Periodic orbits in planar linear systems with input saturation

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***Abstract*—We present sufficient conditions for the existence of periodic orbits of saturating planar systems. We characterize inner and outer sets bounding the periodic orbits. A method to build these bounds, based on the solution to a convex optimization problem, is proposed and illustrated with numerical examples.**

***Index Terms*—Stability of nonlinear systems, Constrained Control**

I. Introduction

Any practical feedback system presents constraints that can be physical, technological or imposed by safety requirements [3]. It is thus mandatory to incorporate these constraints in the model of the system for the closed-loop analysis and design. In a large number of cases, the constraints are magnitude saturations.

Input saturation is at the origin of nonlinear phenomena such as multiple isolated equilibria or isolated periodic trajectories which might occur even in systems of low dimension. An example of limit cycles appearing in planar saturating systems is considered in [3, Chapters 4, 6].

Over the last few decades, several Lyapunov based techniques for analysis and design of systems with input nonlinearities such as saturation [10], [4], quantization [2], backlash [9] and other hysteresis functions [7] have been proposed. Importantly, these systematic approaches are often associated to numerical formulations for the computation of feedback gains. Nonetheless, in most cases, such control design techniques consider only quadratic Lyapunov functions.

On the other hand, analysis and design of periodic trajectories in linear systems with deadzone and saturation on the input has yet to be explored. In this context, a challenging question yet to be answered is: "Is it possible to characterize exactly the periodic trajectories of planar saturating systems as a level set of a function?". Also, in case the system generates periodic trajectory, it is of interest to know its amplitude and frequency.

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This paper presents a first step towards the characterization of periodic trajectories of saturating linear systems. Namely, we set conditions on the data of planar saturating systems for the existence of trajectories between an attractive set and a repulsive one containing the origin. We pay a particular attention to the existence of periodic trajectories. We then propose an optimization based method to obtain two ellipsoids defining a ring that contain periodic trajectories of the system. Moreover the proposed numerical methods does not rely on sector conditions as the nonlinear analysis in [4], [2], [9], [7] in which only the convergence to an invariant set is guaranteed. Furthermore here we show the existence of a periodic orbit by ruling out the existence of equilibrium points other than the origin.

In Section II we present the class of saturating systems studied in this paper. We define a suitable piece-wise affine representation of the system and quadratic functions used for the statement of the main result. In Section III, we show how to build a set in which periodic trajectories of the system will be confined. In Section IV we present algorithms to compute these sets.

Notation. We use $M_{(i,j)}$ to denote the (i, j) entry of a matrix M. We denote by $\Phi(t, x_0)$ the solution to a dynamical system initiated from the point x_0 at time $t = 0$. We also denote the set of positive definite (negative definite) $\mathbb{S}_{>0} = \{M \in \mathbb{R}^{2\times 2} \mid M = M^\top, M > 0\}$ ($\mathbb{S}_{<0} = \{M \in \mathbb{R}^{2\times 2} \mid M = M^\top, M < 0\}$) and positive semi-definite (negative semi-definite) matrices, $\mathbb{S}_{\geq 0} = \{M \in \mathbb{R}^{2\times 2} \mid M = M^\top, M \geq 0\}$ ($\mathbb{S}_{\leq 0} = \{M \in \mathbb{R}^{2\times 2} \mid M = M^\top, M \leq 0\}$). With $P \in \mathbb{S}_{>0}$, we denote the ellipsoidal set $\mathcal{E}(P, \alpha) = \{x \in \mathbb{R}^2 \mid x^\top Px \leq \alpha\}$. The set $\text{Conv}(\mathcal{A})$ is the convex hull of set $\mathcal{A}$ and $\partial\mathcal{A}$ is the boundary of set $\mathcal{A}$.

II. Preliminaries

Consider the class of planar linear saturating systems

$$\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\,\text{sat}(\bar{K}\bar{x}), \tag{1}$$

with $\bar{x} \in \mathbb{R}^2$, $\bar{A} \in \mathbb{R}^{2\times 2}$, $\bar{B} \in \mathbb{R}^{2\times 1}$, $\bar{K} \in \mathbb{R}^{1\times 2}$ and the unit saturation function defined by $sat(u_c) = \text{sign}(u_c)\min(|u_c|, 1)$. With the coordinate transformation $x = T\bar{x}$, with

$$T := \frac{1}{(\bar{K}\bar{K}^\top)}\begin{bmatrix}\bar{K}^\top & \bar{K}^\perp\end{bmatrix}$$

and $\bar{K}^\perp \in \mathbb{R}^{2\times 1}$ satisfying $\bar{K}\bar{K}^\perp = 0$, system (1) becomes

$$\dot{x} = Ax + B\,\text{sat}(Kx) \tag{2}$$

with $A = T^{-1}\bar{A}T$, $B = T^{-1}\bar{B}$ and $K = \begin{bmatrix}1 & 0\end{bmatrix}$.

Remark 2.1: Note that the above system model also accommodates systems with deadzone nonlinearity by using the identity $\mathrm{sat}(u_c) + \mathrm{dz}(u_c) = u_c$.

With the definition of the saturation function , the above system can be put in the following piecewise affine form

$$\dot{x} = \begin{cases} Ax - B & \text{if } x \in \mathcal{R}_L := \{x \in \mathbb{R}^2 \mid x_1 < -1\} \\ A_{CL}x & \text{if } x \in \mathcal{R}_C := \{x \in \mathbb{R}^2 \mid -1 \le x_1 \le 1\} \\ Ax + B & \text{if } x \in \mathcal{R}_R := \{x \in \mathbb{R}^2 \mid 1 < x_1\} \end{cases} \tag{3}$$

with $A_{CL} = A + B\begin{bmatrix}1 & 0\end{bmatrix}$. We also introduce the sets defining the boundaries between $\mathcal{R}_C$, $\mathcal{R}_L$ and $\mathcal{R}_R$ that is $\mathcal{D}_1 := \{x \in \mathbb{R}^2,\ x_1 = 1\}$, $\mathcal{D}_{-1} := \{x \in \mathbb{R}^2,\ x_1 = -1\}$

We consider the following assumption

Assumption 2.1: A and $-A_{CL}$ are Hurwitz matrices.

