# Stabilization of walls in notched ferromagnetic nanowires

Gilles Carbou, David Sanchez

# Stabilization of walls in notched ferromagnetic nanowires

Gilles Carbou* and David Sanchez†

**Abstract**

In this paper we study a one-dimensional model of ferromagnetic nanowire presenting notches. We prove the existence of stable wall profiles even under a small applied magnetic field with the walls localized in notches. Moreover, in order to illustrate wall depinning by applied magnetic field, we prove the non-existence of stationary wall profiles in the presence of a large applied magnetic field.

*Keywords: Landau-Lifshitz equation, ferromagnetism, nanowire, stability*

*MSC:* 35K55, 35Q60

## 1 Introduction

In [10], new applications of ferromagnetic nanowires in the domain of data storage are highlighted. Domain walls formation in such devices allows bits encoding, and walls motion induced by a spin current injection makes data reading faster than in classical devices. In such applications, the stability of walls positions is crucial since an undesired wall motion can deteriorate the information. As it is proved in [5], walls configurations in straight nanowires are stable but not asymptotically stable, so that both chirality and position of walls are not fixed. In addition, (see [6]) in finite length nanowire, walls configurations are unstable. Therefore, a stronger control of walls positions is indispensable. In racetrack memory nanowires, this control is ensured by patterning notches along the wire (see [10]). Then we observe that the domain walls are located at the notches, and between two consecutive notches, the magnetization is almost constant, oriented toward the direction of the wire, in one sense or in the other one. This property is used to encode the data: each bit is encoded by the sense of the magnetization between two consecutive notches.

In this paper, we deal with a one-dimensional model of nanowire obtained by asymptotic analysis in the same spirit as in [4] and [6]. We establish rigorously that walls positions are stabilized by notches. Let us first recall the 3d-model.

We denote by $(e_1, e_2, e_3)$ the canonical basis of $\mathbb{R}^3$. The euclidean scalar product and norm are denoted respectively by $\cdot$ and $|\ |$. The cross product is denoted by $\times$.
The magnetic moment $m(t, \mathbf{x})$ is defined for $t \geq 0$ and $\mathbf{x} \in \Omega \subset \mathbb{R}^3$, where $\Omega$ is the ferromagnetic sample. We assume that the material is saturated so that $m : (t, \mathbf{x}) \mapsto m(t, \mathbf{x})$ takes its values in the unit sphere of $\mathbb{R}^3$. The ferromagnetism energy associated to a configuration $m$ is given by

$$\mathcal{E}_{\text{mic}}(m) = \frac{1}{2}\int_\Omega |\nabla m|^2 d\mathbf{x} + \frac{1}{2}\int_{\mathbb{R}^3} |h_d(m)|^2 d\mathbf{x} - \int_\Omega H_a \cdot m d\mathbf{x},$$

where

- the first term is called the exchange energy,

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- the second term is called the demagnetizing energy. The demagnetizing field $h_d(m)$ is the magnetic field generated by the magnetization, and is given by the following system coupling the static Maxwell equation and the law of faraday:

$$\begin{cases} \operatorname{curl} h_d(m) = 0 \text{ in } \mathbb{R}^3, \\ \\ \operatorname{div}(h_d(m) + \overline{m}) = 0, \text{ where } \overline{m} \text{ is the extension of } m \text{ by zero outside } \Omega. \end{cases}$$

- The last term is the Zeeman energy describing the effects of the applied field $H_a$ on the magnetization.

The variations of $m$ fulfill the Landau Lischitz equation:

$$\frac{\partial m}{\partial t} = -m \times H_{\text{eff}}(m) - \alpha m \times (m \times H_{\text{eff}}(m)), \tag{1.1}$$

where $\alpha > 0$ is called the damping coefficient and where the effective field $H_{\text{eff}}(m)$ is derived from the energy $\mathcal{E}_{\text{mic}}$ by:

$$H_{\text{eff}}(m) = -\partial_m \mathcal{E}_{\text{mic}} = \Delta m + h_d(m) + H_a.$$

The natural boundary conditions is the homogeneous Neumann condition:

$$\frac{\partial m}{\partial \mathbf{n}} = 0 \text{ on } \partial\Omega,$$

where $\mathbf{n}$ is the outward unit normal.

In order to obtain a one-dimensional model of ferromagnetic wire with notches, we consider a ferromagnetic sample $\Omega_\eta$ given by

$$\Omega_\eta = \left\{(x, y, z) \in \mathbb{R}^3,\ x \in \mathbf{I},\ x^2 + y^2 \leq \eta^2 \rho^2(x)\right\},$$

where $\mathbf{I}$ is an interval and $\rho : \mathbf{I} \longrightarrow \mathbb{R}$ is smooth on $\mathbf{I}$ and satisfies:

$$\exists\, \rho_1 > 0,\ \exists\, \rho_2 > 0,\ \forall\, x \in \mathbf{I},\ \ \rho_1 \leq \rho(x) \leq \rho_2.$$

By using the same techniques as in [2, 4, 5], we take the limit of the dynamical model (1.1) when $\eta$ tends to zero, and we obtain the following one-dimensional model: the nanowire is assimilated to the interval $\mathbf{I}$, the magnetization is described by the magnetic moment $m : \mathbb{R}^+ \times \mathbf{I} \longrightarrow \mathbb{R}^3$, which satisfies the saturation constraint:

$$|m(t, x)| = 1 \quad \text{for all } (t, x) \in \mathbb{R}^+ \times \mathbf{I}. \tag{1.2}$$

The one-dimensional ferromagnetic energy is given by

$$\mathcal{E}_{\text{mic}} = \frac{1}{2}\int_{\mathbf{I}} \mathbf{a}(x)|\partial_x m|^2 dx + \frac{1}{4}\int_{\mathbf{I}} \mathbf{a}(x)\left(|m_2|^2 + |m_3|^2\right) dx - \int_{\mathbf{I}} \mathbf{a}(x) m \cdot \mathbf{h}_a dx,$$

where

- $m_i$ are the coordinates of $m$,
- $\mathbf{a}(x) = \pi(\rho(x))^2$ is the area of the wire section at the point $x$,
- $\mathbf{h}_a(x)$ is the resulting applied field, obtained by taking the limit when $\eta$ tends to zero, of the mean value of the applied field $H_a$ on the cross section:

$$\mathbf{h}_a(x) = \lim_{\eta \longrightarrow 0} \frac{1}{\eta^2 \mathbf{a}(x)} \int_{(y,z),\, y^2+z^2 \leq \eta^2\rho^2} H_a(x, y, z)\, dy\, dz.$$

As remarked in several papers on ferromagnetic-nanowire modeling [9, 4, 5], the limit demagnetizing field reduces to an anisotropy term for which the wire axis $\mathbb{R}e_1$ is the easy axis.

The variations of $m$ satisfy the following Landau-Lifshitz type equation:

$$\frac{\partial m}{\partial t} = -m \times \mathcal{H}_e(m) - \alpha m \times (m \times \mathcal{H}_e(m)) \text{ for } (t,x) \in \mathbb{R}^+ \times \mathbf{I}, \tag{1.3}$$

where the resulting effective field $\mathcal{H}_e(m)$ is given by

$$\mathcal{H}_e(m) = \partial_{xx} m + \frac{\mathbf{a}'}{\mathbf{a}} \partial_x m - \frac{1}{2}(m_2 e_2 + m_3 e_3) + \mathbf{h}_a \tag{1.4}$$

(we denote by $\mathbf{a}'$ the derivative of $\mathbf{a}$ with respect to $x$). In the case of a finite wire $[a,b]$ we add at the ends of the wire the homogeneous Neumann boundary conditions:

$$\forall\, t \in \mathbb{R}, \quad \partial_x m(t,a) = \partial_x m(t,b) = 0. \tag{1.5}$$

**Remark 1.1.** *The model we consider is invariant by rotation around the wire axis, that is: if $m$ satisfies* (1.3)-(1.4) *and eventually the boundary conditions* (1.5)*, then $(t,x) \mapsto \mathbf{R}_\varphi m(t,x)$ is also solution of the same system, where $\mathbf{R}_\varphi$ is the rotation around the axis $\mathbb{R}e_1$ defined by:*

$$\mathbf{R}_\varphi = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix}.$$

At first, we will consider an infinite-length nanowire with one notch. It is assimilated to the interval $\mathbf{I} = \mathbb{R}$. The pinched zone is supposed to be symmetric and centered at 0, so that the radius of the wire section, denoted by $\overline{\rho} : \mathbb{R} \longrightarrow \mathbb{R}$, fulfills:

$$\begin{cases} \overline{\rho} = 1 \text{ outside } [-l_0, l_0], \\ \\ \overline{\rho} \text{ is even and non decreasing on } [0, l_0], \\ \\ 0 < \rho_1 \leq \overline{\rho}(x) \leq 1, \end{cases} \tag{1.6}$$

*i.e.* the notch is restricted to the domain $[-l_0, l_0] \subset \mathbb{R}$, where $l_0 > 0$ is fixed. We denote $\bar{\mathbf{a}}(x) = \pi(\bar{\rho}(x))^2$. We assume that the applied field vanishes, so we deal with the equation:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times \mathbf{h}(m) - \alpha m \times (m \times \mathbf{h}(m)) \text{ on } \mathbb{R}^+ \times \mathbb{R}, \\ \\ \mathbf{h}(m) = \partial_{xx} m + \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}} \partial_x m - \dfrac{1}{2}(m_2 e_2 + m_3 e_3). \end{cases} \tag{1.7}$$

We look for stationary magnetization distributions describing one wall separating a left hand side $-e_1$-domain to a right hand side $+e_1$-domain, *i.e.* with the limit condition:

$$m(x) \longrightarrow -e_1 \text{ when } x \longrightarrow -\infty \quad \text{and} \quad m(x) \longrightarrow e_1 \text{ when } x \longrightarrow +\infty. \tag{1.8}$$

The first question we address is the existence of such one-wall profile which is a stationary solution for (1.3)-(1.4) with vanishing $h_a$. Once this question solved, the second problem we tackle is to prove the stability of this wall and the asymptotic stability of its position. The following stability result establishes that the wall is pinned at the notch:

**Theorem 1.1.** *There exists a stationary solution $\mathbf{m}_0$ for* (1.2)-(1.7)-(1.8)*. This solution is stable and asymptotically stable modulo rotations around the wire axis, that is: for all $\varepsilon > 0$, there exists $\eta > 0$ such that for all solution $m$ for* (1.2)-(1.7) *satisfying $\|m(0,\cdot) - \mathbf{m}_0\|_{H^1(\mathbb{R})} \leq \eta$, then*

- $\forall t \geq 0, \|m(t,\cdot) - \mathbf{m}_0\|_{H^1(\mathbb{R})} \leq \varepsilon,$

- *there exists $\varphi_\infty$ such that $\|m(t,\cdot) - \mathbf{R}_{\varphi_\infty}\mathbf{m}_0\|_{H^1(\mathbb{R})} \longrightarrow 0$ when $t \longrightarrow 0$.*

We study now the effects of a magnetic field $\mathbf{h}_a$ applied in the wire direction: $\mathbf{h}_a = he_1$. We deal with the system:

$$\left\{\begin{array}{l} \dfrac{\partial m}{\partial t} = -m \times \mathbf{h}(m) - \alpha m \times (m \times \mathbf{h}(m)) \text{ on } \mathbb{R}^+ \times \mathbb{R}, \\ \\ \mathbf{h}(m) = \partial_{xx} m + \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x m - \dfrac{1}{2}\left(m_2 e_2 + m_3 e_3\right) + he_1. \end{array}\right. \tag{1.9}$$

In non pinched wire, an applied field of the form $\mathbf{h}_a = h_a e_1$ induces a motion of the wall (see [4] and [6]). In the case of pinched wire, we prove that it is not the case, since the wall is stuck in the notch for small applied fields:

**Theorem 1.2.** *There exists $h_{max} > 0$ such that for all $h \in ]-h_{max}, h_{max}[$, there exists $\mathbf{m}_h : \mathbb{R} \longrightarrow S^2$ such that:*

- *for all $h \in ]-h_{max}, h_{max}[$, $\mathbf{m}_h$ is a static solution for* (1.9) *with limit conditions* (1.8)*,*
- *$h \mapsto \mathbf{m}_h$ is $\mathcal{C}^1$ for the $H^2$ norm,*
- *$\mathbf{m}_0$ is the solution for* (1.7) *given by Theorem 1.1,*
- *for all $h \in ]-h_{max}, h_{max}[$, $\mathbf{m}_h$ is stable and asymptotically stable modulo rotation around the $e_1$-axis for* (1.9)-(1.8)*.*

The previous result confirms that in infinite wires, a wall is pinned by the notch, even in presence of a small applied field. If the applied field is strong enough, wall depinning is stated in the following theorem:

**Theorem 1.3.** *There exists $h_0 \in ]0,1[$ such that if $|h| \geq h_0$ there is no stationary solution for* (1.9) *presenting a magnetization switching, i.e. satisfies* (1.8)*.*

Now we aim to consider a wire with several notches. Our goal is to prove that if the length between two consecutive notches is large enough, whatever the data, we can encode it in such device. We introduce $l_1 > 0$ such that $\mathbf{m}_0^1$, the first coordinate of $\mathbf{m}_0$ given by Theorem 1.1, satisfies:

$$\forall\, x \leq -l_1,\ \mathbf{m}_0^1(x) \leq -\frac{3}{4} \quad \text{and} \quad \forall\, x \geq l_1,\ \mathbf{m}_0^1(x) \geq \frac{3}{4}. \tag{1.10}$$

We consider a finite-length wire with $N-1$ notches and we denote by $L$ the distance between two consecutive notches. We assume that $L > 2\max\{l_0, l_1\}$ and that each notch has the same profile as the notch we considered in the infinite-wire case, so that the cross-section radius is given by $\rho \in \mathcal{C}^\infty([0, NL])$:

$$\rho(x) = \left\{\begin{array}{c} 1 \text{ if } x \in [0, \dfrac{L}{2}], \\ \\ \overline{\rho}(x - kL) \text{ if } |x - kL| \leq \dfrac{L}{2},\ k \in \{1, \ldots, N-1\}, \\ \\ 1 \text{ if } x \in [NL - \dfrac{L}{2}, NL]. \end{array}\right. \tag{1.11}$$

We define $\mathbf{a}$ by:

$$\mathbf{a}(x) = \pi(\rho(x))^2. \tag{1.12}$$

We deal with the following model:

$$
\begin{cases}
\dfrac{\partial m}{\partial t} = -m \times \mathcal{H}_e(m) - \alpha m \times (m \times \mathcal{H}_e(m)) \text{ in } \mathbb{R}^+ \times [0, NL], \\[2ex]
\mathcal{H}_e(m) = \partial_{xx} m + \dfrac{\mathbf{a}'}{\mathbf{a}} \partial_x m - \dfrac{1}{2}\left(m_2 e_2 + m_3 e_3\right) + h_a e_1, \\[2ex]
\partial_x m(t,0) = \partial_x m(t, NL) = 0.
\end{cases}
\tag{1.13}
$$

**Definition 1.1.** *Let $D \in \{0,1\}^N$. Let $\mathbf{m} : [0, NL] \longrightarrow S^2$ be a static solution of (1.13). We denote by $\mathbf{m}^1$ its first coordinate. We say that $\mathbf{m}$ encodes $D$ if for all $k \in \{1, \ldots, N\}$, we have:*

$$
D(k) = 0 \implies \forall\, x \in [(k-1)L + l_1, kL - l_1],\ \mathbf{m}^1(x) < -\frac{1}{2},
$$
*and*
$$
D(k) = 1 \implies \forall\, x \in [(k-1)L + l_1, kL - l_1],\ \mathbf{m}^1(x) > \frac{1}{2}.
$$

**Theorem 1.4.** *Let $N$ be in $\mathbb{N}^*$. There exists $L_{min} > 2\max\{l_0, l_1\}$ such that if $L > L_{min}$, then for all data $D \in \{0,1\}^N$, there exists a stationary solution $\mathbf{m}$ of (1.2)-(1.13) with $h_a = 0$ encoding the data $D$. In addition, this solution is asymptotically stable modulo rotation around the wire axis $\mathbb{R}e_1$ for system (1.2)-(1.13) with $h_a = 0$.*

As in the one-wall case, we can prove that a small applied field does not deteriorate the information.

