Anisotropic membrane in large displacements and small strains with application to sail design

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Anisotropic membrane in large displacements and small strains. Application to sail design.

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We recall a non-linear membrane model in large displacements and small strains. The anisotropic constitutive material is approached by the superposition of several layers, one being isotropic, the others being made out of different unidirectional materials. The equilibrium equation is solved by the finite element method. The application part concerns the computation of a Genoa under a given wind pressure.

Membrane Layered composite Non-linear elasticity Finite element method Sail

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Introduction.

A membrane is a very thin structure. The strains are supposed constant over its thickness, therefore the bending effects are neglected. We study the large displacements case. This needs a non-linear strain measure, which gives a non-linear partial differential equation (PDE) to solve. We consider small strains over the membrane, the stiffness of the constitutive material being high in respect to the applied stress.

Our main idea to compute anisotropic membrane is to approximate its material, like a tissue, with the superposition of an isotropic layer with several layers built in different unidirectional fibers. Therefore we have to lie the real elastic coefficients with the virtual elastic coefficients of each layer.

First we recall the continuous membrane problem. Afterwards four finite element assumptions are done to solve the above PDE numerically. We recover the Haug-Powell (1971b) finite element. Finally an orthotropic genoa is computed for a given aerodynamical lift. The orthotropic sailcloth is approached by a three layered composite positioned along the warp-weft directions.

1. The anisotropic non-linear membrane model.

The mechanical variables are integrated through the thickness of the membrane. The Young modulus and stress unit is the Newton / meter (N/m).

Let ω be a surface of the natural space E_3, which represents the initial position of a membrane. Its boundary $\Gamma = \partial\omega$ consists of two subsets :

$$\partial\omega = \Gamma_0 \cup \Gamma_1 \qquad (1.1)$$

where Γ_0 is the fixed part of the membrane. The membrane is loaded by the force field f defined on ω. We study the dead load case : f is independent of the structural displacement. We use the lagrangian variables. A membrane displacement u is called admissible if the boundary condition $u \mid \Gamma_0 = 0$ holds. For such displacements, however large they are, we measure the strain with the Green tensor x :

$$\forall\, m \in \omega \qquad x(u)(m) = \frac{1}{2}\left(\pi\, du + \overline{\pi\, du} + \overline{du}\; du \right)(m) \qquad (1.2)$$

where du is the differential of u, $\overline{du}$ is the transpose of du, and π the orthogonal projection on $T(m)$ the tangent plane at ω in m.

We model the anisotropic constitutive material by the superposition of several layers. The first one is isotropic, the n others are made of unidirectional fibers oriented by the fiber directions of the real material. $F_i(m)$ are the unit vector field, defined on ω, tangent at ω, which indicates the fiber direction of the layer i ($1 \le i \le n$) at m. The deformation of the fiber layer i is given by the real

$$\forall\, m \in \omega \qquad x_i(u)(m) = < x(u) \,.\, F_i \,,\, F_i > (m) \qquad (1.3)$$

where $< , >$ denotes the scalar product of E_3.

The plane stress in the isotropic layer is measured by the second Piola-Kirchhoff stress tensor σ defined in terms of $x(u)$ via the constitutive law

$$\sigma(u)(m) = \frac{E}{1+\nu}\left\{ x(u)(m) + \frac{\nu}{1-2\nu} \mathrm{tr}\,(x(u)(m))\, 1_{T(m)} \right\} \quad . \qquad (1.4)$$

E is the Young modulus of the isotropic layer, ν the Poisson ratio, $1_{T(m)}$ is the identity operator of $T(m)$ and tr is the trace operator.

The stress in the fiber layer i is measured at $m \in \omega$ by the real

$$\sigma_i(u)(m) = E_i \,.\, x_i(u)(m) \qquad . \qquad (1.5)$$

E_i is the Young modulus of the fiber layer i .The stress-strain relations (1.4) and (1.5) are linear, which is valid only in the limit of small strains.

