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Quantitative Convergence and Stability of Seismic Inverse Problems.

Florian Faucher¹,
Hélène Barucq¹, Henri Calandra² and Guy Chavent¹.

Reconstruction Methods for Inverse Problems
Indam Workshop, Roma, Italy,
May 28th – June 1st, 2018.



Istituto Nazionale di Alta Matematica

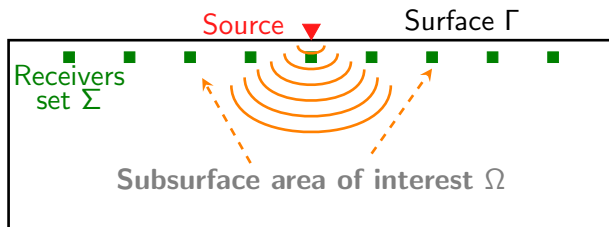


¹Inria Bordeaux Sud-Ouest, Project-Team Magique-3D.

²Total E&P

Seismic inverse problem

Reconstruction of subsurface Earth properties from seismic campaign: collection of **wave** propagation data at the surface.



- ▶ Reflection (back-scattered) partial data,
- ▶ only from the surface of the (large) domain,

nonlinear, ill-posed inverse problem.

Overview

- 1 Time-Harmonic Inverse Problem, FWI
- 2 Quantitative stability and convergence of FWI
 - Finite Curvature/Limited Deflection problem
 - Numerical convergence estimates
 - Numerical stability estimates
- 3 Numerical experiments
- 4 Conclusion

Plan

1 Time-Harmonic Inverse Problem, FWI

Time-harmonic wave equation

The forward problem wave equation depends on the medium:

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- ▶ acoustic isotropic (c)

$$(-\omega^2 c^{-2} - \Delta)p = 0,$$

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- ▶ acoustic isotropic (c)

$$(-\omega^2 c^{-2} - \Delta)p = 0,$$

- ▶ elastic isotropic (λ, μ, ρ)

$$-\rho\omega^2 \mathbf{u} - \nabla(\lambda \nabla \cdot \mathbf{u}) - \nabla \cdot (\mu [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]) = 0.$$

Time-harmonic wave equation

The forward problem wave equation depends on the medium:

- ▶ acoustic isotropic (c)

$$(-\omega^2 c^{-2} - \Delta)p = 0,$$

- ▶ elastic isotropic (λ, μ, ρ)

$$-\rho\omega^2 \mathbf{u} - \nabla(\lambda \nabla \cdot \mathbf{u}) - \nabla \cdot (\mu [\nabla \mathbf{u} + (\nabla \mathbf{u})^T]) = 0.$$

- ▶ anisotropy (stiffness tensor, ρ)

$$-\omega^2 \rho \mathbf{u} - \nabla \cdot (\underline{\underline{C}} \epsilon(\mathbf{u})) = 0.$$

Viscous behavior are considered with complex coefficients.

Full Waveform Inversion (FWI)

FWI provides a quantitative reconstruction of the subsurface by solving a minimization problem,

$$\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \|F(m) - d\|^2.$$

- ▶ d are the observed data,
- ▶ $F(m)$ represents the simulation using an initial model m :

$$F : m \rightarrow \{p(\mathbf{x}_1), p(\mathbf{x}_2), \dots, p(\mathbf{x}_{n_{rcv}})\}.$$



P. Lailly

The seismic inverse problem as a sequence of before stack migrations
[Conference on Inverse Scattering: Theory and Application, SIAM, 1983](#)



A. Tarantola

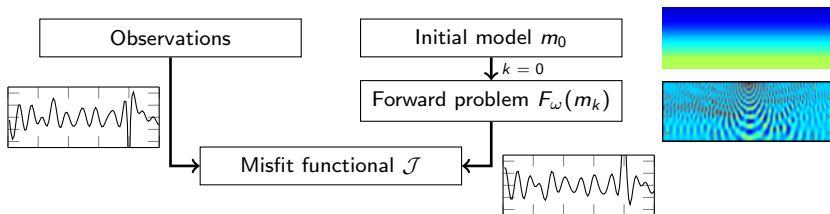
Inversion of seismic reflection data in the acoustic approximation
[Geophysics, 1984](#)