**Theorem 1.5.** *Let $N$ be in $\mathbb{N}^*$. There exists $h_{max}$ such that whatever $L > L_{min}$, whatever $D \in \{0,1\}^N$, there exists a one-parameter family $h_a \mapsto \mathbf{m}(h_a)$, defined for $|h_a| \leq h_{max}$, such that $\mathbf{m}(h_a)$ is a static solutions for (1.2)-(1.13) encoding $D$, and asymptotically stable modulo rotation around the wire axis.*

The paper is organized as follows. Section 2 is devoted to the construction of a stationary solution in the infinite-wire case with vanishing applied field. We use a shooting method on an equivalent pendulum-type equation. We then study the Lyapounov stability of the solution by studying a small perturbation of the magnetization in section 3. In order to take into account the saturation constraint (1.2), we rewrite the perturbations of $\mathbf{m}_0$ in a mobile frame as in [5]. The key point lies in the study of the linearized part of the Landau-Lifshitz equation. We indeed have to take into account the invariance by rotation around the wire's axis of the solution. In Section 4, we address the existence and stability of solutions in the presence of an applied magnetic field. When the applied magnetic field is small enough, the existence of a static solution is deduced from the vanishing-applied-field case thanks to the implicit function theorem and the stability proof is easily adapted. We also prove that for large enough applied magnetic field there does not exist stationary solution to the problem (see Section 5).
In Section 6 we detail the general case of a finite wire with multiple notches. The main difficulty is the construction of the static solution for $L$ great enough. A data being given, using the results of the infinite case, we construct an approximate solution encoding the data. Using IMS formula we obtain the coercivity for the linearization around this approximate solution and we construct the exact solution by a fixed point theorem applied in a neighborhood of the approximate solution.

# 2 Existence of stationary profiles for infinite wire with one notch

In this section, we consider an infinite wire with one notch, and we assume that the applied field vanishes, *i.e.* we deal with the equation (1.7).

We look for stationary profiles $\mathbf{m}_0 : \mathbb{R} \to \mathbb{S}^2$ for (1.7) where one switching of the magnetization occurs. We write $\mathbf{m}_0$ under the form

$$\mathbf{m}_0(x) = \begin{pmatrix} \sin\theta_0(x) \\ \cos\theta_0(x) \\ 0 \end{pmatrix},$$

where $\theta_0 \in \mathcal{C}^2(\mathbb{R})$ is non decreasing and tends to $-\frac{\pi}{2}$ (resp. $+\frac{\pi}{2}$) when $x$ tends to $-\infty$ (resp. $+\infty$). We assume that $\mathbf{m}_0$ is a stationary solution of (1.7), *i.e.* that

$$\mathbf{m}_0 \times \left(\partial_{xx}\mathbf{m}_0 + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x\mathbf{m}_0 - \frac{1}{2}(\mathbf{m}_{0,2}e_2 + \mathbf{m}_{0,3}e_3)\right) = 0,$$

where $\mathbf{m}_{0,i}$ is the $i$th coordinate of $\mathbf{m}_0$, and we obtain that $\theta_0$ satisfies:

$$\theta_0'' + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\theta_0' + \frac{1}{2}\sin\theta_0\cos\theta_0 = 0. \tag{2.14}$$

We claim the following result:

**Proposition 2.1.** *There exists a non decreasing odd function $\theta_0 \in \mathcal{C}^2(\mathbb{R})$ such that*

$$\lim_{x \longrightarrow +\infty} \theta_0 = \frac{\pi}{2},$$

*and satisfying* (2.14) *on* $\mathbb{R}$.

*Proof.* We prove the existence of $\theta_0$ by a shooting method. We denote by $\Psi(p,\cdot)$ the solution of the Cauchy problem coupling (2.14) with the initial condition $\Psi(p,0) = 0$ and $\partial_x\Psi(p,0) = p$:

$$\begin{cases} \partial_x^2\Psi(p,\cdot) + \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x\Psi(p,\cdot) + \dfrac{1}{2}\sin\Psi(p,\cdot)\cos\Psi(p,\cdot) = 0, \\[2ex] \Psi(p,0) = 0, \quad \partial_x\Psi(p,0) = p, \end{cases} \tag{2.15}$$

Our goal is to find $p_0$ such that $x \mapsto \Psi(p_0,x)$ is non decreasing on $\mathbb{R}$ and tends to $+\frac{\pi}{2}$ when $x$ tends to $+\infty$ (since $\bar{\mathbf{a}}$ is even, the solutions of (2.15) are odd by standard argument).

We set

$$\mathcal{E}(p,x) = (\partial_x\Psi(p,x))^2 + \frac{1}{2}\sin^2\Psi(p,x). \tag{2.16}$$

Using (2.15), we remark that

$$\frac{\partial\mathcal{E}}{\partial x}(p,x) = 2\partial_x\Psi(p,x)\left(\partial_{xx}\Psi(p,x) + \frac{1}{2}\sin\Psi(p,x)\cos\Psi(p,x)\right) = -2\frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\left(\partial_x\Psi(p,x)\right)^2,$$

so, since $\bar{\mathbf{a}}$ is non decreasing in $[0,l_0]$ and constant in $[l_0,+\infty]$, $\mathcal{E}$ is non increasing in $[0,l_0]$ and constant in $[l_0,+\infty]$.

If $\Psi(p,x)$ tends to $+\frac{\pi}{2}$ when $x$ tends to $+\infty$, then $\mathcal{E}(p,x)$ tends to $\dfrac{1}{2}$ when $x$ tends to $+\infty$, so $\mathcal{E}(p,x) = \dfrac{1}{2}$ for $x \geq l_0$.

We remark that $p \mapsto \mathcal{E}(p,l_0)$ is continuous (using the continuity of the solution of an o.d.e. with respect to the initial data).

On the one hand, $\mathcal{E}(0,l_0) = 0$, since $\Psi(0,\cdot) \equiv 0$. On the other hand,

$$\partial_x\mathcal{E} = -\frac{2\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}(\partial_x\Psi)^2 = -\frac{2\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\mathcal{E} + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\sin^2\Psi,$$

so

$$\partial_x\mathcal{E} + \frac{2\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\mathcal{E} = \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\sin^2\Psi \geq 0 \quad \text{on } [0,l_0],$$

so $x \mapsto (\bar{\mathbf{a}}(x))^2 \mathcal{E}(p, x)$ is increasing on $[0, l_0]$, so

$$(\bar{\mathbf{a}}(0))^2 \mathcal{E}(p, 0) \leq (\bar{\mathbf{a}}(l_0))^2 \mathcal{E}(p, l_0),$$

that is

$$(\bar{\mathbf{a}}(0))^2 p^2 \leq \pi^2 \mathcal{E}(p, l_0).$$

Thus, since $\bar{\mathbf{a}}(0) > 0$, for $p$ large enough, $\mathcal{E}(p, l_0) > \dfrac{1}{2}$. Therefore, there exists $p \geq 0$ such that $\mathcal{E}(p, l_0) = \dfrac{1}{2}$. We denote by $p_0$ the minimum of these $p$:

$$p_0 = \min\{p,\ \mathcal{E}(p, l_0) = \frac{1}{2}\}.$$

Let us prove that $\theta_0 := \Psi(p_0, \cdot)$ is a solution of our problem.
For all $p < p_0$, $\mathcal{E}(p, l_0) < \dfrac{1}{2}$, so that $(\partial_x \Psi(p, l_0))^2 < \dfrac{1}{2} \cos^2 \Psi(p, l_0)$. Thus, $(\Psi(p, l_0), \partial_x \Psi(p, l_0))$ is between the separatrix of the pendulum equations, *i.e.* is in one connected cell $c_k$ with:

$$c_k = \{(\theta, p),\ \theta \in ]-\frac{\pi}{2} + k\pi, \frac{\pi}{2} + k\pi[,\ |p| < \frac{1}{\sqrt{2}} |\cos\theta|\}.$$

We remark that $(\Psi(0, l_0), \partial_x \Psi(0, l_0)) = (0, 0)$ is in the cell $c_0$, so by continuity arguments, for all $p < p_0$, $(\Psi(p, l_0), \partial_x \Psi(p, l_0))$ is in the cell $c_0$. In particular, we obtain that

$$(\Psi(p_0, l_0), \partial_x \Psi(p_0, l_0)) \in \overline{c_0},$$

and

$$-\frac{\pi}{2} \leq \Psi(p_0, l_0) \leq \frac{\pi}{2}.$$

If $\Psi(p_0, l_0) = \frac{\pi}{2}$, since $\mathcal{E}(p_0, l_0) = \frac{1}{2}$, we have $\partial_x \Psi(p_0, l_0) = 0$. So, since $x \mapsto \Psi(p_0, x)$ satisfies (2.14), $x \mapsto \Psi(p_0, x)$ is constant, which is impossible since $\Psi(p_0, 0) = 0$. With the same argument, we obtain that

$$-\frac{\pi}{2} < \Psi(p_0, l_0) < \frac{\pi}{2}. \tag{2.17}$$

On $[0, l_0[$, $\mathcal{E}(p_0, x)$ is non increasing so that $\mathcal{E}(p_0, x) > \frac{1}{2}$. Thus, since $(\Psi(p_0, 0), \partial_x \Psi(p_0, 0)) = (0, p)$ with $p > 0$, by continuity argument, $(\Psi(p_0, x), \partial_x \Psi(p_0, x))$ remains in the domain $p > \frac{1}{\sqrt{2}} |\cos\theta|$ for $x \in [0, l_0]$. In particular, $\partial_x \Psi(p_0, x) > 0$ on $[0, l_0]$, so, using (2.17),

$$\forall x \in [0, l_0], 0 \leq \Psi(p_0, x) < \frac{\pi}{2}.$$

For $x \geq l_0$, $\bar{\mathbf{a}}(x) = \pi$ and $x \mapsto \Psi(p_0, x)$ satisfies the pendulum equation $\theta'' + \frac{1}{2} \cos\theta \sin\theta = 0$, so $x \mapsto (\Psi(p_0, x), \partial_x \Psi(p_0, x))$ is a trajectory on the separatrix. Therefore, $x \mapsto \Psi(p_0, x)$ is non decreasing on $[l_0 + \infty[$ and tends to $\frac{\pi}{2}$ when $x$ tends to $+\infty$.
Since $x \mapsto \Psi(p_0, x)$ is odd, we conclude that $\theta_0 := \Psi(p_0, x)$ is a solution of our problem.

**Remark 2.1.** *The uniqueness of $\theta_0$ remains open.*

# 3 Stability

Let $\mathbf{m}_0$, given by

$$\mathbf{m}_0(x) = \begin{pmatrix} \sin\theta_0(x) \\ \cos\theta_0(x) \\ 0 \end{pmatrix},$$

be the stationary solution of (1.7) given by Proposition 2.1. We are interested in the Lyapounov stability of $\mathbf{m}_0$ for the Landau-Lifschitz Equation (1.7).

### 3.1 New formulations

In order to deal with perturbations of $\mathbf{m}_0$ satisfying the saturation constraint (1.2), we use the mobile frame technique introduced in [5].
We consider the direct orthonormal frame $(M_0(x), M_1(x), M_2)$ given by:

$$M_0(x) = \mathbf{m}_0(x), \quad M_1(x) = \begin{pmatrix} -\cos\theta_0(x) \\ \sin\theta_0(x) \\ 0 \end{pmatrix} \quad \text{and} \quad M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix}.$$

While a perturbation $m$ of $\mathbf{m}_0$ satisfies $\|m - \mathbf{m}_0\|_{L^\infty} < \sqrt{2}$, we can describe $m$ in the mobile frame $(M_0(x), M_1(x), M_2)$ writing:

$$m(t,x) = M_0(x) + r_1(t,x)M_1(x) + r_2(t,x)M_2 + \mu_0(r(t,x))M_0(x), \tag{3.18}$$

where $\mu_0(\xi_1,\xi_2) = \sqrt{1-(\xi_1)^2-(\xi_2)^2} - 1$, so that the constraint $|m| = 1$ is automatically fulfilled.
Plugging (3.18) in (1.3), we obtain that $m$ satisfies (1.3) if and only if $(r_1, r_2)$ is solution of

$$\partial_t r = \Lambda r + F(x, r, \partial_x r, \partial_x^2 r), \tag{3.19}$$

where

- $\Lambda r = \begin{pmatrix} -\alpha & -1 \\ 1 & -\alpha \end{pmatrix} \begin{pmatrix} L_1 r_1 \\ L_2 r_2 \end{pmatrix}$,
- $L_1(r_1) = -\partial_x^2 r_1 - \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x r_1 + \dfrac{1}{2}(\sin^2\theta_0 - \cos^2\theta_0) r_1$,
- $L_2(r_2) = -\partial_x^2 r_2 - \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x r_2 + (\dfrac{1}{2}\sin^2\theta_0 - (\theta_0')^2) r_2$,
- the non-linear part $F$ writes

$$F = H_1(r)(\partial_x^2 r) + H_2(x,r)\partial_x r + H_3(r)(\partial_x r, \partial_x r) + H_4(x,r), \tag{3.20}$$

with

$$\begin{aligned}
H_1(r)(\partial_{xx} r) &= \begin{pmatrix} -\alpha r_1^2 & \mu_0 - \alpha r_1 r_2 \\ -\mu_0 - \alpha r_1 r_2 & -\alpha r_2^2 \end{pmatrix} \partial_{xx} r - \begin{pmatrix} r_2 + \alpha(1+\mu_0) r_1 \\ -r_1 + \alpha(1+\mu_0) r_2 \end{pmatrix} \mathrm{d}\mu_0(r)(\partial_{xx} r), \\
H_2(x,r)(\partial_x r) &= \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}} \begin{pmatrix} -\alpha r_1^2 & \mu_0 - \alpha r_1 r_2 \\ -\mu_0 - \alpha r_1 r_2 & -\alpha r_2^2 \end{pmatrix} \partial_x r + 2\theta_0' \begin{pmatrix} -\alpha(1 - r_1^2) \\ 1 + \mu_0 + \alpha r_1 r_2 \end{pmatrix} \mathrm{d}\mu_0(r)(\partial_x r) \\
&\quad - \begin{pmatrix} r_2 + \alpha(1+\mu_0) r_1 \\ -r_1 + \alpha(1+\mu_0) r_2 \end{pmatrix} \left( 2\theta_0' \partial_x r_1 + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}} \mathrm{d}\mu_0(r)(\partial_x r) \right), \\
H_3(r)(\xi_1, \xi_2) &= - \begin{pmatrix} r_2 + \alpha(1+\mu_0) r_1 \\ -r_1 + \alpha(1+\mu_0) r_2 \end{pmatrix} \mathrm{d}^2\mu_0(r)(\partial_x r, \partial_x r), \\
H_4(x,r) &= (\frac{1}{2}\sin^2\theta_0 + \theta_0'^2) \begin{pmatrix} -\alpha r_1^3 \\ -\mu_0 r_1 - \alpha r_1^2 r_2 \end{pmatrix} - \frac{1}{2} r_2 \begin{pmatrix} \mu_0(r) - \alpha r_1 r_2 \\ -\alpha (r_2)^2 \end{pmatrix} \\
&\quad + \left( r_1 \sin\theta_0 \cos\theta_0 + (\theta_0'^2 + \frac{1}{2}\cos^2\theta_0)\mu_0(r) \right) \begin{pmatrix} r_2 + \alpha(1+\mu_0) r_1 \\ -r_1 + \alpha(1+\mu_0) r_2 \end{pmatrix}.
\end{aligned}$$

We endow $L^2(\mathbb{R})$ with the following weighted scalar product:

$$\left\langle u \middle| v \right\rangle_{\bar{\mathbf{a}}} = \int_{\mathbb{R}} \bar{\mathbf{a}}(x) u(x) v(x)\, dx,$$

associated to the norme $\|\cdot\|_{L^2_{\bar{\mathbf{a}}}}$ defined by

$$\|u\|_{L^2_{\bar{\mathbf{a}}}} = \left( \int_{\mathbb{R}} \bar{\mathbf{a}}(x) |u(x)|^2\, dx \right)^{1/2}.$$

**Remark 3.1.** *Since* $\bar{\mathbf{a}} \left( -\partial_x^2 - \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}} \partial_x \right) = -\partial_x (\bar{\mathbf{a}} \partial_x)$, *the operators* $L_1$ *and* $L_2$ *are self-adjoint for the inner product* $\left\langle \cdot \middle| \cdot \right\rangle_{\bar{\mathbf{a}}}$.