Let the internal strain energy e stored in the deformed membrane be

$$\forall\, u \text{ (admissible)} \qquad e(u) = e_0(u) + \sum_{i=1}^{n} e_i(u) \qquad , \qquad (1.6)$$

where e_0 is the internal strain energy of the isotropic layer

$$e_0(u) = \frac{1}{2} \int_{\omega} \mathrm{tr}\,(\sigma.x)(u)(m) \quad d\omega \qquad (1.7)$$

and e_i is the internal strain energy of the layer i

$$e_i(u) = \frac{1}{2} \int_{\omega} (\sigma_i.x_i)(u)(m) \quad d\omega \qquad . \qquad (1.8)$$

The potential energy lost by the applied force f is given by the linear functional l

$$l(u) = \int_{\omega} < f,u > (m)\; d\omega \qquad . \qquad (1.9)$$

With the above definitions we obtain the membrane equilibrium equation

$$\exists?\, u \text{ (adm.)} \mid \forall\, v \text{ (adm.)} \quad \frac{d}{du} e(u).v = l(v) \qquad . \qquad (1.10)$$

This means that u is the displacement of ω under the applied force f if u is a critical point of the total potential energy e - l in the admissible displacement space. Note that the non-linearity of (1.10) stems only from the Green strain tensor x .

For the isotropic membrane case (n = 0) Muttin (1989) justified mathematically the equation (1.10) in respect to the three dimensional elasticity model. This justification stems from an extrapolation of Destuynder (1980) results for linear thin shells.

2. Finite element approximations .

Usually we can not solve analytically the equation (1.10). Approximations are needed to transfer the continuous problem into a discrete one which can be solved numerically. Here we describe the four approximations incorporated in the non-linear quadrilateral layered membrane finite element of Haug and Powell (1971b) (1972) .

2.1 Approximation of ω.

ω is approach by a set of mid-surfaces ω_e , quadrilateral and warped in space,

$$\omega \cong \omega_h = \bigcup_{1}^{ne} \omega_e \qquad \text{ne is the number of finite elements.}$$

The mapping φ^e of each ω_e is described by

$$\varphi^e : \ \hat{m} \in \hat{\omega} = (\,]-1,1[\,)^2 \quad \rightarrow \quad m = \varphi^e(\hat{m}) = \sum_{j=1}^{4} m^e_j \, h^j(\hat{m}) \in E_3 \ ,$$

where m^e_j is the position of the corner j $(1 \le j \le 4)$ of the element e. Each m^e_j is chosen in a set N of nodes judiciously marked on ω (finite element nodes). h^j $(j=1...4)$ are the bilinear basis functions

$$h^j : \quad (\xi^1, \xi^2) \in \hat{\omega} \quad \rightarrow \quad h^j(\xi^1, \xi^2) = \frac{1}{4} (1 \pm \xi^1) (1 \pm \xi^2) \ .$$

We denote by n_t (total) the number of finite element nodes, by n_b (blocked) the number of nodes on Γ_0, and by $n_f = n_t - n_b$ (free) the number of free nodes.

2.2 Approximation of the displacement of ω_h.

We search the displacement field u_h of ω_h under the following form :

$$u_h \mid \Gamma_0 = 0 \ , \qquad \forall e \in \{1..n_e\} \quad u_h \mid \omega_e(\varphi^e) = \sum_{j=1}^{4} u_e(m^e_j) \, h^j \ .$$

$u_e(m^e_j)$ denotes the displacement of the node j of the element e. The same functions h^j define φ^e and $u_h \mid \omega_e$, so the obtained finite element will be isoparametric. Classically one remarks that the unknown u of the continuous membrane problem is converted for the discrete problem in the unknown vector $U = \{ u_e(m^e_j) \ 1 \le e \le n_e \ \ 1 \le j \le 4 \ \ m^e_j \notin \Gamma_0 \}$ of $(E_3)^{n_f}$.

2.3 Numerical integration scheme.

Let f^e be an integrable function defined on ω_e. To compute the sum of f^e on ω_e we use the four node Gauss numerical integration scheme

$$\int_{\omega_e} f^e(m) \, d\omega_e \ \cong \ \sum (f^e \sqrt{g^e}) \circ \varphi^e \left(\pm \frac{1}{\sqrt{3}}, \pm \frac{1}{\sqrt{3}} \right) \ ,$$

where g^e is the determinant of the metric tensor of ω_e.