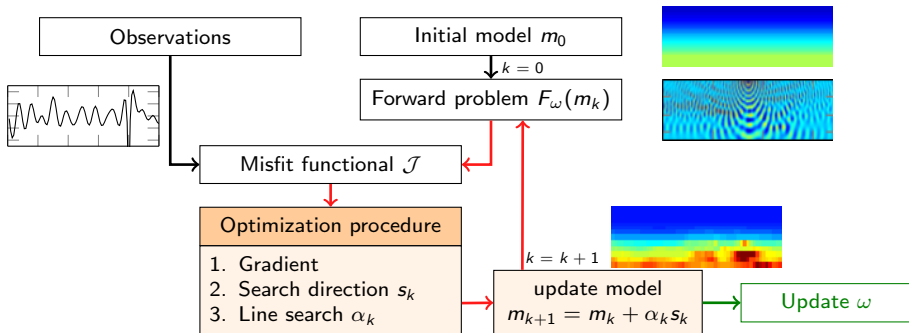
A. Tarantola

Inversion of travel times and seismic waveforms
[Seismic tomography, 1987](#)

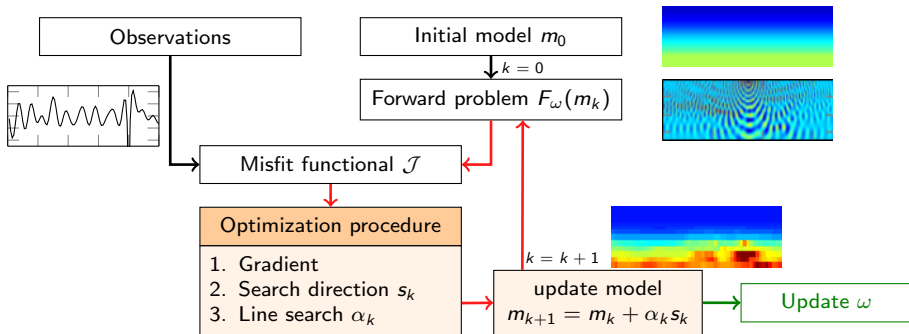
FWI, iterative minimization



FWI, iterative minimization



FWI, iterative minimization



Numerical methods

- ▶ Forward problem resolution with Discontinuous Galerkin methods,
- ▶ parallel computation, HPC, large-scale optimization.
- ▶ Multi-frequency algorithm, stability and convergence.

Plan

- 2 Quantitative stability and convergence of FWI
 - Finite Curvature/Limited Deflection problem
 - Numerical convergence estimates
 - Numerical stability estimates

Stability & Convergence

Helmholtz inverse problem from back-scattered **partial** data,
quantitative reconstruction with iterative optimization.

$$\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \|\mathcal{F}(m) - d\|^2.$$

Convergence radius

- ▶ initial model needs to be within the radius of convergence.

Stability

- ▶ ensures reconstruction accuracy.

Stability & Convergence

Helmholtz inverse problem from back-scattered **partial** data,
quantitative reconstruction with iterative optimization.

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Convergence radius

- ▶ initial model needs to be within the radius of convergence.

Stability

- ▶ ensures reconstruction accuracy.

Numerical estimates to guide the procedure

- ▶ frequency;
- ▶ parametrization;
- ▶ geometry;
- ▶ forward problem, ...

Convergence (Finite Curvature/Limited Deflection)

1/ Convergence radius

- ▶ initial model needs to be within the radius of convergence.

Least squares minimization problem

$$\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \|\mathcal{F}(m) - d\|^2.$$



M. V. de Hoop, L. Qiu, O. Scherzer

An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints

[Numerische Mathematik 2015](#)



G. Chavent

Nonlinear least squares for inverse problems: theoretical foundations and step-by-step guide for applications.

[Springer 2010](#)



G. Chavent and K. Kunisch

On weakly nonlinear inverse problems.

[SIAM Journal on Applied Mathematics 1996](#)

Convergence (Finite Curvature/Limited Deflection)

1/ Convergence radius

- ▶ initial model needs to be within the radius of convergence.

Least squares minimization problem

$$\min_{m \in \mathcal{M}} \mathcal{J}(m) = \frac{1}{2} \|\mathcal{F}(m) - d\|^2.$$

Finite Curvature/Limited Deflection guarantees **uniqueness** of the solution and **unimodality**: no local minimum.



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An analysis of a multi-level projected steepest descent iteration for nonlinear inverse problems in Banach spaces subject to stability constraints

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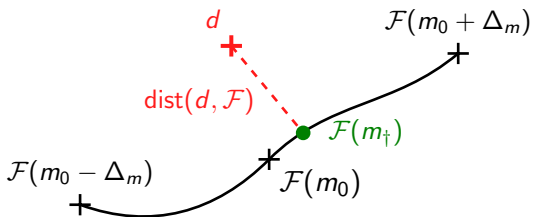


G. Chavent and K. Kunisch

On weakly nonlinear inverse problems.