As already said in Remark 1.1, Equation (1.7) is invariant by rotation around the wire axis. So for all $\varphi \in \mathbb{R}$, $x \mapsto \mathbf{R}_\varphi \mathbf{m}_0(x)$ is a stationary solution for (1.7). Projecting this solution on the mobile frame $(M_1(x), M_2)$, we define $\rho$ by:

$$\rho(\varphi, x) = \begin{pmatrix} \mathbf{R}_\varphi(M_0(x)) \cdot M_1(x) \\ \\ \mathbf{R}_\varphi(M_0(x)) \cdot M_2 \end{pmatrix} = \begin{pmatrix} \sin\theta_0(x) \cos\theta_0(x) \left(\cos\varphi - 1\right) \\ \\ \cos\theta_0(x) \sin\varphi \end{pmatrix}. \tag{3.21}$$

For all $\varphi \in \mathbb{R}$ small enough, $x \mapsto \rho(\varphi, x)$ is a stationary solution (3.19), that is:

$$\Lambda \rho(\varphi, \cdot) + F(\cdot, \rho(\varphi, \cdot), \partial_x \rho(\varphi, \cdot), \partial_x^2 \rho(\varphi, \cdot)) = 0. \tag{3.22}$$

We remark that

$$\partial_\varphi \rho(0, x) = \begin{pmatrix} 0 \\ \cos\theta_0(x) \end{pmatrix},$$

and by differentiating (3.22) with respect to $\varphi$ at $\varphi = 0$, we obtain that $L_2 \cos\theta_0 = 0$. We decompose $r$ as

$$r(t, x) = \rho(\varphi(t), x) + w(t, x), \tag{3.23}$$

where the second coordinate $w_2$ of $w$ satisfies: $\left\langle w_2 \middle| \cos\theta_0 \right\rangle_{\bar{\mathbf{a}}} = 0$. We remark that for $r(t, \cdot)$ in a neighborhood of 0, this decomposition is unique. Indeed, taking the inner product of $r_2(t, \cdot)$ with $\cos\theta_0$, by the orthogonality condition, we obtain that

$$\left\langle r_2(t, \cdot) \middle| \cos\theta_0 \right\rangle_{\bar{\mathbf{a}}} = \left\langle \rho(\varphi, \cdot) \middle| \cos\theta_0 \right\rangle_{\bar{\mathbf{a}}} = \sin\varphi \int_{\mathbb{R}} \bar{\mathbf{a}}(x) \cos^2\theta_0(x) dx.$$

Thus for $r_2(t, \cdot)$ small enough (for the $L^2_{\bar{\mathbf{a}}}$-norm), $\varphi(t)$ is uniquely defined by

$$\varphi(t) = \arcsin\left( \frac{\left\langle r_2(t, \cdot) \middle| \cos\theta_0 \right\rangle_{\bar{\mathbf{a}}}}{\|\cos\theta_0\|^2_{L^2_{\bar{\mathbf{a}}}}} \right), \tag{3.24}$$

and $w$ is then uniquely defined by subtraction.
Plugging (3.23) in (3.19), using (3.22), we obtain that

$$\varphi'(t) \partial_\varphi \rho(\varphi, x) + \partial_t w = \Lambda w + G(x, \varphi, w, \partial_x w, \partial_x^2 w), \tag{3.25}$$

where $\Lambda$ appears in (3.19) and

$$G = F(x, \rho + w, \partial_x(\rho + w), \partial_x^2(\rho + w)) - F(x, \rho, \partial_x \rho, \partial_x^2 \rho). \tag{3.26}$$

Taking the inner product of the second component of the obtained equation with $\cos\theta_0$, using that $L_2$ is self-adjoint and that $L_2(\cos\theta_0) = 0$, we obtain

$$\varphi' = \Gamma(\varphi, w), \tag{3.27}$$

where

$$\Gamma(\varphi, w) = \frac{1}{\cos\varphi \Big\langle \cos\theta_0 \Big| \cos\theta_0 \Big\rangle_{\bar{\mathbf{a}}}} \left( \Big\langle L_1 w_1 \Big| \cos\theta_0 \Big\rangle_{\bar{\mathbf{a}}} + \Big\langle G_2 \Big| \cos\theta_0 \Big\rangle_{\bar{\mathbf{a}}} \right), \tag{3.28}$$

where $G_2$ is the second component of $G$, and by subtraction, we have:

$$\partial_t w = \Lambda w + G + \widetilde{G}, \tag{3.29}$$

with

$$\widetilde{G} = -\Gamma(\varphi, w)\partial_\varphi \rho(\varphi, w). \tag{3.30}$$

In order to ensure the validity of the coordinates $(\varphi, w)$ and the condition $|r| < 1$, which ensures that (3.19) is equivalent to (1.3), we fix $\nu_0 > 0$ such that while $|\varphi(t)| \le \nu_0$ and $\|w(t)\|_{H^1} \le \nu_0$, then System (3.27)-(3.29) remains equivalent to (1.3).

## 3.2 Estimates on the linear part

### 3.2.1 Study of $L_2$

The operator $L_2$, defined for $v \in H^2(\mathbb{R})$ by

$$L_2(v) = -\partial_x^2 v - \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x v + (\frac{1}{2}\sin^2\theta_0 - (\theta_0')^2)v$$

is self-adjoint for the weighted scalar product $\Big\langle \cdot \Big| \cdot \Big\rangle_{\bar{\mathbf{a}}}$ and non-negative since $L_2 = \ell^* \circ \ell$, with

$$\ell v = \partial_x v + \theta_0' \tan\theta_0 v \quad \text{ and } \quad \ell^* v = -\frac{1}{\bar{\mathbf{a}}}\partial_x(\bar{\mathbf{a}} v) + \theta_0' \tan\theta_0 v.$$

As $x$ goes to $+\infty$, $\bar{\mathbf{a}}' = 0$ and $\frac{1}{2}\sin^2\theta_0 - \theta_0'^2$ tends to $1/2$, so the essential spectrum of $L_2$ is $[1/2, +\infty[$. We remark that $L_2(\cos\theta_0) = 0$. In addition, $\cos\theta_0 \in L^2_{\bar{\mathbf{a}}}(\mathbb{R})$. Indeed, for all $|x| \ge l$, $\theta_0'(x) = \frac{1}{\sqrt{2}}\cos\theta_0(x)$. This implies that

$$\int_{|x|\ge a} \bar{\mathbf{a}}(x)\cos^2\theta_0(x)\,dx = \int_{|x|\ge a} \pi\sqrt{2}\cos\theta_0(x)\theta_0'(x)\,dx = \sqrt{2}\pi\,(2 - \sin\theta_0(a) + \sin\theta_0(-a)) < +\infty.$$

Then $\cos\theta_0 \in H^2(\mathbb{R})$. Since $\ell v = 0$ implies $v = K\cos\theta_0$, all the other eigenvalues of $L_2$ are positive and there exists $c_2 \in \left]0, \frac{1}{2}\right]$ such that

$$\forall v \in (\cos\theta_0)^\perp, \quad c_2\,\|v\|^2_{L^2_{\bar{\mathbf{a}}}} \le \Big\langle L_2 v \Big| v \Big\rangle_{\bar{\mathbf{a}}}. \tag{3.31}$$

By Cauchy-Schwartz inequality, we obtain also that

$$\forall v \in (\cos\theta_0)^\perp, \quad c_2\,\|v\|_{L^2_{\bar{\mathbf{a}}}} \le \|L_2 v\|_{L^2_{\bar{\mathbf{a}}}} \text{ and } c_2 \Big\langle L_2 v \Big| v \Big\rangle_{\bar{\mathbf{a}}} \le \|L_2 v\|^2_{L^2_{\bar{\mathbf{a}}}}. \tag{3.32}$$

### 3.2.2 Study of $L_1$

Let us show thanks to a *reductio ad absurdum* that

$$\exists\, c_1 > 0, \quad \forall u \in H^1(\mathbb{R}), \quad c_1\,\|u\|^2_{L^2_{\bar{\mathbf{a}}}} \le \Big\langle L_1 u \Big| u \Big\rangle_{\bar{\mathbf{a}}}. \tag{3.33}$$

Otherwise there exists a sequence $(u_n)_{n\in\mathbb{N}}$ in $\left(H^1(\mathbb{R})\right)^{\mathbb{N}}$ such that $\|u_n\|_{L^2_{\bar{\mathbf{a}}}} = 1$ et $\Big\langle L_1 u_n \Big| u_n \Big\rangle_{\bar{\mathbf{a}}} < \frac{1}{n+1}$.

We write $u_n$ in the form $u_n = v_n + \delta_n \cos\theta_0$, where $v_n \in (\cos\theta_0)^\perp$. We then have

$$\|v_n\|^2_{L^2_{\bar{\mathbf{a}}}} + \delta_n^2 \|\cos\theta_0\|^2_{L^2_{\bar{\mathbf{a}}}} = 1$$

and

$$\begin{aligned}
\left\langle L_2 u_n + ((\theta_0')^2 - \frac{1}{2}\cos^2\theta_0)u_n \middle| u_n \right\rangle_{\bar{\mathbf{a}}} &= \left\langle L_2 u_n \middle| u_n \right\rangle_{\bar{\mathbf{a}}} + \left\langle ((\theta_0')^2 - \frac{1}{2}\cos^2\theta_0)u_n \middle| u_n \right\rangle_{\bar{\mathbf{a}}} \\
&= \left\langle L_2 v_n \middle| v_n \right\rangle_{\bar{\mathbf{a}}} + \left\langle ((\theta_0')^2 - \cos^2\theta_0)u_n \middle| u_n \right\rangle_{\bar{\mathbf{a}}} \\
&\leq \frac{1}{n+1}
\end{aligned}$$

Since $((\theta_0')^2 - \cos^2\theta_0) \geq 0$ on $\mathbb{R}$ we deduce that $\left\langle L_2 v_n \middle| v_n \right\rangle_{\bar{\mathbf{a}}} \leq \frac{1}{n+1}$, and then $v_n \to 0$ in $H^1(\mathbb{R})$. Up to a subsequence of $(\delta_n)_{n\in\mathbb{N}}$ we can assume that $\delta_n \to \delta$ in $\mathbb{R}$. Since $\left\langle ((\theta_0')^2 - \frac{1}{2}\cos^2\theta_0)u_n \middle| u_n \right\rangle_{\bar{\mathbf{a}}} \leq \frac{1}{n+1}$, we get by taking the limit $\left\langle ((\theta_0')^2 - \frac{1}{2}\cos^2\theta_0)\cos\theta_0 \middle| \cos\theta_0 \right\rangle_{\bar{\mathbf{a}}} \delta^2 = 0$. So $\delta = 0$. Therefore, $u_n \to 0$ in $H^1(\mathbb{R})$ strongly which conflicts with $\|u_n\|_{L^2_{\bar{\mathbf{a}}}} = 1$ for all $n \in \mathbb{N}$. Then (3.33) is fulfilled.

By Cauchy-Schwartz inequality, we obtain from (3.33) that

$$\forall u \in H^1(\mathbb{R}), \quad c_1 \|u\|_{L^2_{\bar{\mathbf{a}}}} \leq \|L_1 u\|_{L^2_{\bar{\mathbf{a}}}} \quad \text{and} \quad c_1 \left\langle L_1 u \middle| u \right\rangle_{\bar{\mathbf{a}}} \leq \|L_1 u\|_{L^2_{\bar{\mathbf{a}}}}. \tag{3.34}$$

### 3.2.3 Equivalence of norms

**Proposition 3.1.** *There exists $K_1 > 0$ and $K_2 > 0$ such that*

$$\forall v \in H^1(\mathbb{R}) \text{ such that } \left\langle v \middle| \cos\theta_0 \right\rangle_{\bar{\mathbf{a}}} = 0, \quad K_1 \|v\|_{H^1} \leq \sqrt{\left\langle L_2 v \middle| v \right\rangle_{\bar{\mathbf{a}}}} \leq K_2 \|v\|_{H^1},$$

$$\forall v \in H^2(\mathbb{R}) \text{ such that } \left\langle v \middle| \cos\theta_0 \right\rangle_{\bar{\mathbf{a}}} = 0, \quad K_1 \|v\|_{H^2} \leq \|L_2 v\|_{L^2_{\bar{\mathbf{a}}}} \leq K_2 \|v\|_{H^2},$$

$$\forall v \in H^1(\mathbb{R}), \quad K_1 \|v\|_{H^1} \leq \sqrt{\left\langle L_1 v \middle| v \right\rangle_{\bar{\mathbf{a}}}} \leq K_2 \|v\|_{H^1},$$

$$\forall v \in H^2(\mathbb{R}), \quad K_1 \|v\|_{H^2} \leq \|L_1 v\|_{L^2_{\bar{\mathbf{a}}}} \leq K_2 \|v\|_{H^2}.$$

*Proof.* Since $\left\langle L_2 v \middle| v \right\rangle_{\bar{\mathbf{a}}} = \|\partial_x v\|^2_{L^2_{\bar{\mathbf{a}}}} + \left\langle (\sin^2\theta_0 - (\theta_0')^2)v \middle| v \right\rangle_{\bar{\mathbf{a}}}$, by Estimate (3.31) we obtain the existence of a constant $C > 0$ such that

$$\|\partial_x v\|_{L^2_{\bar{\mathbf{a}}}} \leq C \left( \left\langle L_2 v \middle| v \right\rangle_{\bar{\mathbf{a}}} \right)^{1/2}.$$

