



HAL
open science

Mixed Hodge structures and representations of fundamental groups of algebraic varieties

Louis-Clément Lefèvre

► **To cite this version:**

Louis-Clément Lefèvre. Mixed Hodge structures and representations of fundamental groups of algebraic varieties. *Advances in Mathematics*, 2019, 349, pp.869-910. 10.1016/j.aim.2019.04.028. hal-01809625

HAL Id: hal-01809625

<https://hal.science/hal-01809625v1>

Submitted on 6 Jun 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

MIXED HODGE STRUCTURES AND REPRESENTATIONS OF FUNDAMENTAL GROUPS OF ALGEBRAIC VARIETIES

LOUIS-CLÉMENT LEFÈVRE

ABSTRACT. Given a complex variety X , a linear algebraic group G and a representation ρ of the fundamental group $\pi_1(X, x)$ into G , we develop a framework for constructing a functorial mixed Hodge structure on the formal local ring of the representation variety of $\pi_1(X, x)$ into G at ρ using mixed Hodge diagrams and methods of L_∞ algebras. We apply it in two geometric situations: either when X is compact Kähler and ρ is the monodromy of a variation of Hodge structure, or when X is smooth quasi-projective and ρ has finite image.

CONTENTS

1. Introduction	1
2. Deformation functor of a DG Lie algebra	4
3. The theory of Goldman and Millson	6
4. L_∞ algebras and pro-representability	8
5. L_∞ algebra structure on the mapping cone	11
6. Mixed Hodge structures	13
7. Mixed Hodge complexes	14
8. Mixed Hodge diagrams of L_∞ algebras	16
9. Bar construction on mixed Hodge diagrams of L_∞ algebras	19
10. Construction of mixed Hodge diagrams: the compact case	23
11. Construction of mixed Hodge diagrams: the non-compact case	26
12. Description of the mixed Hodge structure	31
References	33

1. INTRODUCTION

Classical Hodge theory provides the cohomology groups of compact Kähler manifolds with additional structure. This structure has been vastly used as a tool to study and restrict the possible topology of these manifolds. It was generalized by Deligne [Del71] into the notion of mixed Hodge structure for complex algebraic varieties. Since then, mixed Hodge structures have been constructed on many other topological invariants. In particular Morgan [Mor78] and Hain [Hai87] constructed mixed Hodge structures on the rational homotopy groups of such varieties.

In this vein, in this article we construct mixed Hodge structures on invariants associated to linear representations of the fundamental group. Let X be either a compact Kähler manifold or a smooth complex quasi-projective algebraic variety, with a base point x . In both cases the fundamental group $\pi_1(X, x)$ is finitely presentable. Let G be a linear algebraic group over some field $\mathbf{k} \subset \mathbb{R}$ or \mathbb{C} . The set

of group morphisms from $\pi_1(X, x)$ into $G(\mathbf{k})$ is the set of points of an affine scheme of finite type over \mathbf{k} that we denote by $\mathrm{Hom}(\pi_1(X, x), G)$ and call the *representation variety*. Given a representation ρ of $\pi_1(X, x)$, seen as a \mathbf{k} -point of the representation variety, we denote by $\widehat{\mathcal{O}}_\rho$ the formal local ring at ρ . This ring pro-represents the functor of deformations of ρ and its structure tells about the topology of X .

The study of $\widehat{\mathcal{O}}_\rho$ was done first by Goldman and Millson [GM88] when X is a compact Kähler manifold: they show that it has a quadratic presentation. This was later reviewed by Eyssidieux-Simpson [ES11] who constructed a functorial mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ when ρ is the monodromy of a variation of Hodge structure. In case where X is non-compact and ρ has finite image, Kapovich and Millson [KM98] constructed only quasi-homogeneous gradings on $\widehat{\mathcal{O}}_\rho$. They apply it to describe explicitly a class of finitely presented groups that cannot be fundamental groups of smooth algebraic varieties. We improve their construction to a mixed Hodge structure:

Theorem. *There is a mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ in the following situations:*

- (1) *Either X is compact Kähler and ρ is the monodromy of a polarized variation of Hodge structure.*
- (2) *Either X is smooth quasi-projective and ρ has finite image.*

This mixed Hodge structure is defined over the field \mathbf{k} if ρ is (this includes the case $\mathbf{k} = \mathbb{C}$, where one works with complex mixed Hodge structures). It is functorial in X, x, ρ . The weight zero part is the formal local ring defining the orbit by conjugation of ρ inside $\mathrm{Hom}(\pi_1(X, x), G)$.

By the way we present a framework which unifies the previous constructions. Namely in their work, Eyssidieux-Simpson make strong use of the special properties of analysis on compact Kähler manifolds and nothing can be directly generalized if X is not compact. On the other hand, Kapovich-Millson make strong use of the theory of minimal models of Morgan and they only get a grading and not a mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$; furthermore this construction is not functorial. We propose here a different approach in which in all cases the mixed Hodge structure comes directly from the H^0 of an appropriate mixed Hodge complex whose construction depends on the geometric situation, but the functoriality and the dependence on the base point is very explicit. Let us explain.

The theory of Goldman-Millson was one of the starting point of the use of differential graded (DG) Lie algebras in deformation theory. Let L be the DG Lie algebra of differential forms over X with values in the adjoint bundle of the flat principal G -bundle induced by the monodromy of ρ . To L one can associate a deformation functor Def_L which is related to the functor of formal deformations of ρ . We review this in sections 2 and 3. The classical theorems known to Goldman and Millson are that Def_L is invariant under quasi-isomorphisms on L and is pro-representable if $H^0(L) = 0$. In our compact case, L has the special property of being formal (i.e. quasi-isomorphic to its cohomology) and the deformation functor simplifies much.

Since then the theory of formal deformations via DG Lie algebras has received many contributions by many people, including Hinich, Kontsevich, Manetti, Pridham, Lurie. See the survey [Toë17]. It is then very interesting to review the classical theory used by Goldman-Millson and Kapovich-Millson in the light of the much more modern tools developed: derived deformation theory and L_∞ algebras.

In particular this theory furnishes a canonical algebra pro-representing the deformation functor of L (in case $H^0(L) = 0$) and invariant under quasi-isomorphisms. This algebra is obtained by dualizing a coalgebra which is the H^0 of a DG co-algebra $\mathcal{C}(L)$ known as the *bar construction* or *Chevalley-Eilenberg complex*. We explain this in section 4. In our non-compact case L is not formal and using this construction is the first part of our unifying framework.

The second part of the unification is that in all our geometric situations, L has a structure of mixed Hodge complex (a notion introduced by Deligne in order to construct mixed Hodge structures on the cohomology of a complex) compatible with the multiplicative structure. We call this a *mixed Hodge diagram of Lie algebras*. In the compact case this is almost an immediate consequence of the work of Deligne-Zucker [Zuc79] constructing Hodge structures on the cohomology with coefficients in a variation of Hodge structure. In the non-compact case the ideas are already contained in the work of Kapovich-Millson but they use the method of Morgan and we re-write them with the more powerful and more functorial method of Navarro Aznar [Nav87]. From the data of a mixed Hodge diagram of Lie algebras only, we would like to get the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$. Hence we delay the construction of these diagrams to the sections 10 (compact case) and 11 (non-compact case).

However in the theory of Goldman-Millson L controls the deformation theory of ρ only up to conjugation. If we want to eliminate conjugations, we have to work with L together with its augmentation ε_x to \mathfrak{g} the Lie algebra of G given by evaluating degree zero forms at x . To this data is associated by Eyssidieux-Simpson an *augmented deformation functor* $\text{Def}_{L,\varepsilon_x}$ which is a slight variation of Def_L and is isomorphic to the functor of formal deformation of ρ , pro-represented by $\widehat{\mathcal{O}}_\rho$. Thus any algebra pro-representing $\text{Def}_{L,\varepsilon_x}$ will be canonically isomorphic to $\widehat{\mathcal{O}}_\rho$. It also coincides with the deformation functor associated to the DG Lie algebra $\text{Ker}(\varepsilon_x)$ but this may not be anymore a mixed Hodge diagram. In the category of mixed Hodge complexes it is better to work with the mapping cone, but the mapping cone of an augmented DG Lie algebra may not be anymore a DG Lie algebra. Despite our efforts, it is not possible to find an object that would be at the same time DG Lie, mixed Hodge complex, and with $H^0 = 0$.

We find a solution to this issue by working with L_∞ algebras. Briefly, these are DG Lie algebras *up to homotopy* that have the same properties as DG Lie algebras at the level of cohomology but are less rigid to construct (for example, any DG vector space with a quasi-isomorphism to a DG Lie algebra inherits a structure of L_∞ algebra). They also have an associated deformation functor that extends the classical one of DG Lie algebras. In an article of Fiorenza-Manetti [FM07] the mapping cone of a morphism between DG Lie algebras is shown to carry a canonical L_∞ algebra structure and the associated deformation functor is studied. We explain this in section 5. The crucial claim for us is that for augmented DG Lie algebras the deformation functor of the cone coincides with the augmented deformation functor introduced independently by Eyssidieux-Simpson.

The plan of our approach is now clear. Start with the algebraic data of a mixed Hodge diagram of Lie algebras L together with an augmentation ε to a Lie algebra \mathfrak{g} (we review the tools we need from Hodge theory in sections 6 and 7). We show first in section 8 that on the mapping cone C of ε the structures of mixed Hodge complex and of L_∞ algebra are compatible. We call the resulting object a *mixed Hodge diagram of L_∞ algebras*. Then C has $H^0 = 0$ so its deformation functor

is pro-representable, again by the algebra dual to $H^0(\mathcal{C})$ where the functor \mathcal{C} is naturally extended from DG Lie algebras to L_∞ algebras. We then show in section 9 that $\mathcal{C}(C)$ is a *mixed Hodge diagram of coalgebras*. This step is very similar to Hain's *bar construction on mixed Hodge complexes*. And then its H^0 and the dual algebra, which is canonically isomorphic to $\widehat{\mathcal{O}}_\rho$ by its pro-representability property, have mixed Hodge structures. The functoriality of this construction is clear and the dependence on the base point is explicit. In section 12 we extract some description of the mixed Hodge structure we constructed by algebraic methods, especially we describe it at the level of the cotangent space and we describe the weight zero part.

The reason why we are restricted to representations with finite image when X is non-compact is that until now we are unable to construct a mixed Hodge diagram in more general cases, for example when ρ is the monodromy of a variation of Hodge structure. We strongly believe it is possible to deal with this case with our methods. The case of singular varieties may also be accessible. Combined with a further study of group theory, this could lead to new examples of finitely presentable groups not isomorphic to fundamental groups of smooth algebraic varieties.

Acknowledgment. This is part of my Ph.D. thesis. I am very grateful to my supervisor Philippe Eyssidieux and to the thesis reviewers Nero Budur and Carlos Simpson. I also thank the project ANR Hodgefun.

2. DEFORMATION FUNCTOR OF A DG LIE ALGEBRA

We start by recalling the classical framework for deformation theory: functors on local Artin algebras and DG Lie algebras. We fix a field \mathbf{k} of characteristic zero.

Definition 2.1. A *local Artin algebra* over \mathbf{k} is an algebra A over \mathbf{k} , local with maximal ideal \mathfrak{m}_A , with residue field $A/\mathfrak{m}_A = \mathbf{k}$, and finite-dimensional over \mathbf{k} . These form a category $\mathbf{Art}_{\mathbf{k}}$, where the morphisms are morphisms of algebras over \mathbf{k} that are required to preserve the maximal ideals.

This definition implies that \mathfrak{m}_A is a nilpotent ideal and so A is local complete. By *deformation functors* we will mean certain functors from $\mathbf{Art}_{\mathbf{k}}$ to the category of sets \mathbf{Set} .

Definition 2.2. A *pro-Artin algebra* over \mathbf{k} is a complete local algebra R over \mathbf{k} such that all quotients $R/(\mathfrak{m}_R)^n$ are local Artin algebras (so R is a projective limit of local Artin algebras). These form a category $\mathbf{ProArt}_{\mathbf{k}}$ where morphisms have to preserve the maximal ideal.

The most basic examples are the formal power series algebras $\mathbf{k}[[X_1, \dots, X_r]]$. The category $\mathbf{Art}_{\mathbf{k}}$ is a full subcategory of $\mathbf{ProArt}_{\mathbf{k}}$, which is itself a full subcategory of the category of complete local algebras with residue field \mathbf{k} . Namely in all these three categories, morphisms are morphisms of algebras that preserve the maximal ideal. Among all complete local algebras, those that we call pro-Artin are characterized by the fact that their cotangent space $\mathfrak{m}_R/(\mathfrak{m}_R)^2$ is finite-dimensional over \mathbf{k} , or equivalently they are characterized by the Noetherian property.

Any such algebra defines a deformation functor:

Definition 2.3. A functor $F : \mathbf{Art}_{\mathbf{k}} \rightarrow \mathbf{Set}$ is said to be *pro-representable* if F is isomorphic to

$$(2.1) \quad A \longmapsto \underset{\mathbf{ProArt}}{\mathrm{Hom}}(R, A)$$

for some pro-Artin algebra R over \mathbf{k} .

By the pro-Yoneda lemma ([Sch68, § 2], [GM88, 3.1]), such an algebra R is unique up to a unique isomorphism.

We then refer to the lecture notes [Man04, § V] for DG Lie algebras. Such a DG Lie algebra L has a grading $L = \bigoplus_{n \in \mathbb{Z}} L^n$, a differential

$$(2.2) \quad d : L^n \longrightarrow L^{n+1}$$

and an anti-symmetric Lie bracket

$$(2.3) \quad [-, -] : L^i \otimes L^j \longrightarrow L^{i+j}$$

such that d is a derivation for the Lie bracket and the bracket satisfies the graded Jacobi identity. Its set of *Maurer-Cartan elements* is

$$(2.4) \quad \text{MC}(L) := \left\{ x \in L^1 \mid d(x) + \frac{1}{2}[x, x] = 0 \right\}.$$

If A is a local Artin algebra with maximal ideal \mathfrak{m}_A then $L \otimes \mathfrak{m}_A$ is a *nilpotent* DG Lie algebra with bracket

$$(2.5) \quad [u \otimes a, v \otimes b] := [u, v] \otimes ab, \quad u, v \in L, \quad a, b \in \mathfrak{m}_A$$

and differential

$$(2.6) \quad d(u \otimes a) := d(u) \otimes a.$$

On $L^0 \otimes \mathfrak{m}_A$, which is a nilpotent Lie algebra, the Baker-Campbell-Hausdorff formula defines a group structure that we denote by $(\exp(L^0 \otimes \mathfrak{m}_A), *)$, whose elements are denoted by e^α for $\alpha \in L^0 \otimes \mathfrak{m}_A$. This groups acts on $\text{MC}(L \otimes \mathfrak{m}_A)$ by *gauge transformations*.

Definition 2.4. The *deformation functor* of a DG Lie algebra L over \mathbf{k} is the functor

$$(2.7) \quad \begin{array}{ccc} \text{Def}_L : \mathbf{Art}_{\mathbf{k}} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \frac{\text{MC}(L \otimes \mathfrak{m}_A)}{\exp(L^0 \otimes \mathfrak{m}_A)}. \end{array}$$

If a deformation functor F is isomorphic to some Def_L we say that L *controls* the deformation problem given by F .

