(0,T)\times\omega}\chi^{4+2(i+1)}e^{-2s\alpha}\xi^{28-3(i+1)}|D^{i+1}\partial_{xy}\varphi^\varepsilon|^2\Bigg). \quad (5.9)$$

The exponents of s, ξ and λ in (5.8) might look strange. The reason is that we have $s^4\lambda^5$ in the last local term of (5.7) instead of $s^3\lambda^4$, which is the usual term.

In order to get in the right-hand side of (5.8) only a weighted local L^2-norm of φ_1^ε we must treat $\partial_{xy}\varphi_1^\varepsilon$ and $\partial_{xy}\varphi_2^\varepsilon$ differently. As for $\partial_{xy}\varphi_1^\varepsilon$, we can deal with it quite easily. Indeed, when we

integrate by parts twice, we have that:

$$s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\chi^4e^{-2s\alpha}\xi^{28}|\partial_{xy}\varphi_1^\varepsilon|^2=s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\partial_{xy}\left(\chi^4e^{-2s\alpha}\xi^{28}\right)\partial_{xy}\varphi_1^\varepsilon\varphi_1^\varepsilon$$
$$+s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\partial_{x}\left(\chi^4e^{-2s\alpha}\xi^{28}\right)\partial_{xyy}\varphi_1^\varepsilon\varphi_1^\varepsilon+s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\partial_{y}\left(\chi^4e^{-2s\alpha}\xi^{28}\right)\partial_{xxy}\varphi_1^\varepsilon\varphi_1^\varepsilon$$
$$+s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\chi^4e^{-2s\alpha}\xi^{28}\partial_{xxyy}\varphi_1^\varepsilon\varphi_1^\varepsilon. \quad (5.10)$$

We can deal with all the term of (5.10) as usual. In the end, for $\delta>0$ as small as needed, after an absorption, we get that:

$$s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\chi^4e^{-2s\alpha}\xi^{28}|\partial_{xy}\varphi_1^\varepsilon|^2\leq C_\delta s^{34}\lambda^{35}\iint\limits_{(0,T)\times\omega}e^{-2s\alpha}\xi^{34}|\varphi_1^\varepsilon|^2$$
$$+\delta\left(s^{22}\lambda^{23}\iint\limits_{(0,T)\times\omega}\chi^8e^{-2s\alpha}\xi^{22}|\partial_{xxyy}\varphi_1^\varepsilon|^2+s^{24}\lambda^{25}\iint\limits_{(0,T)\times\omega}\chi^6e^{-2s\alpha}\xi^{24}|D^1\partial_{xy}\varphi_1^\varepsilon|^2\right). \quad (5.11)$$

Finally, we have to estimate the term of $\partial_{xy}\varphi_2^\varepsilon$. Indeed, by (1.6), we find out that:

$$s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\chi^4e^{-2s\alpha}\xi^{28}|\partial_{xy}\varphi_2^\varepsilon|^2$$
$$=s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\chi^4e^{-2s\alpha}\xi^{28}\partial_{xy}\varphi_2^\varepsilon(-\varepsilon\partial_t\varphi_1^\varepsilon-(1+\varepsilon)\partial_{xx}\varphi_1^\varepsilon-\varepsilon\partial_{yy}\varphi_1^\varepsilon).$$

We can deal with the term in the right-hand side integrating by parts in space and time and using weighted Cauchy-Schwarz inequalities. In order to deal with the term of $\varepsilon\partial_{txy}\varphi_2^\varepsilon$ that appears after the integration by parts, we have to consider that:

$$\varepsilon\partial_{txy}\varphi_2^\varepsilon=-\left(\varepsilon\partial_{xxxy}\varphi_2^\varepsilon+(1+\varepsilon)\partial_{xyyy}\varphi_2^\varepsilon+\partial_{xxyy}\varphi_1^\varepsilon\right).$$

Consequently, we get that, after an absorption:

$$s^{28}\lambda^{29}\iint\limits_{(0,T)\times\omega}\chi^4e^{-2s\alpha}\xi^{28}|\partial_{xy}\varphi_2^\varepsilon|^2\leq C_\delta s^{34}\lambda^{35}\iint\limits_{(0,T)\times\omega}e^{-2s\alpha}\xi^{34}|\varphi_1^\varepsilon|^2$$
$$+\delta\left(s^{22}\lambda^{23}\iint\limits_{(0,T)\times\omega}\chi^8e^{-2s\alpha}\xi^{22}|D^2\partial_{xy}\varphi^\varepsilon|^2+s^{24}\lambda^{25}\iint\limits_{(0,T)\times\omega}\chi^6e^{-2s\alpha}\xi^{24}|D^1\partial_{xy}\varphi^\varepsilon|^2\right). \quad (5.12)$$

Summing up, if $m\geq 8$, $\lambda\geq\lambda_0$, and $s\geq e^{C\lambda}(T^m+T^{2m})$, combining (5.8), (5.11) and (5.12) we get (1.9).

A Existence, uniqueness and regularity of (2.6)

In this section we first prove Lemma 2.6 and then prove Lemma 2.8. Our proofs are classical, since they use Galerkin method and elliptic estimates (see Lemma A.1 below). We follow the steps of [15, Chapter 7.1], but we do the necessary adaptations due to the different boundary conditions.