Note that the above assumption does not require either the eigenvalues of A or A_{CL} to be complex conjugate. Note that if the pair $(\bar{A}, \bar{B})$ is controllable (hence pair (A, B) is controllable) and $\bar{A}$ is Hurwitz it is always possible to obtain $\bar{K}$ such that $\bar{A} + \bar{B}\bar{K}$ gives A_{CL} such that the pair A and A_{CL} satisfy Assumption 2.1. The lemma below concerns the equilibria of (3).

Lemma 2.1: Under Assumption 2.1, the origin is the only equilibrium point of system (3).

Proof: Denote $A = \left[\begin{smallmatrix} a_{11} & a_{12} \\ a_{21} & a_{22}\end{smallmatrix}\right]$, $B = \left[\begin{smallmatrix} b_1 \\ b_2\end{smallmatrix}\right]$ yielding $A_{CL} = \left[\begin{smallmatrix} a_{11}+b_1 & a_{12} \\ a_{21}+b_2 & a_{22}\end{smallmatrix}\right]$.

From Assumption 2.1, $\mathrm{trace}(A) < 0$, $\det(A) > 0$ and $\mathrm{trace}(A_{CL}) > 0$, $\det(A_{CL}) > 0$, that is

$$\begin{cases} a_{11} + a_{22} < 0 \\ a_{11}a_{22} - a_{21}a_{12} > 0 \end{cases}, \quad \begin{cases} a_{11} + b_1 + a_{22} > 0 \\ (a_{11} + b_1)a_{22} \\ \quad -(a_{21} + b_2)a_{12} > 0. \end{cases} \tag{4}$$

From (3) the possible equilibria are $x_L = A^{-1}B$ if $x_L \in \mathcal{R}_L$, $x_C = 0$ and $x_R = -A^{-1}B$ if $x_R \in \mathcal{R}_R$.

Note that $x_L \in \mathcal{R}_L \Leftrightarrow KA^{-1}B < -1$ and $x_R \in \mathcal{R}_R \Leftrightarrow -KA^{-1}B > 1$. Let us show that $-KA^{-1}B > 1$ can not hold. We have

$$KA^{-1}B = \frac{a_{22}b_1 - a_{12}b_2}{\det(A)}.$$

Suppose that $-KA^{-1}B > 1$ holds, that is $\det(A) + a_{22}b_1 - a_{12}b_2 < 0$ and $(a_{11} + b_1)a_{22} - (a_{21} + b_2)a_{12} < 0$ which contradicts (4). Thus, $-KA^{-1}B > 1$ can not hold. It follows that $KA^{-1}B < -1$ can not hold either and we conclude that the origin is the only equilibrium point of (3). ■

Under Assumption 2.1, given $Q_1 \in \mathbb{S}_{<0}$ and $Q_{2_{CL}} \in \mathbb{S}_{>0}$, $\exists P_1, P_2 \in \mathbb{S}_{>0}$ satisfying the Lyapunov equations

$$A^\top P_1 + P_1A = Q_1 \tag{5a}$$

$$A_{CL}^\top P_2 + P_2A_{CL} = Q_{2_{CL}}. \tag{5b}$$

With P_1, P_2 satisfying the above equations, define

$$Q_{1_{CL}} := A_{CL}^\top P_1 + P_1A_{CL} \tag{6a}$$

$$Q_2 := A^\top P_2 + P_2A. \tag{6b}$$

Lemma 2.2: Q_2 and $Q_{1_{CL}}$ are not sign definite and are invertible.

Proof: We have $x^\top Q_{i_{CL}}x = x^\top Q_ix + 2x_1B^\top P_ix$, for $i \in \{1, 2\}$, take $x = \begin{bmatrix}0 \\ 1\end{bmatrix}$ to obtain

$$\begin{bmatrix}0 & 1\end{bmatrix} Q_{i_{CL}} \begin{bmatrix}0 \\ 1\end{bmatrix} = \begin{bmatrix}0 & 1\end{bmatrix} Q_i \begin{bmatrix}0 \\ 1\end{bmatrix} = Q_{i(2,2)}. \tag{7}$$

From (7), since $Q_{2_{CL}} \in \mathbb{S}_{>0}$ we have that $Q_{2(2,2)} > 0$, hence $Q_2 \notin \mathbb{S}_{\le 0}$. By Assumption 2.1 A is Hurwitz, thus since $P_2 > 0$, $Q_2 \notin \mathbb{S}_{\ge 0}$. To prove this claim, suppose $Q_2 \in \mathbb{S}_{\ge 0}$. Consider a quadratic function $V(x) = x^TP_2x$, with P_2 in (5b). Differentiating $V(x)$ along the trajectories of $\dot{x} = Ax$ and using (6b) gives $\dot{V}(x) = x^TQ_2x \ge 0$, which implies that $V(x(t)) \ge V(x(0))\ \forall t$, therefore contradicting $\lim_{t\to\infty} x(t) = 0$ for $\dot{x} = Ax$ which stems from the fact that A is Hurwitz. We thus conclude that Q_2 can not be positive semi-definite. Since $Q_2 \notin \mathbb{S}_{\ge 0}$, $Q_2 \notin \mathbb{S}_{\le 0}$ and $n = 2$, both eigenvalues of Q_2 are not zero (Q_2^{-1} exists), and have opposite sign (Q_2 is not sign definite).