We also have

$$\begin{aligned}
\left\| \partial_x^2 v \right\|_{L^2_{\bar{\mathbf{a}}}} &= \left\| -L_2 v - \tfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}} \partial_x v + (\tfrac{1}{2}\sin^2\theta_0 - (\theta_0')^2)v \right\|_{L^2_{\bar{\mathbf{a}}}} \\
&\leq \|L_2 v\|_{L^2_{\bar{\mathbf{a}}}} + \|\tfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\|_\infty \|\partial_x v\|_{L^2_{\bar{\mathbf{a}}}} + \|\tfrac{1}{2}\sin^2\theta_0 - (\theta_0')^2\|_\infty \|v\|_{L^2_{\bar{\mathbf{a}}}}.
\end{aligned}$$

These two inequalities provide us the two first estimates of the proposition since the domination by the $H^1$ and $H^2$ norms are obvious. We prove the estimates about $L_1$ in the same way, using (3.33). □

## 3.3 Proof of the stability

In order to measure the $H^1$ and the $H^2$ norms of $w$, using Proposition 3.1, we define $\mathcal{N}_1$ and $\mathcal{N}_2$ by:

$$\mathcal{N}_1(w) = \left( \left\langle L_1 w_1 \middle| w_1 \right\rangle_{\bar{\mathbf{a}}} + \left\langle L_2 w_2 \middle| w_2 \right\rangle_{\bar{\mathbf{a}}} \right)^{\frac{1}{2}},$$

$$\mathcal{N}_2(w) = \left( \|L_1 w_1\|^2_{L^2_{\bar{\mathbf{a}}}} + \|L_2 w_2\|^2_{L^2_{\bar{\mathbf{a}}}} \right)^{\frac{1}{2}}.$$

The nonlinear right-hand-side terms in (3.29) and the right-hand-side term in (3.27) are estimated in the following proposition:

**Proposition 3.2.** *There exists $\nu_1 > 0$, with $\nu_1 < \nu_0$, and a constant $K$ such that while $|\varphi(t)| \le \nu_1$ and $\mathcal{N}_1(w) \le \nu_1$, then*

$$\|G\|_{L^2_{\bar{\mathbf{a}}}} \le K \left(|\varphi| + \mathcal{N}_1(w)\right) \mathcal{N}_2(w),$$

$$\left| \left\langle \tilde{G} \middle| \begin{pmatrix} L_1 w_1 \\ L_2 w_2 \end{pmatrix} \right\rangle_{\bar{\mathbf{a}}} \right| \le K|\varphi|\mathcal{N}_1(w)\mathcal{N}_2(w)$$

*and*

$$|\Gamma(\varphi, w)| \le K\mathcal{N}_1(w).$$

For the convenience of the reader, the proof of this proposition is postponed into Section 3.4.

We perform estimates on $w$ by taking the inner product of (3.29) with $\begin{pmatrix} L_1 w_1 \\ L_2 w_2 \end{pmatrix}$. We get:

$$\frac{1}{2}\frac{d}{dt}\left( \left\langle L_1 w_1 \middle| w_1 \right\rangle_{\bar{\mathbf{a}}} + \left\langle L_2 w_2 \middle| w_2 \right\rangle_{\bar{\mathbf{a}}} \right) + \alpha \left( \|L_1 w_1\|^2_{L^2_{\bar{\mathbf{a}}}} + \|L_2 w_2\|^2_{L^2_{\bar{\mathbf{a}}}} \right) =$$

$$\left\langle G_1 + \widetilde{G}_1 \middle| L_1 w_1 \right\rangle_{\bar{\mathbf{a}}} + \left\langle G_2 + \widetilde{G}_2 \middle| L_2 w_2 \right\rangle_{\bar{\mathbf{a}}}.$$

Thanks to Proposition 3.2, while $|\varphi(t)| \le \nu_1$ and $\mathcal{N}_1(w) \le \nu_1$, then

$$\frac{1}{2}\frac{d}{dt}(\mathcal{N}_1(w))^2 + \alpha(\mathcal{N}_2(w))^2 \le 2K\left(|\varphi| + \mathcal{N}_1(w)\right)(\mathcal{N}_2(w))^2,$$

and so:

$$\frac{1}{2}\frac{d}{dt}(\mathcal{N}_1(w))^2 + (\mathcal{N}_2(w))^2\left(\alpha - 2K|\varphi| - K\mathcal{N}_1(w)\right) \le 0.$$

We set

$$\nu_2 = \min\{\nu_1, \frac{\alpha}{8K}\}.$$

While $|\varphi(t)| \le \nu_2$ and $\mathcal{N}_1(w(t)) \le \nu_2$, then

$$\frac{1}{2}\frac{d}{dt}(\mathcal{N}_1(w))^2 + (\mathcal{N}_2(w))^2\frac{\alpha}{2} \le 0,$$

so, denoting $c = \min\{c_1, c_2\}$, using (3.32) and (3.34), we obtain that

$$\frac{1}{2}\frac{d}{dt}(\mathcal{N}_1(w))^2 + \frac{\alpha c}{2}(\mathcal{N}_1(w))^2 \le 0.$$

By comparison argument, we obtain that

$$\text{while } |\varphi(t)| \le \nu_2 \text{ and } \mathcal{N}_1(w(t)) \le \nu_2, \quad \mathcal{N}_1(w(t)) \le \mathcal{N}_1(w(0))e^{-\frac{\alpha c t}{2}}. \tag{3.35}$$

On the other hand, integrating (3.27), using Proposition (3.2) and the previous estimate on $\mathcal{N}_1(w(t))$, we obtain that:

$$\text{while } |\varphi(t)| \le \nu_2 \text{ and } \mathcal{N}_1(w(t)) \le \nu_2, \quad |\varphi(t)| \le |\varphi(0)| + K\frac{2}{\alpha c}\mathcal{N}_1(w(0)). \tag{3.36}$$

We define $\nu_3$ by:

$$\nu_3 = \nu_2 \min\{\frac{1}{4}, \frac{\alpha c}{16K}\}.$$

We assume that $|\varphi(0)| \leq \nu_3$ and $\mathcal{N}_1(w(0)) \leq \nu_3$. Let us prove that for all $t \geq 0$, $|\varphi(t)| < \nu_2$ and $\mathcal{N}_1(w(t)) < \nu_2$. This is true in a neighbourhood of 0 by continuity argument. If it is false at a time $t_1 > 0$, we introduce $t_2$, $0 < t_2 \leq t_1$ the first time in which the property is false. We have then

$$\forall\, t < t_2, \quad |\varphi(t)| < \nu_2 \text{ and } \mathcal{N}_1(w(t)) < \nu_2, \tag{3.37}$$

and

$$|\varphi(t_2)| = \nu_2 \text{ or } \mathcal{N}_1(w(t_2)) = \nu_2. \tag{3.38}$$

By (3.37), (3.35) and (3.36) yield:

$$\mathcal{N}_1(w(t)) \leq (\mathcal{N}_1(w(0))) \leq \nu_3 \leq \frac{\nu_2}{4},$$

and

$$|\varphi(t)| \leq |\varphi(0)| + K\frac{2}{\alpha c}\mathcal{N}_1(w(0)) \leq \nu_3 + K\frac{2}{\alpha c}\nu_3 \leq \frac{\nu_2}{2}.$$

So, by continuity arguments, we have:

$$\mathcal{N}_1(w(t_2)) \leq \frac{\nu_2}{4} \text{ and } |\varphi(t_2)| \leq \frac{\nu_2}{2},$$

which is contradictory with (3.38).

Therefore,

$$\forall\, t \geq 0, \quad |\varphi(t)| < \nu_2 \text{ and } \mathcal{N}_1(w(t)) < \nu_2,$$

so by (3.35):

$$\forall\, t \geq 0, \quad \mathcal{N}_1(w(t)) \leq \mathcal{N}_1(w(0))e^{-\frac{\alpha c t}{2}},$$

*i.e.* $w(t)$ tends to zero in $H^1(\mathbb{R})$ when $t$ tends to $+\infty$. In addition, for all $t \geq 0$,

$$|\varphi'(t)| \leq K\mathcal{N}_1(w(0))e^{-\frac{\alpha c}{2}t},$$

thus $\varphi'$ is integrable on $\mathbb{R}^+$ and $\varphi(t)$ tends to a finite limit $\varphi_\infty$ when $t$ tends to $+\infty$. $\square$

## 3.4 Proof of Proposition 3.2

By (3.21), there exists $C$ such that for all $\phi \in \mathbb{R}$,

$$\|\rho(\phi,\cdot)\|_{W^{2,\infty}(\mathbb{R})} \leq C|\phi|. \tag{3.39}$$

We fix $\nu_1 > 0$ such that for all $\phi \in \mathbb{R}$ and $w = (w_1, w_2) \in H^2(\mathbb{R})$ with $\left\langle w_2 \middle| \cos\theta_0 \right\rangle_{\bar{\mathbf{a}}} = 0$,

$$|\phi| \leq \nu_1 \text{ and } \mathcal{N}_1(w) \leq \nu_1 \Longrightarrow \|\rho(\phi,\cdot) + w(\cdot)\|_{L^\infty} \leq \frac{1}{2} \text{ and } \|\rho(\phi,\cdot) + w(\cdot)\|_{H^1(\mathbb{R})} \leq 1.$$

We assume in addition that $\nu_1 \leq \frac{\pi}{4}$ (so that $\Gamma$ is well defined, see (3.28)). Using that $G$, the right-hand-side nonlinear term in (3.25), is defined by $G = F(x, \rho(\varphi) + w, \partial_x(\rho(\varphi) + w), \partial_x^2(\rho(\varphi) + w)) - F(x, \rho(\varphi), \partial_x\rho(\varphi), \partial_x^2\rho)$, using the Taylor expansion of $F$, we rewrite $G$ as:

$$G = K_1(x,\varphi,w)\partial_x^2 w + K_2(x,\varphi)(\partial_x w, \partial_x w) + K_3(x,\varphi,w)(\partial_x w) + K_4(x,\rho,w),$$

where

$$\begin{aligned}
K_1(x,\varphi,w)\partial_x^2 w = \quad & H_1(\rho(\varphi)+w)(\partial_x^2 w),\\
K_2(x,\varphi)(\partial_x w,\partial_x w) = \quad & H_3(\rho(\varphi)+w)(\partial_x w,\partial_x, w),\\
K_3(x,\varphi,w)(\partial_x w) = \quad & \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}} H_1(\rho(\varphi)+w)(\partial_x w) + H_2(x,\rho(\varphi)+w)(\partial_x w)\\
& +2H_3(\rho(\varphi)+w)(\partial_x\rho(\varphi),\partial_x w),\\
K_4(x,\varphi,w) = \quad & \widetilde{H}_1(\rho(\varphi),w)(w)\left(\partial_x^2\rho(\varphi)+\tfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x\rho(\varphi)\right)+\widetilde{H}_2(\rho(\varphi),w)(w)(\partial_x\rho(\varphi))\\
& +\widetilde{H}_3(\rho(\varphi),w)(w)(\partial_x\rho(\varphi),\partial_x\rho(\varphi))+\widetilde{H}_4(x,\rho(\varphi),w)(w),
\end{aligned}$$

where

- $\widetilde{H}_1(\rho,w) = \displaystyle\int_0^1 \mathrm{d}_r H_1(\rho+sw)\,ds \in \mathcal{L}(\mathbb{R}^2;\mathcal{M}_2(\mathbb{R})),$
- $\widetilde{H}_2(x,\rho,w) = \displaystyle\int_0^1 \mathrm{d}_r H_2(x,\rho+sw)\,ds \in \mathcal{L}(\mathbb{R}^2;\mathcal{M}_2(\mathbb{R})),$
- $\widetilde{H}_3(\rho,w) = \displaystyle\int_0^1 \mathrm{d}_r H_3(\rho+sw)\,ds \in \mathcal{L}(\mathbb{R}^2;(\mathcal{L}_2(\mathbb{R}^2))^2),$
- $\widetilde{H}_4(x,\rho,w) = \displaystyle\int_0^1 \mathrm{d}_r H_4(\rho+sw)\,ds \in \mathcal{L}(\mathbb{R}^2;\mathbb{R}^2).$

(we denote by $\mathrm{d}_r$ the derivative with respect to $r$. For instance, $\mathrm{d}_r H_1(r) \in \mathcal{L}(\mathbb{R}^2;\mathcal{M}_2(\mathbb{R}))$).

Let us estimate the terms $H_1,\dots,H_4$ given in (3.20). We remark that $H_1 \in \mathcal{C}^\infty(B(0,1);\mathcal{M}_2(\mathbb{R}))$, $H_2 \in \mathcal{C}^\infty(\mathbb{R}\times B(0,1);\mathcal{M}_2(\mathbb{R}))$, $H_3 \in \mathcal{C}^\infty\left(B(0,1);\left(\mathcal{L}_2(\mathbb{R}^2)\right)^2\right)$, $H_4 \in \mathcal{C}^\infty\left(\mathbb{R}\times B(0,1);\mathbb{R}^2\right)$, so there exists a constant $C$ such that for all $x\in\mathbb{R}$ and all $r\in B(0,\frac{1}{2})$, we have:

$$\begin{aligned}
&|H_1(r)| \le C|r|^2, \quad |\mathrm{d}_r H_1(r)| \le C|r|,\\
&|H_2(x,r)| \le C|r|, \quad |\mathrm{d}_r H_2(x,r)| \le C,\\
&|H_3(r)(\xi_1,\xi_2)| \le C|r||\xi_1||\xi_2| \text{ and } |\mathrm{d}_r H_3(r)(\xi)(\xi_1,\xi_2)| \le C|\xi||\xi_1||\xi_2|,\\
&|H_4(x,r)| \le C|r|^2 \text{ and } |\mathrm{d}_r H_4(x,r)| \le C|r|
\end{aligned} \tag{3.40}$$

Under the assumptions $|\varphi|\le\nu_1$ and $\mathcal{N}_1(w)\le\nu_1$, using (3.39), (3.40), we obtain that there exists a constant $C$ such that

$$\begin{aligned}
&\|K_1(\cdot,\varphi,w)\|_{L^\infty} \le C\|\rho(\varphi)+w\|_{L^\infty} \le C(|\varphi|+\mathcal{N}_1(w)),\\
&\|K_2(\cdot,\varphi)\|_{L^\infty} \le C,\\
&\|K_3(\cdot,\varphi,w)\|_{L^\infty} \le C\|\rho(\varphi)+w\|_{L^\infty} \le C(|\varphi|+\mathcal{N}_1(w)).
\end{aligned}$$

Therefore, we obtain that

$$\left\|K_1(\cdot,\varphi,w)\partial_x^2 w\right\|_{L^2_{\bar{\mathbf{a}}}} \leq \|K_1(\cdot,\varphi,w)\|_{L^\infty} \left\|\partial_x^2 w\right\|_{L^2_{\bar{\mathbf{a}}}} \leq C(|\varphi|+\mathcal{N}_1(w))\mathcal{N}_2(w),$$

$$\|K_2(\cdot,\varphi)(\partial_x w,\partial_x w)\|_{L^2_{\bar{\mathbf{a}}}} \leq \pi\|K_2(\cdot,\varphi)\|_{L^\infty}\|\partial_x w\|^2_{L^4(\mathbb{R})} \leq C\|w\|_{L^\infty(\mathbb{R})}\|\partial_x^2 w\|_{L^2(\mathbb{R})} \leq C\mathcal{N}_1(w)\mathcal{N}_2(w)$$