Proof of Lemma 2.6: uniqueness. In order to prove the uniqueness, we just have to show that for $f = 0$, $h = 0$ and $v^0 = 0$, the unique solution is $v^\varepsilon = 0$. Indeed, by multiplying by v^ε the first equation of (2.6) and by integrating in $(0,t) \times \Omega$ (by parts), we have, for all $t \in [0,T]$:

$$\int_\Omega \frac{|v^\varepsilon(t,\cdot)|^2}{2} + \iint_{[0,t]\times\Omega} |\nabla v^\varepsilon|^2 + \varepsilon \iint_{[0,t]\times\Omega} |q^\varepsilon|^2 = 0,$$

which implies that $v^\varepsilon = 0$.

Proof of Lemma 2.6: existence. As for the existence, we consider the Galerkin method. It is well-know that there is a set of eigenvalues $\{\lambda_i\}_{i\in\mathbb{N}} \to +\infty$ and a set of $\mathbf{L}^2(\Omega)$-orthonormal and $\mathbf{H}^1(\Omega)$-orthogonal eigenvectors w_i such that $\{w_i\}_{i\in\mathbb{N}} \subset \mathbf{H}^2(\Omega)$ and that $-\Delta w_i = \lambda_i w_i$. In that sense, for $u \in \mathbf{L}^2(\Omega)$, we denote $\mathbb{P}_n u$ the orthonormal projection of u into $\langle w_1, \dots, w_n\rangle$. We consider the Galerkin sub-problems, for $n \in \mathbb{N}$, $t \in [0,T]$:

$$\int_\Omega \partial_t v_n^\varepsilon(t,\cdot)\cdot w_i + \int_\Omega \nabla v_n^\varepsilon(t,\cdot) : \nabla w_i + \int_\Omega \frac{(\nabla\cdot v_n^\varepsilon(t,\cdot))(\nabla\cdot w_i)}{\varepsilon} = \int_\Omega f(t,\cdot)\cdot w_i + \int_{\partial\Omega} h(t,\cdot)\cdot w_i, \quad \text{(A.1)}$$

for all $i = 1, \dots, n$. We look for a solution which belongs to $\mathbf{C}^1\big([0,T]; \langle w_1, \dots, w_n\rangle\big)$; that is, we look for $a_{i,n}^\varepsilon \in C^1([0,T])$ such that $a_{i,n}^\varepsilon(0) = \langle v^0, w_i\rangle_{\mathbf{L}^2(\Omega)}$ and that $v_n^\varepsilon(t,x) := \sum_{i=1}^n a_{i,n}^\varepsilon(t) w_i(x)$ is a solution of (A.1).

Energy estimates. It is easy to see that each set of components $(a_{i,n}^\varepsilon)_{i=1}^n$ is the solution of a linear ordinary differential equation of n equations and n unknowns. Therefore, system (A.1) together with the initial condition has a well-defined solution. Moreover, adding up (A.1) multiplied by the coefficients $a_{i,n}^\varepsilon$ and integrating in time, we get that, for any $t \in [0,T]$, provided that v_n^ε is defined:

$$\int_\Omega \frac{|v_n^\varepsilon|^2(t,\cdot)}{2} + \iint_{[0,t]\times\Omega} |\nabla v_n^\varepsilon|^2 + \iint_{[0,t]\times\Omega} \frac{(\nabla\cdot v_n^\varepsilon)^2}{\varepsilon} = \iint_{[0,t]\times\Omega} f\cdot v_n^\varepsilon + \iint_{[0,t]\times\partial\Omega} h\cdot v_n^\varepsilon + \int_\Omega \frac{|\mathbb{P}_n v_0|^2}{2}.$$

We have to consider that:

$$\iint_{[0,t]\times\Omega} f\cdot v_n^\varepsilon \le C(1+T) \iint_{[0,t]\times\Omega} |f|^2 + \frac{1}{4(1+T)} \iint_{[0,t]\times\Omega} |v_n^\varepsilon|^2.$$

Moreover, we have that:

$$\iint\limits_{[0,t]\times\partial\Omega} h\cdot v_n^\varepsilon \leq C(1+T)\iint\limits_{[0,t]\times\partial\Omega}|h|^2+\frac{1}{4(1+T)}\iint\limits_{[0,t]\times\Omega}\left(|v_n^\varepsilon|^2+|\nabla v_n^\varepsilon|^2\right).$$

Consequently, due to Gronwall's inequality and usual absorptions, we get that:

$$\int_\Omega |v_n^\varepsilon|^2(t,\cdot)+\iint\limits_{[0,t]\times\Omega}|\nabla v_n^\varepsilon|^2+\iint\limits_{[0,t]\times\Omega}\frac{(\nabla\cdot v_n^\varepsilon)^2}{\varepsilon}$$
$$\leq C\left((1+T)\iint\limits_{[0,t]\times\Omega}|f|^2+(1+T)\iint\limits_{[0,t]\times\partial\Omega}|h|^2+\int_\Omega|\mathbb{P}_n v^0|^2\right). \quad \text{(A.2)}$$

Thanks to estimate (A.2), all the solutions $(v_n^\varepsilon)_{n\in\mathbb{N}}$ are bounded uniformly in $C^0([0,t];\mathbf{L}^2(\Omega))\cap L^2(0,t;\mathbf{H}^1(\Omega))$. This implies, due to extension theorems related with the Cauchy-Lipschitz systems, that all the v_n^ε are defined in $[0,T]$ and that we can take a weak limit in $L^2(0,T;\mathbf{H}^1(\Omega))$.