The proof is similar for $Q_{1_{CL}}$ and is thus omitted. ■

Let us now define quadratic functions $V_i(x) := x^\top P_ix$, $i \in \{1, 2\}$ functions associated to the solutions P_1, P_2 of (5) and introduce $\dot{V}_{iL}$, $\dot{V}_{iC}$, $\dot{V}_{iR} \colon \mathbb{R}^2 \to \mathbb{R}$

$$\begin{aligned} \dot{V}_{iL}(x) &:= x^\top Q_ix - 2x^\top P_iB \\ \dot{V}_{iC}(x) &:= x^\top Q_{i_{CL}}x \\ \dot{V}_{iR}(x) &:= x^\top Q_ix + 2x^\top P_iB. \end{aligned} \tag{8}$$

Using these notations, the time-derivative of the functions V_i along trajectories of (3) is given by

$$\dot{V}_i(x) = \begin{cases} 2x^\top P_i(Ax - B) = \dot{V}_{iL}(x) & \text{if } x \in \mathcal{R}_L \\ 2x^\top P_iA_{CL}x \qquad = \dot{V}_{iC}(x) & \text{if } x \in \mathcal{R}_C \\ 2x^\top P_i(Ax + B) = \dot{V}_{iR}(x) & \text{if } x \in \mathcal{R}_R \end{cases}$$

Note that $\dot{V}_i \in \mathcal{C}^0$ since the vector field (3) is continuous.

Now define the sets $\mathcal{Z}_i := \{x \in \mathbb{R}^2 \mid \dot{V}_i(x) = 0\}$, $i \in \{1, 2\}$ and introduce

$$\begin{aligned} \mathcal{Z}_{iL} &:= \{x \in \mathbb{R}^2 \mid \dot{V}_{iL}(x) = 0\} \\ \mathcal{Z}_{iC} &:= \{x \in \mathbb{R}^2 \mid \dot{V}_{iC}(x) = 0\} \\ \mathcal{Z}_{iR} &:= \{x \in \mathbb{R}^2 \mid \dot{V}_{iR}(x) = 0\} \end{aligned}$$

such that with the partition $\{\mathcal{R}_L, \mathcal{R}_C, \mathcal{R}_R\}$ of $\mathbb{R}^2$ we have

$$\mathcal{Z}_i = (\mathcal{Z}_{iL} \cap \mathcal{R}_L) \cup (\mathcal{Z}_{iC} \cap \mathcal{R}_C) \cup (\mathcal{Z}_{iR} \cap \mathcal{R}_R).$$

By continuity of $\dot{V}_i$, we have $\mathcal{Z}_{iL} \cap \mathcal{D}_{-1} = \mathcal{Z}_{iC} \cap \mathcal{D}_{-1}$, $\mathcal{Z}_{iR} \cap \mathcal{D}_1 = \mathcal{Z}_{iC} \cap \mathcal{D}_1$.

Let us characterize the sets $\mathcal{Z}_{1L}$, $\mathcal{Z}_{1C}$ and $\mathcal{Z}_{1R}$. According to (8), we have

$$\begin{aligned} \dot{V}_{1L}(x) &= x^\top Q_1x - 2x^\top P_1B \\ &\qquad + B^\top P_1Q_1^{-1}P_1B - B^\top P_1Q_1^{-1}P_1B \\ &= W_{1L}(x) - d_1 \end{aligned}$$

where $W_{1L}(x) := (x - Q_1^{-1}P_1B)^\top Q_1(x - Q_1^{-1}P_1B)$ and $d_1 := B^\top P_1Q_1^{-1}P_1B$. From the above identity, the set $\mathcal{Z}_{1L} = \{x \in \mathbb{R}^2 \mid \dot{V}_{1L}(x) = 0\} = \{x \in \mathbb{R}^2 \mid W_{1L}(x) = d_1\}$

is an ellipse centred at $x_{c_1} := Q_1^{-1} P_1 B$.
Proceeding the same way for $\dot{V}_{1R}(x)$, we obtain $\dot{V}_{1R}(x) = W_{1R}(x) - d_1$ with $W_{1R}(x) := (x + x_{c_1})^\top Q_1 (x + x_{c_1})$ and the set $\mathcal{Z}_{1R} = \{x \in \mathbb{R}^2 \mid \dot{V}_{1R}(x) = 0\} = \{x \in \mathbb{R}^2 \mid W_{1R}(x) = d_1\}$ is the image of $\mathcal{Z}_{1L}$ by central symmetry, yielding an ellipse centred at $-x_{c_1}$.
Finally, let us characterize $\mathcal{Z}_{1C} = \{x \in \mathbb{R}^2 \mid \dot{V}_{1C}(x) = 0\} = \{x \in \mathbb{R}^2 \mid x^\top Q_{1_{CL}} x = 0\}$. From Lemma 2.2, the matrix $Q_{1_{CL}}$ is not sign definite and $\det(Q_{1_{CL}}) < 0$. Thus the set $\mathcal{Z}_{1C}$ consists of two lines passing through the origin. From (7), $Q_{1_{CL}(2,2)} \neq 0$ since $\begin{bmatrix} 0 & 1 \end{bmatrix} Q_{1_{CL}} \begin{bmatrix} 0 \\ 1 \end{bmatrix} < 0$. The solution to $x^\top Q_{1_{CL}} x = 0$, subject to $x \in \mathcal{D}_1$ is

$$\begin{bmatrix} 1 \\ \frac{-Q_{1_{CL}(1,2)} - \sqrt{-det(Q_{1_{CL}})}}{Q_{1_{CL}(2,2)}} \end{bmatrix}, \begin{bmatrix} 1 \\ \frac{-Q_{1_{CL}(1,2)} + \sqrt{-det(Q_{1_{CL}})}}{Q_{1_{CL}(2,2)}} \end{bmatrix},$$

from which we obtain the two lines defining $\mathcal{Z}_{1C}$ and the coordinates of the intersection $\mathcal{Z}_{1C} \cap \mathcal{D}_1$. The sets $\mathcal{Z}_{1L}$, $\mathcal{Z}_{1C}$ and $\mathcal{Z}_{1R}$ are depicted in Figure 1. Using the