[SIAM Journal on Applied Mathematics 1996](#)

Condition 1/2: Finite Curvature problem



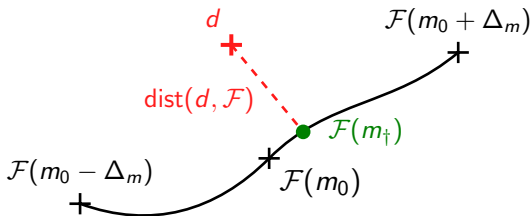
$$\forall m_0, \Delta_m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m).$$



G. Chavent

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Springer 2010

Condition 1/2: Finite Curvature problem



$$\forall m_0, \Delta_m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m).$$

Finite Curvature controls the distance between data and attainable set,

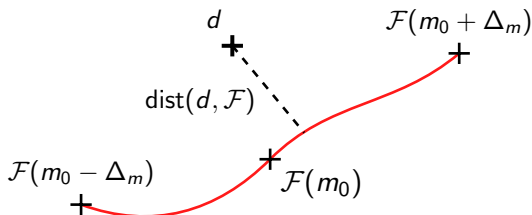
$$\text{dist}(d, \mathcal{F}) < \frac{\|P'(t)\|^2}{\|P''(t)\|}.$$



G. Chavent

Nonlinear least squares for inverse problems: theoretical foundations and step-by-step guide for applications.
Springer 2010

Condition 2/2: Limited Deflection problem



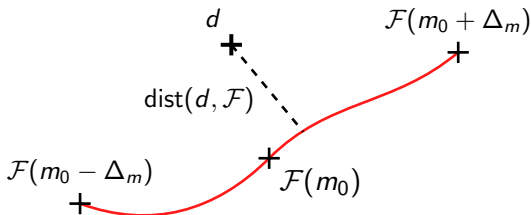
$$\forall m_0, \Delta_m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m).$$



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Springer 2010

Condition 2/2: Limited Deflection problem



$$\forall m_0, \Delta_m \in \mathcal{M}, \quad P : t \in [0, 1] \rightarrow \mathcal{F}(m_0 + (2t - 1)\Delta_m).$$

Limited Deflection property controls the attainable set and model space,

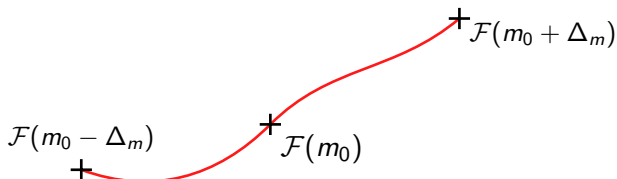
$$\Theta(P) \leq \int_0^1 \frac{\|P''(t)\|}{\|P'(t)\|} dt. \quad \text{FC/LD requires } \Theta(P) \leq \frac{\pi}{2}.$$



G. Chavent

Nonlinear least squares for inverse problems: theoretical foundations and step-by-step guide for applications.
Springer 2010

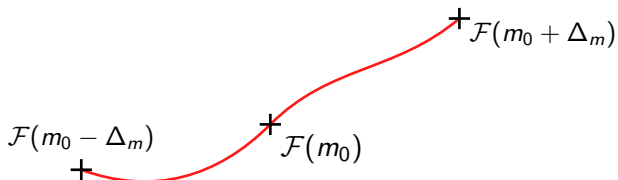
Model space size estimation via Limited Deflection



For a given m_0 , we estimate the maximal distance Δ_m that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

Larger size are required when missing a priori info.

Model space size estimation via Limited Deflection



For a given m_0 , we estimate the maximal distance Δ_m that still verifies the Limited Deflection property $\Theta(P) \leq \frac{\pi}{2}$.

- ▶ Δ_m is estimated in the direction δ_k ,

$$\|\Delta_m^{\delta_k}\| \leq \frac{\pi}{4} \frac{\|\mathcal{F}'(m_0)(\delta_k)\|}{\|\mathcal{F}''(m_0)(\delta_k, \delta_k)\|}.$$

Larger size are required when missing a priori info.

Estimation of the basin of attraction

Numerical estimate of the size $\Delta_m^{\delta_k}$

- 1 with direction δ_k (the geometry of the unknown),
- 2 with the frequency ω ,
- 3 with the parametrization.

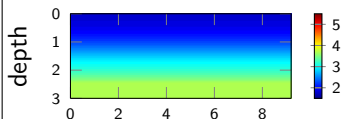
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_{\delta_k}^m$

- 1 with direction δ_k (the geometry of the unknown),
- 2 with the frequency ω ,
- 3 with the parametrization.

context: Helmholtz equation with back-scattered data

$$(-\omega^2 m - \Delta)p = 0,$$



m_0 is a smooth velocity

- ▶ reflection data (top free surface),
- ▶ absorbing conditions on the sides,
- ▶ piecewise constant m .