Estimates on $\mathbf{H}^{1,1}(Q)$. Next, in order to take limits in (A.1), we need to prove that the v_n^ε are also uniformly bounded in $H^1(0,T;\mathbf{L}^2(\Omega))$. By multiplying (A.1) by $(a_{i,n}^\varepsilon)'$, adding all up and integrating in time, we have that:

$$\iint_Q|\partial_t v_n^\varepsilon|^2+\int_0^T\frac{\partial_t}{2}\int_\Omega|\nabla v_n^\varepsilon|^2+\int_0^T\frac{\partial_t}{2}\int_\Omega\frac{(\nabla\cdot v_n^\varepsilon)^2}{\varepsilon}=\iint_Q f\cdot\partial_t v_n^\varepsilon+\iint_\Sigma h\cdot\partial_t v_n^\varepsilon.$$

In order to deal with the term $\iint_\Sigma h\cdot\partial_t v_n^\varepsilon$ we have to integrate by parts in time. Then, using also (A.2), we get for a constant C that does not depend on n:

$$\|v_n^\varepsilon\|_{\mathbf{H}^{1,1}(Q)}\leq C\sqrt{1+T}\big(\|f\|_{\mathbf{L}^2(Q)}+\|h\|_{\mathbf{H}^{1,1/2}(\Sigma)}\big)+$$
$$C\left(\|v^0\|_{\mathbf{H}^1(\Omega)}+\left\|\frac{\nabla\cdot v^0}{\varepsilon}\right\|_{\mathbf{L}^2(\Omega)}+\|h(0,\cdot)\|_{\mathbf{L}^2(\Sigma)}+\|h(T,\cdot)\|_{\mathbf{L}^2(\Sigma)}\right). \quad \text{(A.3)}$$

So, up to extracting a subsequence, we have that $(v_n^\varepsilon)_{n\in\mathbb{N}}$ converges weakly in $\mathbf{H}^{1,1}(Q)$ to some function v^ε which satisfies:

$$\|v^\varepsilon\|_{\mathbf{H}^{1,1}(Q)}\leq C\sqrt{1+T}\big(\|f\|_{\mathbf{L}^2(Q)}+\|h\|_{\mathbf{H}^{1,1/2}(\Sigma)}\big)+$$
$$C\left(\|v^0\|_{\mathbf{H}^1(\Omega)}+\left\|\frac{\nabla\cdot v^0}{\varepsilon}\right\|_{\mathbf{L}^2(\Omega)}+\|h(0,\cdot)\|_{\mathbf{L}^2(\Sigma)}+\|h(T,\cdot)\|_{\mathbf{L}^2(\Sigma)}\right). \quad \text{(A.4)}$$

Thus, we can take limits in (A.1). Indeed, we have for every $i\in\mathbb{N}$, as functions of $L^2(0,T)$:

$$\int_\Omega v_t^\varepsilon\cdot w_i+\int_\Omega\nabla v^\varepsilon:\nabla w_i+\int_\Omega\frac{(\nabla\cdot v^\varepsilon)(\nabla\cdot w_i)}{\varepsilon}=\int_\Omega f\cdot w_i+\int_{\partial\Omega}h\cdot w_i. \quad \text{(A.5)}$$

We recall that $H^1(0,T;L^2(\Omega))$ is compactly embedded in $C^0([0,T];H^{-1}(\Omega))$. Thus, weak convergence in $H^1(0,T;L^2(\Omega))$ implies strong convergence in $C^0([0,T];H^{-1}(\Omega))$, so $v^\varepsilon(0,\cdot)=v^0$.

Estimates on $\mathbf{H}^{1,2}(Q)$. In order to prove that the solution is in $L^2(0,T;\mathbf{H}^2(\Omega))$, we use that it satisfies for any $i\in\mathbb{N}$, as functions of $L^2(0,T)$, that:

$$\int_\Omega \nabla v^\varepsilon : \nabla w_i + \int_\Omega \frac{(\nabla\cdot v^\varepsilon)(\nabla\cdot w_i)}{\varepsilon} = \int_\Omega (f - v_t^\varepsilon)\cdot w_i + \int_{\partial\Omega} h\cdot w_i. \tag{A.6}$$

So, we use the following lemma, whose proof can be found in [5, Theorem IV.7.1]:

Lemma A.1. *Let us consider $\Omega\in C^2$ and the system:*

$$\begin{cases} -\Delta u + \nabla g = f_1 & \text{in } \Omega,\\ \nabla\cdot u = f_2 & \text{in } \Omega,\\ \partial_n u - gn = f_3 & \text{on } \partial\Omega,\\ \int_\Omega u = 0, \end{cases} \tag{A.7}$$

for $f_1\in\mathbf{L}^2(\Omega)$, $f_2\in L^2(\Omega)$ and $f_3\in\mathbf{H}^{1/2}(\partial\Omega)$. Then, if we have as a vector equation:

$$\int_\Omega f_1 + \int_{\partial\Omega} f_3 = 0, \tag{A.8}$$

the solution (u,g) of (A.7) is unique and

$$\|D^2u\|_{\mathbf{L}^2(\Omega)} + \|g\|_{H^1(\Omega)} \le C\left(\|f_1\|_{\mathbf{L}^2(\Omega)} + \|f_2\|_{H^1(\Omega)} + \|f_3\|_{\mathbf{H}^{1/2}(\partial\Omega)}\right).$$