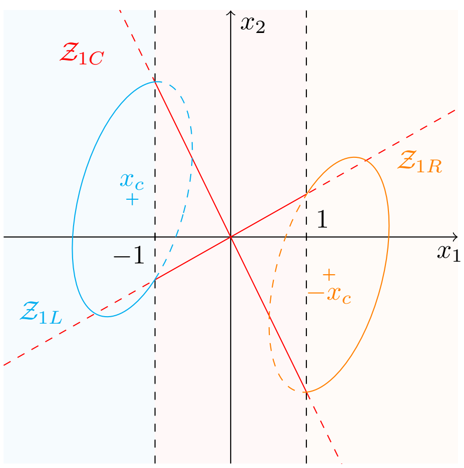

Figure 1: The sets $\mathcal{Z}_{1L}$, $\mathcal{Z}_{1C}$ and $\mathcal{Z}_{1R}$.

central symmetry between $\mathcal{Z}_{1L}$ and $\mathcal{Z}_{1R}$ and the continuity property of $\dot{V}_1$, we obtain the sets where $\dot{V}_1(x) > 0$ and $\dot{V}_1(x) < 0$ as depicted in Figure 2.

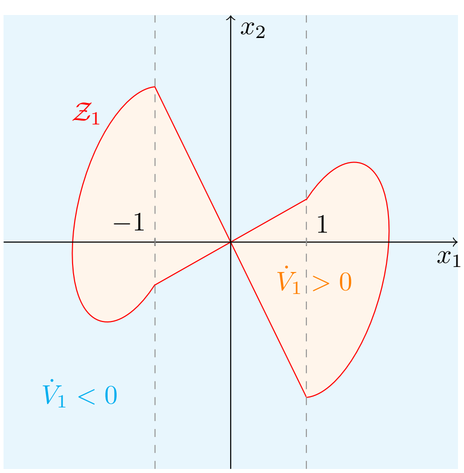

Figure 2: The partition of $\mathbb{R}^2$ based on the sign of $\dot{V}_1$.

We now study sets $\mathcal{Z}_{2L}$, $\mathcal{Z}_{2C}$ and $\mathcal{Z}_{2R}$. Similar to $\dot{V}_{1L}(x)$, we have $\dot{V}_{2L}(x) = W_{2L}(x) - d_2$ where $W_{2L}(x) := (x - Q_2^{-1} P_2 B)^\top Q_2 (x - Q_2^{-1} P_2 B)$ and $d_2 := B^\top P_2 Q_2^{-1} P_2 B$. Since Q_2 is not sign definite the set $\mathcal{Z}_{2L} = \{x \in \mathbb{R}^2 \mid \dot{V}_{2L}(x) = 0\} = \{x \in \mathbb{R}^2 \mid W_{2L} = d_2\}$ is an hyperbole centred at $x_{c_2} := Q_2^{-1} P_2 B$.

In the same way $\dot{V}_{2R}(x) = W_{2R}(x) - d_2$ with $W_{2R}(x) := (x + x_{c_2})^\top Q_2 (x + x_{c_2})$ and the set $\mathcal{Z}_{2R} = \{x \in \mathbb{R}^2 \mid \dot{V}_{2R}(x) = 0\} = \{x \in \mathbb{R}^2 \mid W_{2R} = d_2\}$ is the image of $\mathcal{Z}_{2L}$ by central symmetry, yielding an hyperbole centred at $-x_{c_2}$.

According to Assumption 2.1, $Q_{2_{CL}} = A_{CL}^\top P_2 + P_2 A_{CL} > 0$. Hence $\forall x \in \mathcal{R}_C \backslash \{0\}$, $\dot{V}_2(x) > 0$ and $\mathcal{Z}_{2C} = \{0\}$. In particular, by continuity of $\dot{V}_2$ we have $\dot{V}_2(x) = \dot{V}_{2C}(x) = \dot{V}_{2R}(x) > 0$, $\forall x \in \mathcal{D}_1$, $\dot{V}_2(x) = \dot{V}_{2C}(x) = \dot{V}_{2L}(x) > 0$, $\forall x \in \mathcal{D}_{-1}$. Thus, $\mathcal{D}_1$ and $\mathcal{D}_{-1}$ do not intersect the hyperboles $\mathcal{Z}_{2L}$, $\mathcal{Z}_{2R}$. The regions where $\dot{V}_2(x) > 0$ and $\dot{V}_2(x) < 0$ are depicted in Figure 3.

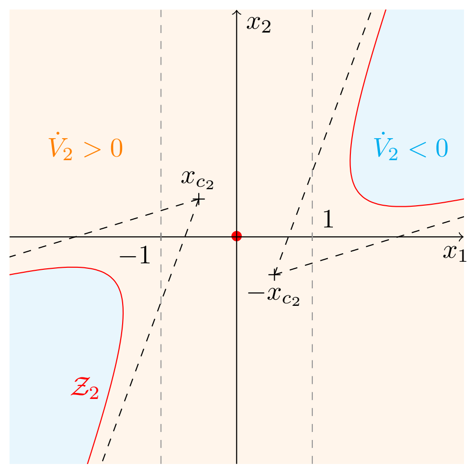

Figure 3: The partition of $\mathbb{R}^2$ based on the sign of $\dot{V}_2$.

Let us apply the above results to an example, that verifies Assumption 2.1.

Example 2.1 (Analysis): We consider system (3) with

$$A = \begin{bmatrix} -1 & -1 \\ 1 & 0 \end{bmatrix} B = \begin{bmatrix} 2 \\ 0.5 \end{bmatrix} \text{which gives } A_{CL} = \begin{bmatrix} 1 & -1 \\ 1.5 & 0 \end{bmatrix}$$

With $Q_1 = -I_2$, $Q_{2_{CL}} = I_2$ we solve (5) to obtain P_1 and P_2. The corresponding sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$ are depicted in Figure 4.