2.4 Fiber orientation in a finite element midsurface ω_e.

The fiber orientations $F_i(m)$ used to formulate the continuous membrane problem are converted in the following set of reals to formulate the discrete membrane problem

$$A = \{ \ \alpha^e_i(0,0) \in [0,2\pi] \quad 1 \le e \le n_e \quad 1 \le i \le n \ \} \ ,$$

where $\alpha^e_i(0,0)$ is at the centroid $m = \varphi^e(0,0)$ of the element e, the angle between the fiber layer i, and the first basis vector $\frac{\partial \varphi^e}{\partial \xi^1}(0,0)$ of the tangent plane at ω_e.

Starting from the data A, Haug and Powell [1971b] define the fiber orientation $\alpha^e_i(\xi^1,\xi^2)$ of the fiber layer i in any point $\varphi^e(\hat{m})$ of ω_e, with the following relation

$$\beta^e(\xi^1,0) - \alpha^e{}_i(\xi^1,\xi^2) = \beta^e(0,0) - \alpha^e{}_i(0,0) \qquad (2.1)$$

where $\beta^e(\xi^1,\xi^2)$ is the angle between the two basis vector $\frac{\partial \varphi^e}{\partial \xi^1}(\xi^1,\xi^2)$ $\frac{\partial \varphi^e}{\partial \xi^1}(\xi^1,\xi^2)$ of the tangent plane $T^e(\xi^1,\xi^2)$ at ω_e in $\varphi^e(\xi^1,\xi^2)$. The unit vector in $T^e(\xi^1,\xi^2)$, tangent at the fiber layer i in $\varphi^e(\xi^1,\xi^2)$ is noted $F^e{}_i(\xi^1,\xi^2)$.

3. Formulation of the membrane discrete equilibrium equation and its solution.

3.1 Formulation of the discrete problem.

An approximation of the continuous problem is made by the following statements.

Let $\omega_h(N), \Gamma_0, f_h, E, \nu, E_i, \nu_i, A = \{\alpha^e{}_i(0,0)\}, n, n_e$

$\exists?\ u_h$ (adm.) $\mid\ \forall\ v_h$ (adm.)

$$\frac{d}{du_h} e_{h,0}[u_h].v_h + \sum_{i=1}^{n} \frac{d}{du_h} e_{h,i}[u_h].v_h = l_h[v_h] \ , \qquad (3.1)$$

where

$$e_{h,0}(u_h) = \sum_{e=1}^{ne} e_{0,e}(u_{h|\omega_e})$$

$$e_{h,i}(u_h) = \sum_{e=1}^{ne} e_{i,e}(u_{h|\omega_e})$$

$$l_h(u_h) = \sum_{e=1}^{n} l_e(u_{h|\omega_e})$$

with $(u_{h|\omega_e} = u_e)$

$$e_{0,e}(u_e) = \sum \{ tr(\sigma.x)(u_e).\sqrt{g^e} \} \circ \varphi^e(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$$

$$e_{i,e}(u_e) = \sum \{ (\sigma_i.x_i)(u_e).\sqrt{g^e} \} \circ \varphi^e(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$$

$$l_e(u_e) = \sum \{ < f_{h|\omega_e}, u_e >.\sqrt{g^e} \} \circ \varphi^e(\pm\frac{1}{\sqrt{3}}, \pm\frac{1}{\sqrt{3}})$$

$x(u_e)$ and $\sigma(u_e)$ are defined by equations (1.2) and (1.4) applied to ω_e. $x_i(u_e)$ and $\sigma_i(u_e)$ are the strain and the stress in the fiber layer i. They are defined by equations (1.3) and (1.5) in which F_i is remplaced by $F^e{}_i$. f_h is the force field applied on ω_h.

Now we put the discrete problem under a more intuitive form. Note that $e_{h,0}$ and $e_{h,i}$ are twice continually differentiable and that l_h is linear. Obviously $e_{0,e}$, $e_{i,e}$ and l_e have the same properties. On one hand it is equivalent to know the displacement field u_h and the vector

$U \in (E_3)^{nf}$ of the free node displacements. On the other hand the functions $\frac{d}{du_h} e_h[u_h].v_h$ and $l_h[v_h]$ are linear in v_h. Hence we obtain

$$\forall\ u_h, v_h \text{ (adm.)} : \exists!\ U, V, F_{int,0}(U), F_{int,i}(U), F_{ext} \in (E_3)^{nf} \mid$$

$$\frac{d}{du_h} e_{h,0}[u_h].v_h = < F_{int,0}(U), V > \qquad (3.2)$$

$$\frac{d}{du_h} e_{h,i}[u_h].v_h = < F_{int,i}(U), V > \quad \forall i \in \{1 \ldots n\} \qquad (3.3)$$

$$l_h[v_h] = < F_{ext}, V > \quad . \qquad (3.4)$$

Starting with these three relations, we obtain two fundamental interpretations.