(by Gagliardo-Niremberg inequality),

$$\|K_3(\cdot,\varphi,w)(\partial_x w)\|_{L^2_{\bar{\mathbf{a}}}} \leq \|K_3(\cdot,\varphi,w)\|_{L^\infty}\|\partial_x w\|_{L^2_{\bar{\mathbf{a}}}} \leq C(|\varphi|+\mathcal{N}_1(w))\mathcal{N}_2(w).$$

In addition, using (3.39), we have

$$\|K_4(\cdot,\varphi,w)\|_{L^2_{\bar{\mathbf{a}}}} \leq C\|w\|_{L^2_{\bar{\mathbf{a}}}}|\varphi| \leq C|\varphi|\mathcal{N}_2(w).$$

Therefore, there exists a constant $C$ such that if $|\varphi| \leq \nu_1$ and $\mathcal{N}_1(w) \leq \nu_1$, then

$$\|G\|_{L^2_{\bar{\mathbf{a}}}} \leq C(|\varphi|+\mathcal{N}_1(w))\mathcal{N}_2(w). \tag{3.41}$$

From (3.28), we have, since $|\varphi| \leq \frac{\pi}{4}$:

$$|\Gamma(\varphi,w)| \leq C\left(\left|\left\langle L_1 w\middle|\cos\theta_0\right\rangle_{\bar{\mathbf{a}}}\right| + \left|\left\langle G\middle|\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix}\right\rangle_{\bar{\mathbf{a}}}\right|\right).$$

We have:

$$\left|\left\langle L_1 w\middle|\cos\theta_0\right\rangle_{\bar{\mathbf{a}}}\right| = |\left\langle w\middle|L_1\cos\theta_0\right\rangle_{\bar{\mathbf{a}}}| \leq C\mathcal{N}_1(w).$$

In addition,

$$\begin{aligned}
\left\langle K_1(x,\varphi,w)\partial_x^2 w\middle|\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix}\right\rangle_{\bar{\mathbf{a}}} &= \int_{\mathbb{R}} \bar{\mathbf{a}}(x)H_1(\rho(\varphi)+w)\partial_x^2 w\cdot\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix},\\
&= \int_{\mathbb{R}} \bar{\mathbf{a}}\partial_x^2 w\cdot {}^tK_1(\varphi,w)\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix},\\
&= -\int_{\mathbb{R}} \partial_x w\cdot\partial_x\left(\bar{\mathbf{a}}{}^tH_1(\rho(\varphi)+w)\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix}\right),\\
&= -\int_{\mathbb{R}} \partial_x w\cdot{}^tH_1(\rho(\varphi)+w)\begin{pmatrix}0\\ \partial_x(\bar{\mathbf{a}}\cos\theta_0)\end{pmatrix}\\
&\quad -\int_{\mathbb{R}} \partial_x w\cdot{}^t(\mathrm{d}_r H_1(\rho(\varphi)+w)(\partial_x\rho(\varphi)+\partial_x w))\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix}.
\end{aligned}$$

By the estimates on $H_1$ (see (3.40)), we obtain that if $|\varphi| \leq \nu_1$ and $\mathcal{N}_1(w) \leq \nu_1$, then

$$|\left\langle K_1(\varphi,w)\partial_x^2 w\middle|\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix}\right\rangle_{\bar{\mathbf{a}}} \leq C\mathcal{N}_1(w).$$

Furthermore, assuming that $|\varphi| \leq \nu_1$ and $\mathcal{N}_1(w) \leq \nu_1$, then

$$\begin{aligned}
\left|\left\langle K_2(\varphi)(\partial_x w,\partial_x w)\middle|\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix}\right\rangle_{\bar{\mathbf{a}}}\right| &\leq \|H_3(\rho(\varphi)+w)\|_{L^\infty}\|\partial_x w\|^2_{L^2_{\bar{\mathbf{a}}}}\\
&\leq C\mathcal{N}_1(w),
\end{aligned}$$

and

$$\left|\left\langle K_3(\cdot,\varphi,w)(\partial_x w)+K_4(\cdot,\varphi,w)\middle|\begin{pmatrix}0\\ \cos\theta_0\end{pmatrix}\right\rangle_{\bar{\mathbf{a}}}\right| \leq C\left(\|\partial_x w\|_{L^2_{\bar{\mathbf{a}}}}+\|w\|_{L^2_{\bar{\mathbf{a}}}}\right)\|\cos\theta_0\|_{L^2_{\bar{\mathbf{a}}}} \leq C\mathcal{N}_1(w).$$

Therefore, there exists a constant $C$ such that if $|\varphi| \leq \nu_1$ and $\mathcal{N}_1(w) \leq \nu_1$, then

$$|\Gamma(\varphi, w)| \leq C\mathcal{N}_1(w). \tag{3.42}$$

We have:

$$\partial_\varphi \rho(\phi, \cdot) = \begin{pmatrix} -\sin\varphi \sin\theta_0 \cos\theta_0 \\ \\ \cos\varphi \cos\theta_0 \end{pmatrix}.$$

On the one hand:

$$\left| \left\langle -\sin\varphi \sin\theta_0 \cos\theta_0 \middle| L_1 w_1 \right\rangle_{\bar{\mathbf{a}}} \right| \leq C|\varphi|\mathcal{N}_2(w),$$

on the other hand:

$$\left\langle \cos\varphi \cos\theta_0 \middle| L_2 w_2 \right\rangle_{\bar{\mathbf{a}}} = \cos\varphi \left\langle L_2 \cos\theta_0 \middle| w_2 \right\rangle_{\bar{\mathbf{a}}} = 0.$$

Therefore, using (3.42), we obtain that

$$\left| \left\langle \tilde{G} \middle| \begin{pmatrix} L_1 w_1 \\ L_2 w_2 \end{pmatrix} \right\rangle_{\bar{\mathbf{a}}} \right| \leq C\mathcal{N}_1(w)|\varphi|\mathcal{N}_2(w).$$

This concludes the proof of Proposition 3.2. □

# 4 Existence and stability of stationary profile under an applied magnetic field

In the presence of an applied magnetic field in the form $H_a = he_1$, the magnetization fulfills

$$\partial_t m = m \times (\mathcal{H}_e(m) + he_1) - m \times (m \times (\mathcal{H}_e(m) + he_1)). \tag{4.43}$$

Looking for a stationary solution $\mathbf{m}_h$ of the form $x \mapsto \begin{pmatrix} \sin\theta_h(x) \\ \cos\theta_h(x) \\ 0 \end{pmatrix}$, the equation for $\theta_h$ writes:

$$\theta_h'' + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\theta_h' + \frac{1}{2}\sin\theta_h \cos\theta_h + h\cos\theta_h = 0 \tag{4.44}$$

Let $\theta_0 \in \mathcal{C}^2(\mathbb{R})$ be the solution to (2.14) given by Proposition 2.1. We look for $\theta_h$ on the form $\theta_h = \theta_0 + g_h$ with $g_h \in H^2(\mathbb{R})$. Then

$$g_h'' + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}g_h' + \frac{1}{2}\sin(\theta_0 + g_h)\cos(\theta_0 + g_h) + h\cos(\theta_0 + g_h) + \theta_0'' + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\theta_0' = 0.$$

We define $\Psi \ : \ \mathbb{R} \times H^2(\mathbb{R}) \to L^2(\mathbb{R})$ by

$$\Psi(h, g) = g'' + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}g' + \frac{1}{2}\sin(\theta_0 + g)\cos(\theta_0 + g) + h\cos(\theta_0 + g) + \theta_0'' + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\theta_0'.$$

We then have that $\Psi(0, 0) = 0$ since $\theta_0$ is solution to (2.14) and

$$D_g\Psi(0, 0)(u) = u'' + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}u' + \frac{1}{2}\left(\cos^2\theta_0 - \sin^2\theta_0\right)u = -L_1 u.$$

Since $L_1$ is coercive on $H^2(\mathbb{R})$ we can apply the implicit function theorem and we obtain the existence of $h_0 > 0$ and a function $v \ : \ ]-h_0, h_0[ \to H^2(\mathbb{R})$ such that for all $h \in ]-h_0, h_0[$ $\Psi(h, v(h)) = 0$. Moreover for all $h \in ]-h_0, h_0[$ we classically have that $v(h) \in \mathcal{C}^2(\mathbb{R})$ as solution of a regular ordinary equation and then $\theta_h = \theta_0 + v(h) \in \mathcal{C}^2(\mathbb{R})$ satisfies (4.44).

We aim to study the Lyapunov stability of the constructed solution. We prove the stability of $\mathbf{m}_h = \begin{pmatrix} \sin\theta_h \\ \cos\theta_h \\ 0 \end{pmatrix}$ using the same moving-frame method as in Section 3. We introduce $M_1^h$ and $M_2$ given by:

$$M_1^h(x) = \begin{pmatrix} -\cos\theta_h(x) \\ \sin\theta_h(x) \\ 0 \end{pmatrix} \text{ and } M_2 = \begin{pmatrix} 0 \\ 0 \\ 1 \end{pmatrix},$$

and we write a perturbation $m$ of $\mathbf{m}_h$ as:

$$m(t,x) = r_1(t,x)M_1^h(x) + r_2(t,x)M_2 + (1+\mu_0(r(t,x))\mathbf{m}_h(x).$$

In this case, the equivalent formulation of (4.43) in the moving frame rewrites:

$$\partial_t r = \Lambda_h r + F^h(x, r, \partial_x r, \partial_x^2 r) + hM^h(x,r), \tag{4.45}$$

where

- $\Lambda_h r = \begin{pmatrix} -1 & -1 \\ 1 & -1 \end{pmatrix} \begin{pmatrix} L_1^h r_1 \\ L_2^h r_2 \end{pmatrix}$, with

$$L_1^h(r_1) = -\partial_x^2 r_1 - \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x r_1 + \frac{1}{2}(\sin^2\theta_h - \cos^2\theta_h)r_1 + h\sin\theta_h r_1,$$

$$L_2^h(r_2) = -\partial_x^2 r_2 - \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\partial_x r_2 + (\frac{1}{2}\sin^2\theta_h - (\theta_h')^2)r_2 + h\sin\theta_h r_2,$$

- the non-linear part $F^h$ as the same form as $F$ (see (3.20)) replacing $\theta_0$ by $\theta_h$,
- $M^h(x,r) = \begin{pmatrix} \cos\theta_h\left(\mu_0 + r_1^2 + r_1 r_2\right) - \mu_0 r_1 \sin\theta_h \\ \\ -\cos\theta_h\left(\mu_0 + \mu_0^2 + r_1^2 - r_1 r_2\right) + \mu_0 r_2 \sin\theta_h \end{pmatrix}.$

As in Section 3.1, in order to take into account the invariance of the Landau-Lifschitz equation by translation in the variable $x$, we split $r$ into:

$$r(t,x) = \rho_h(\varphi(t), x) + w(t,x), \tag{4.46}$$

where $\rho_h(\varphi,\cdot)$ is the projection of $x \mapsto \mathbf{R}_\varphi \mathbf{m}_h(x)$ on the mobile frame:

$$\rho_h(\varphi, x) = \begin{pmatrix} \mathbf{R}_\varphi(\mathbf{m}_h(x)) \cdot M_1^h \\ \\ \mathbf{R}_\varphi(\mathbf{m}_h(x)) \cdot M_2 \end{pmatrix} = \begin{pmatrix} \sin\theta_h(x)\cos\theta_h(x)\left(\cos\varphi - 1\right) \\ \\ \cos\theta_h(x)\sin\varphi \end{pmatrix}, \tag{4.47}$$

and where the second coordinate of $w$ satisfies the orthogonality condition:

$$\left\langle w_2(t,\cdot) \middle| \cos\theta_h \right\rangle_{\bar{\mathbf{a}}} = 0.$$

As in Section 3.1, we obtain then an equivalent system for the new unknown $(\varphi, w)$ on the form:

$$\begin{cases} \varphi' = \Gamma_h(\varphi, w), \\ \\ \partial_t w = \Lambda_h w + G_h + \tilde{G}_h, \end{cases} \tag{4.48}$$

where $\Gamma_h$, $G_h$ and $\tilde{G}_h$ satisfy the same properties as $\Gamma$, $G$ and $\tilde{G}$ in Section 3.4. The key point is now to study the coercivity of the linear operators $L_1^h$ and $L_2^h$.

Concerning $L_2^h$, as in Section 3.2.1, we prove that we can factorize it as $L_2^h = \ell_h^* \circ \ell_h$, with

$$\ell_h v = \partial_x v + \theta_h' \tan\theta_h v \quad \text{and} \quad \ell_h^* v = -\frac{1}{\bar{\mathbf{a}}}\partial_x(\bar{\mathbf{a}} v) + \theta_h' \tan\theta_h v.$$

and we obtain that the kernel of $L_2^h$ is one-dimensional and is generated by $\cos\theta_h$. We assume that $|h| < \frac{1}{2}$. As $x$ goes to $\pm\infty$, $\bar{\mathbf{a}}' = 0$ and $\dfrac{1}{2}\sin^2\theta_h - (\theta_h')^2 + h\sin\theta_h$ tends to $1/2 \pm h$ so the essential spectrum of $L_2^h$ is $[1/2 - |h|, +\infty[$. The others eigenvalues of $L_2^h$ are positive, so there exists a constant $c_2^h \in \left]0, \frac{1}{2} - |h|\right]$ such that for all $u \in (\cos\theta_h)^\perp$,

$$c_2^h \|u\|_{L^2_{\bar{\mathbf{a}}}}^2 \leq \left\langle L_2^h u \middle| u \right\rangle_{\bar{\mathbf{a}}}.$$

In order to prove the coercivity of $L_1^h$, we write:

$$L_1^h = L_1 + \phi_1^h(x), \quad \text{with } \phi_1^h(x) = \frac{1}{2}(\sin^2\theta_h(x) - \sin^2\theta_0(x)) - \frac{1}{2}(\cos^2\theta_h(x) - \cos^2\theta_0(x)) + h\sin\theta_h.$$

Since $h \mapsto \theta_h$ is continuous with values in $H^2(\mathbb{R})$, when $h$ tends to 0, $\phi_1^h$ tends to zero in $L^\infty(\mathbb{R})$. So the coercivity inequality (3.33) yields that for $h$ small enough: for all $u \in H^1(\mathbb{R})$,

$$\frac{c_1}{2} \|u\|_{L^2_{\bar{\mathbf{a}}}}^2 \leq \left\langle L_1^h u \middle| u \right\rangle_{\bar{\mathbf{a}}}.$$

Once this coercivity established, the stability proof for System (4.48) is the same as for System (3.27)-(3.29).