III. Main results

We now provide a set of definitions required to introduce the main result of this section. To characterize the asymptotic behaviour of $\Phi(t, x_0)$, $x_0 \neq 0$ we introduce the following definitions.

Definition 3.1: The set $\mathcal{L}$ is said to be *invariant* with respect to system $\dot{x} = f(x)$ provided

$$x_0 = \Phi(0, x_0) \in \mathcal{L} \Rightarrow x(t) = \Phi(t, x_0) \in \mathcal{L}, \ \forall t \in \mathbb{R}.$$

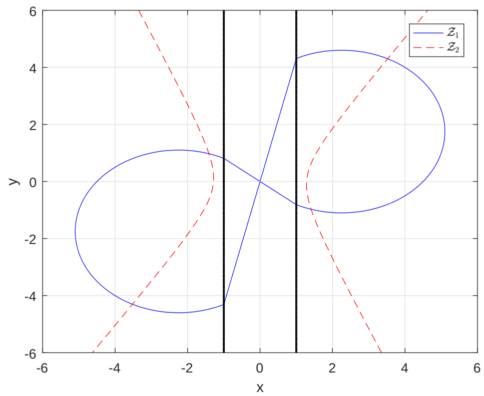

Figure 4: Sets $\mathcal{Z}_1$ and $\mathcal{Z}_2$ with $Q_1 = -I_2$, $Q_{2_{CL}} = I_2$.

If the previous implication only holds for all $t \geq 0$, $\mathcal{L}$ is said to be *positively invariant*.

Definition 3.2: A set $\mathcal{L}$ is said to be *finite-time attractive* with respect to system $\dot{x} = f(x)$ provided for any trajectory $\Phi(t, x_0)$, $x_0 \in \mathbb{R}^2$, $\exists t^* \geq 0$ such that $\Phi(t, x_0) \in \mathcal{L}, \forall t \geq t^*$.

Definition 3.3: A set $\mathcal{L}$ is said to be *finite-time repulsive* with respect to system $\dot{x} = f(x)$ provided that for any trajectory $\Phi(t, x_0)$, $\exists t^* \geq 0$ such that $\Phi(t, x_0) \notin \mathcal{L}, \forall t \geq t^*$.

Definition 3.4: A point y is an *ω-limit point* of the trajectory $\Phi(t, x_0)$ if there exists a sequence $(t_n)_{n\in\mathbb{N}}$ in $\mathbb{R}$ so that $\lim_{n\to\infty} t_n = +\infty$ and $\lim_{n\to\infty} \Phi(t_n, x_0) = y$. The set of all *ω-limit point* of $\Phi(t, x_0)$ is called *ω-limit set*.

We now exploit the set $\mathcal{Z}_1$ to prove the existence of finite-time attractive sets.

Lemma 3.1: For any scalar $\beta > \max_{\mathcal{Z}_1} V_1(x)$, the set $\mathcal{E}(P_1, \beta)$ is finite-time attractive with respect to (3).

Proof: Since $\beta > \max_{\mathcal{Z}_1} V_1(x)$ we have $\mathcal{Z}_1 \subset \mathcal{E}(P_1, \beta)$. From the definition of $\mathcal{Z}_1$, $\forall x \in \partial\mathcal{E}(P_1, \beta)$, $\frac{\partial V_1}{\partial x}\dot{x}(t) = \dot{V}_1(x(t)) < 0$, that is, in its boundary, the vector field in (3) points inwards the set $\mathcal{E}(P_1, \beta)$. Thus $\mathcal{E}(P_1, \beta)$ is positively invariant with respect to (3).

Consider an arbitrary initial condition x_0 and denote $\beta_0 = V_1(x_0)$. If $\beta_0 \leq \beta$, we have $x_0 \in \mathcal{E}(P_1, \beta)$ and finite-time attractiveness holds with $t^* = 0$. If $\beta_0 > \beta$, consider the set $\Gamma_1 = \{x | \beta \leq V_1(x) \leq \beta_0\}$, which is a compact set containing x_0 and take $\delta = -\max_{\Gamma_1} \dot{V}_1(x)$ which verifies $\delta > 0$ since $\dot{V}_1 < 0$, $\forall x \in \Gamma_1$. Now suppose that $\Phi(t, x_0) \in \Gamma_1$ $\forall t \geq 0$, that is, suppose $\beta \leq V_1(\Phi(t, x_0)) \leq \beta_0$ $\forall t \geq 0$. Using δ defined above and the differentiability of V_1 we have that $\forall t \geq 0$, $\beta \leq V_1(\Phi(t, x_0))$ and $V_1(\Phi(t, x_0)) = V_1(\Phi(0, x_0)) + \int_0^t \dot{V}_1(\Phi(\theta, x_0))d\theta = \beta_0 + \int_0^t \dot{V}_1(\Phi(\theta, x_0))d\theta \leq \beta_0 - \delta t$. Thus $\forall t > \frac{\beta_0 - \beta}{\delta}$ the inequality above is no longer satisfied hence $\exists t^*$ such that $\Phi(t^*, x_0) \notin \Gamma_1$. Since $\mathcal{E}(P_1, \beta_0)$ is also a positively invariant set, the trajectory $\Phi(t, x_0)$ has to leave Γ_1 by its inner boundary that is $\partial\mathcal{E}(P_1, \beta)$ and we have $\Phi(t, x_0) \in \mathcal{E}(P_1, \beta)$ $\forall t > t^*$. Thus, the set $\mathcal{E}(P_1, \beta)$ is finite-time attractive. ■

We now exploit the sets delimited by $\mathcal{Z}_2$ to obtain finite-time repulsive sets.

Lemma 3.2: For any scalar $\alpha < \min_{\mathcal{Z}_2\setminus\{0\}} V_2(x)$, the set $\mathcal{E}(P_2, \alpha)\setminus\{0\}$ is finite-time repulsive.