(1) The components of the vector $F_{int,0}(U) + \sum_{i=1}^{n} F_{int,i}(U)$ are the forces that we have to apply at the free nodes to balance the internal stress in the set of layers when the displacement of the free nodes is U.

(2) The components of the vector F_{ext} are the resultants at the free nodes of the force field f_h applied on ω_h.

According to the equations (3.2) (3.3) and (3.4), the discrete problem becomes

$$\exists?\ U \in (E_3)^{nf} \mid F_{int,0}(U) + \sum_{i=1}^{n} F_{int,i}(U) = F_{ext} \qquad (3.5).$$

The equilibrium of the discretized membrane is obtained when the nodal forces, which balance the internal stress, are the applied forces.

3.3 Solution of the discrete problem.

The non-linear equation (3.5) is solved by the Newton-Raphson method. The iterative sequence (U^j) is constructed by

$$U^0 = 0$$

$$\forall j \geq 1 \quad [\frac{dF_{int,0}}{dU} + \sum_{i=1}^{n} \frac{dF_{int,i}}{dU}](U^{j-1}) \cdot \{ U^j - U^{j-1} \} =$$

$$- \{ F_{int,0}(U^{j-1}) + \sum_{i=1}^{n} F_{int,i}(U^{j-1}) - F_{ext} \} .$$

$[\frac{dF_{int,0}}{dU}(U^{j-1})]$ $[\frac{dF_{int,i}}{dU}(U^{j-1})]$ are respectively the tangent stiffness operator of the isotropic layer and of the unidirectional layer i, when the displacement of the free nodes is U^{j-1}.

4. Description of a genoa as an orthotropic membrane.

A genoa is a lightweight structure which may have great displacements under the wind pressure. The chosen sailcloth is an isotropic thin sheet in terphane thermoforming with a tissue in Kevlar. This material is represented by an isotropic layer and two unidirectional layers oriented along the warp and the weft.

(1) Geometry : A genoa is approximatively a bulging triangle with concave bounds. The shape ω is given by the CAO software FABRIC (c) CRAIN . Figure (4.1) shows ω . Figure (4.2) shows ω_h . It is a genoa number 3 of a 12 J.I. meters.

(2) Boundary conditions : The boat and the rig of the genoa are taken as rigid. The backward boundary (leech) and the bottom (foot) are free. Along the foremost boundary (luff) the normal component of u_h at this curved boundary is set to zero (the tangential component is free). The tack point is blocked. The backward corner (clew) is connected to a prestressed cable named the sheet. This one is represented by a single Haug-Powell [1971a] non-linear cable finite element. The head point is also connected to a prestressed cable, the Halyard.

(3) Wind pressure: We do not take into account the wind/structure interaction. To compute the wind pressure, we consider the sail as rigid and we use its initial shape. The wind pressure is then considered as a "constant load field". It is computed by the software AEROK (c) CRAIN , author M. Kermarec (1986). The real wind speed is 24 knots with an angle of 32 degrees to the boat direction.
Inertial effects are neglected in the applied forces.

(4) Fiber layers orientation : Figure (4.3) shows the warp orientation (layer i=1) of the constitutive orthotropic material on each finite element. The weft and the warp are orthogonal. To obtain this orientation, we have computed this sail with a virtual isotropic material of high modulus, which gives the natural orientations of the two principal stress (eigen vectors of σ) for small sail strain. The high modulus of Kevlar fibers do not give small strain in the genoa, if there is shear strain in the tissue.