In order to apply Lemma A.1 it suffices to take $f_1(t,\cdot) := f(t,\cdot) - v_t^\varepsilon(t,\cdot)$, $f_2(t,\cdot) := \nabla\cdot v^\varepsilon(t,\cdot)$ and $f_3(t,\cdot) := h(t,\cdot)$. In addition to that, (A.8) is satisfied because:

$$\int_\Omega v_t^\varepsilon = \int_\Omega (\Delta v^\varepsilon - \nabla q) + \int_\Omega f = \int_{\partial\Omega}(\partial_n v^\varepsilon - qn) + \int_\Omega f = \int_{\partial\Omega} h + \int_\Omega f.$$

Thus, since:

$$u(t,\cdot) = v^\varepsilon(t,\cdot) - \tfrac{1}{|\Omega|}\textstyle\int_\Omega v^\varepsilon(t,\cdot) \quad\text{and}\quad g(t,\cdot) = \tfrac{\nabla\cdot v^\varepsilon}{\varepsilon}(t,\cdot) = q^\varepsilon(t,\cdot),$$

if we combine (A.4) and Lemma A.1, remarking that $D^2u = D^2v^\varepsilon$, we get the estimate:

$$\begin{aligned}\|v^\varepsilon\|_{\mathbf{H}^{1,2}(Q)} + \|q^\varepsilon\|_{H^{0,1}(Q)} &\le C\sqrt{1+T}\big(\|f\|_{\mathbf{L}^2(Q)} + \|h\|_{\mathbf{H}^{1,1/2}(\Sigma)}\big)\\ &+ C\left(\|v^0\|_{\mathbf{H}^1(\Omega)} + \|\nabla\cdot v^\varepsilon\|_{H^{0,1}(Q)} + \left\|\frac{\nabla\cdot v^0}{\varepsilon}\right\|_{\mathbf{L}^2(\Omega)} + \|h(0,\cdot)\|_{\mathbf{L}^2(\Sigma)} + \|h(T,\cdot)\|_{\mathbf{L}^2(\Sigma)}\right).\end{aligned} \tag{A.9}$$

This expression can be simplified since for ε small enough we can absorb the term $\|\nabla\cdot v^\varepsilon\|_{H^{0,1}(Q)}$ by $\|q^\varepsilon\|_{H^{0,1}(Q)}$. So estimate (2.7) is established.

Proof of Lemma 2.8. As for the proof of Lemma 2.8, it consists of repeating the Galerkin method for v_t^ε, since v_t^ε is a solution of (2.6) with $(f, h, 0)$ replaced by $(f_t, h_t, 0)$. Indeed, we first get an estimate for each $\partial_t v_n^\varepsilon$ and then pass to the limit. Finally, we use a more complete version of Lemma A.1, which can be found in [5, Theorem IV.7.1].

B Proof of Proposition 2.13

Throughout this proof we consider $\widehat{\omega}$ some open subdomain of Ω compactly contained in $\widetilde{\omega}$ such that $\overline{\omega_0} \subset \widehat{\omega}$ (see Proposition 2.13 for the definition of $\widetilde{\omega}$). In order to make the reading of the proof more comfortable we split it in several steps: first, we bound left of (2.15) by a trace and a local term with the help of the rotational; then, we deal with the trace and local terms as usual.

Step 1: Bounding by a trace and a local term.
To begin with, we have that $\nabla \times \varphi^\varepsilon$ is a solution of the heat equation, since $\nabla \times (\nabla \pi^\varepsilon) = 0$. So, using Lemma 2.11 for $r = -1$ and $\delta = 1$, we get that if $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$s^2\lambda^3 \iint_Q e^{-2s\alpha}\xi^2|\nabla \times \varphi^\varepsilon|^2 + \lambda \iint_Q e^{-2s\alpha}|\nabla(\nabla \times \varphi^\varepsilon)|^2 \\ \leq C\left(\iint_\Sigma e^{-2s\alpha}|\partial_n(\nabla \times \varphi^\varepsilon)|^2 + s^2\lambda^3 \iint_{(0,T)\times\widehat{\omega}} e^{-2s\alpha}\xi^2|\nabla \times \varphi^\varepsilon|^2\right). \quad \text{(B.1)}$$

Next, we consider that the divergence satisfies:

$$\nabla(\nabla \cdot \varphi^\varepsilon) = \Delta\varphi^\varepsilon + \nabla \times (\nabla \times \varphi^\varepsilon).$$

This implies that φ^ε satisfies:

$$-\frac{\varepsilon}{1+\varepsilon}\partial_t\varphi^\varepsilon - \Delta\varphi^\varepsilon = \frac{1}{1+\varepsilon}(\nabla \times (\nabla \times \varphi^\varepsilon)).$$

Thus, using again Lemma 2.11 for $\widehat{\omega}$ defined as before, $r = 0$ and now $\delta = \frac{\varepsilon}{1+\varepsilon}$, we get that if $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2 + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi^\varepsilon|^2 \\ \leq C\left(s^3\lambda^4 \iint_{(0,T)\times\widehat{\omega}} e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2 + \iint_Q e^{-2s\alpha}|\nabla(\nabla \times \varphi^\varepsilon)|^2 + s\lambda \iint_\Sigma e^{-2s\alpha}\xi|\partial_n\varphi^\varepsilon|^2\right). \quad \text{(B.2)}$$