The main theorem of deformation theory is:

Theorem 2.5 ([Man04, V.52]). *Let $\varphi : L \xrightarrow{\sim} L'$ be a quasi-isomorphism between DG Lie algebras. Then the induced morphism*

$$(2.8) \quad \text{Def}_L \xrightarrow{\sim} \text{Def}_{L'}$$

is an isomorphism.

When reviewing the theory of Goldman-Millson, Eyssidieux and Simpson introduce a small variation of the classical deformation functor for augmented DG Lie algebras.

Definition 2.6 ([ES11, § 2.1.1]). Let $\varepsilon : L \rightarrow \mathfrak{g}$ be an augmentation of the DG Lie algebra L to the Lie algebra \mathfrak{g} (that one can see as a DG Lie algebra concentrated in degree zero). The *augmented deformation functor* is the functor

$$(2.9) \quad \text{Def}_{L, \varepsilon} : \mathbf{Art}_{\mathbf{k}} \longrightarrow \mathbf{Set}$$

defined on $A \in \mathbf{Art}_{\mathbf{k}}$ as the quotient of the set

$$(2.10) \quad \left\{ (x, e^\alpha) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(\mathfrak{g} \otimes \mathfrak{m}_A) \mid d(x) + \frac{1}{2}[x, x] = 0 \right\}$$

by the action of $\exp(L^0 \otimes \mathfrak{m}_A)$ given by

$$(2.11) \quad e^\lambda.(x, e^\alpha) := (e^\lambda.x, e^\alpha * e^{-\varepsilon(\lambda)}).$$

The $*$ on the right-hand side of (2.11) is the product given by the Baker-Campbell-Hausdorff formula. For this group law $e^{-\varepsilon(\lambda)}$ is the inverse of $e^{\varepsilon(\lambda)}$, thus this is really a left action of the group $\exp(L^0 \otimes \mathfrak{m}_A)$.

Remark 2.7. This is not exactly the same action as in [ES11], but we make this choice so as to be compatible with the construction developed in section 5.

The fundamental Theorem 2.5 has a small variation for the augmented deformation functor.

Lemma 2.8 (See [ES11, 2.7]). *Let $\varphi : L \xrightarrow{\approx} L'$ be a quasi-isomorphism of augmented DG Lie algebras commuting with the augmentations:*

$$(2.12) \quad \begin{array}{ccc} L & \xrightarrow[\varphi]{\approx} & L' \\ & \searrow \varepsilon & \swarrow \varepsilon' \\ & \mathfrak{g} & \end{array}$$

Then φ induces an isomorphism of deformation functors

$$(2.13) \quad \mathrm{Def}_{L, \varepsilon} \xrightarrow{\cong} \mathrm{Def}_{L', \varepsilon'}.$$

3. THE THEORY OF GOLDMAN AND MILLSON

Then we give a brief account of [GM88]. We fix a linear algebraic group G over $\mathbf{k} = \mathbb{Q}, \mathbb{R}$ or \mathbb{C} . We think of G as a representable functor

$$(3.1) \quad G : \mathbf{Alg}_{\mathbf{k}} \longrightarrow \mathbf{Grp}$$

from the category of algebras over \mathbf{k} to the category of groups. We are interested in studying a group Γ by looking at its representations into $G(\mathbf{k})$, and actually into all $G(A)$ for varying algebras A over \mathbf{k} .

Definition 3.1 ([LM85]). For any finitely generated group Γ the functor

$$(3.2) \quad \begin{array}{ccc} \mathrm{Hom}(\Gamma, G) : \mathbf{Alg}_{\mathbf{k}} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \mathrm{Hom}_{\mathbf{Grp}}(\Gamma, G(A)) \end{array}$$

is represented by an affine scheme of finite type over \mathbf{k} . We denote it again by $\mathrm{Hom}(\Gamma, G)$ (we think of it as a scheme structure on the set $\mathrm{Hom}(\Gamma, G(\mathbf{k}))$) and call it the *representation variety* of Γ into G . If ρ is a representation of Γ into $G(\mathbf{k})$, seen as a point over \mathbf{k} of $\mathrm{Hom}(\Gamma, G)$, we denote by $\widehat{\mathcal{O}}_\rho$ the completion of the local ring of $\mathrm{Hom}(\Gamma, G)$ at ρ .

The local ring $\widehat{\mathcal{O}}_\rho$ will be one of our main objects of study. It is a pro-Artin algebra whose corresponding pro-representable functor on local Artin algebras over \mathbf{k}

$$(3.3) \quad \begin{array}{ccc} \mathrm{Hom}(\widehat{\mathcal{O}}_\rho, -) : \mathbf{Art}_{\mathbf{k}} & \longrightarrow & \mathbf{Set} \\ A & \longmapsto & \mathrm{Hom}_{\mathbf{ProArt}}(\widehat{\mathcal{O}}_\rho, A) \end{array}$$

is canonically isomorphic to the functor of *formal deformations of ρ*

$$(3.4) \quad \text{Def}_\rho : A \longrightarrow \{ \tilde{\rho} : \Gamma \rightarrow G(A) \mid \tilde{\rho} = \rho \bmod \mathfrak{m}_A \}.$$

Define G^0 to be the functor

$$(3.5) \quad \begin{array}{ccc} G^0 : \mathbf{Art}_{\mathbf{k}} & \longrightarrow & \mathbf{Grp} \\ A & \longrightarrow & \{ g \in G(A) \mid g = 1_{G(\mathbf{k})} \bmod \mathfrak{m}_A \} \end{array}$$

then $G^0(A)$ acts on $\text{Def}_\rho(A)$ by conjugation, functorially in A . We simply denote by Def_ρ/G^0 the quotient functor from $\mathbf{Art}_{\mathbf{k}}$ to \mathbf{Set} . If \mathfrak{g} is the Lie algebra of G then $G^0(A)$ has $\mathfrak{g} \otimes \mathfrak{m}_A$ as Lie algebra and actually one can construct it as

$$(3.6) \quad G^0(A) = \exp(\mathfrak{g} \otimes \mathfrak{m}_A).$$

Let now X be a manifold whose fundamental group is finitely presentable. This hypothesis encompasses both the compact Kähler manifolds and the smooth algebraic varieties over \mathbb{C} . Let x be a base point of X . We are interested in the deformation theory of representations of the fundamental group $\pi_1(X, x)$. Over both fields \mathbb{R} and \mathbb{C} , $G(\mathbf{k})$ is a Lie group. To a representation

$$(3.7) \quad \rho : \pi_1(X, x) \longrightarrow G(\mathbf{k})$$

corresponds a flat principal $G(\mathbf{k})$ -bundle

$$(3.8) \quad P := \tilde{X} \times_{\pi_1(X, x)} G(\mathbf{k})$$

(where \tilde{X} is the universal cover of X , seen as a principal $\pi_1(X, x)$ -bundle). The adjoint bundle is

$$(3.9) \quad \text{Ad}(P) := P \times_{G(\mathbf{k})} \mathfrak{g} = \tilde{X} \times_{\pi_1(X, x)} \mathfrak{g}$$

where $\pi_1(X, x)$ acts on \mathfrak{g} via $\text{Ad} \circ \rho$. Thus $\text{Ad}(P)$ is a local system of Lie algebras with fiber at x canonically identified to \mathfrak{g} . Let now

$$(3.10) \quad L := \mathcal{E}^\bullet(X, \text{Ad}(P))$$

be the DG Lie algebra of \mathcal{C}^∞ differential forms with values in $\text{Ad}(P)$. Locally an element of L is given by a sum $\sum \alpha_i \otimes u_i$ where α_i is a differential form and u_i is an element of \mathfrak{g} . The differential is given by

$$(3.11) \quad d(\alpha \otimes u) := d(\alpha) \otimes u$$

and the Lie bracket is given by

$$(3.12) \quad [\alpha \otimes u, \beta \otimes v] := (\alpha \wedge \beta) \otimes [u, v].$$

Part of the main theorem of Goldman and Millson can be stated as follows:

Theorem 3.2 ([GM88]). *There is a canonical isomorphism of deformation functors*

$$(3.13) \quad \text{Def}_L \xrightarrow{\cong} \text{Def}_\rho / G^0.$$

Roughly, this is a deformed version of the usual correspondence between representations of the fundamental group and principal bundles with a flat connection: a representation of $\pi_1(X, x)$ modulo conjugation corresponds to a flat principal bundle modulo automorphisms, or equivalently to a fixed principal bundle with a flat connection modulo the action of the automorphism group on flat connections.

However we are interested in the functor Def_ρ and not his quotient by conjugation by G^0 , because Def_ρ is pro-represented by the pro-Artin algebra $\widehat{\mathcal{O}}_\rho$. For this we introduce the augmentation

$$(3.14) \quad \begin{array}{ll} \varepsilon_x : L & \longrightarrow \mathfrak{g} \\ \omega \in L^0 & \longmapsto \omega(x) \\ \omega \in L^{>0} & \longmapsto 0. \end{array}$$

Then the correspondence of Goldman-Millson becomes the following:

Theorem 3.3 ([GM88], [ES11, 2.7]). *The isomorphism of Theorem 3.2 induces a canonical isomorphism of deformation functors*

$$(3.15) \quad \text{Def}_{L, \varepsilon_x} \xrightarrow{\simeq} \text{Def}_\rho.$$

When X is a compact Kähler manifold and ρ has image contained in a compact subgroup of $G(\mathbb{R})$, or ρ is the monodromy of a polarized variation of Hodge structure over X , then L is formal (i.e. quasi-isomorphic to a DG Lie algebra with zero differential) and the functor $\text{Def}_{L, \varepsilon_x}$ simplifies much thanks to Theorem 2.5 (or its augmented variant, Lemma 2.8). This allows Goldman and Millson to give a description of $\widehat{\mathcal{O}}_\rho$ via its property that it pro-represents the deformation functor $\text{Def}_{L, \varepsilon}$ and one can construct quite easily such an object when L has $d = 0$. However to deal with more general geometric situations where L is not formal we must study more carefully how to pro-represent $\text{Def}_{L, \varepsilon_x}$.

4. L_∞ ALGEBRAS AND PRO-REPRESENTABILITY

We will need to work with L_∞ algebras for two well-distinct reasons. The first one is for giving a pro-representability theorem for the deformation functor of DG Lie algebras. The second one is for explaining how the augmented deformation functor of $\varepsilon : L \rightarrow \mathfrak{g}$ comes from a L_∞ algebra structure on the mapping cone of ε . Our reference is again the lectures notes [Man04, § VIII–IX]. Through the sections 8–9 we will show that these constructions have good compatibilities with Hodge theory.

Briefly, L_∞ algebras are weakened versions of DG Lie algebras in which the Jacobi identity only holds *up to homotopy*. Such a L_∞ algebra is given by a graded vector space L with a sequence of anti-symmetric linear maps

$$(4.1) \quad \ell_r : L^{\wedge r} \longrightarrow L \quad (r \geq 1)$$

where ℓ_r has degree $2 - r$. The precise definition is best stated in terms of codifferential on a cofree conilpotent coalgebra and we will need this point of view.

So let us say some words on coalgebras first. In the following we will work with DG coalgebras X , equipped with a comultiplication

$$(4.2) \quad \Delta : X \longrightarrow X \otimes X$$

that is always assumed to be coassociative and cocomutative, and with a differential d on X that is a coderivation for Δ ; it is called the *codifferential*.

Definition 4.1. On a graded coalgebra X , there is a *canonical filtration* indexed by \mathbb{N} given by

$$(4.3) \quad X_n := \text{Ker}(\Delta^n : X \rightarrow X^{\otimes(n+1)})$$

where Δ^n is the iterated comultiplication. The coalgebra is said to be *conilpotent* (terminology of [LV12, 1.2.4]; these are called *locally conilpotent* in [Man04]) if the canonical filtration is exhaustive, i.e.

$$(4.4) \quad X = \bigcup_{n \geq 0} X_n.$$

The dual of a coalgebra, in our sense, is a commutative algebra without unit. Conilpotent coalgebras are dual to maximal ideals of complete local algebras and conilpotent coalgebras with finite canonical filtration are dual to maximal ideals of local algebras with nilpotent maximal ideal. So if X is such a coalgebra then $\mathbf{k} \oplus X^*$ is an algebra with unit and with maximal ideal X^* .

Lemma 4.2. *If X is a conilpotent coalgebra whose canonical filtration is by finite-dimensional sub-coalgebras, then its dual X^* is the maximal ideal of a pro-Artin algebra and conversely the dual of a pro-Artin algebra is such a conilpotent coalgebra.*

Proof. For such an X , $R := \mathbf{k} \oplus X^*$ is a complete local algebra with maximal ideal X^* whose quotients are finite-dimensional, hence R is pro-Artin. Conversely if R is pro-Artin then R is a projective limit of finite-dimensional algebras, which dualizes to an inductive limit of coalgebras, defining a conilpotent coalgebra X with filtration by finite-dimensional sub-coalgebras. \square

Definition 4.3 ([Man04, VIII.24]). Let V be a graded vector space. The *cofree (conilpotent) coalgebra* on V is the graded vector space given by the reduced symmetric algebra

$$(4.5) \quad \mathcal{F}^c(V) := \text{Sym}^+(V) = \bigoplus_{r \geq 1} V^{\odot r}$$

(we denote by \odot the symmetric product) endowed with the comultiplication

$$(4.6) \quad \Delta(v_1 \odot \cdots \odot v_r) := \sum_{p=0}^r \sum_{\tau \in \mathfrak{S}(p, r-p)} \varepsilon(\tau; v_1, \dots, v_r) (v_{\tau(1)} \odot \cdots \odot v_{\tau(p)}) \otimes (v_{\tau(p+1)} \odot \cdots \odot v_{\tau(r)}).$$

where $\mathfrak{S}(p, r-p) \subset \mathfrak{S}(r)$ is the set of permutations of $\{1, \dots, r\}$ that are increasing on $\{1, \dots, p\}$ and on $\{p+1, \dots, r\}$.

This construction is a right adjoint to the forgetful functor from graded conilpotent coalgebras to graded vector spaces. The canonical filtration here is simply the natural increasing filtration of the symmetric algebra:

$$(4.7) \quad \mathcal{F}^c(V)_n := \bigoplus_{r=1}^n V^{\odot r}.$$

A coderivation Q on V is determined uniquely by its components ([Man04, VIII.34])

$$(4.8) \quad q_r : V^{\odot r} \longrightarrow V \quad (r \geq 1).$$

For any DG vector space V and integer $i \in \mathbb{Z}$ we define the *shift* $V[i]$ which has

$$(4.9) \quad V[i]^n := V^{n+i}$$

and differential $d_{V[i]} = (-1)^i d_V$. We denote by $x[i]$ an element of $V[i]$ obtained by shifting the degree by i from an element x of V (if x has degree n in V then $x[i]$ is the element x with degree $n-i$ in $V[i]$, since $V[i]^{n-i} = V^n$).

Definition 4.4. A L_∞ algebra is the data of a graded vector space L and a codifferential Q on $\mathcal{F}^c(V[1])$. We denote by $\mathcal{C}(L)$ this coalgebra.