(5) Elastic coefficients of the isotropic, warp and weft layers : Experimental stress-strain measures on sailcloth give the elastic coefficients $E_1, E_2, G_{12}, \nu_{12}, \nu_{21}$ which lie stress $(\sigma_{11},\sigma_{22},\sigma_s)$ and strain $(\varepsilon_{11},\varepsilon_{22},\varepsilon_s)$ in an orthonormal basis :

$$\begin{pmatrix} \varepsilon_{,11} \\ \varepsilon_{,22} \\ \varepsilon_{,s} \end{pmatrix} = \begin{bmatrix} \frac{1}{E_1} & \frac{-\nu_{12}}{E_2} & 0 \\ \frac{-\nu_{21}}{E_1} & \frac{1}{E_2} & 0 \\ 0 & 0 & \frac{1}{G_{12}} \end{bmatrix} \begin{pmatrix} \sigma_{,11} \\ \sigma_{,22} \\ \sigma_{,s} \end{pmatrix}$$

We can identify the elastic coefficient E, ν of the isotropic layer and E_1 , E_2 of the warp and weft layers by using the following relation ($m = 1 - \nu_{12}\nu_{21}$) :

$$\begin{bmatrix} \frac{E_1}{m} & \frac{\nu_{21}E_2}{m} & 0 \\ \frac{\nu_{12}E_1}{m} & \frac{E_2}{m} & 0 \\ 0 & 0 & G_{12} \end{bmatrix} = \begin{bmatrix} \frac{E}{1-\nu^2} & \frac{\nu E}{1-\nu^2} & 0 \\ \frac{\nu E}{1-\nu^2} & \frac{E}{1-\nu^2} & 0 \\ 0 & 0 & \frac{E}{2(1+\nu)} \end{bmatrix} + \begin{bmatrix} E_1 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \end{bmatrix} + \begin{bmatrix} 0 & 0 & 0 \\ 0 & E_2 & 0 \\ 0 & 0 & 0 \end{bmatrix}$$

After some algebraic calculation we obtain

$$\nu = \frac{\nu_{12} E_1}{\nu_{12} E_1 + 2 G_{12} m} \qquad E = 2(1+\nu) G_{12}$$

$$E_1 = \frac{E_1}{m} - \frac{E}{1-\nu^2} \qquad E_2 = \frac{E_2}{m} - \frac{E}{1-\nu^2}$$

This sail is made out of several tissues which correspond to the following coefficients.

Tissue number	1	2	3	4	5	6	7
E (N/m)	3E+6	3E+5	3E+5	3E+5	3E+5	3E+5	3E+5
ν	0.26	0.26	0.26	0.26	0.26	0.26	0.26
E_1 (N/m)	0	3E+6	2.5E+6	2E+6	2.5E+6	2E+6	1.5E+6
E_2 (N/m)	0	5E+5	5E+5	5E+5	1E+6	2E+6	1.5E+6

Figure (4.4) shows the patchwork.

5. Numerical results.

When the displacement increment $(U^j - U^{j-1})$, especially for the first iteration, is to large, we multiply its components by a small scalar to put them within a data bound, typically 5 to 10 cm.

Unphysical compressive stress can be obtained with the model. It is the case for σ_2 near the tack. Thus the localisation off the warp stress concentration near the tack is not in agreement with real-life observations.

Figure (5.1) and (5.2) show the contour-values of the warp and the weft stress σ_1 and σ_2. The main flow stress concerns the warp. It starts from the head, decreases along the leech, to increase near the clew. At the front of the sail the warp and the weft stress are equals. Note that elsewhere the stress is mainly unidirectional. So in sail design unidirectional tissues are often used. We obtain the following internal forces at the three corners of the sail, halyard : 1175 Kg., sheet : 1925 Kg., tack : 570 Kg. The equilibrium shape $\omega_h + u_h$ is in good agreement with real-life observations.

Such computations have been used by the sailmakers of the two French syndicates for the America Cup 87 : "French Kiss" and "Challenge France".