Next, we remark that the term of $\nabla(\nabla\times\varphi^\varepsilon)$ on the right-hand side of (B.2) can be absorbed by the left-hand side of (B.1) for $\lambda\geq\lambda_0$. Thus, we have that:

$$s^3\lambda^4\iint_Q e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2+s\lambda^2\iint_Q e^{-2s\alpha}\xi|\nabla\varphi^\varepsilon|^2+s^2\lambda^3\iint_Q e^{-2s\alpha}\xi^2|\nabla\times\varphi^\varepsilon|^2$$
$$+\lambda\iint_Q e^{-2s\alpha}|\nabla(\nabla\times\varphi^\varepsilon)|^2\leq C\Bigg(s^3\lambda^4\iint\limits_{(0,T)\times\widehat{\omega}} e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2$$
$$+s^2\lambda^3\iint\limits_{(0,T)\times\widehat{\omega}} e^{-2s\alpha}\xi^2|\nabla\times\varphi^\varepsilon|^2+s\lambda\iint_\Sigma e^{-2s\alpha}\xi|\partial_n\varphi^\varepsilon|^2+\iint_\Sigma e^{-2s\alpha}|\partial_n(\nabla\times\varphi^\varepsilon)|^2\Bigg). \quad \text{(B.3)}$$

Step 2: Absorption of the trace.
In this step we absorb the traces with the estimates established in Lemma 2.6. We recall that on $\partial\Omega$: $\alpha=\alpha^*$ and $\xi=\xi^*$.

Let us first bound the third integral on the right-hand side of (B.3). First, we consider that, integrating by parts:

$$s\lambda\iint_\Sigma e^{-2s\alpha^*}\xi^*|\partial_n\varphi^\varepsilon|^2\leq C\|(s\xi^*)^{5/4-1/m}\lambda^2e^{-s\alpha^*}\varphi^\varepsilon\|^{1/2}_{\mathbf{L}^2(Q)}\|(s\xi^*)^{1/4+1/m}e^{-s\alpha^*}\varphi^\varepsilon\|^{3/2}_{\mathbf{H}^{0,2}(Q)}.$$

Using Young's inequality we get that:

$$s\lambda\iint_\Sigma e^{-2s\alpha^*}\xi^*|\partial_n\varphi^\varepsilon|^2\leq C\left(\|(s\xi^*)^{5/4-1/m}\lambda^2e^{-s\alpha^*}\varphi^\varepsilon\|^2_{\mathbf{L}^2(Q)}+\|(s\xi^*)^{1/4+1/m}e^{-s\alpha^*}\varphi^\varepsilon\|^2_{\mathbf{H}^{0,2}(Q)}\right). \quad \text{(B.4)}$$

We can absorb the first term on the right-hand side of (B.4) by the left-hand side of (B.3) by taking $s\geq CT^{2m}$ and $\lambda\geq 1$.

We can bound the fourth integral at the right-hand side of (B.3) similarly. Indeed, integrating by parts, we get that, if $s\geq CT^{2m}$:

$$\iint_\Sigma e^{-2s\alpha^*}|\partial_n(\nabla\times\varphi^\varepsilon)|^2\leq C\|(s\xi^*)^{1/4}e^{-s\alpha^*}\varphi^\varepsilon\|^{3/2}_{\mathbf{H}^{0,2}(Q)}\|(s\xi^*)^{-3/4}e^{-s\alpha^*}\varphi^\varepsilon\|^{1/2}_{\mathbf{H}^{0,4}(Q)}$$
$$\leq C\left(\|(s\xi^*)^{1/4}e^{-s\alpha^*}\varphi^\varepsilon\|^2_{\mathbf{H}^{0,2}(Q)}+\|(s\xi^*)^{-3/4}e^{-s\alpha^*}\varphi^\varepsilon\|^2_{\mathbf{H}^{0,4}(Q)}\right). \quad \text{(B.5)}$$

So, we first deal with the term $\|\eta\varphi^\varepsilon\|^2_{\mathbf{H}^{1,2}(Q)}$ (see (2.13) for the definition of η). We remark that

$$(t,x)\mapsto\eta(T-t)\varphi^\varepsilon(T-t,x)$$

is a solution of (2.6) with null initial value, force $-\eta'(T-t)\varphi^\varepsilon(T-t,x)$ and boundary Neumann term $\eta(T-t)h(T-t,x)$. Consequently, because of (2.7), we get that:

$$\|\eta\varphi^\varepsilon\|^2_{\mathbf{H}^{1,2}(Q)}\leq C(1+T)\left(\|\eta'\varphi^\varepsilon\|^2_{\mathbf{L}^2(Q)}+\|\eta h\|^2_{\mathbf{H}^{1,1/2}(\Sigma)}\right). \quad \text{(B.6)}$$

Moreover, the term of $\eta'\varphi^\varepsilon$ can be absorbed by the left of (B.3) if $m \geq 8$, $\lambda \geq \lambda_0$, and $s \geq e^{C\lambda}(T^m + T^{2m})$, since in that case:

$$(1+T^{1/2})|\eta'| \leq C(s\xi^*)^{1+1/4+2/m}e^{-s\alpha^*} \leq C(s\xi^*)^{3/2}e^{-s\alpha^*}. \tag{B.7}$$