Thus the structure of L_∞ algebra on L is determined either by the maps

$$(4.10) \quad q_r : (L[1])^{\odot r} \longrightarrow L[1]$$

of degree 1, either by the maps ℓ_r of (4.1) obtained as shifts of q_r (since we take symmetric and exterior products in the graded sense, there is a canonical isomorphism $(L[1])^{\odot r} \simeq L^{\wedge r}[r]$), with all axioms encoded in the relation $Q \circ Q = 0$. DG Lie algebras correspond exactly to L_∞ algebras with $\ell_r = 0$ for $r \geq 3$, then ℓ_1 is the differential of L and ℓ_2 is the Lie bracket. The construction \mathcal{C} is known in the literature under the names of *Quillen \mathcal{C} functor*, *bar construction* or (homological) *Chevalley-Eilenberg complex*. We prefer the term of bar construction and we call the canonical filtration (4.7) of $\mathcal{C}(L)$ the *bar filtration*.

Definition 4.5. A (strong) *morphism* between L_∞ algebras L, M is a morphism of graded vector spaces $L \rightarrow M$ that commutes with all the operations ℓ_r . A L_∞ -*morphism* is a morphism of graded coalgebras $\mathcal{C}(L) \rightarrow \mathcal{C}(M)$.

Any morphism induces canonically an L_∞ -morphism, and a L_∞ -morphism has a *linear part* which is a morphism of DG vector spaces (i.e. commuting with ℓ_1 , the differential).

Theorem 4.6 ([Man04, IX.9]). *A L_∞ -morphism from L to M whose linear part is a quasi-isomorphism induces a quasi-isomorphism of DG coalgebras from $\mathcal{C}(L)$ to $\mathcal{C}(M)$.*

We call such morphisms L_∞ -*quasi-isomorphisms*. The fundamental theorem of deformation theory with L_∞ algebras takes the following form.

Theorem 4.7 ([Man04, § IX]). *To any L_∞ algebra L over \mathbf{k} is associated a deformation functor*

$$(4.11) \quad \text{Def}_L : \mathbf{Art}_{\mathbf{k}} \longrightarrow \mathbf{Set}$$

which coincides for DG Lie algebras with the usual deformation functor of Definition 2.4 and is invariant under L_∞ -quasi-isomorphisms.

Then we state the pro-representability theorem we will use. This appears in various places: it is stated for example in the survey by Toën [Toë17], in the article of Hinich [Hin01, § 9.3], and in the lecture notes of Kontsevich [Kon94]. But we never find it written exactly in the form we want in the literature, especially because of the duality and the finite-dimensionality issue, so we give here the simplest possible proof using only the above theorems.

Theorem 4.8 (Pro-representability). *Let L be a L_∞ algebra with $H^n(L) = 0$ for $n \leq 0$ and $H^1(L)$ finite-dimensional. Then $H^0(\mathcal{C}(L))$ is a coalgebra satisfying the hypothesis of Lemma 4.2 and*

$$(4.12) \quad R := \mathbf{k} \oplus H^0(\mathcal{C}(L))^*$$

is a pro-Artin algebra that pro-represents the deformation functor of L and is invariant under L_∞ -quasi-isomorphisms.

Proof. First, Theorem 4.6 states that $H^0(\mathcal{C}(L))$ is invariant under L_∞ -quasi-isomorphisms, and so is R . The L_∞ algebra structure on L transfers, up to L_∞ -quasi-isomorphism, to a L_∞ algebra structure on its cohomology ([LV12, § 10.3]). Thus we can assume that L has $L^n = 0$ for $n \leq 0$ and L^1 finite-dimensional. The deformation functor of L is then simply given by the solutions of the Maurer-Cartan equation

$$(4.13) \quad A \in \mathbf{Art}_{\mathbf{k}} \longmapsto \left\{ x \in L^1 \otimes \mathfrak{m}_A \mid 0 = \sum_{r \geq 1} \frac{1}{r!} \ell_r(x \wedge \cdots \wedge x) \right\}$$

(this sum is finite since \mathfrak{m}_A is nilpotent, and when $\ell_r = 0$ for $r \geq 3$ this is the usual Maurer-Cartan equation for DG Lie algebras), without any quotient. But by a simple calculation ([Man04, VIII.27]) this functor is the same as

$$(4.14) \quad A \in \mathbf{Art}_{\mathbf{k}} \longmapsto \mathrm{Hom}((\mathfrak{m}_A)^*, H^0(\mathcal{C}(L))).$$

Assuming that L^1 is finite-dimensional, the canonical filtration of $\mathcal{C}(L)$ is by finite-dimensional sub-coalgebras and similarly for $H^0(\mathcal{C}(L))$. Thus one can dualize as in Lemma 4.2 and the above functor is isomorphic to

$$(4.15) \quad A \in \mathbf{Art}_{\mathbf{k}} \longmapsto \mathrm{Hom}(H^0(\mathcal{C}(L))^*, \mathfrak{m}_A)$$

which is also $\mathrm{Hom}(R, A)$ with the Hom in the category of local algebras. \square

Remark 4.9. When L is a DG Lie algebra and is formal, the above theorem is a simple calculation combined with the invariance of Def_L under quasi-isomorphisms. In the general case however, and even when L is a DG Lie algebra, this is really a theorem in derived deformation theory and for us it improves much the classical theory (compare with [GM90], [Man99, § 4]).

5. L_∞ ALGEBRA STRUCTURE ON THE MAPPING CONE

After the pro-representability theorem, the second main point of our approach is the article of Fiorenza-Manetti [FM07] showing that the mapping cone of a morphism between DG Lie algebras has a canonical L_∞ algebra structure and describing explicitly the deformation functor.

Definition 5.1. Let $\varepsilon : L \rightarrow M$ be a morphism between DG vector spaces. The *mapping cone* of ε is the DG vector space $\mathrm{Cone}(\varepsilon)$ with

$$(5.1) \quad \mathrm{Cone}(\varepsilon)^n := L^{n+1} \oplus M^n$$

and differential

$$(5.2) \quad d_{\mathrm{Cone}(\varepsilon)}^n(x, y) := (-d_L^{n+1}(x), d_M^n(y) - \varepsilon(x)).$$

It is sometimes more natural to work with the *desuspended mapping cone* $\mathrm{Cone}(\varepsilon)[-1]$, which has

$$(5.3) \quad \mathrm{Cone}(\varepsilon)[-1]^n = L^n \oplus M^{n-1}$$

and differential

$$(5.4) \quad d_{\mathrm{Cone}(\varepsilon)[-1]}^n(x, y) = (d_L^n(x), \varepsilon(x) - d_M^{n-1}(y)).$$

If ε is a morphism between DG Lie algebras, then on the desuspended mapping cone one defines a *naive bracket* which is a bilinear map of degree zero given by

$$(5.5) \quad [(x, u), (y, v)] := \left([x, y], \frac{1}{2} [u, \varepsilon(y)] + \frac{(-1)^{|x|}}{2} [\varepsilon(x), v] \right)$$

for $x, y \in L, u, v \in M$.

The naive bracket is anti-symmetric and the differential is a derivation for the bracket, but in general it doesn't satisfy the Jacobi identity. Let us sum up all what we need directly in one theorem.

Theorem 5.2 ([FM07, Main Thm.]). *Let $\varepsilon : L \rightarrow M$ be a morphism between DG Lie algebras and let C be its desuspended mapping cone. Then C has a L_∞ algebra structure, with ℓ_1 the usual differential of the cone and ℓ_2 the naive bracket.*

The higher brackets of C , given as the components q_r of a codifferential Q on $\mathcal{C}(C)$, are given as follows (we write x, y, \dots for elements of L and u, v, \dots for elements of M). If $r + k \geq 3$ and $k \neq 1$ then

$$(5.6) \quad q_{r+k}(u_1 \odot \cdots \odot u_r \otimes x_1 \odot \cdots \odot x_k) = 0$$

and for $r \geq 2$

$$(5.7) \quad q_{r+1}(u_1 \odot \cdots \odot u_r \otimes x) = -(-1)^{\sum_{i=1}^r |u_i|} \frac{B_r}{r!} \sum_{\tau \in \mathfrak{S}(r)} \varepsilon(\tau, u_1, \dots, u_r) [u_{\tau(1)}, [u_{\tau(2)}, \dots, [u_{\tau(r)}, \varepsilon(x)] \dots]]$$

where the B_n are the Bernoulli numbers.

This L_∞ algebra structure is functorial from the category of morphisms between DG Lie algebras to the category of L_∞ algebras with their strong morphisms.

The associated deformation functor on \mathbf{Art}_k is isomorphic to the quotient functor of the set of objects (for $A \in \mathbf{Art}_k$)

$$(5.8) \quad \left\{ (x, e^\alpha) \in (L^1 \otimes \mathfrak{m}_A) \times \exp(M^0 \otimes \mathfrak{m}_A) \mid d(x) + \frac{1}{2}[x, x] = 0, e^\alpha * \varepsilon(x) = 0 \right\}$$

by the relation

$$(5.9) \quad (x, e^\alpha) \sim (y, e^\beta) \iff \exists (\lambda, \mu) \in (L^0 \otimes \mathfrak{m}_A) \times (M^{-1} \otimes \mathfrak{m}_A), \quad e^\lambda \cdot x = y, \quad e^\beta = e^{d(\mu)} * e^\alpha * e^{-\varepsilon(\lambda)}.$$

And our main remark for this section is:

Lemma 5.3. *Applied to an augmentation $\varepsilon : L \rightarrow \mathfrak{g}$ to a Lie algebra \mathfrak{g} , the above deformation functor associated to the cone of ε introduced by Fiorenza-Manetti coincides with the augmented deformation functor defined by Eyssidieux-Simpson.*

Proof. Apply the theorem with M concentrated in degree zero: $\varepsilon(x) = 0$ and also $\mu = 0$ so $e^{d(\mu)}$ is the identity of $\exp(L^0 \otimes \mathfrak{m}_A)$. Then we see immediately that the set of objects (2.10) and (5.8) are the same and the equivalence relations in (2.11) and (5.9) are also the same. \square

6. MIXED HODGE STRUCTURES

We now turn to Hodge theory and we adopt all the classical notations and conventions from Deligne [Del71] concerning filtrations and Hodge structures. In particular we always denote by W_\bullet an increasing filtration and by F^\bullet a decreasing filtration. They are all indexed by \mathbb{Z} . Filtrations of vector spaces are assumed to be finite and filtrations of DG vector spaces are assumed to be biregular (i.e. induce a finite filtration on each component) unless we state explicitly that we work with inductive or projective limits of such objects. We will work with a fixed field $\mathbf{k} \subset \mathbb{R}$.

First recall the classical definitions of pure and mixed Hodge structures.

Definition 6.1. A (pure) *Hodge structure* of weight k over the field \mathbf{k} is the data of a finite-dimensional vector space K over \mathbf{k} and a bigrading over \mathbb{C}

$$(6.1) \quad K_{\mathbb{C}} := K \otimes_{\mathbf{k}} \mathbb{C} = \bigoplus_{p+q=k} K^{p,q}$$

with the complex conjugation exchanging $K^{p,q}$ and $K^{q,p}$. The *Hodge filtration* is then the decreasing filtration F^\bullet of $K_{\mathbb{C}}$ defined by

$$(6.2) \quad F^p K_{\mathbb{C}} := \bigoplus_{p' \geq p} K^{p',q}.$$

Definition 6.2. A *mixed Hodge structure* over \mathbf{k} is the data of a finite-dimensional vector space K over \mathbf{k} with an increasing filtration W_\bullet called the *weight filtration* and a decreasing filtration F^\bullet on $K_{\mathbb{C}}$ such that for each $k \in \mathbb{Z}$ the graded piece

$$(6.3) \quad \mathrm{Gr}_k^W(K) := W_k(K)/W_{k-1}(K)$$

with the induced filtration F over \mathbb{C} forms a pure Hodge structure of weight k .

Pure and mixed Hodge structures are abelian categories equipped with a tensor product and internal homs. In particular one can talk about various kinds of algebras (resp. of coalgebras) carrying a compatible mixed Hodge structure: this is simply the data of a mixed Hodge structure K with a multiplication map $K \otimes K \rightarrow K$ (resp. a comultiplication $K \rightarrow K \otimes K$) that is a morphism of mixed Hodge structures and satisfies the corresponding axioms of algebra or coalgebra.

We will also have to deal with inductive and projective limits of these: by definition a pro-Artin algebra with a mixed Hodge structure is the data of a pro-Artin algebra R with a mixed Hodge structure on each $R/(\mathfrak{m}_R)^n$ (which is finite-dimensional) such that the canonical morphisms

$$(6.4) \quad R/(\mathfrak{m}_R)^{n+1} \longrightarrow R/(\mathfrak{m}_R)^n$$

are morphisms of mixed Hodge structures. Dually a conilpotent coalgebra with a mixed Hodge structure is a conilpotent coalgebra X with a mixed Hodge structure on each term X_n of the canonical filtration, compatible with the inclusion $X_n \subset X_{n+1}$.

A mixed Hodge structure on K defines (in several ways) a bigrading of $K_{\mathbb{C}}$.

Definition 6.3 ([PS08, 3.4]). Let K be a mixed Hodge structure. Let $K^{p,q}$ be the (p, q) -component of $\mathrm{Gr}_{p+q}^W(K)$. Define the subspace of $K_{\mathbb{C}}$

$$(6.5) \quad I^{p,q} := F^p \cap W_{p+q} \cap \left(\overline{F}^q \cap W_{p+q} \cap \sum_{j \geq 2} (\overline{F}^{q-j+1} \cap W_{p+q-j}) \right).$$

Then this defines a bigrading

$$(6.6) \quad K_{\mathbb{C}} = \bigoplus_{p,q} I^{p,q}$$

such that the canonical projection $K_{\mathbb{C}} \rightarrow \mathrm{Gr}_{p+q}^W(K_{\mathbb{C}})$ induces an isomorphism

$$(6.7) \quad I^{p,q} \simeq K^{p,q}.$$

We call it the *Deligne splitting*. It is functorial and compatible with duals and tensor products.

Actually we will use only the grading by weight $K_{\mathbb{C}} = \bigoplus_k I^k$ with

$$(6.8) \quad I^k = \bigoplus_{p+q=k} I^{p,q}.$$

7. MIXED HODGE COMPLEXES

For us the basic tool for constructing mixed Hodge structures will be Deligne's notion of mixed Hodge complex, introduced in [Del74, § 8]. This consists of several complexes related by filtered quasi-isomorphisms and axioms implying that all cohomology groups carry mixed Hodge structures.

Definition 7.1. If (K, W) and (L, W) are filtered DG vector spaces, a *filtered quasi-isomorphism* is a morphism $f : K \rightarrow L$ of filtered DG vector spaces that induces a quasi-isomorphism

$$(7.1) \quad \mathrm{Gr}_k^W(K) \xrightarrow{\simeq} \mathrm{Gr}_k^W(L)$$

for all k . Since the filtrations are biregular, f is in particular a quasi-isomorphism.

If (K, W, F) and (L, W, F) are bifiltered DG vector spaces (equipped with two filtrations, W increasing and F decreasing) a *bifiltered quasi-isomorphism* is a morphism of bifiltered DG vector spaces $f : K \rightarrow L$ that induces a quasi-isomorphism

$$(7.2) \quad \mathrm{Gr}_k^W \mathrm{Gr}_F^p(K) \xrightarrow{\simeq} \mathrm{Gr}_k^W \mathrm{Gr}_F^p(L)$$

for all k, p . By the Zassenhaus lemma ([Del71, 1.2.1]) the two graded pieces $\mathrm{Gr}_k^W \mathrm{Gr}_F^p$ and $\mathrm{Gr}_F^p \mathrm{Gr}_W^k$ are canonically isomorphic and one can invert them in (7.2). Again, f is in particular a quasi-isomorphism.