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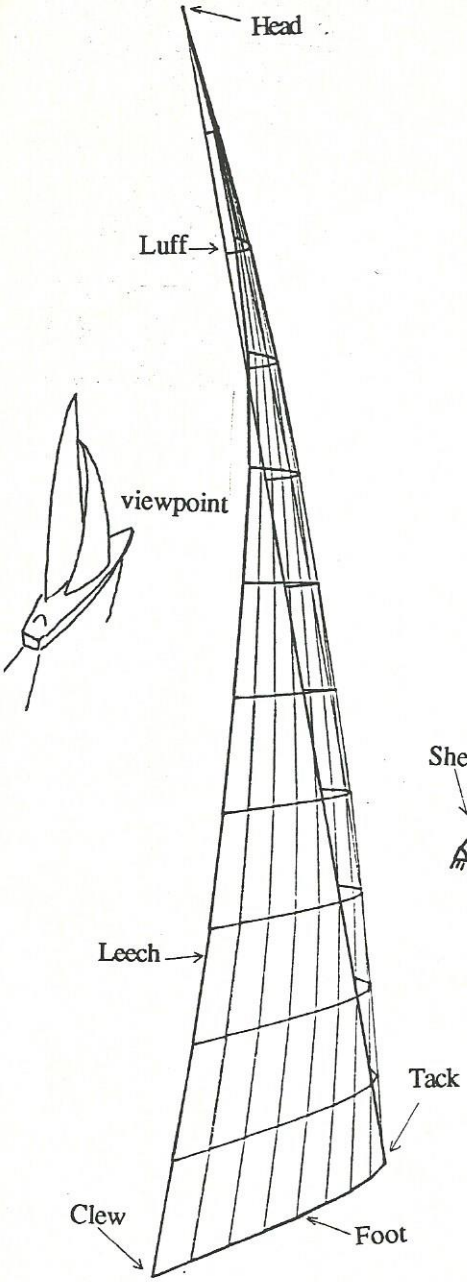


Figure 4.1 The midsurface ω of the Genoa .

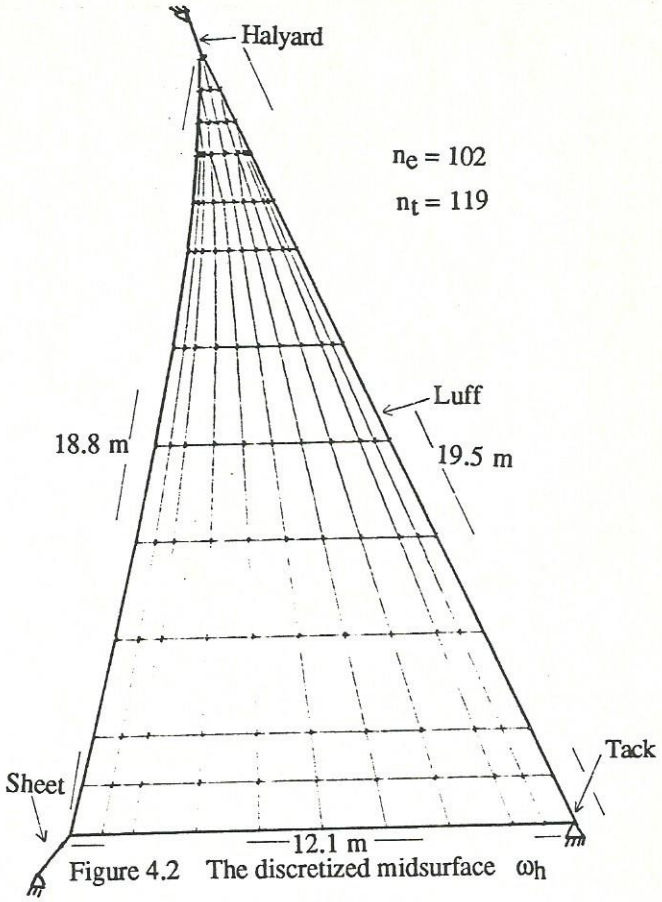


Figure 4.2 The discretized midsurface ω_h

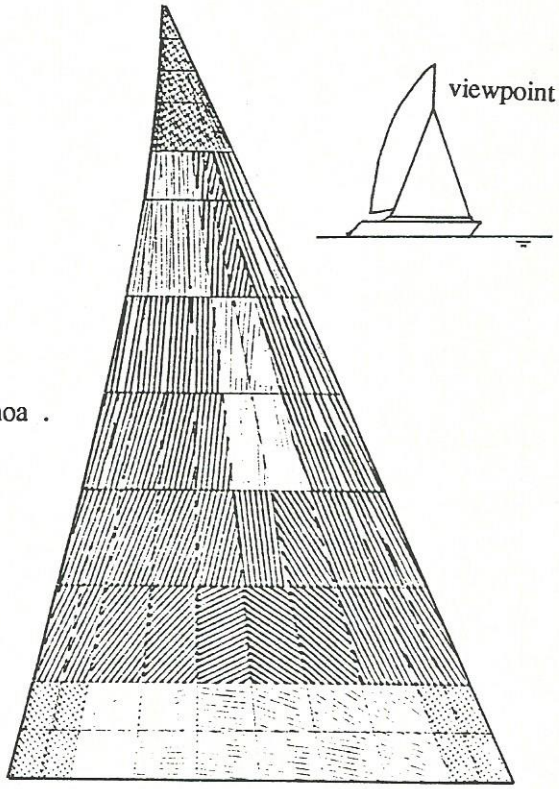


Figure 4.3 Orientation of the Fibers of the warp layer .