Let us now estimate the term $\|\tilde{\eta}\varphi^\varepsilon\|^2_{\mathbf{H}^{0,4}(Q)}$ (see (2.14) for the definition of $\tilde{\eta}$). We have that

$$(t,x) \mapsto \tilde{\eta}(T-t)\varphi^\varepsilon(T-t,x)$$

is a solution of (2.6) with null initial value, force $-\tilde{\eta}'(T-t)\varphi^\varepsilon(T-t,x)$ and boundary Neumann term $\tilde{\eta}(T-t)h(T-t,x)$. Consequently, if we use (2.8), we get that:

$$\|\tilde{\eta}\varphi^\varepsilon\|^2_{\mathbf{H}^{0,4}(Q)} \leq C(1+T)\left(\|\tilde{\eta}'\varphi^\varepsilon\|^2_{\mathbf{H}^{1,2}(Q)} + \|\tilde{\eta}h\|^2_{\mathbf{H}^{2,5/2}(\Sigma)}\right). \tag{B.8}$$

Let us now estimate the first norm at the right-hand side of (B.8). To begin with, since $m \geq 8$ and $s \geq e^{C\lambda}(T^m + T^{2m})$ imply that $(1+T^{1/2})|\tilde{\eta}'| \leq C\eta$, we have that:

$$(1+T)\|\tilde{\eta}'\varphi^\varepsilon\|^2_{\mathbf{H}^{0,2}(Q)} \leq C\|\eta\varphi^\varepsilon\|^2_{\mathbf{H}^{0,2}(Q)}, \tag{B.9}$$

which is estimated by the left-hand side of (B.6). To continue with, we have that, if $m \geq 8$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$(1+T)\|\tilde{\eta}''\varphi^\varepsilon\|^2_{\mathbf{L}^2(Q)} \leq C\|(s\xi^*)^{1+1/4+2/m}e^{-s\alpha^*}\varphi^\varepsilon\|^2_{\mathbf{L}^2(Q)} \leq C\|(s\xi^*)^{3/2}e^{-s\alpha^*}\varphi^\varepsilon\|^2_{\mathbf{L}^2(Q)}, \tag{B.10}$$

a term which can be absorbed by the left-hand side of (B.3) for λ large enough. Finally, we have that, if $m \geq 8$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$(1+T^{1/2})|\tilde{\eta}'\varphi_t| \leq |\eta\varphi^\varepsilon_t| \leq |(\eta\varphi^\varepsilon)_t| + |\eta'\varphi^\varepsilon|,$$

which implies that:

$$(1+T)\|\tilde{\eta}'\varphi^\varepsilon_t\|^2_{\mathbf{L}^2(Q)} \leq C\big(\|\eta\varphi^\varepsilon\|^2_{\mathbf{H}^{1,0}(Q)} + \|\eta'\varphi^\varepsilon\|^2_{\mathbf{L}^2(Q)}\big), \tag{B.11}$$

terms which can be estimated by the left-hand side of (B.6) and (B.3) respectively.

Summing up, if we combine (B.3)-(B.11) we get that, if $m \geq 8$, $\lambda \geq \lambda_0$ and $s \geq e^{C\lambda}(T^m + T^{2m})$:

$$\begin{aligned} s^3\lambda^4 \iint_Q e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2 + s\lambda^2 \iint_Q e^{-2s\alpha}\xi|\nabla\varphi^\varepsilon|^2 + s^2\lambda^3 \iint_Q e^{-2s\alpha}\xi^2|\nabla\times\varphi^\varepsilon|^2 \\ + \lambda \iint_Q e^{-2s\alpha}|\nabla(\nabla\times\varphi^\varepsilon)|^2 \leq C\Bigg(s^3\lambda^4 \iint\limits_{(0,T)\times\widehat{\omega}} e^{-2s\alpha}\xi^3|\varphi^\varepsilon|^2 \\ + s^2\lambda^3 \iint\limits_{(0,T)\times\widehat{\omega}} e^{-2s\alpha}\xi^2|\nabla\times\varphi^\varepsilon|^2 + (1+T)\left(\|\eta h\|^2_{\mathbf{H}^{1,1/2}(\Sigma)} + \|\tilde{\eta}h\|^2_{\mathbf{H}^{2,5/2}(\Sigma)}\right)\Bigg). \end{aligned}$$

Finally, we remove the derivative from the local terms. We do it with the usual localizing techniques: we multiply by a cut-off function χ, integrate by parts and use Cauchy-Schwarz weighted inequalities. So, if $\lambda \geq \lambda_0$, $s \geq e^{C\lambda}(T^m + T^{2m})$ and $m \geq 8$, we get estimate (2.15).