Definition 7.2 ([Del74, 8.1.5]). A *mixed Hodge complex* over \mathbf{k} is the data of a filtered bounded-below DG vector space $(K_{\mathbf{k}}, W_{\bullet})$ over \mathbf{k} and a bifiltered bounded-below DG vector space $(K_{\mathbb{C}}, W_{\bullet}, F^{\bullet})$, together with a chain of filtered quasi-isomorphisms

$$(7.3) \quad (K_{\mathbf{k}}, W) \otimes \mathbb{C} \xleftarrow{\simeq} (K_{\mathbb{C}}, W)$$

satisfying the following axioms:

- (1) For all $n \in \mathbb{Z}$, $H^n(K)$ is finite-dimensional.
- (2) For all $k \in \mathbb{Z}$, the differential of $\mathrm{Gr}_k^W(K_{\mathbb{C}})$ is strictly compatible with the filtration F .
- (3) For all $n \in \mathbb{Z}$ and all $k \in \mathbb{Z}$, the filtration F induced on $H^n(\mathrm{Gr}_k^W(K_{\mathbb{C}}))$ and the form $H^n(\mathrm{Gr}_k^W(K_{\mathbf{k}}))$ over \mathbf{k} are part of a pure Hodge structure of weight $k + n$ over \mathbf{k} on $H^n(\mathrm{Gr}_k^W(K))$.

Remark 7.3. In the following, we will work with mixed Hodge complexes with a fixed chain of quasi-isomorphisms from $K_{\mathbf{k}}$ to $K_{\mathbb{C}}$, via intermediate components over \mathbb{C} that we denote by K_i (i is an index belonging to some indexing category) carrying a filtration W . The morphisms $K_i \rightarrow K_j$ are called the *comparison quasi-isomorphisms*. See [CG16] and [Cir15] for this point of view.

Theorem 7.4 (Deligne). *If K is a mixed Hodge complex, then on each $H^n(K)$ the induced filtrations F and $W[n]$ (with $W[n]_k := W_{k-n}$) define a mixed Hodge structure.*

Definition 7.5. A *morphism* f between mixed Hodge complexes K, L is the data of a collection of morphisms $f_i : K_i \rightarrow L_i$ of filtered DG vector spaces, and $f_{\mathbb{C}} : K_{\mathbb{C}} \rightarrow L_{\mathbb{C}}$ of bifiltered DG vector spaces, commuting with the comparison quasi-isomorphisms. It is said to be a *quasi-isomorphism* if f_i is a filtered quasi-isomorphism and $f_{\mathbb{C}}$ is a bifiltered quasi-isomorphism.

Let us describe three useful constructions on mixed Hodge complexes.

Proposition 7.6 ([Del74, 8.1.24], [PS08, 3.20]). *The tensor product of two mixed Hodge complexes K, L , defined level-wise, is again a mixed Hodge complex. Its cohomology computes (via the Künneth formula) the tensor product of the mixed Hodge structures on cohomology.*

Similarly, the exterior and symmetric products of mixed Hodge complexes are again mixed Hodge complexes, as well as the whole tensor, exterior and symmetric algebras.

Definition 7.7. We call *mixed Hodge diagram* a mixed Hodge complex K having an additional structure of algebra (commutative algebra, or Lie algebra, or coalgebra).

Equivalently this means that all components K_i are algebras and the comparison quasi-isomorphisms are morphisms of algebras.

Remark 7.8. Algebraically, this is just a little bit more than a mixed Hodge complex, for example this is what Hain calls *multiplicative mixed Hodge complex* ([Hai87, § 3.1]). Geometrically however, the construction of a mixed Hodge diagram of algebras whose cohomology computes the whole cohomology algebra of a given variety together with its mixed Hodge structure is much more difficult. This is the object of section 11.

Proposition 7.9. *If K is a mixed Hodge complex and $r \in \mathbb{Z}$, then the shift $K[r]$, which has in each component $K_i[r]^n := K_i^{n+r}$, equipped with the filtrations $W[r]$ and F , is again a mixed Hodge complex.*

This is the natural way of shifting a complex together with its mixed Hodge structure on cohomology. We will use it mainly for $r = 1$ or $r = -1$.

Proposition 7.10 ([PS08, 3.22]). *The mapping cone of a morphism $f : K \rightarrow L$ between mixed Hodge complexes is again a mixed Hodge complex, with filtration W in the component i given by*

$$(7.4) \quad W_k \text{Cone}(f)_i^n := W_{k-1}K_i^{n+1} \oplus W_kL_i^n$$

and in the component over \mathbb{C} carrying also the filtration F

$$(7.5) \quad F^p \text{Cone}(f)_{\mathbb{C}}^n := F^p K_{\mathbb{C}}^n \oplus F^p L_{\mathbb{C}}^n.$$

Combined with the preceding construction, the desuspended mapping cone is a mixed Hodge complex with

$$(7.6) \quad W_k(\text{Cone}(f)[-1])_i^n := W_k K_i^n \oplus W_{k+1} L_i^{n-1}$$

and

$$(7.7) \quad F^p(\text{Cone}(f)[-1])_{\mathbb{C}}^n := F^p K_{\mathbb{C}}^n \oplus F^p L_{\mathbb{C}}^{n-1}.$$

The long exact sequence

$$(7.8) \quad \cdots \longrightarrow H^n(K) \longrightarrow H^n(L) \longrightarrow H^n(\text{Cone}(f)) \longrightarrow H^{n+1}(K) \longrightarrow H^{n+1}(L) \longrightarrow \cdots$$

(where one can replace $H^n(\text{Cone}(f))$ by $H^{n+1}(\text{Cone}(f)[-1])$) is a long exact sequence of mixed Hodge structures.

8. MIXED HODGE DIAGRAMS OF L_{∞} ALGEBRAS

In this section we show the compatibility of the construction of Fiorenza-Manetti with mixed Hodge complexes. The main object of study that we introduce will be the following.

Definition 8.1. A *mixed Hodge diagram of L_{∞} algebras* is the data of a mixed Hodge complex L where all components L_i are L_{∞} algebras and such that the operations ℓ_r in r variables of degree $2-r$ ($r \geq 1$) are compatible with the filtrations, in the sense that

$$(8.1) \quad \ell_r((W_{k_1} L_i^{n_1}) \wedge \cdots \wedge (W_{k_r} L_i^{n_r})) \subset W_{k_1+\cdots+k_r} L_i^{n_1+\cdots+n_r+2-r}$$

and

$$(8.2) \quad \ell_r((F^{p_1} L_{\mathbb{C}}^{n_1}) \wedge \cdots \wedge (F^{p_r} L_{\mathbb{C}}^{n_r})) \subset F^{p_1+\cdots+p_r} L_{\mathbb{C}}^{n_1+\cdots+n_r+2-r}.$$

Furthermore the comparison quasi-isomorphisms are required to be strong morphisms of L_{∞} algebras. If in each component $\ell_r = 0$ for $r \geq 3$ this is simply a *mixed Hodge diagram of Lie algebras*. A *morphism* between mixed Hodge diagrams of L_{∞} algebras L, M is given by a morphism of mixed Hodge complexes commuting with the operations ℓ_r .

At the level of cohomology, this behaves exactly as one would expect from mixed Hodge diagrams of Lie algebras.

Proposition 8.2. *If L is a mixed Hodge diagram of L_{∞} algebras (a fortiori, of Lie algebras) then on cohomology the induced Lie bracket is a morphism of mixed Hodge structures.*

Proof. By passing to cohomology we forget all the operations ℓ_r for $r \neq 2$ and ℓ_2 becomes a Lie bracket $[-, -]$. The statement that the Lie bracket respects F is clear because the condition

$$(8.3) \quad \ell_2((F^p L_{\mathbb{C}}^n) \wedge (F^q L_{\mathbb{C}}^m)) \subset F^{p+q} L_{\mathbb{C}}^{n+m}$$

directly induces on cohomology

$$(8.4) \quad [F^p H^n(L_{\mathbb{C}}), F^q H^m(L_{\mathbb{C}})] \subset F^{p+q} H^{n+m}(L_{\mathbb{C}}).$$

For W , in any of the components L_i , one has to be more careful. Take cohomology classes

$$(8.5) \quad [u] \in W[n]_k H^n(L_i), \quad [v] \in W[m]_{\ell} H^m(L_i).$$

Then $[u]$ comes from an element $u \in W_{k-n}L_i^n$, and $[v]$ comes from $v \in W_{\ell-m}L_i^m$. So

$$(8.6) \quad \ell_2(u \wedge v) \in W_{(k-n)+(\ell-m)}L_i^{n+m}$$

and this corresponds to

$$(8.7) \quad [[u], [v]] \in W[n+m]_{k+\ell}H^{n+m}(L_i).$$

This proves that the Lie bracket is a morphism of mixed Hodge structures. \square

Definition 8.3. An *augmented mixed Hodge diagram of Lie algebras* is the data of a mixed Hodge diagram of Lie algebras L and a Lie algebra \mathfrak{g} carrying a mixed Hodge structure, seen as a mixed Hodge diagram of Lie algebras concentrated in degree zero, together with a morphism

$$(8.8) \quad \varepsilon : L \longrightarrow \mathfrak{g}$$

of mixed Hodge diagrams of Lie algebras.

Now we can state and prove the main theorem of this section.

Theorem 8.4. *Let $\varepsilon : L \rightarrow \mathfrak{g}$ be an augmented mixed Hodge diagram of Lie algebras. Assume that in each of the components L_i the filtration W has only non-negative weights (i.e. $W_k L_i = 0$ for $k < 0$) and that the mixed Hodge structure on \mathfrak{g} is pure of weight zero. Then the desuspended mapping cone C of ε with its L_∞ algebra structure constructed by Fiorenza-Manetti is a mixed Hodge diagram of L_∞ algebras.*

Proof. It is practical to consider \mathfrak{g} as a mixed Hodge diagram of Lie algebras concentrated in degree 0. So we will write terms \mathfrak{g}_i^n that are zero for $n \neq 0$. The structure of mixed Hodge complex on C is written in Proposition 7.10 (with $C = \text{Cone}(\varepsilon)[-1]$), the axioms to be checked are in Definition 8.1 using the operations of L_∞ algebra on C described in Theorem 5.2.

Let Q be the codifferential on the cofree coalgebra on $C[1]$ which gives the structure of L_∞ algebra to C , with its components q_r ($r \geq 1$), and ℓ_r are the corresponding operations on C . Up to sign and shift of grading, the operations q_r and ℓ_r are given by the same algebraic formulas.

First check the compatibility for W in some component C_i .

For ℓ_1 , which is the differential of the desuspended mapping cone: take $(x, u) \in W_k C_i^n$, so that $x \in W_k L_i^n$ and $u \in W_{k+1} \mathfrak{g}_i^{n-1}$. Then we know that

$$(8.9) \quad \ell_1(x, u) = (d(x), \varepsilon(x) - d(u))$$

(actually $d = 0$ on \mathfrak{g}_i). But $d(x) \in W_k L_i^{n+1}$, $\varepsilon(x) \in W_k \mathfrak{g}_i^n \subset W_{k+1} \mathfrak{g}_i^n$ and $d(u) \in W_{k+1} \mathfrak{g}_i^n$. So one sees that

$$(8.10) \quad \ell_1(x, u) \in W_k C_i^{n+1}.$$

For ℓ_2 , which is the naive bracket: take $(x, u) \in W_k C_i^n$, $(y, v) \in W_\ell C_i^m$, so that $x \in W_k L_i^n$, $u \in W_{k+1} \mathfrak{g}_i^{n-1}$, $y \in W_\ell L_i^m$, $v \in W_{\ell+1} \mathfrak{g}_i^{m-1}$. We want to show that

$$\ell_2((x, u) \wedge (y, v)) \in W_{k+\ell} C_i^{n+m}.$$

For the part $\ell_2(x \wedge y)$ this is given (up to sign) by $[x, y]$, and it is in $W_{k+\ell} L_i^{n+m}$. For $\ell_2(u \otimes v)$ this is given up to sign by $[u, \varepsilon(y)]$ which is in $W_{(k+1)+\ell} \mathfrak{g}_i^{(n-1)+m}$. This proves the compatibility for ℓ_2 .

Now for the higher operations ℓ_r with $r \geq 3$ there is only one compatibility in the relation (5.7) to check, and up to sign this is just an iterated bracket. So take r elements

$$(8.11) \quad (x_j, u_j) \in W_{k_j} C_i^{n_j}, \quad j = 1, \dots, r$$

so that $x_j \in W_{k_j} L^{n_j}$ and $u_j \in W_{k_j+1} \mathfrak{g}^{n_j-1}$. In computing $\ell_r((x_1, u_1) \wedge \dots \wedge (x_r, u_r))$, the only nonzero part is when we multiply only one of the x_j with the others u_j ; call it x_s . Since \mathfrak{g}_i is concentrated in degree 0, this is zero if all u are not of degree zero or if x_s is not of degree 0. So we can assume $n_s = 0$ and $n_j = 1$ for $j \neq s$. Then for the iterated bracket, and for a permutation $\{t_1, \dots, t_{r-1}\}$ of $\{1, \dots, \hat{s}, \dots, r\}$,

$$(8.12) \quad [u_{t_1}, [u_{t_2}, \dots, [u_{t_{r-1}}, \varepsilon(x_s)] \dots]] \\ \in W_{(k_{t_1}+1)+\dots+(k_{t_{r-1}}+1)+k_s} \mathfrak{g}_i^{(n_{t_1}-1)+\dots+(n_{t_{r-1}}-1)+n_s} \\ = W_{k_1+\dots+k_r+(r-1)} \mathfrak{g}_i^{n_1+\dots+n_r-r+1} = W_{k_1+\dots+k_r+r-1}(\mathfrak{g}_i).$$

One would like

$$(8.13) \quad \ell_r((x_1, u_1) \wedge \dots \wedge (x_r, u_r)) \in W_{k_1+\dots+k_r} C_i^{n_1+\dots+n_r+2-r} = W_{k_1+\dots+k_r} C_i^1$$

so that the iterated bracket in (8.12) would land in $W_{k_1+\dots+k_r+1}(\mathfrak{g}_i)$. But if we assume that \mathfrak{g}_i has pure weight zero and since $r \geq 3$, the condition

$$W_{k_1+\dots+k_r+r-1}(\mathfrak{g}_i) \subset W_{k_1+\dots+k_r+1}(\mathfrak{g}_i) \subset \mathfrak{g}_i$$

is realized by an equality as soon as $k_1 + \dots + k_r + 1 \geq 0$. So, under the assumption that L_i has only non-negative weights, one can reduce the compatibility checking to $k_1, \dots, k_r \geq 0$ and this equality is realized.