# 5 Non-existence of stationary profiles with a large magnetic field

**Proposition 5.1.** *There exists $h_0 \in ]0, \frac{1}{2}[$ such that for all $h \in \mathbb{R}$ fulfilling $|h| \geq h_0$ there does not exist stationary profiles to* (4.43) *with a magnetization switching, i.e. such that*

$$\begin{cases} \theta'' + \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\theta' + \dfrac{1}{2}\sin\theta\cos\theta + h\cos\theta = 0 \quad \text{on } \mathbb{R}, \\ \lim\limits_{x\to-\infty} \theta(x) = -\dfrac{\pi}{2}, \\ \lim\limits_{x\to+\infty} \theta(x) = \dfrac{\pi}{2}, \\ \theta' \geq 0 \text{ on } \mathbb{R}. \end{cases}$$

*Proof.* Let us assume that there exists a stationary solution $\theta$. We assume first that $h > 0$. From (4.43) we obtain as in Prop. 2.1 the energy equation:

$$\partial_x \mathcal{E} = -\frac{2\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}(\theta')^2, \tag{5.49}$$

where $\mathcal{E} = (\theta')^2 + \dfrac{1}{2}(\sin\theta + 2h)^2$. On $[-l_0, 0]$, $\bar{\mathbf{a}}' \leq 0$, so

$$-\frac{2\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}(\theta')^2 \leq -\frac{2\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\mathcal{E}.$$

Therefore we have:

$$\partial_x \mathcal{E} \leq -\frac{2\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\mathcal{E} \text{ on } [-l_0, 0],$$

and multiplying by $\bar{\mathbf{a}}^2$, we obtain that:

$$\partial_x(\bar{\mathbf{a}}^2\mathcal{E}) \leq 0 \text{ on } [-l_0, 0].$$

Therefore,

$$(\bar{\mathbf{a}}(0))^2\mathcal{E}(0) \le \pi^2\mathcal{E}(-l_0).$$

In addition, from (5.49), $\mathcal{E}$ is non increasing on $[0, l_0]$ since $\bar{\mathbf{a}}' \ge 0$ on this interval. Therefore,

$$\mathcal{E}(l_0) \le \mathcal{E}(0) \le \frac{\pi^2}{(\bar{\mathbf{a}}(0))^2}\mathcal{E}(-l_0).$$

Now, on $[l_0, +\infty[$, $\mathcal{E}$ is constant and since $\theta(x)$ tends to $\frac{\pi}{2}$ when $x$ tends to $+\infty$, this constant equals $\frac{1}{2}(1+2h)^2$. In the same way, on $]-\infty, -l_0]$, $\mathcal{E} = \frac{1}{2}(1-2h)^2$. Therefore, we obtain:

$$(1+2h)^2 \le \frac{\pi^2}{(\bar{\mathbf{a}}(0))^2}(1-2h)^2,$$

so

$$h \le \frac{1}{2}\frac{\pi - \bar{\mathbf{a}}(0)}{\pi + \bar{\mathbf{a}}(0)}.$$

Let us assume now that $h < 0$. We set $\tau(x) = -\theta(-x)$. Then, since $\bar{\mathbf{a}}$ is even, $\tau$ satisfies:

$$\begin{cases} \tau'' + \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}}\tau' + \dfrac{1}{2}\sin\tau\cos\tau - h\cos\tau = 0 \ \text{ on } \mathbb{R}, \\ \lim\limits_{x\to-\infty} \tau(x) = -\dfrac{\pi}{2}, \\ \lim\limits_{x\to+\infty} \tau(x) = \dfrac{\pi}{2}, \\ \tau' \ge 0 \text{ on } \mathbb{R}. \end{cases}$$

Since $-h \ge 0$, we can apply the first case and we obtain that

$$-h \le \frac{1}{2}\frac{\pi - \bar{\mathbf{a}}(0)}{\pi + \bar{\mathbf{a}}(0)}.$$

This conclude the proof of Proposition 5.1 we setting $h_0 = \dfrac{1}{2}\dfrac{\pi - \bar{\mathbf{a}}(0)}{\pi + \bar{\mathbf{a}}(0)}$.

□

# 6 Finite wire with multiple notches

In this section, we consider a wire of length $NL$ with $N-1$ notches. The area of the cross section is described by $x \mapsto \mathbf{a}(x)$ given by (1.12). The magnetization in this wire is modeled by $m : \mathbb{R}^+_t \times [0, NL] \longrightarrow S^2 \subset \mathbb{R}^3$. We assume first that the applied field vanishes so that we consider the system:

$$\begin{cases} \dfrac{\partial m}{\partial t} = -m \times \mathcal{H}_e(m) - \alpha m \times (m \times \mathcal{H}_e(m)) \text{ in } \mathbb{R}^+ \times [0, NL], \\ \\ \mathcal{H}_e(m) = \partial_{xx}m + \dfrac{\mathbf{a}'}{\mathbf{a}}\partial_x m - \dfrac{1}{2}\left(m_2e_2 + m_3e_3\right), \\ \\ \partial_x m(t, 0) = \partial_x m(t, NL) = 0. \end{cases} \tag{6.50}$$

For $u \in H^2([0, NL]; \mathbb{R})$, we denote $F(u) = u'' + \frac{\mathbf{a}'}{\mathbf{a}}u' + \frac{1}{2}\sin u\cos u$, so that $m : [0, NL] \longrightarrow S^2$ of the form $x \mapsto \begin{pmatrix} \sin\theta(x) \\ \cos\theta(x) \\ 0 \end{pmatrix}$ is a stationary solution for (6.50) if and only if

$$\begin{cases} F(\theta) = 0, \\ \\ \theta'(0) = \theta'(NL) = 0. \end{cases} \tag{6.51}$$

A datum $D \in \{0,1\}^N$ being given, we look for a stationary solution for (6.50) encoding $D$ on the form:

$$\mathbf{m}(x) = \begin{pmatrix} \sin\theta(x) \\ \cos\theta(x) \\ 0 \end{pmatrix},$$

so we look for $\theta : [0, NL] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}] \subset \mathbb{R}$ satisfying (6.51) so that $\mathbf{m}$ satisfies (6.50), and such that for all $k \in \{1, \dots, N\}$, if $D(k) = 0$ (resp. $D(k) = 1$), then for all $x \in [(k-1)L + l_1, kL - l_1]$, $\theta(x) < -\frac{\pi}{6}$ (resp. $\theta(x) > \frac{\pi}{6}$), so that $\mathbf{m}$ encodes $D$.

The scheme of the proof is the following: first we construct an approximate solution $\Theta^L_{app} : [0, NL] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$, with $F(\Theta^L_{app})$ close to zero when $L$ is large enough. Then we look for $\theta$ writing $\theta = \Theta^L_{app} + v$, so that, writing the Taylor expansion of $F$ around $\Theta^L_{app}$, we look for $v$ satisfying:

$$0 = F(\Theta^L_{app} + v) = F(\Theta^L_{app}) + \partial_v F(\Theta^L_{app})(v) + C(\Theta^L_{app}, v)v^2$$

where

$$\begin{aligned} \partial_v F(\Theta^L_{app})(v) &= -v'' - \frac{\mathbf{a}'}{\mathbf{a}} v' - \frac{1}{2}(\cos^2 \Theta^L_{app} - \sin^2 \Theta^L_{app}) v \\ C(\Theta^L_{app}, v) &= \int_0^1 (1-s) \sin(2(\Theta^L_{app} + sv))\, ds. \end{aligned}$$

The key point is now to prove that $\partial_v F(\Theta^L_{app})$ is invertible if $L$ is large enough (see Lemma 6.3 in Section 6.2). Then $\theta = \Theta^L_{app} + v$ is solution if and only if $v$ fulfills

$$v = \left[\partial_v F(\Theta^L_{app})\right]^{-1} \left(-F(\Theta^L_{app}) - C(\Theta^L_{app}, v)v^2\right) := \Phi_L(v).$$

The existence of $v$ satisfying the previous equation is established by proving that $\Phi_L$ admits a fixed point.

## 6.1 Construction of an approximate solution

We assume that $L \geq 3\max\{l_0, l_1\}$. Let $\theta_0$ be the solution obtained in the infinite-wire case in Section 2. Let $\psi : \mathbb{R} \longrightarrow [0,1]$ be a smooth non decreasing map such that $\psi(x) = 0$ for $x \leq \frac{1}{3}$ and $\psi(x) = 1$ for $x \geq \frac{1}{2}$. We define $J_L : [-\frac{L}{2}, \frac{L}{2}] \longrightarrow [-\frac{\pi}{2}, \frac{\pi}{2}]$ such that

- $J_L$ is smooth and odd,
- $J_L(x) = (1 - \psi(\frac{x}{L}))\theta_0(x) + \psi(\frac{x}{L})\frac{\pi}{2}$,

so that $J_L(x) = \theta_0(x)$ for $x \in [-\frac{L}{3}, \frac{L}{3}]$ and realizes on $[\frac{L}{3}, \frac{L}{2}]$ (resp. $[-\frac{L}{2}, -\frac{L}{3}]$) a smooth junction between $\theta_0(\frac{L}{3})$ and $\frac{\pi}{2}$ (resp. $-\frac{\pi}{2}$ and $\theta_0(-\frac{L}{3})$) .
For $u : \mathbb{R} \longrightarrow \mathbb{R}$, we denote $\bar{F}(u) = u'' + \frac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}} u' + \frac{1}{2}\sin u \cos u$. We claim the following lemma:

**Lemma 6.1.** *There exists a constant $C$ such that for all $L$ satisfying $L \geq 3\max\{l_0, l_1\}$,*

$$\|\bar{F}(J_L)\|_{L^2([-\frac{L}{2}, \frac{L}{2}])} \leq C e^{-\frac{L}{3\sqrt{2}}}.$$

*In addition,*

$$\forall\, x \leq -l_1, \quad \sin J_L(x) < -\frac{3}{4} \quad \textit{and} \quad \forall\, x \geq l_1, \quad \sin J_L(x) > \frac{3}{4}.$$

*Proof.* For $x \in [-\frac{L}{3}, \frac{L}{3}]$, $J_L(x) = \theta_0(x)$ so $\bar{F}(J_L)(x) = 0$.

For $x \geq l_0$, $\bar{\mathbf{a}}'(x) = 0$. So on $[l_0, +\infty[$, $\theta_0$ satisfies $\theta_0' = \frac{1}{\sqrt{2}} \cos\theta_0$ and by solving the pendulum equation, there exists $x_0$ such that:

$$\forall x \geq l_0, \quad \theta_0(x) = \arcsin \tanh\left(\frac{1}{\sqrt{2}}(x - x_0)\right).$$

Then when $x$ tends to $+\infty$,

$$\theta_0(x) = \frac{\pi}{2} + \mathcal{O}(e^{-\frac{x}{\sqrt{2}}}), \quad \theta_0'(x) = \cos\theta_0(x) = \mathcal{O}(e^{-\frac{x}{\sqrt{2}}}), \text{ and } \theta_0''(x) = -\frac{1}{2}\cos\theta_0\sin\theta_0 = \mathcal{O}(e^{-\frac{x}{\sqrt{2}}}).$$

Therefore there exists a constant $C$ such that for all $L \geq 3l_0$, for all $x \in [\frac{L}{3}, \frac{L}{2}]$,

$$|\theta_0(x) - \frac{\pi}{2}| + |\theta_0'(x)| + |\theta_0''(x)| \leq Ce^{-\frac{L}{3\sqrt{2}}}. \tag{6.52}$$

Now, we have:

$$\begin{aligned}
J_L(x) &= \psi(\tfrac{x}{L})\tfrac{\pi}{2} + (1 - \psi(\tfrac{x}{L}))\theta_0(x), \\
J_L'(x) &= \tfrac{1}{L}\psi'(\tfrac{x}{L})(\tfrac{\pi}{2} - \theta_0(x)) + (1 - \psi(\tfrac{x}{L}))\theta_0'(x), \\
J_L''(x) &= \tfrac{1}{L^2}\psi''(\tfrac{x}{L})(\tfrac{\pi}{2} - \theta_0(x)) - \tfrac{1}{L}\psi'(\tfrac{x}{L})\theta_0'(x) + (1 - \psi(\tfrac{x}{L}))\theta_0''(x).
\end{aligned}$$

So using (6.52), there exists a constant $C$ such that for $x \geq l_0$, ,

$$\left|J_L(x) - \frac{\pi}{2}\right| \leq Ce^{-\frac{x}{\sqrt{2}}} \text{ and } \left|J_L''(x)\right| \leq Ce^{-\frac{x}{\sqrt{2}}},$$

and thus

$$\left|\bar{F}(J_L(x))\right| \leq Ce^{-\frac{x}{\sqrt{2}}}.$$

Therefore, for $L$ such that $\frac{L}{3} \geq l_0$, we have:

$$\|\bar{F}(J_L(x))\|^2_{L^2([0,\frac{L}{2}])} \leq C\int_{\frac{L}{3}}^{\frac{L}{2}} e^{-\frac{2x}{\sqrt{2}}}dx \leq Ce^{-\frac{2L}{3\sqrt{2}}}.$$

By oddness arguments, we obtain the same estimate for $x \in [-\frac{L}{2}, -\frac{L}{3}]$. Therefore, there exists a constant $C$ such that for all $L \geq 3\max\{l_0, l_1\}$,

$$\|\bar{F}(J_L(x))\|_{L^2([-\frac{L}{2},\frac{L}{2}])} \leq Ce^{-\frac{L}{3\sqrt{2}}}.$$

Moreover, if $x \geq l_1$, $0 < \theta_0(x) \leq J_L(x) \leq \frac{\pi}{2}$ and since $\sin\theta_0(x) \geq \frac{3}{4}$ (see (1.10)), then $\sin J_L(x) > \frac{3}{4}$. In the same way, if $x \leq -l_1$, $\sin J_L(x) < -\frac{3}{4}$.

□

The data $D \in \{0,1\}^N$ being given, we define $\Theta^L_{app}$ as follows:

- for $x$ in the left boundary cell $[0, \frac{L}{2}]$, if $D(1) = 0$ (resp. $D(1) = 1$), then $\Theta^L_{app}(x) = -\frac{\pi}{2}$ (resp. $\Theta^L_{app}(x) = \frac{\pi}{2}$),
- for $k \in \{1, \dots, N-1\}$ such that $D(k) = D(k+1) = 0$ (resp. $D(k) = D(k+1) = 1$), then for $x$ in the cell $[kL - \frac{L}{2}, kL + \frac{L}{2}]$ around the $k$-th notch, $\Theta^L_{app}(x) = -\frac{\pi}{2}$ (resp. $\Theta^L_{app}(x) = \frac{\pi}{2}$),
- for $k \in \{1, \dots, N-1\}$ such that $D(k) = 0$ and $D(k+1) = 1$ (resp. $D(k) = 1$ and $D(k+1) = 0$), then for $x \in [kL - \frac{L}{2}, kL + \frac{L}{2}]$, $\Theta^L_{app}(x) = J_L(x - kL)$ (resp. $\Theta^L_{app}(x) = -J_L(x - kL)$), where $J_L$ is defined above.
- for $x$ in the right boundary cell $[NL - \frac{L}{2}, NL]$, if $D(N) = 0$ (resp. $D(N) = 1$), then $\Theta^L_{app}(x) = -\frac{\pi}{2}$ (resp. $\Theta^L_{app}(x) = \frac{\pi}{2}$).

**Remark 6.1.** *For the sake of simplicity we assume that the wire is finite and that all the notches are regularly spaced. The construction of the approximate solution could be adapted to the case of different space lengths between consecutive notches, or by adding a semi-finite wire at one end of the wire.*

**Lemma 6.2.** *There exists a constant $K_1$ such that for all $L \geq 3l_0$,*

$$\|F(\Theta^L_{app})\|_{L^2([0,NL])} \leq K_1 \sqrt{N}\, e^{-\frac{L}{3\sqrt{2}}}.$$

*In addition, for all $k \in [1, N]$, if $D(k) = 0$ (resp. $D(k) = 1$), then for all $x \in [(k-1)L + l_1, kL - l_1]$, $\sin \Theta^L_{app} \leq -\frac{3}{4}$ (resp. $\sin \Theta^L_{app} \geq \frac{3}{4}$).*

*Proof.* For $x \in [0, L/2] \cup [NL - L/2, NL]$, $\Theta^L_{app}(x) = \pm\frac{\pi}{2}$, so $F(\Theta^L_{app})(x) = 0$.