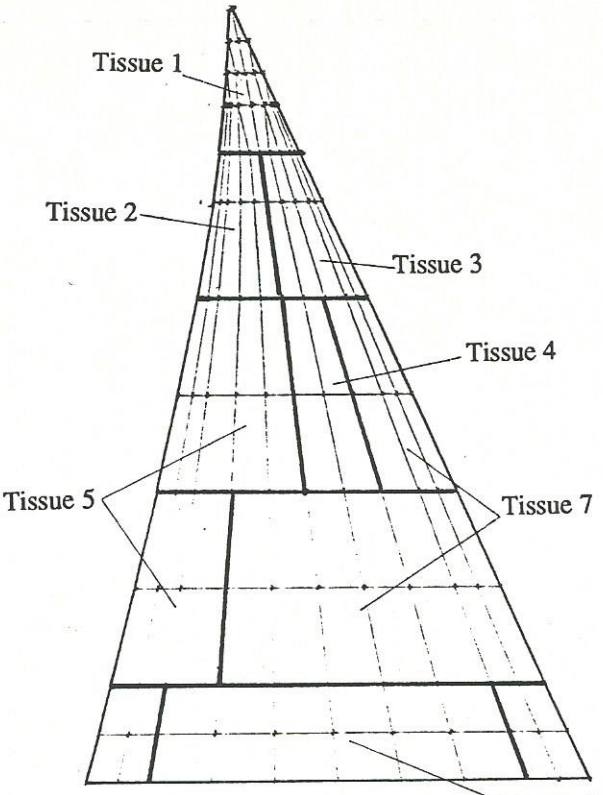
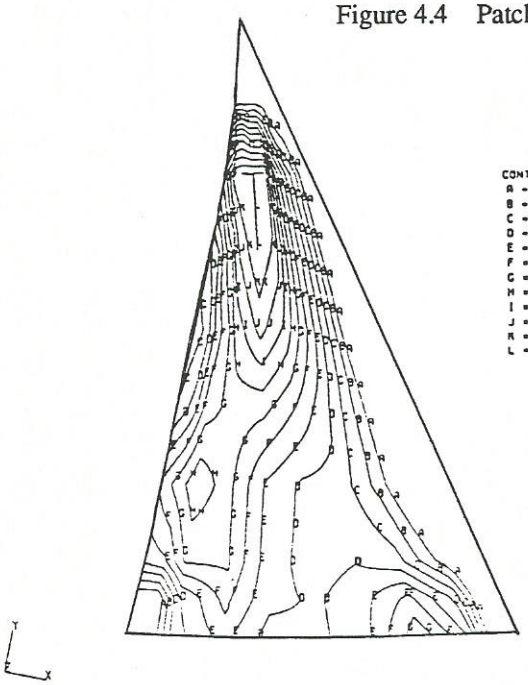
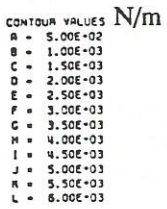


Figure 4.4 Patchwork of the Genoa





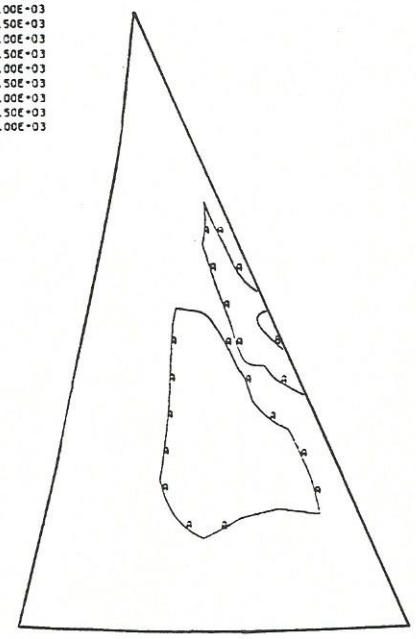


Figure 5.1 Stress in the warp layer.

Figure 5.2 Stress in the weft layer.