The condition to check for F on $C_{\mathbb{C}}$ is much easier because there is no shift in the filtration. One sees directly that $(x, u) \in F^p C_{\mathbb{C}}^n$ means $x \in F^p L_{\mathbb{C}}^n$, $u \in F^p \mathfrak{g}_{\mathbb{C}}^{n-1}$, so that $d(x) \in F^p L_{\mathbb{C}}^{n+1}$, $\varepsilon(x) \in F^p \mathfrak{g}_{\mathbb{C}}^n$ and $d(u) \in F^p \mathfrak{g}_{\mathbb{C}}^n$ so

$$(8.14) \quad \ell_1(x, u) \in F^p C_{\mathbb{C}}^{n+1}.$$

For ℓ_2 then $(x, u) \in F^p C_{\mathbb{C}}^n$, $(y, v) \in F^q C_{\mathbb{C}}^m$ means that $x \in F^p L_{\mathbb{C}}^n$, $u \in F^p \mathfrak{g}_{\mathbb{C}}^{n-1}$, $y \in F^q L_{\mathbb{C}}^m$, $v \in F^q \mathfrak{g}_{\mathbb{C}}^{m-1}$. So

$$(8.15) \quad \ell_2((x, u) \wedge (y, v)) = \left([x, y], \frac{1}{2} [u, \varepsilon(y)] + \frac{(-1)^{|x|}}{2} [\varepsilon(x), v] \right) \\ \in F^{p+q} L_{\mathbb{C}}^{n+m} \oplus F^{p+q} \mathfrak{g}_{\mathbb{C}}^{n+m-1} = F^{p+q} C_{\mathbb{C}}^{n+m}.$$

Finally for the higher operations, take again $(x_j, u_j) \in F^{p_j} C_{\mathbb{C}}^{n_j}$ so that $x_j \in F^{p_j} L_{\mathbb{C}}^{n_j}$ and $u_j \in F^{p_j} \mathfrak{g}_{\mathbb{C}}^{n_j-1}$. Again, select one element x_s and consider a permutation $\{t_1, \dots, t_{r-1}\}$ of $\{1, \dots, \hat{s}, \dots, r\}$. Then

$$(8.16) \quad [u_{t_1}, [u_{t_2}, \dots, [u_{t_{r-1}}, \varepsilon(x_s)] \dots]] \\ \in F^{p_{t_1}+\dots+p_{t_{r-1}}+p_s} \mathfrak{g}_{\mathbb{C}}^{(n_{t_1}-1)+\dots+(n_{t_{r-1}}-1)+n_s} = F^{p_1+\dots+p_r} \mathfrak{g}_{\mathbb{C}}^{n_1+\dots+n_r-r+1}.$$

This checks directly that

$$(8.17) \quad \ell_r((x_1, u_1) \wedge \dots \wedge (x_r, u_r)) \in F^{p_1+\dots+p_r} C_{\mathbb{C}}^{n_1+\dots+n_r+2-r}.$$

□

Our theorem needs to be completed by a study of quasi-isomorphisms because we will have to work with mixed Hodge diagrams defined up to quasi-isomorphism.

Lemma 8.5. *Let $\varphi : L \xrightarrow{\approx} L'$ be a quasi-isomorphism of mixed Hodge diagrams of augmented Lie algebras over the same Lie algebra \mathfrak{g} with a mixed Hodge structure, satisfying both the hypothesis of Theorem 8.4. Then the induced morphism $\psi : C \rightarrow C'$ between the desuspended mapping cones is a quasi-isomorphism of mixed Hodge diagrams of L_∞ algebras.*

Remark 8.6. Without taking into account mixed Hodge complexes, the fact that ψ is a quasi-isomorphism of L_∞ algebras follows from the functoriality of the structure of Fiorenza-Manetti (showing that ψ is a morphism of L_∞ algebras) and from the five lemma applied to the long exact of the mapping cone (showing that ψ is a quasi-isomorphism). And by the way one recovers the Lemma 2.8 by invariance of the deformation functor under quasi-isomorphism.

Proof of Lemma 8.5. By functoriality ψ is already a morphism of mixed Hodge diagrams of L_∞ algebras. Then the argument is the same as in the above remark but applying first the functor Gr : in the component i by definition φ induces a quasi-isomorphism $\text{Gr}_k^W(L_i) \xrightarrow{\approx} \text{Gr}_k^W(L'_i)$. So apply the five lemma to the long exact sequence of the mapping cones of $\text{Gr}_k^W(L_i) \rightarrow \text{Gr}_k^W(\mathfrak{g})$ and $\text{Gr}_k^W(L'_i) \rightarrow \text{Gr}_k^W(\mathfrak{g})$ then ψ is a filtered quasi-isomorphism in the component i . Similarly over \mathbb{C} with the two filtrations W, F then ψ is a bifiltered quasi-isomorphism. \square

9. BAR CONSTRUCTION ON MIXED HODGE DIAGRAMS OF L_∞ ALGEBRAS

Then we show the compatibility of the bar construction with the mixed Hodge diagrams of L_∞ algebras. We will show that if L is such a mixed Hodge diagram then $\mathcal{C}(L)$ (defined by applying the functor \mathcal{C} to each component of the diagram) is a (inductive limit of) mixed Hodge diagram of coalgebras, and its H^0 is then a coalgebra with a mixed Hodge structure.

Recall that $\mathcal{C}(L)$ has a canonical increasing filtration given by

$$(9.1) \quad \mathcal{C}_s(L) := \bigoplus_{r=1}^s (L[1])^{\odot r}, \quad s \geq 1$$

by sub-DG coalgebras that we call the *bar filtration* and $\mathcal{C}(L)$ is the inductive limit of the $\mathcal{C}_s(L)$. We will work with a fixed index s and we will always consider $\mathcal{C}(L)$ as an inductive limit.

Definition 9.1. Let (L, W) be a filtered L_∞ algebra (i.e. W satisfies equation (8.1) of Definition 8.1). The filtration induced by W on $\mathcal{C}(L)$ via $W[1]$ on $L[1]$ and then by multiplicative extension to $(L[1])^{\odot r}$ is called the *bar-weight filtration*. We denote it by $\mathcal{C}W$.

As in Hain's work [Hai87, § 3.2], the bar-weight filtration is a convolution of the weight filtration and the bar filtration.

Remark 9.2. The bar-weight filtration on $\mathcal{C}(L)$ may not be biregular, however it is biregular on each $\mathcal{C}_s(L)$ if L is bounded-below because it involves only a finite number of symmetric products.

Lemma 9.3. *Let (L, W) be a filtered L_∞ algebra, bounded-below. Then $\mathcal{C}_s(L)$ is a filtered DG coalgebra for the bar-weight filtration.*

Proof. As $\mathcal{C}W$ is induced by $W[1]$ on $L[1]$ and then by multiplicative extension, and seeing the algebraic formula for the coproduct of the cofree coalgebra (Definition 4.3 equation (4.6)), it is clear that $\mathcal{C}W$ is compatible with the graded coalgebra structure. Then we have to show that the codifferential Q of $\mathcal{C}_s(L)$ respects the filtration, and it is enough to check it for its components $q_r : (L[1])^{\odot r} \rightarrow L[1]$ ($r \geq 1$) because of the explicit formula for recovering Q from its components ([Man04, VIII.34]).

So take r elements

$$(9.2) \quad x_j[1] \in W[1]_{k_j} L_i[1]^{n_j}, \quad j = 1, \dots, r$$

which means $x_j \in W_{k_j-1} L_i^{n_j+1}$. Then

$$(9.3) \quad \begin{aligned} q_r(x_1[1] \odot \dots \odot x_r[1]) &= \pm \ell_r(x_1 \wedge \dots \wedge x_r) \\ &\in W_{(k_1-1)+\dots+(k_r-1)} L_i^{(n_1+1)+\dots+(n_r+1)+(2-r)} \\ &\subset W_{k_1+\dots+k_r-1} L_i^{n_1+\dots+n_r+2} = W[1]_{k_1+\dots+k_r} L_i[1]^{n_1+\dots+n_r+1}. \end{aligned}$$

This is the desired compatibility. \square

If L is a bifiltered L_∞ algebra (i.e. equipped with filtration W, F as is $L_{\mathbb{C}}$ in Definition 8.1) then there is an induced filtration F on $\mathcal{C}_s(L)$ defined by the induced F (without shifting) to $L[1]$ and then by multiplicative extension. It is then easier to see that $\mathcal{C}_s(L)$ is also a filtered coalgebra for F : take r elements

$$(9.4) \quad x_j[1] \in F^{p_j} L_{\mathbb{C}}[1]^{n_j}, \quad j = 1, \dots, r$$

which means $x_j \in F^{p_j} L_{\mathbb{C}}^{n_j+1}$ and then directly

$$(9.5) \quad \begin{aligned} q_r(x_1[1] \odot \dots \odot x_r[1]) &= \pm \ell_r(x_1 \wedge \dots \wedge x_r) \\ &\in F^{p_1+\dots+p_r} L_{\mathbb{C}}^{(n_1+1)+\dots+(n_r+1)+(2-r)} \\ &= F^{p_1+\dots+p_r} L_{\mathbb{C}}^{n_1+\dots+n_r+2} = F^{p_1+\dots+p_r} L_{\mathbb{C}}[1]^{n_1+\dots+n_r+1}. \end{aligned}$$

To sum up, if L is a mixed Hodge diagram of L_∞ algebras whose components L_i are bounded-below, then $\mathcal{C}_s(L)$ is a diagram consisting of a filtered DG coalgebra $(\mathcal{C}_s(L_{\mathbf{k}}), \mathcal{C}W)$ over \mathbf{k} and a bifiltered DG coalgebra $(\mathcal{C}_s(L_{\mathbb{C}}), \mathcal{C}W, F)$ over \mathbb{C} , related by a chain of morphisms

$$(9.6) \quad (\mathcal{C}_s(L_{\mathbf{k}}), \mathcal{C}W) \otimes_{\mathbf{k}} \mathbb{C} = (\mathcal{C}_s(L_{\mathbf{k}} \otimes_{\mathbf{k}} \mathbb{C}), \mathcal{C}W) \longleftrightarrow (\mathcal{C}_s(L_{\mathbb{C}}), \mathcal{C}W)$$

of DG coalgebras filtered by $\mathcal{C}W$.

We arrive finally at our main goal. For this we follow closely the method of Hain [Hai87, § 3] for commutative DG algebras, re-writing it for L_∞ algebras. This is also re-written in [PS08, § 8.7].

Theorem 9.4. *Let L be a mixed Hodge diagram of L_∞ algebras. Assume that each component L_i is non-negatively graded ($L_i^n = 0$ for $n < 0$) and that $H^0(L) = 0$. Then $\mathcal{C}_s(L)$ is a mixed Hodge diagram of coalgebras for any $s \geq 1$ and $\mathcal{C}(L)$ is an inductive limit of mixed Hodge diagrams of coalgebras.*

Proof. We need to check the axioms of Definition 7.2 for $\mathcal{C}_s(L)$. The fact that $\mathcal{C}(L)$ will be an inductive limit in the category of mixed Hodge diagrams will then be clear.

We fix temporarily a component L_i . Since we took the precaution to work with inductive limits, $\mathcal{C}_s(L_i)$ is a bounded-below complex because it is obtained by a

finite numbers of symmetric powers from $L_i[1]$ which is bounded-below. Also, the filtration induced by W is biregular.

The axiom 1 is almost checked during the proof of Theorem 4.8: using transfer of structure to the cohomology, up to L_∞ -quasi-isomorphism (which does not change the cohomology of $\mathcal{C}_s(L)$), one can assume that L_i has $L_i^n = 0$ for $n \leq 0$ and other terms L_i^n finite-dimensional. Then $L_i[1]^n = 0$ for $n < 0$ and $\mathcal{C}_s(L_i)$ is a finite sum of a finite number of symmetric powers of such $L_i[1]$ so is finite-dimensional in each degree and so is $H^\bullet(\mathcal{C}_s(L_i))$.

To go further we need to compute the spectral sequence for the bar-weight filtration on $\mathcal{C}_s(L)$ and relate it to the spectral sequence for the weight filtration on L .

Since W and $\mathcal{C}W$ are decreasing filtrations, we work with $-k$ instead of k . We denote by $\mathcal{C}_s(L_i)^m$ the component of total degree m in $\mathcal{C}_s(L_i)$. Then by definition of the spectral sequence

$$(9.7) \quad \mathcal{C}W E_0^{-k,q}(\mathcal{C}_s(L_i)) = \mathrm{Gr}_k^{\mathcal{C}W} \mathcal{C}_s(L_i)^{-k+q}.$$

Since

$$(9.8) \quad \begin{aligned} \mathcal{C}W_k(L_i[1])^{\odot r} &= \bigoplus_{k_1+\dots+k_s=k} W[1]_{k_1} L_i[1] \odot \dots \odot W[1]_{k_s} L_i[1] \\ &= \bigoplus_{k_1+\dots+k_r=k-r} W_{k_1} L_i[1] \odot \dots \odot W_{k_s} L_i[1] \end{aligned}$$

it follows that

$$(9.9) \quad \mathrm{Gr}_k^{\mathcal{C}W} \mathcal{C}_s(L_i)^{-k+q} = \bigoplus_{r=1}^s \mathrm{Gr}_{k-r}^W ((L_i[1])^{\odot r})^{-k+q} = \bigoplus_{r=1}^s \mathrm{Gr}_{k-r}^W (L_i^{\wedge r})^{-k+q+r}.$$

So we recognize

$$(9.10) \quad \mathcal{C}W E_0^{-k,q}(\mathcal{C}_s(L_i)) = \bigoplus_{r=1}^s W E_0^{-k+r,q}(L_i^{\wedge r}).$$

The differential d_0 on $\mathcal{C}W E_0^{-k,\bullet}(\mathcal{C}_s(L_i))$ is induced by the codifferential $Q := \sum q_r$ of $\mathcal{C}(L_i)$. Crucial here is the equation (9.3) appearing in the preceding proof, which shows that q_r is zero on $\mathrm{Gr}_k^{\mathcal{C}W} \mathcal{C}(L_i)$ for all $r \geq 2$. Thus d_0 is only induced by q_1 , which is up to sign $d[1]$, and it is the sum of the d_0 's appearing on the right side of (9.10). But by Proposition 7.6 $L^{\wedge r}$ is a mixed Hodge complex and this right side is a direct sum of terms $W E_0$ of mixed Hodge complexes.

This computation allows us to check that the comparisons morphisms are quasi-isomorphisms. Let

$$(9.11) \quad \varphi : (L_i, W) \xrightarrow{\cong} (L_j, W)$$

be some comparison morphism between the two components L_i, L_j , which by hypothesis is a filtered quasi-isomorphism. By the Künneth formula (combined with the fact that we work with bounded below complexes) φ induces a filtered quasi-isomorphism

$$(9.12) \quad ((L_i)^{\wedge r}, W^{\wedge r}) \xrightarrow{\cong} ((L_j)^{\wedge r}, W^{\wedge r})$$

so it induces an isomorphism

$$(9.13) \quad W E_0^{-k+r,q}(L_i^{\wedge r}) \xrightarrow{\cong} W E_0^{-k+r,q}(L_j^{\wedge r}).$$

So equation (9.10) tells us precisely that φ induces a filtered quasi-isomorphism

$$(9.14) \quad (\mathcal{C}_s(L_i), \mathcal{C}W) \xrightarrow{\sim} (\mathcal{C}_s(L_j), \mathcal{C}W).$$

Then we can check the axiom 2, in the component over \mathbb{C} carrying also the filtration F . By this axiom for $L_{\mathbb{C}}^{\wedge r}$ the differential of ${}_W E_0^{-k+r, \bullet}(L_{\mathbb{C}}^{\wedge r})$ is strictly compatible with the induced filtration F , so from equation (9.10) the differential of ${}_{\mathcal{C}W} E_0^{k, \bullet}(\mathcal{C}_r(L_{\mathbb{C}}))$ is the direct sum of these and is also strictly compatible with F .