For all $k \in \{1, \dots, N-1\}$, either $\Theta^L_{app}(x) = \pm\frac{\pi}{2}$ for all $x \in [kL - \frac{L}{2}, kL + \frac{L}{2}]$, or $\Theta^L_{app}(x) = \pm J_L(x - kL)$. In the first case, $F(\Theta^L_{app}) = 0$ on $[kL - \frac{L}{2}, kL + \frac{L}{2}]$. In the second case,

$$\forall\, x \in [kL - \frac{L}{2}, kL + \frac{L}{2}], \quad F(\Theta^L_{app}(x)) = \pm \bar{F}(J_L(x - kL)).$$

By applying Lemma 6.1, we obtain that

$$\|F(\Theta^L_{app})\|^2_{L^2([kL - \frac{L}{2}, kL + \frac{L}{2}])} \leq C e^{-\frac{2L}{3\sqrt{2}}}.$$

Therefore

$$\|F(\Theta^L_{app})\|^2_{L^2([0,NL])} \leq CN e^{-\frac{2L}{3\sqrt{2}}},$$

and denoting $K_1 = \sqrt{C}$, we have:

$$\|F(\Theta^L_{app})\|_{L^2([0,NL])} \leq K_1 \sqrt{N} e^{-\frac{L}{3\sqrt{2}}}.$$

Let $k \in \{1, \dots N\}$. If $D(k) = 0$, then on $[(k-1)L, (k-\frac{1}{2})L[$ either $\Theta^L_{app}(x) = -\frac{\pi}{2}$ or $\Theta^L_{app}(x) = -J_L(x - (k-1)L$. In both cases, for $x \in [(k-1)L + l_1, (k-\frac{1}{2})L[$, $\sin \Theta^L_{app} \leq \frac{3}{4}$ by Lemma 6.1. On $[(k-\frac{1}{2})L, kL]$, either $\Theta^L_{app}(x) = -\frac{\pi}{2}$ or $\Theta^L_{app}(x) = J_L(x - kL)$. In both cases, for $x \in [(k-\frac{1}{2})L, kL - l_1]$, $\sin \Theta^L_{app} \leq \frac{3}{4}$. We adress the case $D(k) = 1$ with the same argument, which concludes the proof of Lemma 6.2.

□

## 6.2 Existence of stationary profiles

We endow $L^2([0, NL])$ with the weighted inner product:

$$\Big\langle u \Big| v \Big\rangle_{\mathbf{a}} = \int_{[0,NL]} \mathbf{a}(s) u(s) v(s) ds,$$

and we denote by $\|\cdot\|_{L^2_{\mathbf{a}}}$ the associated norm.

We let $F(u) = -u'' - \frac{\mathbf{a}'}{\mathbf{a}} u' - \frac{1}{2} \sin u \cos u$. We aim to prove the existence of $\theta \in H^2([0, NL])$ satisfying (6.51). We look for $\theta$ as a perturbation of the approximate solution. We denote by $\mathbf{V}$ the space:

$$\mathbf{V} = \{v \in H^2([0, NL]),\ \partial_x v(0) = \partial_x v(NL) = 0\}.$$

We let $\theta = \Theta^L_{app} + v$, with $v \in \mathbf{V}$. Then we have

$$0 = F(\Theta^L_{app} + v) = F(\Theta^L_{app}) + \partial_v F(\Theta^L_{app})(v) + C(\Theta^L_{app}, v) v^2,$$

where

$$\begin{aligned}
\partial_v F(\Theta^L_{app})(v) &= -v'' - \frac{\mathbf{a}'}{\mathbf{a}} v' - \frac{1}{2}(\cos^2 \Theta^L_{app} - \sin^2 \Theta^L_{app}) v, \\
C(\Theta^L_{app}, v) &= \int_0^1 (1-s) \sin(2(\Theta^L_{app} + sv))\, ds.
\end{aligned}$$

Let us study the operator $\partial_v F(\Theta^L_{app})$, defined on the domain $\mathbf{V}$. This operator is self-adjoint for the weighted inner product $\Big\langle \cdot \Big| \cdot \Big\rangle_{\mathbf{a}}$. We claim the following:

**Lemma 6.3.** *If $L$ is large enough, $\partial_v F(\Theta^L_{app}) \; : \; \mathbf{V} \to L^2([0,NL])$ is invertible and there exists $(d,d_1,d_2) \in (\mathbb{R}^{+*})^3$ such that for all $v \in \mathbf{V}$*

$$\left\langle \partial_v F(\Theta^L_{app})(v) \middle| v \right\rangle_{\mathbf{a}} \geq d \, \|v\|^2_{L^2_{\bar{\mathbf{a}}}}, \quad \left\| \partial_v F(\Theta^L_{app})(v) \right\|_{L^2_{\bar{\mathbf{a}}}} \geq d \, \|v\|_{L^2_{\bar{\mathbf{a}}}},$$

$$d_1 \|v\|_{H^2} \leq \left\| \partial_v F(\Theta^L_{app})(v) \right\|_{L^2_{\bar{\mathbf{a}}}} \leq d_2 \|v\|_{H^2}.$$

*Proof.* We write $\partial_v F(\Theta^L_{app})(v) = -v'' - \frac{\mathbf{a}'}{\mathbf{a}} v' + f(x) v$ with $f(x) = \dfrac{1}{2}\left(\sin^2 \Theta^L_{app} - \cos^2 \Theta^L_{app}\right)$.
When $\Theta^L_{app} = \pm\frac{\pi}{2}$, we have

$$f(x) = 1/2.$$

In a junction $[kL - L/3, kL + L/3]$ occurring around $kL$ we have

$$f(x) = \frac{1}{2}\left(\sin^2 \theta_0(x - kL) - \cos^2 \theta_0(x - kL)\right).$$

We remind (see (3.33)) that $L_1 v = -v'' - \dfrac{\bar{\mathbf{a}}'}{\bar{\mathbf{a}}} v' + \dfrac{1}{2}(\sin^2\theta_0 - \cos^2\theta_0) v$ fulfills $\left\langle L_1 v \middle| v \right\rangle_{\bar{\mathbf{a}}} \geq c_1 \|v\|^2_{L^2_{\bar{\mathbf{a}}}}$ with $c_1 > 0$.

To study the behavior of the linearized part we use the IMS formula to highlight the behavior in each junction. We let $\mu \in \mathcal{C}^\infty(\mathbb{R})$ such that for all $x \in \mathbb{R}$, $0 \leq \mu(x) \leq 1$, $\mu(x) = 1$ if $x \in [-1/6, 1/6]$ and $\mu(x) = 0$ if $|x| \geq 1/3$. Let $\nu_0 = \sqrt{1 - \mu^2}$.
We now define $K = \{i \in \{1, \ldots, N-1\}, D(i) \neq D(i+1)\}$ the set of indexes where are located the junctions.
For every $k \in K$, we set $\chi_k(x) = \mu\left(\frac{x - kL}{L}\right)$ and

$$\chi_0(x) = \begin{cases} 1 & \text{if } x \notin \bigcup_{k \in K} [kL - L/2, kL + L/2] \\ \nu_0\left(\frac{x - kL}{L}\right) & \text{if } x \in [kL - L/2, kL + L/2] \text{ and } k \in K. \end{cases}$$

We then have $\displaystyle\sum_{k \in K \cup \{0\}} \chi_k^2 = 1$ on $[0, NL]$, and so:

$$\left\langle \partial_v F(\Theta^L_{app})(v) \middle| v \right\rangle_{\mathbf{a}} = \left\langle \partial_v F(\Theta^L_{app})(v) \middle| \sum_{k \in K \cup \{0\}} \chi_k^2 v \right\rangle_{\bar{\mathbf{a}}} = \sum_{k \in K \cup \{0\}} \left\langle \partial_v F(\Theta^L_{app})(v) \middle| \chi_k^2 v \right\rangle_{\bar{\mathbf{a}}}$$

For all $k \in K \cup \{0\}$,

$$\begin{aligned} \left\langle \partial_v F(\Theta^L_{app})(v) \middle| \chi_k^2 v \right\rangle_{\mathbf{a}} &= \int_0^{NL} (-\mathbf{a} v'' - \mathbf{a}' v' + \mathbf{a} f v) \chi_k^2 v \, dx \\ &= \left\langle \partial_x F(\Theta^L_{app})(\chi_k v) \middle| \chi_k v \right\rangle_{\mathbf{a}} + \int_0^{NL} (\chi_k' v)(\chi_k v) \mathbf{a}' \, dx \\ &\quad + \int_0^{NL} (2\chi_k' v' + \chi_k'' v)(\chi_k v) \mathbf{a} \, dx. \end{aligned}$$

Since $\sum_{k \in K \cup \{0\}} \chi_k^2 = 1$ we obtain $\sum_{k \in K \cup \{0\}} \chi_k \chi_k' = 0$ and

$$\left\langle \partial_v F(\Theta^L_{app})(v) \middle| v \right\rangle_{\mathbf{a}} = \sum_{k \in K \cup \{0\}} \left( \left\langle \partial_x F(\Theta^L_{app})(\chi_k v) \middle| \chi_k v \right\rangle_{\mathbf{a}} + \int_0^{NL} (\chi_k'' v)(\chi_k v) \mathbf{a} \, dx \right).$$

Moreover

$$\begin{aligned} |\chi_k \chi_k''| &\leq \frac{1}{L^2} \|\mu''\|_\infty \mathbf{1}_{[kL - L/3, kL - L/6] \cup [kL + L/6, kL + L/3]} \quad \forall k \in K, \\ |\chi_0 \chi_0''| &\leq \frac{1}{L^2} \|\nu_0''\|_\infty \sum_{k \in K} \mathbf{1}_{[kL - L/3, kL - L/6] \cup [kL + L/6, kL + L/3]}. \end{aligned}$$

So

$$\left|\sum_{k\in K\cup\{0\}}\int_0^{NL}(\chi_k''v)(\chi_k v)\mathbf{a}\,dx\right| \le \frac{1}{L^2}\left(\|\mu''\|_\infty+\|\nu_0''\|_\infty\right)\int_0^{NL}\mathbf{a}v^2\,dx.$$

Since $|\Theta_{app}^L|\in[\theta_0(L/6),\pi/2]$ for all $x\in\mathrm{supp}\chi_0$ we have

$$f(x)=\frac{1}{2}\left(\sin^2(\Theta_{app}^L)-\cos^2(\Theta_{app}^L)\right)=-\frac{1}{2}\cos(2\Theta_{app}^L)\ge-\frac{1}{2}\cos(2\theta_0(L/6)),$$

and

$$\begin{aligned}\left\langle \partial_x F(\Theta_{app}^L)(\chi_0 v)\middle|\chi_0 v\right\rangle_{\mathbf{a}} &= \int_0^{NL}(-\mathbf{a}(\chi_0 v)''-\mathbf{a}'(\chi_0 v)'+\mathbf{a}f(x)(\chi_0 v))\chi_0 v\,dx,\\ &= \int_0^{NL}\mathbf{a}(\chi_0 v')^2+\mathbf{a}f(x)(\chi_0 v)^2\,dx,\\ &\ge \int_0^{NL}\mathbf{a}f(x)(\chi_0 v)^2\,dx\ge-\frac{1}{2}\cos 2\theta_0\left(\frac{L}{6}\right)\|\chi_0 v\|^2_{L^2_{\mathbf{a}}}.\end{aligned}$$

Thanks to Prop. 3.1, for all $k\in K$

$$\left\langle \partial_x F(\Theta_{app}^L)(\chi_k v)\middle|\chi_k v\right\rangle_{\mathbf{a}}\ge c_1\|\chi_k v\|^2_{L^2_{\mathbf{a}}}.$$

Since $\theta_0\to+\dfrac{\pi}{2}$ as $x\to+\infty$. If $L$ is large enough, $-\dfrac{1}{2}\cos 2\theta_0(L/6)\ge c_1$ and

$$\left\langle \partial_v F(\Theta_{app}^L)(v)\middle|v\right\rangle_{\mathbf{a}}\ge c_1\sum_{k\in K\cup\{0\}}\|\chi_k v\|^2_{L^2_{\mathbf{a}}}-\frac{C}{L^2}\|v\|^2_{L^2_{\mathbf{a}}}\ge\left(c_1-\frac{C}{L^2}\right)\|v\|^2_{L^2_{\mathbf{a}}}.$$

If $L$ is large enough then

$$\left\langle \partial_v F(\Theta_{app}^L)(v)\middle|v\right\rangle_{\mathbf{a}}\ge\frac{c_1}{2}\|v\|^2_{L^2_{\mathbf{a}}}.$$

The end of the proof follows the proof of Prop. 3.1. □

We aim to obtain uniform estimates with respect to $L$. We have the following lemma:

**Lemma 6.4.** *For all $L\ge 1$, for all $N$,*

$$\forall u\in H^1([0,NL]),\quad \|u\|_{L^\infty}([0,NL])\le\|u\|_{L^2([0,NL])}+\|u'\|_{L^2([0,NL])}.$$

*Proof.* For $u\in\mathcal{C}^1([0,NL])$, for all $x\in[0,1]$, for all $y\in[0,NL]$, we have:

$$(u(y))^2=(u(x))^2+2\int_x^y u(s)u'(s)ds\le(u(x))^2+2\|u\|_{L^2([0,NL])}\|u'\|_{L^2([0,NL)}.$$

We integrate this estimate for $x\in[0,1]\subset[0,NL]$ (since $L\ge 1$). We obtain that for all $y$,

$$\begin{aligned}(u(y))^2\le&\quad\int_0^1(u(x))^2dx+2\|u\|_{L^2([0,NL])}\|u'\|_{L^2([0,NL)}\\ \le&\quad\|u\|^2_{L^2([0,NL])}+2\|u\|_{L^2([0,NL])}\|u'\|_{L^2([0,NL)}\\ \le&\quad\left(\|u\|_{L^2([0,NL])}+\|u'\|_{L^2([0,NL)}\right)^2.\end{aligned}$$

So, we obtain that for $u\in\mathcal{C}^1([0,NL])$,

$$\|u\|_{L^\infty([0,NL])}\le\|u\|_{L^2([0,NL])}+\|u'\|_{L^2([0,NL)}.$$

We conclude the proof of Lemma 6.4 by density argument.