Proof: Since $\alpha < \min_{\mathcal{Z}_2\setminus\{0\}} V_2(x)$ we have $\mathcal{E}(P_2, \alpha) \cap \mathcal{Z}_2 = \{0\}$. From the definition of $\mathcal{Z}_2$, $\forall x \in \partial\mathcal{E}(P_2, \alpha)$, $\frac{\partial V_2}{\partial x}\dot{x}(t) = \dot{V}_2(x(t)) > 0$, that is, in its boundary, the vector field in (3) points outwards the set $\mathcal{E}(P_2, \alpha)$. Thus, $\mathcal{E}^c(P_2, \alpha) := \mathbb{R}^2 \setminus \mathcal{E}(P_2, \alpha)$ is positively invariant with respect to (3).

Consider an arbitrary initial condition $x_0 \neq 0$ and denote $\alpha_0 = V_2(x_0)$. If $\alpha_0 \geq \alpha$, we have $x_0 \in \mathcal{E}^c(P_2, \alpha)$ and finite-time repulsiveness holds with $t^* = 0$. If $\alpha_0 \leq \alpha$, consider the set $\Gamma_2 = \{x \mid \alpha_0 \leq V_2(x) \leq \alpha\}$, which is a compact set containing x_0. Take $\delta = \min_{\Gamma_2} \dot{V}_2(x)$ which verifies $\delta > 0$ since $\dot{V}_2 > 0$ $\forall x \in \Gamma_2$, that is, suppose $\alpha_0 \leq V_2(\Phi(t, x_0)) \leq \alpha$ $\forall t \geq 0$. Using δ obtained above and the differentiability of V_2 we have $\forall t \geq 0$, $\alpha \geq V_2(\Phi(t, x_0))$ and $V_2(\Phi(t, x_0)) = V_2(\Phi(0, x_0)) + \int_0^t \dot{V}_2(\Phi(\theta, x_0))d\theta = \alpha_0 + \int_0^t \dot{V}_2(\Phi(\theta, x_0))d\theta \geq \alpha_0 + \delta t$. Thus, $\forall t > \frac{\alpha - \alpha_0}{\delta}$ the inequality is no longer satisfied hence $\exists t^*$ such that $\Phi(t^*, x_0) \notin \Gamma_2$. Since $\mathcal{E}^c(P_2, \alpha_0)$ is also a positively invariant set, the trajectory $\Phi(t, x_0)$ has to leave Γ_2 by its outer boundary that is $\partial\mathcal{E}(P_2, \alpha)$ and we have $\Phi(t, x_0) \notin \mathcal{E}(P_2, \alpha)$ $\forall t > t^*$. Thus, the set $\mathcal{E}(P_2, \alpha)$ is finite-time repulsive. ■

From the above results, we can obtain the following property.

Proposition 3.1: For any $x_0 \neq 0$, the trajectory $\Phi(t, x_0)$, solution of (3) verifying Assumption 2.1 converges to a periodic orbit that encircles the origin.

Proof: From lemmas 3.1 and 3.2 there exists an finite-time attractive set $\mathcal{E}(P_1, \beta)$ and a finite-time repulsive set $\mathcal{E}(P_2, \alpha)$ with $\beta > \max_{\mathcal{Z}_1} V_1(x)$ and $\alpha < \min_{\mathcal{Z}_2\setminus\{0\}} V_2(x)$. Thus, $\forall x_0 \in \mathbb{R}^2\setminus\{0\}$, $\exists t^*$ such that $\forall t \geq t^*$, $\Phi(t, x_0) \in \mathcal{E}(P_1, \beta)$, $\Phi(t, x_0) \notin \mathcal{E}(P_2, \alpha)$. That is, $\forall t \geq t^*$, $\Phi(t, x_0) \in \mathcal{R} := \mathcal{E}(P_1, \beta) \setminus \mathcal{E}(P_2, \alpha)$. The set $\mathcal{R}$ is finite-time attractive and does not contain the origin, which is the only equilibrium point of (3) (from Lemma 2.1). Hence, from [5, Theorem 7.1 on p.290], the *ω-limit set* L^+ of $\Phi(t, x_0)$ is a periodic orbit. Since inside any periodic orbit of a planar system there must exists at least one equilibrium point (see [5, Corollary 7.1 on p.299]), we conclude that L^+ encircles the origin. ■

Remark 3.1: Since $L^+ \in \mathcal{R}$ and L^+ encircles the origin, the set inclusion $\mathcal{E}(P_2, \alpha) \subset \mathcal{E}(P_1, \beta)$ must hold.

A. Systems with dead-zone

Consider the system with unit dead-zone function $\dot{\bar{x}} = \bar{A}\bar{x} + \bar{B}\,\mathrm{dz}(\bar{K}\bar{x})$. Following Remark 2.1, the above system can be written as (1), that is $\dot{\bar{x}} = (\bar{A} + \bar{B}\bar{K})x - \bar{B}\,\mathrm{sat}(\bar{K}x)$. Thus, Assumption 2.1 applied to the above system becomes

Assumption 3.1 (Deadzone systems): $-A$ and A_{CL} are Hurwitz matrices.

In view of Proposition 3.1, for open loop unstable systems (that is $-A$ is Hurwitz) with dead-zone on its input, a stabilising feedback gain K (that is $A+BK$ is Hurwitz) guarantees that any trajectory with $x_0 \neq 0$ converges to a periodic orbit.