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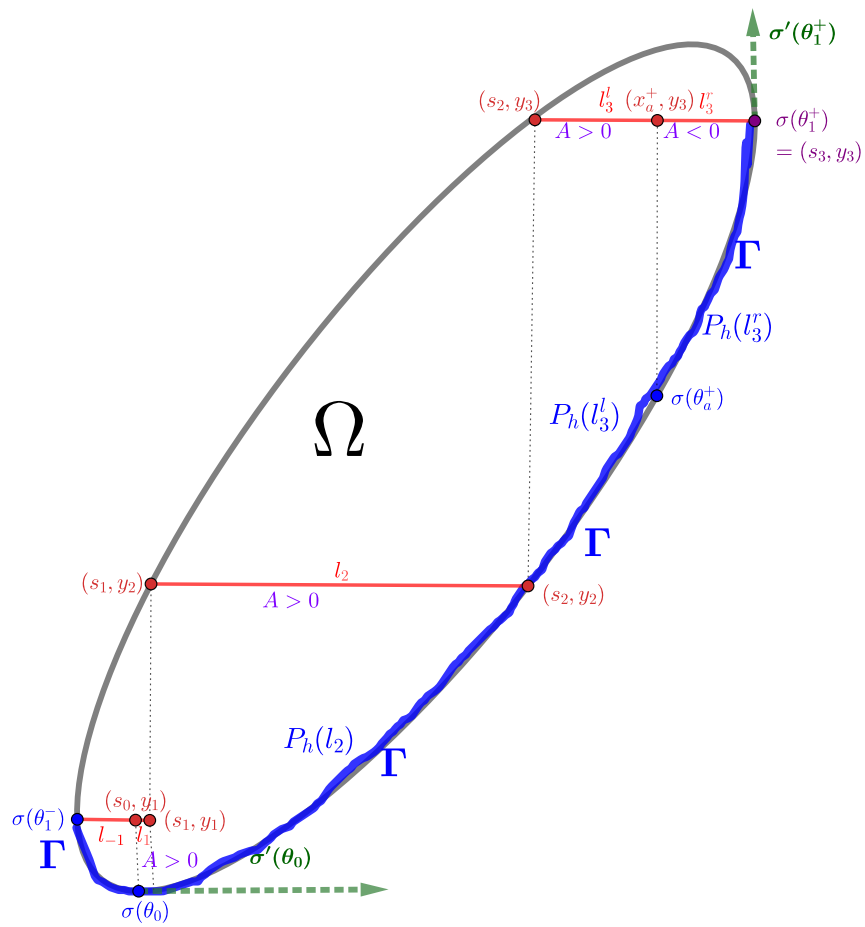