Finally to check the axiom 3 we compute the spectral sequence at the page E_1 . By definition

$$(9.15) \quad {}_{\mathcal{C}W} E_1^{-k, q}(\mathcal{C}_s(L_i)) = H^{-k+q}(\mathrm{Gr}_k^{\mathcal{C}W} \mathcal{C}_s(L_i))$$

where the cohomology is computed with respect to d_0 . Then using (9.10)

$$(9.16) \quad H^{-k+q}(\mathrm{Gr}_k^{\mathcal{C}W} \mathcal{C}_s(L_i)) = \bigoplus_{r=1}^s H^{-k+q}(\mathrm{Gr}_{k-r}^W(L_i^{\wedge r})^{\bullet+r}) = \bigoplus_{r=1}^s {}_W E_1^{-k+r, q}(L_i^{\wedge r}).$$

So, put together,

$$(9.17) \quad {}_{\mathcal{C}W} E_1^{-k, q}(\mathcal{C}_s(L_i)) = \bigoplus_{r=1}^s {}_W E_1^{-k+r, q}(L_i^{\wedge r}).$$

Since $L^{\wedge r}$ is a mixed Hodge complex, in equation (9.17) the terms on the right side

$$(9.18) \quad {}_W E_1^{-k+r, q}(L_i^{\wedge r}) = H^{-k+r+q}(\mathrm{Gr}_{k-r}^W L_i^{\wedge r})$$

define, when varying i , a pure Hodge structure of weight q . So does their direct sum and this proves that the terms

$$(9.19) \quad {}_{\mathcal{C}W} E_1^{-k, q}(\mathcal{C}_s(L_i)) = H^{-k+q}(\mathrm{Gr}_k^{\mathcal{C}W} \mathcal{C}_s(L_i))$$

define a pure Hodge structure of weight q . \square

As for the main theorem of the preceding section we need to complete our theorem by a study of quasi-isomorphisms. It is already clear that this construction is functorial.

Lemma 9.5 (Compare with Thm. 4.6). *Let $\varphi : L \xrightarrow{\sim} L'$ be a quasi-isomorphism of mixed Hodge diagrams of L_{∞} algebras satisfying both the hypothesis of Theorem 9.4. Then the induced morphism $\mathcal{C}(\varphi)$ is a quasi-isomorphism of mixed Hodge diagrams of coalgebras.*

Proof. By the functoriality and explicit nature of the bar construction, it is already clear that $\mathcal{C}(\varphi)$ is a morphism of diagrams of filtered DG coalgebras, compatible with the bar filtration. Then, following the proof of the preceding theorem, we see that in the component i and in equation (9.17) the hypothesis tells us that φ_i induces an isomorphism on the right-hand side, so it induces an isomorphism on the left-hand side. Similarly for the bifiltered part we repeat the arguments replacing $L_{\mathbb{C}}$ by $\mathrm{Gr}_F^p(L_{\mathbb{C}})$. \square

Let us sum up what we will need.

Corollary 9.6. *If L is a mixed Hodge diagram of L_{∞} algebras satisfying the hypothesis of Theorem 9.4 then the pro-Artin algebra*

$$(9.20) \quad R := \mathbf{k} \oplus H^0(\mathcal{C}(L))^*$$

of the pro-representability Theorem 4.8 has a mixed Hodge structure, functorial in L , independent of L up to quasi-isomorphism.

Proof. Since $\mathcal{C}_s(L)$ is a mixed Hodge diagram, its H^0 has a mixed Hodge structure and so has its dual. This defines a mixed Hodge structure on the coalgebra $H^0(\mathcal{C}(L))$ and a mixed Hodge structure on the pro-Artin algebra R . \square

10. CONSTRUCTION OF MIXED HODGE DIAGRAMS: THE COMPACT CASE

In the two following sections we present several different geometric situations concerning a complex manifold X and a representation ρ of its fundamental group into a linear algebraic group G . In each of them we construct an appropriate augmented mixed Hodge diagram of Lie algebras that controls the deformation theory of ρ . Then the machinery we developed in the two preceding sections gives us directly and functorially a mixed Hodge structure on the complete local ring $\widehat{\mathcal{O}}_\rho$ of the representation variety $\text{Hom}(\pi_1(X, x), G)$ at ρ .

The compact case is much easier to deal with because the construction of a mixed Hodge diagram over \mathbb{R} computing the cohomology of a variety is straightforward using the algebra of differential forms. So let X be a compact Kähler manifold, for example a smooth complex projective algebraic variety.

Definition 10.1. A real *polarized variation of Hodge structure* of weight k on X is the data of a local system of finite-dimensional real vector spaces V on X with a decreasing filtration of the associated holomorphic vector bundle by holomorphic sub-vector bundles $\mathcal{F}^\bullet \subset V \otimes \mathcal{O}_X$, a flat bilinear map $Q : V \otimes V \rightarrow \mathbb{R}$, and a flat connection $\nabla : V \otimes \mathcal{O}_X \rightarrow V \otimes \Omega_X^1$ such that at each point $x \in X$ the data $(V_x, \mathcal{F}_x^\bullet, Q_x)$ forms a polarized Hodge structure of weight k . Furthermore ∇ is required to satisfy Griffiths' transversality $\nabla(\mathcal{F}^p) \subset \mathcal{F}^{p-1} \otimes \Omega_X^1$.

Let x be a base point of X . Let

$$(10.1) \quad \rho : \pi_1(X, x) \longrightarrow GL(V_x)$$

be a representation which is the monodromy of a real polarized variation of Hodge structure $(V, \mathcal{F}^\bullet, \nabla, Q)$ of weight k on X . The local system of Lie algebras associated to ρ is now $\text{End}(V)$, which by the usual linear algebraic constructions is a polarized variation of Hodge structures of weight zero. Explicitly

$$(10.2) \quad \mathcal{F}^p \text{End}(V \otimes \mathcal{O}_X) = \{f : V \otimes \mathcal{O}_X \rightarrow V \otimes \mathcal{O}_X \mid f(\mathcal{F}^\bullet) \subset \mathcal{F}^{\bullet+p}\}.$$

One constructs a real mixed Hodge diagram as follows. Let

$$(10.3) \quad L_{\mathbb{R}} := \mathcal{E}^\bullet(X, \text{End}(V))$$

and

$$(10.4) \quad L_{\mathbb{C}} := \mathcal{E}^\bullet(X, \text{End}(V \otimes \mathbb{C})).$$

One defines a filtration W which is the trivial one (everything has weight zero) and a filtration F on $L_{\mathbb{C}}$ by

$$(10.5) \quad F^p L_{\mathbb{C}}^n := \bigoplus_{r+s \geq p} \mathcal{E}^{r, n-r}(X, \mathcal{F}^s \text{End}(V)).$$

The DG Lie algebra structure is given as usual in (3.11) and (3.12).

Then we define an augmentation ε_x from $(L_{\mathbb{R}}, L_{\mathbb{C}}, W, F)$ to the mixed Hodge diagram of Lie algebras formed by the Hodge structure on the Lie algebra $\mathfrak{g} := \text{End}(V_x)$ with

$$(10.6) \quad \mathfrak{g}_{\mathbb{C}} = \text{End}(V_x \otimes \mathbb{C}) = \text{End}(V_x) \otimes \mathbb{C}$$

and the Hodge filtration is simply $F^{\bullet} = \mathcal{F}_x^{\bullet}$, by evaluating forms of degree zero at x as in (3.14).

Theorem 10.2. *The data*

$$(10.7) \quad (L_{\mathbb{R}}, L_{\mathbb{C}}, W, F)$$

forms a real mixed Hodge diagram of Lie algebras. Together with $\varepsilon_x : L \rightarrow \mathfrak{g}$ this is an augmented mixed Hodge diagram of Lie algebras satisfying the hypothesis of Theorem 8.4.

Proof. The fact that L is a real mixed Hodge complex is essentially the classical Hodge theory with values in a variation of Hodge structure of Deligne-Zucker [Zuc79, § 2] and follows from the Kähler identities with twisted coefficients. And by construction the Lie bracket and ε_x are compatible with the filtrations. \square

So we apply for the first time the method we developed.

Theorem 10.3. *If X is a compact Kähler manifold and ρ is the monodromy of a real polarized variation of Hodge structure V on X , then there is a real mixed Hodge structure on the complete local ring $\widehat{\mathcal{O}}_{\rho}$ of the representation variety $\text{Hom}(\pi_1(X, x), GL(V_x))$ at ρ which is functorial in X, x, ρ .*

Proof. Over both fields \mathbb{R} and \mathbb{C} , L and its augmentation control the deformation theory of ρ : this is the main theorem of Goldman-Millson (Theorem 3.3). We will say that the functor of deformations of ρ is controlled by the mixed Hodge diagram of augmented Lie algebras L . By Lemma 5.3, this deformation functor is associated with the L_{∞} algebra structure on the desuspended mapping cone C of ε_x .

We need to check that $H^0(C) = 0$. By definition of the cone, a closed element of C^0 is given by a \mathcal{C}^{∞} section ω of $\text{End}(V)$ such that $d(\omega) = 0$, so that ω is locally constant, and such that $\omega(x) = 0$. So $\omega = 0$ globally.

Then the deformation functor of ρ is pro-represented by a pro-Artin algebra that we denote by R as in Theorem 4.8. By the pro-Yoneda lemma this R is canonically isomorphic to $\widehat{\mathcal{O}}_{\rho}$. Again all this construction commutes with the change of base field so we work with C as mixed Hodge diagram. The augmented mixed Hodge diagram of Lie algebras we just have defined satisfies the hypothesis of our Theorem 8.4. So C is a mixed Hodge diagram of L_{∞} algebras. Then we apply the Theorem 9.4 (or its Corollary 9.6) to get a mixed Hodge structure on the pro-Artin algebra R , which as we said is canonically isomorphic to $\widehat{\mathcal{O}}_{\rho}$ (as pro-Artin algebra, so that each quotients by powers of the maximal ideals are isomorphic). And this induces the mixed Hodge structure on $\widehat{\mathcal{O}}_{\rho}$. \square

With the same method of proof, there are several possible variations. First one can work with representations taking values in a real linear algebraic group.

Proposition 10.4. *Let G be a real linear algebraic group. Let $\rho : \pi_1(X, x) \rightarrow G(\mathbb{R})$ be a representation which is the monodromy of a real polarized variation of Hodge structure on X . Then there is a functorial mixed Hodge structure on the local ring $\widehat{\mathcal{O}}_{\rho}$ of the representation variety $\text{Hom}(\pi_1(X, x), G)$ at ρ .*

Proof. Re-write the proof of Theorem 10.3 by replacing $GL(V_x)$ by G and $\text{End}(V_x)$ by the Lie algebra \mathfrak{g} of G . Then again one gets L that is an augmented mixed Hodge diagram Lie algebras over \mathfrak{g} . \square

One can also work with mixed Hodge structures over \mathbb{C} , as in [ES11].

Definition 10.5 ([ES11, 1.1]). A *complex Hodge structure* of weight k is the data of a finite-dimensional vector space K over \mathbb{C} with two filtrations F, \overline{G} , such that K decomposes as $K = \bigoplus_{p+q=k} K^{p,q}$ with

$$(10.8) \quad F^p K = \bigoplus_{p' \geq p} K^{p',q}, \quad \overline{G}^q K = \bigoplus_{q' \geq q} K^{p,q'}.$$

A *complex mixed Hodge structure* is the data of a finite-dimensional vector space K over \mathbb{C} with two decreasing filtrations F, \overline{G} and an increasing filtration W such that each graded part $\text{Gr}_k^W(K)$ with the induced filtrations F, \overline{G} is a complex Hodge structure of weight k .

We refer to [ES11] for the definitions of polarization (Def. 1.1) and of variation of Hodge structure (Def. 1.8) in this context.

For example if K is a mixed Hodge structure over $\mathbf{k} \subset \mathbb{R}$ then $K \otimes \mathbb{C}$ is canonically a mixed Hodge structure over \mathbb{C} with \overline{G} being the conjugate filtration of F . It is polarized if K is.

So one can state the most general result:

Proposition 10.6. *Let G be a complex linear algebraic group. Let $\rho : \pi_1(X, x) \rightarrow G(\mathbb{C})$ be a representation which is the monodromy of a complex polarized variation of Hodge structure on X . Then there is a functorial complex mixed Hodge structure on the local ring $\widehat{\mathcal{O}}_\rho$ of the representation variety $\text{Hom}(\pi_1(X, x), G)$ at ρ .*

Proof. Define a *complex mixed Hodge complex* to be the data of a DG vector space K over \mathbb{C} equipped with filtrations W, F, \overline{G} , satisfying the usual axioms of mixed Hodge complex (Definition 7.2): in the axiom 2 we require the differential of $\text{Gr}_k^W(K)$ to be strictly compatible with *both* filtrations F, \overline{G} and in the axiom 3 we require each term $H^n(\text{Gr}_k^W(K))$ to carry a pure complex Hodge structure with the induced filtrations F, \overline{G} .

Then each term of the cohomology of a complex mixed Hodge complex carries a complex mixed Hodge structure. So we can re-write everything with complex mixed Hodge structures and complex mixed Hodge complexes. \square

We can also show that the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ is defined over $\mathbf{k} \subset \mathbb{R}$ if ρ is (i.e. if the polarized variation of Hodge structure whose monodromy is ρ and the algebraic group G are defined over \mathbf{k}).

Proposition 10.7. *In Theorem 10.3 assume that the variation of Hodge structure V is defined over \mathbf{k} . Then the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ is defined over \mathbf{k} .*

However we will prove this only after studying the non-compact case. For this one needs to construct mixed Hodge diagrams over \mathbb{Q} and this is rational homotopy theory.

11. CONSTRUCTION OF MIXED HODGE DIAGRAMS: THE NON-COMPACT CASE

In the case where X is non-compact, the construction of an appropriate mixed Hodge diagram that computes the cohomology of X is more difficult and depends on the choice of a compactification of X .

So let X be a smooth quasi-projective algebraic variety over \mathbb{C} . Let x be a base point of X . Let G be a linear algebraic group over $\mathbf{k} \subset \mathbb{R}$ with Lie algebra \mathfrak{g} . Let

$$(11.1) \quad \rho : \pi_1(X, x) \longrightarrow G(\mathbf{k})$$

be a representation and we assume that ρ has *finite* image. Under these hypothesis the ideas to construct a controlling mixed Hodge diagram of Lie algebras are already entirely present in the work of Kapovich-Millson [KM98, § 14–15]. However they rely strongly on the theory of minimal models of Morgan [Mor78] which is not completely functorial. So we re-write these ideas using the more powerful construction of mixed Hodge diagrams of Navarro Aznar [Nav87].