IV. Estimation of periodic orbits

In the previous section we showed that the set containing all the periodic trajectories is a ring delimited by two ellipsoids. This section presents results allowing to compute a ring containing the periodic trajectories. That is, to compute ellipsoids $\mathcal{E}(P_1,\beta)$ and $\mathcal{E}(P_2,\alpha)$ by letting either P_1 and P_2 or α and β to be unknowns. We propose algorithms that allow for the 1) minimisation of the outer set $\mathcal{E}(P_1,\beta)$ for P_1 or β given; 2) maximisation of the inner set $\mathcal{E}(P_2,\alpha)$ for P_2 or α given. The lemma below, that relies on the symmetry of both $\mathcal{E}(P_1,\beta)$ and $\mathcal{Z}_1$, (depicted in Figure 2), and the convexity of $\mathcal{E}(P_1,\beta)$, to establish conditions that ensure the set inclusion

$$\mathcal{Z}_1 \subset \mathcal{E}(P_1,\beta). \tag{9}$$

Lemma 4.1: The set inclusion (9) holds if and only if

$$\mathcal{H}_R := \left\{x\in\mathbb{R}^2 \mid\ Kx \geq 1,\ \dot{V}_{1R} \geq 0\right\} \subset \mathcal{E}(P_1,\beta). \tag{10}$$

Proof: From the symmetry with respect to the origin of $\mathcal{E}(P_1,\beta)$, we have that (10) implies $\mathcal{H}_L := \left\{x\in\mathbb{R}^2 \mid\ Kx \leq -1,\ \dot{V}_{1L} \geq 0\right\} \subset \mathcal{E}(P_1,\beta)$. Now, from the convexity of $\mathcal{E}(P_1,\beta)$, we conclude $\mathrm{Conv}(\mathcal{H}_R\cup\mathcal{H}_L) \subset \mathcal{E}(P_1,\beta)$. Since at the same time $\mathcal{Z}_1 \subset \mathrm{Conv}(\mathcal{H}_R\cup\mathcal{H}_L)$, by transitivity we obtain $\mathcal{Z}_1 \subset \mathcal{E}(P_1,\beta)$. ∎

A sufficient condition to verify the inclusion (10) is formulated with the *S-procedure* and is given in the proposition below.

Proposition 4.1: If there exist two positives scalars $\tau_1 \geq 0$, $\tau_2 \geq 0$ such that the matrix inequality

$$M_1(\tau_1,\tau_2) := \begin{bmatrix} -P_1 & 0 \\ 0 & \beta \end{bmatrix} + \tau_1 \begin{bmatrix} -(A^\top P_1 + P_1 A) & -P_1 B \\ -B^\top P_1 & 0 \end{bmatrix} + \tau_2 \begin{bmatrix} 0 & -\frac{1}{2}K^\top \\ -\frac{1}{2}K & 1 \end{bmatrix} \geq 0 \tag{11}$$

holds then inclusion (10) holds.

The proof is a straightforward application of the *S-procedure* [11],[1, Chapter 2.6.3].

Next we present necessary and sufficient condition for the following set inclusion

$$\mathcal{E}(P_2,\alpha) \subset \left\{x \mid \dot{V}_2(x) > 0\right\} \cup \{0\}, \tag{12}$$

to hold. Notice that this inclusion implies $\mathcal{E}(P_2,\alpha)\cap\mathcal{Z}_2 = \{0\}$ (refer to Figure 3 for a depiction of the sets).

Lemma 4.2: The set inclusion $\mathcal{E}(P_2,\alpha) \subset \left\{x \mid \dot{V}_2(x) > 0\right\} \cup \{0\}$ holds if and only if

$$\mathcal{F}_R := \left\{x\in\mathbb{R}^2 \mid\ Kx \geq 1, x^\top P_2 x \leq \alpha\right\} \subset \left\{x\in\mathbb{R}^2 \mid \dot{V}_2(x) > 0\right\}. \tag{13}$$

Proof: Define $\mathcal{F}_L := \left\{x\in\mathbb{R}^2 \mid\ Kx \leq -1, x^\top P_2 x \leq \alpha\right\}$. From the symmetry of $\mathcal{E}(P_2,\alpha)$ with respect to the origin, we have that if (13) holds then $\mathcal{F}_L \subset \left\{x\in\mathbb{R}^2 \mid \dot{V}_2(x) > 0\right\}$ also holds. The set $\mathcal{F}_C := \left\{x\in\mathbb{R}^2 \mid\ -1 < Kx < 1\right\}$ verifies $\mathcal{F}_C \subset \left\{x\in\mathbb{R}^2 \mid \dot{V}_2(x) > 0\right\} \cup \{0\}$. Thus, since $\mathcal{E}(P_2,\alpha) = \mathcal{F}_L \cup \mathcal{F}_C \cup \mathcal{F}_R$, if (13) holds we have $\mathcal{E}(P_2,\alpha) \subset \left\{x \mid \dot{V}_2(x) > 0\right\} \cup \{0\}$. ∎

The proposition below is a sufficient condition to verify the inclusion (13).

Proposition 4.2: If there exist two scalars $\tau_1 \geq 0$, $\tau_2 \geq 0$ such that

$$M_2(\tau_1,\tau_2) := \begin{bmatrix} A^\top P_2 + P_2 A & P_2 B \\ B^\top P_2 & 0 \end{bmatrix} + \tau_1 \begin{bmatrix} P_2 & 0 \\ 0 & -\alpha \end{bmatrix} + \tau_2 \begin{bmatrix} 0 & -\frac{1}{2}K^\top \\ -\frac{1}{2}K & 1 \end{bmatrix} \geq 0 \tag{14}$$

holds, then inclusion (13) holds.

The conditions for the set inclusions (9), (12) to hold, established by propositions 4.1, 4.2 are expressed in terms of matrix inequalities. Therefore they are convenient for the formulation of numerical procedures to compute estimates of the set containing the periodic orbit. Indeed, these inequalities can be used as constraints of optimization problems, namely semi-definite programs (SDP). In case these constraints have an affine dependence on the unknowns, the feasible set is convex. When associated to a linear objective function, a convex optimization problem is cast and its solution can be obtained using freely available software [6], [8]. In the remaining of the section we exploit the set inclusion inequalities (9), (12) to formulate SDP programs thus providing estimates of the periodic trajectories.