Figure 1: An illustration of the strictly convex case

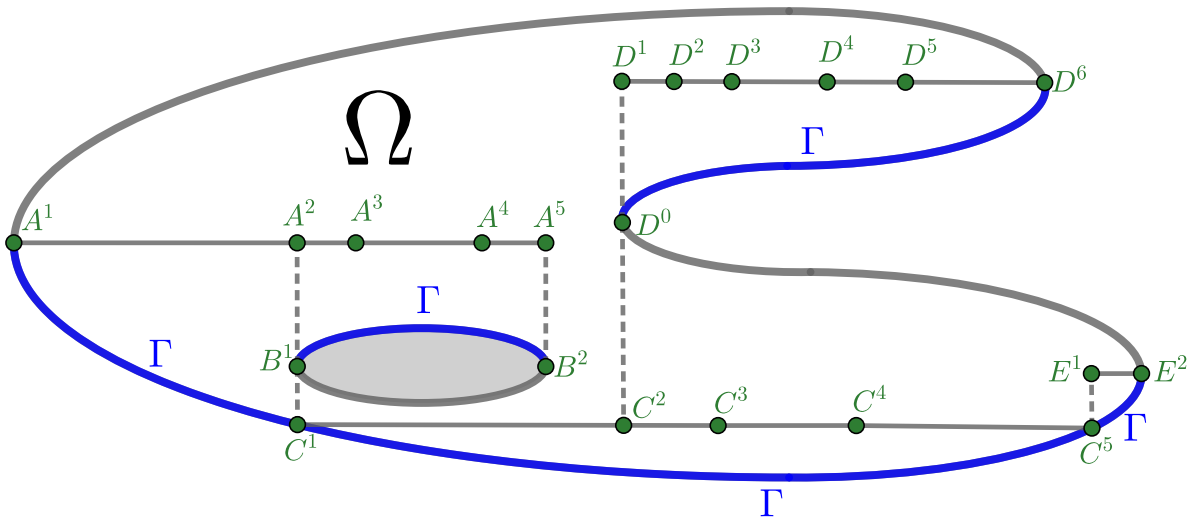

Figure 2: An illustration of S in a non-convex domain

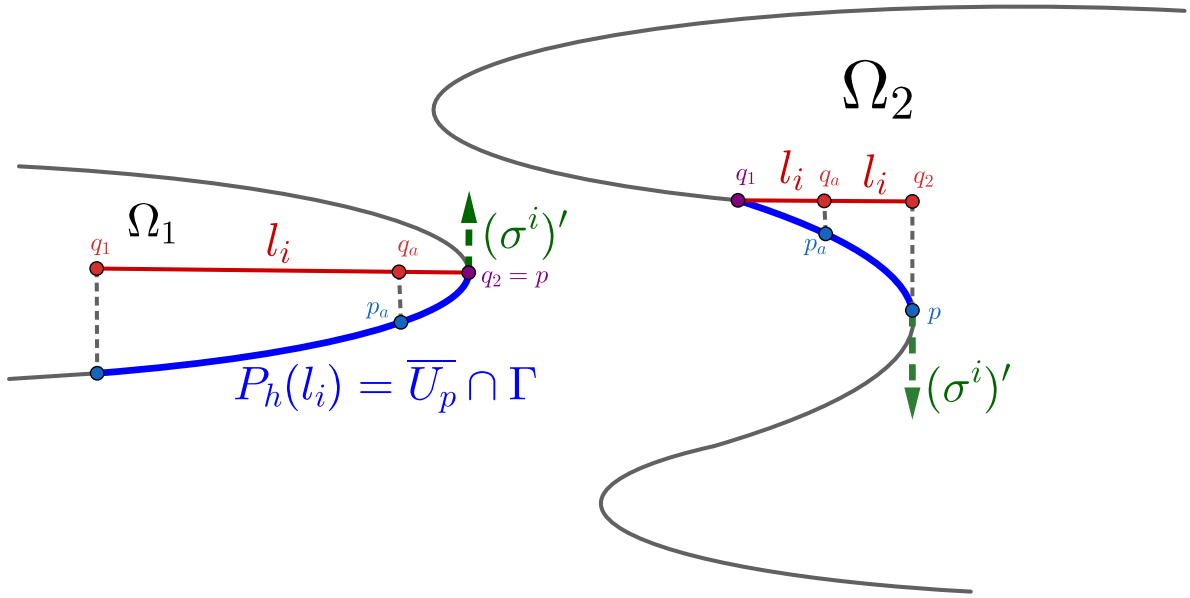

Figure 3: Case 1 of the proof of Theorem 1.8

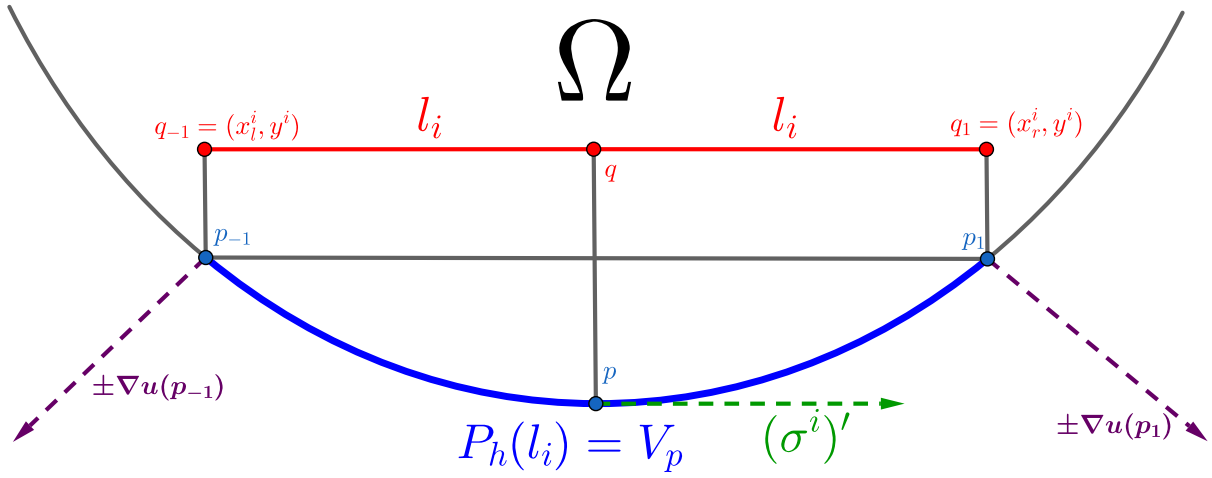

Figure 4: Case 3 of the proof of Theorem 1.8

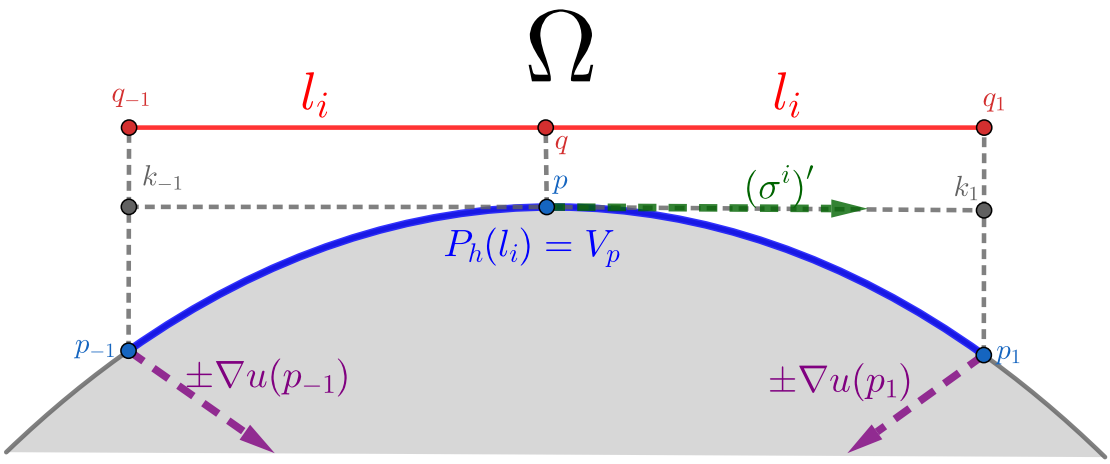

Figure 5: Case 4 of the proof of Theorem 1.8