□

Since we established that $\partial_v F(\Theta^L_{app}) : \mathbf{V} \longrightarrow L^2([0,NL])$ is invertible, $\theta = \Theta^L_{app} + v$ is solution if and only if $v$ fulfills

$$v = \left[\partial_v F(\Theta^L_{app})\right]^{-1} \left(-F(\Theta^L_{app}) - C(\Theta^L_{app}, v)v^2\right) = \Phi_L(v).$$

To prove the existence of $v$ we use a fixed point method in the space $H^1([0,NL])$:

**Lemma 6.5.** *For all $L$, the operator $\Phi_L$ is well defined from $H^1([0,NL])$ into $H^1([0,NL])$. If $L$ is large enough, there exists $\eta_L$ such that $\Phi_L\left(B_{H^1}(0,\eta_L)\right) \subset B_{H^1}(0,\eta_L)$ and for all $(v,w) \in \left(B_{H^1}(0,\eta_L)\right)^2$, $\Phi_L$ is $\frac{1}{2}$-Lipschitz.*

*Proof.* Since $\|v^2\|_{L^2([0,NL])} \leq \|v\|_{L^2([0,NL])}\|v\|_{L^\infty([0,NL])} \leq \|v\|^2_{H^1}$ by Lemma 6.4, we have

$$\| - F(\Theta^L_{app}) - C(\Theta^L_{app}, v)v^2\|_2 \leq \|F(\Theta^L_{app})\|_2 + \|v^2\|_{L^2([0,NL])} \leq K_1\sqrt{N}e^{-\frac{L}{3\sqrt{2}}} + \|v\|^2_{H^1([0,NL])}.$$

Thanks to Lemma 6.3 we have

$$\begin{aligned}
\|\Phi_L(v)\|_{H^1([0,NL])} &\leq \|\Phi_L(v)\|_{H^2([0,NL])} \leq \frac{1}{d_1}\| - F(\Theta^L_{app}) - C(\Theta^L_{app}, v)v^2\|_{L^2([0,NL])} \\
&\leq \frac{1}{d_1}\left(K_1\sqrt{N}e^{-\frac{L}{3\sqrt{2}}} + \|v\|^2_{H^1([0,NL])}\right).
\end{aligned}$$

Moreover, for all $(v,w) \in \left(H^1([0,NL])\right)^2$,

$$\Phi_L(v) - \Phi_L(w) = [\partial_v F(\Theta^L_{app})]^{-1}\left(C(\Theta^L_{app}, w)w^2 - C(\Theta^L_{app}, v)v^2\right),$$

$$\begin{aligned}
|C(\Theta^L_{app}, w) - C(\Theta^L_{app}, v)| &= \left|\int_0^1 (1-s)\left(\sin(2(\Theta^L_{app} + sw)) - \sin(2(\Theta^L_{app} + sv))\right) ds\right| \\
&\leq 2\int_0^1 (1-s)s\, ds|w-v| = \frac{1}{3}|w-v|,
\end{aligned}$$

and

$$\begin{aligned}
\|\Phi_L(v) \; - \; \Phi_L(w)\|_{H^1([0,NL])} &= \|[\partial_v F(\Theta^L_{app})]^{-1}\left(C(\Theta^L_{app}, w)w^2 - C(\Theta^L_{app}, v)v^2\right)\|_{H^1([0,NL])}, \\
&\leq \frac{1}{d_1}\|(C(\Theta^L_{app}, w) - C(\Theta^L_{app}, v))w^2\|_{L^2([0,NL])} + \frac{1}{d_1}\|C(\Theta^L_{app}, v)(w^2 - v^2)\|_{L^2([0,NL])}, \\
&\leq \frac{1}{3d_1}\|w\|^2_{L^\infty([0,NL])}\|w-v\|_{L^2([0,NL])} \\
&\quad + \frac{1}{d_1}\|C(\Theta^L_{app}, v)\|_{L^\infty([0,NL])}\|v+w\|_{L^\infty([0,NL])}\|w-v\|_{L^2([0,NL])}, \\
&\leq \frac{1}{d_1}\left(\frac{2}{3}\|w\|^2_{L^\infty([0,NL])} + \|w\|_{L^\infty([0,NL])} + \|v\|_{L^\infty([0,NL])}\right)\|w-v\|_{H^1([0,NL])}, \\
&\leq \frac{1}{d_1}\left(\frac{2}{3}\|w\|^2_{H^1} + \|w\|_{H^1} + \|v\|_{H^1}\right)\|w-v\|_{H^1}.
\end{aligned}$$

We are now looking for $\eta > 0$ such that if $\|v\|_{H^1([0,NL])} \leq \eta$ then

$$\|\Phi_L(v)\|_{H^1([0,NL])} \leq \frac{1}{d_1}\left(K_1\sqrt{N}e^{-\frac{L}{3\sqrt{2}}} + \eta^2\right) \leq \eta.$$

We fix $\eta_L$ by:

$$\eta_L = 2\frac{K_1}{d_1}\sqrt{N}e^{-\frac{L}{3\sqrt{2}}}. \tag{6.53}$$

We have:

$$\eta_L^2 - d_1\eta_L + K_1\sqrt{N}e^{-\frac{L}{3\sqrt{2}}} = \left(4(K_1/d_1)^2 N e^{-\frac{L}{3\sqrt{2}}} - K_1\sqrt{N}\right)e^{-\frac{L}{3\sqrt{2}}} \leq 0 \text{ for } L \text{ great enough.}$$

So, for $L$ great enough, $\Phi_L$ maps the ball $B_{H^1}(0,\eta_L)$ onto itself. In addition,

$$\begin{aligned}\|\Phi_L(v)-\Phi_L(w)\|_{H^1([0,NL])} &\leq \frac{1}{d_1}\left(\frac{2}{3}\eta_L^2 + 2\eta_L\right)\|w-v\|_{H^1([0,NL])}\\ &\leq \frac{1}{2}\|v-w\|_{H^1([0,NL])} \text{ if } L \text{ is large enough.}\end{aligned}$$

□

**Proposition 6.1.** *Let $N \in \mathbb{N}^*$ and $\mathbf{a} = \pi\rho^2$ as described at the beginning of Section 6. Let $D \in \{0,1\}^N$. There exists $L_0 \geq 3\max\{l_0, l_1\}$ such that for all $L \geq L_0$ there exists $\theta \in H^2([0,NL])$ which fulfills*

- $\theta'' + \frac{\mathbf{a}'}{\mathbf{a}}\theta' + \frac{1}{2}\sin\theta\cos\theta = 0$ *on* $[0,NL]$,
- $\theta'(0) = \theta'(NL) = 0$,
- $\|\theta - \Theta_{app}^L\|_{H^1([0,NL])} \leq \eta_L$ *where $\Theta_{app}^L$ is the approximate solution encoding $D$ defined in section 6.1 and $\eta_L$ is given by (6.53).*
- $\mathbf{m} := \begin{pmatrix}\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$ *encodes the data $D$.*

*Proof.* We can now use a fixed point theorem on $\Phi_L$ and we deduce the existence of $v \in H^1([0,NL]) \cap \mathbf{V}$ such that $F(\Theta_{app}^L + v) = 0$ hence an exact solution in the form $\theta = \Theta_{app}^L + v$ to (2.14). Moreover

$$\|\theta - \Theta_{app}^L\|_{L^\infty([0,NL])} \leq \|\theta - \Theta_{app}^L\|_{H^1([0,NL])} \leq \eta_L.$$

We assume that $\eta_L < \frac{1}{8}$ (true if $L$ is large enough). We fix $k \in \{1,\dots,N\}$. Let us suppose that $D(k) = 0$. For all $x \in [(k-1)N + l_1, kL - l_1]$, using Lemma 6.2,

$$\sin\theta(x) \leq \sin\Theta_{app}^L(x) + \eta_L \leq -\frac{3}{4} + \eta_L \leq -\frac{5}{8} < -\frac{1}{2}. \tag{6.54}$$

In the same way, if $D(k) = 1$, for all $x \in [(k-1)N + l_1, kL - l_1]$, using Lemma 6.2,

$$\sin\theta(x) \geq \sin\Theta_{app}^L(x) - \eta_L \geq \frac{3}{4} - \eta_L \geq \frac{5}{8} > \frac{1}{2}.$$

Therefore, the map $\mathbf{m} : x \mapsto \begin{pmatrix}\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$ encodes the data $D$. This concludes the proof of Proposition 6.1.

□

## 6.3 Stability

For $L$ large enough, we define $\mathbf{m} = \begin{pmatrix}\sin\theta\\ \cos\theta\\ 0\end{pmatrix}$ as in Proposition (6.1). We prove the asymptotic stability modulo rotation of $\mathbf{m}$ for the Landau-Lifschitz model (6.50) using the same method as in Section 3: we introduce the mobile frame $(\mathbf{m}(x), \mathbf{m}_1(x), \mathbf{m}_2)$ with

$$\mathbf{m}_1(x) = \begin{pmatrix}-\cos\theta(x)\\ \sin\theta(x)\\ 0\end{pmatrix} \text{ and } \mathbf{m}_2 = \begin{pmatrix}0\\0\\1\end{pmatrix}.$$

We describe the perturbations $m$ of $\mathbf{m}$ in the mobile frame writing:

$$m(t,x) = r_1(t,x)\mathbf{m}_1(x) + r_2(t,x)\mathbf{m}_2 + (1+\mu_0(r(t,x)))\mathbf{m}(x), \quad \text{with } (1+\mu_0(\xi))^2 = 1-(\xi_1)^2-(\xi_2)^2,$$

and plugging this expression for $m$ in (6.50), by projection on the mobile frame, we obtain an equivalent formulation for the unknown $r=(r_1,r_2)$ on the form:

$$\partial_t r = \Lambda r + F(x, r, \partial_x r, \partial_x^2 r), \tag{6.55}$$

where $\Lambda$ and $F$ are defined as in the inifinite case by (3.19) and (3.20), replacing $\theta_0$ by $\theta$ and $\bar{\mathbf{a}}$ by $\mathbf{a}$.

Equation (6.55) is stated in the finite domain $[0, NL]$ and the Neumann homogeneous boundary conditions of (6.50) yield that:

$$\partial_x r(t,0) = \partial_x r(t,NL) = 0. \tag{6.56}$$

In order to take into account the invariance by rotation of the system, we split $r$ as a rotation of angle $\varphi(t)$ of $\mathbf{m}$ plus a term $w$ such that $\left\langle w_2 \middle| \cos\theta \right\rangle_{\mathbf{a}} = 0$, and we obtain an equivalent system of the form:

$$\begin{cases} \varphi'(t) = \Gamma(\varphi(t), w(t)) \text{ for } t \in \mathbb{R}^+, \\ \\ \partial_t w = \Lambda w + G + \widetilde{G} \text{ on } \mathbb{R}_t^+ \times [0, NL], \\ \\ \partial_x w(t,0) = \partial_x w(t,NL) = 0 \text{ for } t \in \mathbb{R}^+, \end{cases}$$

where $G$, $\Gamma$ and $\widetilde{G}$ are defined as in (3.26), (3.28) and (3.30), replacing $\theta_0$ by $\theta$ and $\bar{\mathbf{a}}$ by $\mathbf{a}$.

Now the only difference lies in the proof of Proposition 3.1 to establish the coercivity of the linear part.

Concerning the operator $L_2$, we remark that it is self-adjoint for the $L^2_{\mathbf{a}}([0,NL])$-inner product, its resolvent is compact. In addition, we have

$$L_2(r) = -\partial_x^2 r - \frac{\mathbf{a}'}{\mathbf{a}}\partial_x r + \left(\frac{1}{2}\sin^2\theta - (\theta')^2\right) r = \frac{1}{\mathbf{a}}\ell^* \circ (\mathbf{a}\ell),$$

with $\ell v = \partial_x v + \theta' \tan\theta v$. This induces that $L_2$ is positive. Its kernel is one-dimensional, and is generated by $\cos\theta$. Since the other eigenvalues of $L_2$ are non negative, there exists $c>0$ such that:

$$\forall\, v \in H^2([0,NL]) \cap (\cos\theta)^\perp,\ \left\langle L_2 v \middle| v \right\rangle_{\bar{\mathbf{a}}} \geq c\|v\|^2_{L^2_{\bar{\mathbf{a}}}([0,NL])}.$$

Concerning $L_1$, we have

$$L_1(r) = -\partial_x^2 r - \frac{\mathbf{a}'}{\mathbf{a}}\partial_x r + \frac{1}{2}\left(\sin^2\theta - \cos^2\theta\right) r = \partial_v F(\Theta^L_{app})(r) + \frac{1}{2}\left(\cos(2\Theta^L_{app}) - \cos(2\theta)\right) r.$$

Since $\|\theta - \Theta^L_{app}\|_\infty \leq \eta$ we have

$$\begin{aligned} \left\langle L_1(r) \middle| r \right\rangle_{\mathbf{a}} &= \left\langle \partial_v F(\Theta^L_{app}) r \middle| r \right\rangle_{\mathbf{a}} + \tfrac{1}{2}\left\langle \left(\cos(2\Theta^L_{app}) - \cos(2\theta)\right) r \middle| r \right\rangle_{\bar{\mathbf{a}}} \\ &\geq \frac{c_1}{2}\|r\|^2_{L^2_{\mathbf{a}}} - \frac{\eta}{2}\|r\|^2_{L^2_{\bar{\mathbf{a}}}} \geq \left(\frac{c_1}{2} - \frac{\eta}{2}\right)\|r\|^2_{L^2_{\mathbf{a}}} \end{aligned}$$

which proves the property if $\eta$ is small enough.

Once the coercivity established for $\Lambda$, we conclude the proof of Theorem 1.4 as in Section 3.

# 7 Proof of Theorem 1.5

We consider $\theta : [0,NL] \longrightarrow \mathbb{R}$ such that $\mathbf{m} = \begin{pmatrix} \sin\theta \\ \cos\theta \\ 0 \end{pmatrix}$ encodes the data $D \in \{0,1\}^N$. For $h_a$ small enough, we look for a static solution of (1.13) of the form

$$\mathbf{m}_{h_a} = \begin{pmatrix} \sin\theta_{h_a} \\ \cos\theta_{h_a} \\ 0 \end{pmatrix},$$

where $\theta_{h_a} : [0, NL] \longrightarrow \mathbb{R}$ satisfies

$$\begin{cases} \theta''_{h_a} + \frac{\mathbf{a}'}{\mathbf{a}} \theta'_{h_a} + \frac{1}{2} \sin \theta_{h_a} \cos \theta_{h_a} + h_a \cos \theta_{h_a} = 0 \text{ on } [0, NL], \\ \theta'_{h_a}(0) = \theta'_{h_a}(NL) = 0. \end{cases} \tag{7.57}$$

As is Section 4, we construct a solution of (7.57) by using the implicit function theorem on the map $\Psi : \mathbb{R} \times \mathbf{V} \longrightarrow L^2([0, NL])$ given by:

$$\Psi(h, v) = v'' + \frac{\mathbf{a}'}{\mathbf{a}} v' + \frac{1}{2} \sin v \cos v + h \cos v.$$

We have $\Psi(0, \theta) = 0$ and:

$$D_v \Psi(0, \theta)(u) = u'' + \frac{\mathbf{a}'}{\mathbf{a}} u' + \frac{1}{2}(\cos^2 \theta - \sin^2 \theta) u = -L_1 u.$$

We have proved above that $L_1$ is coercive on $\mathbf{V}$, so that we can apply the implicit function theorem. By continuity argument, for $h_a$ small enough,

$$\| \sin \theta_{h_a} - \sin \theta \|_{L^\infty} \leq \frac{1}{8}.$$

So, using (6.54), we obtain that if $D(k) = 0$, then $\sin \theta_{h_a} \leq -\frac{1}{2}$ on $[(k-1)L + l_1, kL - l_1]$. With the same arguments, we prove that if $D(k) = 1$, then $\sin \theta_{h_a} \geq \frac{1}{2}$ on $[(k-1)L + l_1, kL - l_1]$. So, $\mathbf{m}_{h_a}$ encodes the data $D$ for $h_a$ small enough.

We obtain the asymptotic stability modulo rotations of the solutions with the same arguments as in Section 4. □

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