Let us first explain briefly the ideas and the notations. Let the finite group

$$(11.2) \quad \Phi := \frac{\pi_1(X, x)}{\text{Ker}(\rho)} \simeq \rho(\pi_1(X, x)).$$

To $\text{Ker}(\rho) \subset \pi_1(X, x)$ corresponds a finite étale Galois cover $\pi : Y \rightarrow X$ with automorphism group Φ that acts simply transitively on the fibers, and equipped with a fixed base point $y \in Y$ over x . It is known that Y is automatically a smooth quasi-projective algebraic variety. The flat principal bundle P induced by the holonomy of ρ is trivial when pulled-back to Y , as well as its adjoint bundle $\text{Ad}(P)$. So the DG Lie algebra of Goldman-Millson is (over \mathbb{R} or \mathbb{C})

$$(11.3) \quad L := \mathcal{E}^\bullet(X, \text{Ad}(P)) = (\mathcal{E}^\bullet(Y, \pi^* \text{Ad}(P)))^\Phi = (\mathcal{E}^\bullet(Y) \otimes \mathfrak{g})^\Phi$$

(where the exponent Φ denotes the invariants by the action of Φ). In order to construct a mixed Hodge diagram that is quasi-isomorphic to this we simply want to find a mixed Hodge diagram for Y equipped with an action of Φ , then tensor it with \mathfrak{g} , then take the invariants by Φ .

For the augmentation, since there is a canonical identification of fibers $\text{Ad}(P)_x \simeq \mathfrak{g}$ one can define an augmentation

$$(11.4) \quad \varepsilon_x : \mathcal{E}^\bullet(X, \text{Ad}(P)) \longrightarrow \mathfrak{g}$$

exactly as in the compact case (3.14) by evaluating degree zero forms at x and higher degree forms to zero. This augmentation can be lifted equivariantly to Y : ε_x comes from the augmentation

$$(11.5) \quad \eta_x : \mathcal{E}^\bullet(Y) \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

defined by

$$(11.6) \quad \eta_x(\omega \otimes u) := \frac{1}{|\Phi|} \sum_{g \in \Phi} \varepsilon_y(g.(\omega \otimes u))$$

where ε_y simply evaluates forms with values in \mathfrak{g} at y . Observe the notations: ε_y depends on y but in the definition of η_x we sum over the whole (finite) fiber of π over x so η_x depends only on x . Then we see that η_x induces ε_x when restricted to the equivariant forms $(\mathcal{E}^\bullet(Y) \otimes \mathfrak{g})^\Phi$.

We will also need an *equivariant completion* of Y . By the theorem of Sumihiro [Sum74] it is possible to compactify $Y \hookrightarrow \overline{Y}'$ so that the action of Φ extends to \overline{Y}' .

And then by the work of Bierstone-Milman on canonical resolutions of singularities [BM97, § 13] one can construct a resolution of singularities $\overline{Y} \rightarrow \overline{Y}'$ to which the action of Φ lifts. This \overline{Y} we call an equivariant completion of Y and $D := \overline{Y} \setminus Y$ is a divisor with simple normal crossings on which Φ acts.

From the data of a smooth quasi-projective variety Y with a smooth compactification \overline{Y} by a divisor with normal crossings D , Navarro Aznar [Nav87] has constructed a functorial mixed Hodge diagram of commutative algebras computing the cohomology algebra of Y over $\mathbf{k} \subset \mathbb{R}$ together with its mixed Hodge structure. Let us denote by $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}$ this diagram. We will only need to know that each component is related by a canonical chain of quasi-isomorphisms to the usual DG algebra computing the cohomology of Y : the component over \mathbb{C} is related to the usual differential forms on Y via the holomorphic forms with logarithmic poles along D , denoted by $\Omega_{\overline{Y}}^{\bullet}(\log D)$, and the component over \mathbf{k} is related to the usual singular cochain complex over \mathbf{k} .

Since these mixed Hodge diagrams are functorial, in our situation the group Φ acts on $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}$. We always denote by an exponent Φ the diagram formed by the elements invariant under the action of Φ .

Lemma 11.1. *The diagram of invariants $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}^{\Phi}$ is again a mixed Hodge diagram that computes canonically the cohomology of X with its mixed Hodge structure.*

Proof. This follows essentially from the fact that taking the invariants by a finite group commutes with cohomology. Hence the cohomology of X is given by the invariant cohomology of Y . Since Φ acts by morphisms of mixed Hodge diagrams (i.e. preserves the structures of DG algebras and the filtrations) it is easy to check that $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}^{\Phi}$ is again a mixed Hodge diagram that computes this invariant cohomology of Y . \square

Now equation (11.3) and the related remarks explain how to construct the controlling DG Lie algebra of Goldman-Millson. In each component of $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}$, which is a commutative DG algebra, one tensors by \mathfrak{g} which is defined over \mathbf{k} . Then the group Φ acts on both the DG algebra and on \mathfrak{g} and we take the invariants. We denote this by

$$(11.7) \quad M := (\text{MHD}(\overline{Y}, D)_{\mathbf{k}} \otimes \mathfrak{g})^{\Phi}.$$

Lemma 11.2. *This M is a mixed Hodge diagram of Lie algebras that is canonically quasi-isomorphic to the DG Lie algebra L of Goldman-Millson controlling the deformation theory of ρ .*

Proof. First, each component of M is a DG Lie algebra. The fact that both taking cohomology and taking the graded pieces of filtrations commute with both the tensor product with \mathfrak{g} (which is concentrated in degree zero and has trivial filtrations) and with the invariants by Φ (which is a finite group that acts preserving all these structures) implies easily that M is a mixed Hodge diagram of Lie algebras.

Then follow the equation (11.3): since $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}$ computes equivariantly the cohomology of Y then $\text{MHD}(\overline{Y}, D)_{\mathbf{k}} \otimes \mathfrak{g}$ computes the cohomology of Y with coefficients in \mathfrak{g} and M computes the cohomology of X with twisted coefficients in $\text{Ad}(P)$. \square

Remark 11.3. We see from the proof that a quasi-isomorphism φ between two such mixed Hodge diagrams associated with two equivariant compactifications $\overline{Y}, \overline{Y}'$, with φ equivariant with respect to Φ , induces a quasi-isomorphism $M \xrightarrow{\sim} M'$ between the corresponding mixed Hodge diagrams of Lie algebras.

The last step is to construct the augmentation ε_x at the level of M and not L . Essentially this follows from the fact that the evaluation of differential forms on Y at a base point y can be defined sheaf-theoretically: it is induced by the natural morphism from the constant sheaf \mathbf{k}_Y to the skyscraper sheaf \mathbf{k}_y which in some sense evaluates sections at y . However at this point we need to enter more into the detailed construction of Navarro Aznar.

Recall that the mixed Hodge diagrams of Navarro Aznar are obtained in two steps. First one defines sheaves of commutative DG algebras equipped with filtrations, forming a *mixed Hodge diagram of sheaves of commutative algebras*. Then one applies the *Thom-Whitney functor* $R_{\text{TW}}\Gamma$ which is quasi-isomorphic to the usual derived functor of global sections $R\Gamma$ and gives here a mixed Hodge diagram of commutative algebras.

The component over \mathbb{C} of $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}$ is the $R_{\text{TW}}(\overline{Y}, -)$ of a certain sheaf of analytic differential forms on \overline{Y} with logarithmic poles along D called the *logarithmic Dolbeaut complex* ([Nav87, § 8]) denoted by $\mathcal{A}_{\overline{Y}}^{\bullet}(\log D)_{\mathbb{C}}$. So it is naturally equipped with an augmentation

$$(11.8) \quad \mu_y : \mathcal{A}_{\overline{Y}}^{\bullet}(\log D)_{\mathbb{C}} \longrightarrow \mathbb{C}_y$$

which by functoriality induces a morphism

$$(11.9) \quad R_{\text{TW}}(\mu_y) : R_{\text{TW}}\Gamma(\overline{Y}, \mathcal{A}_{\overline{Y}}^{\bullet}(\log D)_{\mathbb{C}}) \longrightarrow R_{\text{TW}}\Gamma(\overline{Y}, \mathbb{C}_y).$$

The left-hand side is by definition the component over \mathbb{C} of $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}$ and the right-hand side is simply \mathbb{C} itself. Similarly, the component over \mathbb{R} comes from a certain sheaf $\mathcal{A}_{\overline{Y}}^{\bullet}(\log D)_{\mathbb{R}}$ of real analytic differential forms on \overline{Y} with logarithmic poles along D and comes equipped with a natural augmentation to \mathbb{R}

$$(11.10) \quad R_{\text{TW}}(\mu_y) : R_{\text{TW}}\Gamma(\overline{Y}, \mathcal{A}_{\overline{Y}}^{\bullet}(\log D)_{\mathbb{R}}) \longrightarrow R_{\text{TW}}\Gamma(\overline{Y}, \mathbb{R}_y) \simeq \mathbb{R}.$$

For the component over \mathbf{k} : let us denote by $j : Y \hookrightarrow \overline{Y}$ the compactification then the sheaf of commutative DG algebras forming the mixed Hodge diagram of sheaves on \overline{Y} is $R_{\text{TW}}j_*\mathbf{k}_Y$ (which is quasi-isomorphic to the usual $Rj_*\mathbf{k}_Y$). It is canonically equipped with an augmentation μ_y to \mathbf{k}_y , which induces

$$(11.11) \quad R_{\text{TW}}(\mu_y) : R_{\text{TW}}(\overline{Y}, R_{\text{TW}}j_*\mathbf{k}_Y) \longrightarrow R_{\text{TW}}(\overline{Y}, \mathbf{k}_y) \simeq \mathbf{k}.$$

All this defines an augmentation of the mixed Hodge diagram

$$(11.12) \quad \mu_y : \text{MHD}(\overline{Y}, D)_{\mathbf{k}} \longrightarrow \mathbf{k}.$$

Then we mimic (11.6): we define an augmentation

$$(11.13) \quad \nu_x : \text{MHD}(\overline{Y}, D)_{\mathbf{k}} \otimes \mathfrak{g} \longrightarrow \mathfrak{g}$$

as

$$(11.14) \quad \nu_x(\omega \otimes u) := \frac{1}{|\Phi|} \sum_{g \in \Phi} (R_{\text{TW}}(\mu_y) \otimes \text{id}_{\mathfrak{g}})(g.(\omega \otimes u))$$

where again, in each component, g acts on both the mixed Hodge diagram and on \mathfrak{g} . When restricted to the invariants by Φ , this induces an augmentation (still denoted by ν_x)

$$(11.15) \quad \nu_x : M = (\text{MHD}(\overline{Y}, D)_{\mathbb{R}} \otimes \mathfrak{g})^{\Phi} \longrightarrow \mathfrak{g}.$$

Lemma 11.4. *Via the canonical chain of quasi-isomorphisms relating M to L , this ν_x corresponds to the augmentation ε_x of (11.4).*

By this we mean that the whole canonical chain of quasi-isomorphisms relating M to L has augmentations to \mathfrak{g} , commuting with the quasi-isomorphisms, and relating ν_x to ε_x .

Proof. First on Y , ε_y is also induced by the augmentation at the level of sheaves

$$\mathcal{E}_{Y, \mathbf{k}}^{\bullet} \longrightarrow \mathbf{k}_y$$

then by taking global sections and tensoring with \mathfrak{g} . This augmentation commutes with the chain of quasi-isomorphisms relating \mathcal{E}_Y^{\bullet} to $\mathcal{A}_Y^{\bullet}(\log D)$, via the intermediate augmentation of $\Omega_Y^{\bullet}(\log D)$ which is defined by the same obvious way, evaluating degree zero holomorphic forms at y (important is the fact that y is in Y and not on D). This is enough to prove the claim on Y , since then it is easy to tensor all the chain of augmented quasi-isomorphisms by \mathfrak{g} .

One goes from Y to X by simply comparing the formulas (11.14) and (11.6), from which we see by construction that ν_x corresponds to the augmentation that we denoted by η_x , and then by going to the invariants under Φ we see that ν_x on M corresponds to ε_x on L . \square

Let us sum up.

Theorem 11.5. *The data of M and ε_x is an augmented mixed Hodge diagram of Lie algebras over \mathbf{k} quasi-isomorphic to the augmented DG Lie algebra of Goldman-Millson controlling the deformation theory of ρ . Up to quasi-isomorphism, it depends only on X, x, ρ .*

Proof. Combining Lemma 11.2 and Lemma 11.4 forms the first part of the claim. We need to prove the independence of M on \overline{Y} up to quasi-isomorphism. By the usual Galois correspondence for covering spaces Y, y depend already only on X, x, ρ . And the augmentation does not depend on \overline{Y} . So as in Remark 11.3 it is enough to show that $\text{MHD}(\overline{Y}, D)_{\mathbf{k}}$ is independent of \overline{Y} up to quasi-isomorphism. The argument is well-known ([Del71, 3.2.II.C]) except that we work with equivariant compactifications. So let $\overline{Y}', \overline{Y}''$ be two equivariant compactifications of Y . We look for a third compactification \overline{Y} which dominates both, i.e. with two morphisms of pairs

$$(11.16) \quad \begin{array}{ccc} & (\overline{Y}, D) & \\ j' \swarrow & & \searrow j'' \\ (\overline{Y}', D') & & (\overline{Y}'', D''). \end{array}$$

This \overline{Y} can be obtained as a resolution of singularities of the closure of the image of the diagonal embedding of Y into $\overline{Y}' \times \overline{Y}''$. But by invoking again the combination of the theorem of Sumihiro on equivariant completion combined with the theorem

of Bierstone-Milman on canonical resolutions of singularities, one can find such an \overline{Y} to which the action of Φ lifts. So in the diagram (11.16) the compactification \overline{Y} dominates the two others as equivariant compactifications. Then j' and j'' both induce quasi-isomorphisms of mixed Hodge diagrams. \square

And our conclusion of this section.

Theorem 11.6. *Let X be a smooth complex quasi-projective algebraic variety. Let $\rho : \pi_1(X, x) \rightarrow G(\mathbf{k})$ be a representation with finite image into a linear algebraic group over the field $\mathbf{k} \subset \mathbb{R}$. Then there is a mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ defined over \mathbf{k} that is functorial in X, x, ρ .*

Proof. From this data we constructed in the previous theorem an augmented mixed Hodge diagram of Lie algebras over \mathbf{k} controlling the deformation theory of ρ . It is independent of \overline{Y} up to quasi-isomorphism. So this works exactly as in the proof of Theorem 10.3 except that we also have to invoke Lemma 8.5 and Lemma 9.5 since the controlling mixed Hodge diagram is defined only up to quasi-isomorphism.