We first provide a solution for the following problems for a saturating system given a feedback gain satisfying Assumption 2.1.

Problem 4.1 (Computation of the outer estimate): For P_1 satisfying (5) for a given Q_1, compute the smallest scalar β satisfying (9).

Based on the condition in Propositions 4.1 we propose the following (convex) SDP to solve the above problem

$$\underset{\beta,\tau_1,\tau_2}{\text{minimise}}\ \beta \text{ subject to } \quad M_1 \geq 0, \beta \geq 0, \tau_1 \geq 0, \tau_2 \geq 0.$$

Problem 4.2 (Computation of inner estimate): For P_2 satisfying (5) for a given $Q_{2_{CL}}$, compute the largest scalar α satisfying (12).

Based on the condition in Propositions 4.2 we propose the following (convex) SDP to solve the above problem

$$\underset{\alpha,\tau_1,\tau_2}{\text{maximise}}\ \alpha \text{ subject to } \quad M_2 \geq 0, \alpha \geq 0, \tau_1 \geq 0, \tau_2 \geq 0.$$

Consider again Example 2.1. We solve the above SDPs with $Q_1 = -I_2$ and $Q_{2_{CL}} = I_2$ to obtain inner and outer estimates (depicted in green in Figure 5). A periodic trajectory is also depicted to illustrate the containment in the ring delimited by the two ellipsoids.

Note that the computed estimates are optimal for a given P_i, not necessarily corresponding to the closest inner and outer ellipsoids to the periodic trajectories. Therefore, to obtain better estimates, we consider matrices P_i, related to the ellipsoid shape, as variables. Letting P_i vary requires the corresponding (variables) Q_1 and $Q_{2_{CL}}$ to satisfy

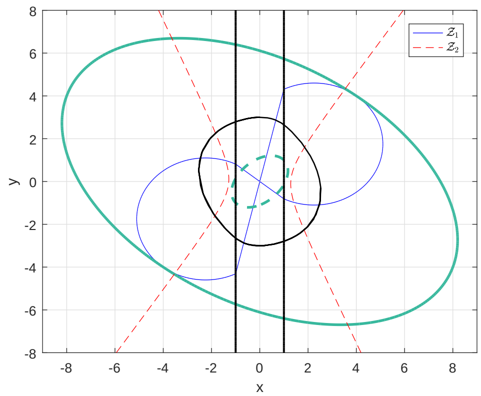

Figure 5: Outer and inner estimates with $Q_1 = -I_2$, $Q_{2_{CL}} = I_2$ for Example 2.1.

$Q_1 \in \mathbb{S}_{<0}$ and $Q_{2_{CL}} \in \mathbb{S}_{>0}$. Such a requirement imposes the inequalities

$$A^\top P_1 + P_1 A < 0 \tag{15}$$

$$A_{CL}^\top P_2 + P_2 A_{CL} > 0. \tag{16}$$

in the formulation of the optimization problem. Whenever P_1 (P_2) is an optimization variable we fix the scalar β (α) defining the set $\mathcal{E}(P_1, \beta)$ ($\mathcal{E}(P_2, \alpha)$). We now propose optimization-based solutions to the problems

Problem 4.3: Given $\beta > 0$, compute a matrix P_1 such that (9) and (15) hold and $\mathcal{E}(P_1, \beta)$ is as close as possible to the periodic orbit.

Problem 4.4: Given $\alpha > 0$, compute a matrix P_2 such that (12) and (16) hold and $\mathcal{E}(P_2, \alpha)$ is as close as possible to the periodic orbit.

To set an SDP to solve the above problems we define a linear function on the decisions variables related to the distance to the periodic trajectories. A possible criteria is the trace of the matrix P_i which is adopted below. Note that whenever P_i is a variable, to obtain a convex optimization problem using (11) (or (14)) parameter τ_1 has to be fixed. Thus, to solve problems 4.3 and 4.4, we perform a line search on parameter τ_1 by solving the SDPs for a fixed values of τ_1

$$\underset{P_1,\tau_2}{\text{maximise}} \ \text{trace}(P_1) \text{ subject to } (11), P_1 > 0, \ \tau_2 \geq 0,$$

$$\underset{P_2,\tau_2}{\text{minimise}} \ \text{trace}(P_2) \text{ subject to } (14), (16), \ P_2 > 0, \tau_2 \geq 0.$$

We consider again Example 2.1. We let P_1 (P_2) be a variable and we solve Problem 4.3 (4.4) by performing a line search on τ_1. The results are depicted in Figure 6.

V. Conclusion

For planar saturating systems we established conditions on the system matrix for the existence of periodic trajectories. Moreover we have characterized a set defined by two ellipsoids which contains periodic trajectories. A parametrization of such a set in terms of matrix inequalities has allowed us to estimate sets containing the periodic trajectories by solving convex optimization problems. We are currently developing strategies for feedback design that generate periodic trajectories.

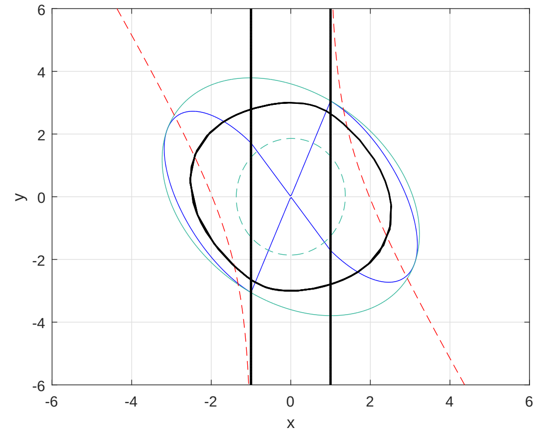

Figure 6: Outer and inner estimates of example 2.1 with $Q_1 = -I_2$ for P_1 variable.

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