We also have to check the functoriality. A morphism

$$(11.17) \quad f : (X_1, x_1, \rho_1) \longrightarrow (X_2, x_2, \rho_2),$$

meaning that we require $f(x_1) = x_2$ and the commutativity of

$$(11.18) \quad \begin{array}{ccc} \pi_1(X_1, x_1) & \xrightarrow{f_*} & \pi_1(X_2, x_2) \\ & \searrow \rho_1 & \swarrow \rho_2 \\ & G(\mathbf{k}) & \end{array}$$

induces a morphism

$$(11.19) \quad \begin{array}{ccc} \Phi_1 \curvearrowright Y_1 & \xrightarrow{h} & Y_2 \curvearrowleft \Phi_2 \\ \pi_1 \downarrow & & \downarrow \pi_2 \\ X_1 & \xrightarrow{f} & X_2 \end{array}$$

compatibly with the base points and equivariant with respect to Φ_1 , that acts on Y_1 and on Y_2 via the induced morphism of groups $\varphi : \Phi_1 \rightarrow \Phi_2$. Then we can upgrade this to a morphism of equivariant compactifications: start with such compactifications $\overline{Y}_1, \overline{Y}_2$ and consider the graph Γ_h of h seen as a subset

$$(11.20) \quad \Gamma_h \subset Y_1 \times Y_2 \subset \overline{Y}_1 \times \overline{Y}_2.$$

By construction, Γ_h is Φ_1 -invariant, where Φ_1 acts diagonally. So is its Zariski closure. This defines a morphism $\overline{Y}_1 \rightarrow \overline{Y}_2$ which is φ -equivariant. Such a morphism induces a morphism of mixed Hodge diagrams

$$(11.21) \quad \text{MHD}(\overline{Y}_2, D_2)_{\mathbf{k}} \longrightarrow \text{MHD}(\overline{Y}_1, D_1)_{\mathbf{k}}$$

and then a morphism of mixed Hodge diagrams of Lie algebras $M_2 \rightarrow M_1$ augmented over \mathfrak{g} . So the canonically induced morphism $\widehat{\mathcal{O}}_{\rho_1} \rightarrow \widehat{\mathcal{O}}_{\rho_2}$ is a morphism of mixed Hodge structures. \square

As in the compact case, one can also re-write this easily for complex mixed Hodge structures.

Proposition 11.7. *Assume that G and ρ are defined over \mathbb{C} . Then $\widehat{\mathcal{O}}_\rho$ has a functorial complex mixed Hodge structure.*

Proof. In this case \mathfrak{g} is defined over \mathbb{C} . In the mixed Hodge diagram of Navarro Aznar, keeping only the part over \mathbb{C} defines a complex mixed Hodge diagram $\text{MHD}(\overline{Y}, D)_{\mathbb{C}}$. The controlling mixed Hodge diagram of Lie algebras is

$$(11.22) \quad M := (\text{MHD}(\overline{Y}, D)_{\mathbb{C}} \otimes \mathfrak{g})^{\Phi}$$

where the tensor product is over \mathbb{C} . □

As announced, the construction of Navarro Aznar allows us to show that in the compact case the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ is defined over \mathbf{k} if ρ is. Recall that in this case ρ is the monodromy of a variation of Hodge structure defined over \mathbf{k} , so that V is a local system of finite-dimensional vector spaces over \mathbf{k} .

Proof of Proposition 10.7. Let \mathcal{L} be the sheaf of sections of $\text{End}(V)$. It is a local system of finite-dimensional Lie algebras over \mathbf{k} . Then $R_{\text{TW}}\Gamma(X, \mathcal{L})$ is a DG Lie algebra over \mathbf{k} (the construction described in [Nav87, § 3] for commutative algebras works as well for Lie algebras). So we use it as the part over \mathbf{k} of the mixed Hodge diagram of Lie algebras L of Theorem 10.2. The augmentation at x comes from the canonical morphism of sheaves $\mathcal{L} \rightarrow \mathcal{L}_x$ where the right-hand side is the sheaf supported at x with stalk the Lie algebra $\text{End}(V_x)$, and with $R_{\text{TW}}\Gamma(X, \mathcal{L}_x) = \text{End}(V_x)$. □

12. DESCRIPTION OF THE MIXED HODGE STRUCTURE

In this final section we extract a description of the mixed Hodge structure we constructed on $\widehat{\mathcal{O}}_\rho$. We use only algebraic methods, without referring to the geometric origin of our mixed Hodge structures. Hence we relax the notations and write X, G, ρ in all cases. We denote by L a controlling mixed Hodge diagram of Lie algebras, ε_x the augmentation at the base point x and C the desuspended mapping cone of ε_x .

The first observation that is straightforward is that the mixed Hodge structure on $\widehat{\mathcal{O}}_\rho$ has only non-positive weights. Namely during the construction of L and C we always work with non-negative weights, and the induced weights on $H^0(\mathcal{C}(C))$ are non-negative, but then we dualize and this produces the non-positive weights.

The second easy observation is that we get an explicit description of the mixed Hodge structure on the cotangent space to $\widehat{\mathcal{O}}_\rho$.

Theorem 12.1. *The mixed Hodge structure on the cotangent space to $\widehat{\mathcal{O}}_\rho$ is dual to the mixed Hodge structure on $H^1(C)$, which fits into a short exact sequence*

$$(12.1) \quad 0 \rightarrow \mathfrak{g}/\varepsilon_x(H^0(L)) \rightarrow H^1(C) \rightarrow H^1(L) \rightarrow 0.$$

Proof. The coalgebra $H^0(\mathcal{C}(C))$ has a canonical filtration which is dual to the sequence of quotients of $\widehat{\mathcal{O}}_\rho$ by powers of the maximal ideal. In particular its first term $H^0(\mathcal{C}_1(C))$ is dual to the cotangent space. But by construction $\mathcal{C}_1(C)$ is simply $C[1]$. The long exact sequence for C yields exactly the exact sequence (12.1) with $H^0(C[1]) = H^1(C)$. □

In our situations this leads us to the following descriptions:

- (1) If X is compact then $H^1(L)$ is pure of weight 1. So $H^1(C)$ has this as weight 1 part and a weight 0 part coming from $\mathfrak{g}/\varepsilon_x(H^0(L))$. We see that our mixed Hodge structure coincides with the one of Eyssidieux-Simpson on the cotangent space.
- (2) If X is smooth quasi-projective then $H^1(L)$ has weights only 1, 2. Furthermore if ρ has finite image then ε_x is surjective and the left-hand side vanishes. So the only weights are 1, 2. Splitting the weight filtration on the cotangent space and lifting a basis to generators of $\widehat{\mathcal{O}}_\rho$ produces generators of weight 1, 2 as in the theorem of Kapovich-Millson.
- (3) As a particular case, if $H^1(L)$ is pure of weight 2 (which happens if the mixed Hodge structure on the H^1 of the finite cover corresponding to ρ is pure of weight 2) then $\widehat{\mathcal{O}}_\rho$ has only homogeneous generators. In this sense one recovers the main theorem of [Lef17].
- (4) In the general non-compact case where ρ is the monodromy of a variation of Hodge structure we expect generators of weight 0, 1, 2 on $\widehat{\mathcal{O}}_\rho$.

In order to recover completely the theorem of Kapovich-Millson, we would like to get a presentation of $\widehat{\mathcal{O}}_\rho$ as a quotient of the formal power series algebra on $H^1(C)$ by an ideal carrying a mixed Hodge structure with weights 2, 3, 4 coming from $H^2(C) = H^2(L)$. We are unable to get this.

Now we identify the weight filtration of $\widehat{\mathcal{O}}_\rho$ globally. The group G acts algebraically on $\mathrm{Hom}(\pi_1(X, x), G)$ and the orbit of ρ defines a reduced closed subscheme Ω_ρ . Formally at ρ , Ω_ρ is defined by an ideal $\mathfrak{j} \subset \widehat{\mathcal{O}}_\rho$. The quotient $\widehat{\mathcal{O}}_\rho/\mathfrak{j}$ is the algebra of formal function on Ω_ρ at ρ .

Theorem 12.2. *The weight zero part of $\widehat{\mathcal{O}}_\rho$ is the formal local ring of the orbit of ρ .*

At the level of the cotangent space, we see this from the previous theorem.

Proof. We will write an explicit morphism from $\widehat{\mathcal{O}}_\rho$ to the formal local ring of Ω_ρ using our methods of cones, functor \mathcal{C} and abstract pro-representability theorems. The kernel of this morphism will be the ideal \mathfrak{j} and the compatibility of this construction with mixed Hodge structures will be clear.

Let us consider the desuspended mapping cone D of $\varepsilon_x : H^0(L) \rightarrow \mathfrak{g}$. This has $D^0 = H^0(L)$, $D^1 = \mathfrak{g}$, with the differential being given by ε_x . Its L_∞ algebra structure is in fact DG Lie since the only non-zero Lie brackets are the one induced on D^0 and the naive bracket between D^0 and D^1 . The deformation functor is given by ($A \in \mathbf{Art}_k$)

$$(12.2) \quad \mathrm{Def}_D(A) = \exp(\mathfrak{g} \otimes \mathfrak{m}_A) / \exp(H^0(L) \otimes \mathfrak{m}_A).$$

Let us consider also the DG Lie algebra E which has only $E^1 = \mathfrak{g}/\varepsilon_x(H^0(L))$ with zero bracket and differential. One can see E as the desuspended mapping cone of $0 \rightarrow \mathfrak{g}/\varepsilon_x(H^0(L))$, the L_∞ algebra structure being trivial in this case. The deformation functor is given by

$$(12.3) \quad \mathrm{Def}_E(A) = (\mathfrak{g}/\varepsilon_x(H^0(L))) \otimes \mathfrak{m}_A.$$

There is a canonical quasi-isomorphism $D \xrightarrow{\sim} E$. One can see this as being induced by a morphism between the morphisms whose D, E are the cones, hence by the functoriality of the L_∞ (in fact DG Lie) structure on the mapping cone this

is morphism of DG Lie algebras, and it induces an isomorphism of the deformation functors. Hence there is a canonical isomorphism

$$(12.4) \quad H^0(\mathcal{C}(D))^* = H^0(\mathcal{C}(E))^*.$$

But the computation of $H^0(\mathcal{C}(E))^*$ is straightforward: it is (up to adding \mathbf{k}) the algebra of power series on the vector space $\mathfrak{g}/\varepsilon_x(H^0(L))$.

Now we follow [ES11, § 2.2.2] where it is explained how this corresponds to the algebra of formal functions on the orbit of ρ . Observe that, following their notations, our D corresponds to their deformation functor h'_1 and E to h_1 . Then there is a natural morphism $D \rightarrow C$ which corresponds to the morphism of deformation functors $h'_1 \rightarrow \text{Def}_\rho$ (the inclusion of the orbit inside the representation variety). It induces a morphism

$$(12.5) \quad p : \mathbf{k} \oplus H^0(\mathcal{C}(C))^* \longrightarrow \mathbf{k} \oplus H^0(\mathcal{C}(D))^*.$$

Via their pro-representability property, the left-hand side is identified with $\widehat{\mathcal{O}}_\rho$ and the right-hand side is identified with the algebra of formal functions of Ω_ρ . So the kernel of p is the ideal \mathfrak{j} defining the orbit inside $\widehat{\mathcal{O}}_\rho$.

If we replace in the right-hand side of (12.5) E by D we see clearly that the mixed Hodge structure is pure of weight zero. By the strict compatibility of the weight filtration with respect to morphisms of mixed Hodge structures we get that $\text{Gr}_0^W(\widehat{\mathcal{O}}_\rho)$ equals this right-hand side. \square

REFERENCES

- [BM97] E. Bierstone and P. D. Milman, *Canonical desingularization in characteristic zero by blowing up the maximum strata of a local invariant*, *Inventiones Mathematicae* **128** (1997), no. 2, 207–302.
- [CG16] J. Cirici and F. Guillén, *Homotopy theory of mixed Hodge complexes*, *Tohoku Mathematical Journal* **68** (2016), no. 3, 349–375.
- [Cir15] J. Cirici, *Cofibrant models of diagrams: mixed Hodge structures in rational homotopy*, *Transactions of the American Mathematical Society* **367** (2015), no. 8, 5935–5970.
- [Del71] P. Deligne, *Théorie de Hodge, II*, *Publications Mathématiques de l’IHES* **40** (1971), 5–57.
- [Del74] ———, *Théorie de Hodge, III*, *Publications Mathématiques de l’IHES* **44** (1974), 5–77.
- [ES11] P. Eyssidieux and C. Simpson, *Variations of mixed Hodge structure attached to the deformation theory of a complex variation of Hodge structures*, *Journal of the European Mathematical Society* **13** (2011), no. 6, 1769–1798.
- [FM07] D. Fiorenza and M. Manetti, *L_∞ structures on mapping cones*, *Algebra & Number Theory* **1** (2007), no. 3, 301–330.
- [GM88] W. M. Goldman and J. J. Millson, *The deformation theory of representations of fundamental groups of compact Kähler manifolds*, *Publications Mathématiques de l’IHES* **67** (1988), 43–96.
- [GM90] ———, *The homotopy invariance of the Kuranishi space*, *Illinois Journal of Mathematics* **34** (1990), no. 2, 337–367.
- [Hai87] R. M. Hain, *The de Rham homotopy theory of complex algebraic varieties I*, *K-Theory* **1** (1987), no. 3, 271–324.
- [Hin01] V. Hinich, *DG coalgebras as formal stacks*, *Journal of Pure and Applied Algebra* **162** (2001), no. 2-3, 209–250.
- [KM98] M. Kapovich and J. J. Millson, *On representation varieties of Artin groups, projective arrangements and the fundamental groups of smooth complex algebraic varieties*, *Publications Mathématiques de l’IHES* **88** (1998), 5–95.
- [Kon94] M. Kontsevich, *Topics in algebra, deformation theory*, *Lectures notes*, 1994.
- [Lef17] L.-C. Lefèvre, *A criterion for quadraticity of a representation of the fundamental group of an algebraic variety*, *Manuscripta Mathematica* **152** (2017), no. 3-4, 381–397.

- [LM85] A. Lubotzky and A. R. Magid, *Varieties of representations of finitely generated groups*, *Memoirs of the American Mathematical Society* **336** (1985).
- [LV12] J.-L. Loday and B. Vallette, *Algebraic operads*, *Grundlehren der mathematischen Wissenschaften*, no. 346, Springer, 2012.
- [Man99] M. Manetti, *Deformation theory via differential graded Lie algebras*, *Seminari di Geometria Algebrica 1998-1999*, Scuola Normale Superiore (1999).
- [Man04] ———, *Lectures on deformations of complex manifolds. Deformations from differential graded viewpoint*, *Rendiconti di Matematica* **24** (2004), no. 1, 1–183.
- [Mor78] J. W. Morgan, *The algebraic topology of smooth algebraic varieties*, *Publications Mathématiques de l’IHES* **48** (1978), 137–204.
- [Nav87] V. Navarro Aznar, *Sur la théorie de Hodge-Deligne*, *Inventiones Mathematicae* **90** (1987), 11–76.
- [PS08] C. A. M. Peters and J. H. M. Steenbrink, *Mixed Hodge structures*, *Ergebnisse der Mathematik und ihrer Grenzgebiete*, no. 52, Springer, 2008.
- [Sch68] M. Schlessinger, *Functors of Artin rings*, *Transactions of the American Mathematical Society* **130** (1968), 208–222.
- [Sum74] H. Sumihiro, *Equivariant completion*, *Journal of Mathematics of Kyoto University* **14** (1974), 1–28.
- [Toë17] B. Toën, *Problèmes de modules formels. Exposé 1111*, *Astérisque. Séminaire Bourbaki*, volume 2015/2016 **390** (2017), 199–244.
- [Zuc79] S. Zucker, *Hodge theory with degenerating coefficients: L_2 cohomology in the Poincaré metric*, *Annals of Mathematics* **109** (1979), 415–476.

E-mail address: louisclément.lefevre@univ-grenoble-alpes.fr

UNIV. GRENOBLE ALPES, CNRS, INSTITUT FOURIER, F-38000 GRENOBLE, FRANCE