Estimation of the basin of attraction

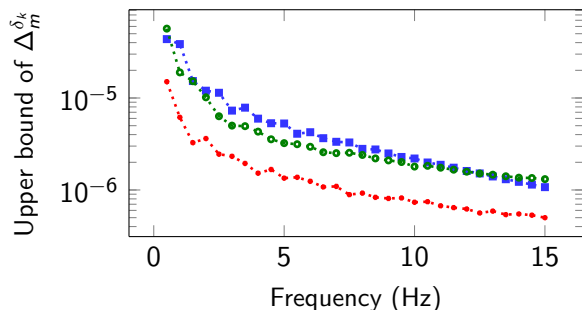
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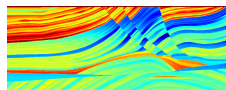
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_m^{\delta_k}$ with

- 1 with direction δ_k (the geometry of the unknown),
- 2 with the frequency ω .



(a) direction δ_1



(b) direction δ_2

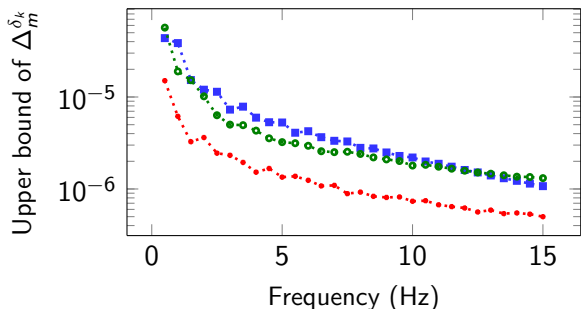


(c) direction δ_3

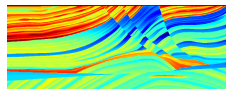
Estimation of the basin of attraction

Numerical estimate of the size $\Delta_m^{\delta_k}$ with

- 1 with direction δ_k (the geometry of the unknown),
- 2 with the frequency ω .



(a) direction δ_1



(b) direction δ_2



(c) direction δ_3

Low frequencies increase the size.

Reflecting objects complicate the procedure.

Complex frequency

$$(-\omega^2 c^{-2} - \Delta)p = 0$$

- ▶ $-\omega^2 = (s + 2i\pi f)^2$
- ▶ $s = 0$: Fourier domain $\omega^2 = (2\pi f)^2$,
- ▶ $f = 0$: Laplace domain $-\omega^2 = s^2$.



C. Shin and Y. H. Cha

Waveform inversion in the laplace domain ; Waveform inversion in the laplace fourier domain
[Geophysical Journal International 2008–2009](#)



W. Ha, S. Pyun, J. Yoo and C. Shin

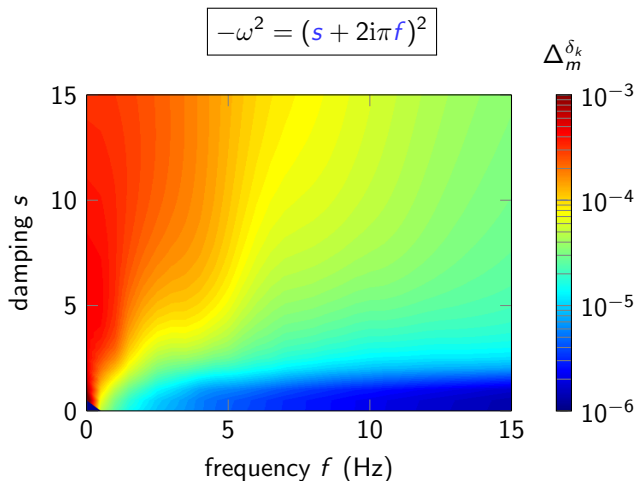
Acoustic full waveform inversion of synthetic land and marine data in the laplace domain ;
[Geophysical Prospecting 2010](#)



P. V. Petrov and G. A. Newman

Three-dimensional inverse modelling of damped elastic wave propagation in the fourier domain.
[Geophysical Journal International 2014](#)

Complex frequency



Selection of complex frequency.

Stability with frequency

2/ Stability

- ▶ ensures reconstruction accuracy.

The stability indicates how minimizing the data recovers the model

$$\|m_1 - m_2\| \leq C \|\mathcal{F}(m_1) - \mathcal{F}(m_2)\|.$$

The stability constant C depends on the frequency and the number of unknowns.



G. Alessandrini

Stable determination of conductivity by boundary measurement
[Applicable Analysis 1988](#)



E. Beretta, M. V. de Hoop, F. and O. Scherzer

Inverse boundary value problem for the Helmholtz equation: quantitative conditional Lipschitz stability estimates.
[SIAM J. Math. Analysis 2016](#)



G. Alessandrini, M.V. de Hoop, R. Gaburro and E. Sincich

Lipschitz stability for a piecewise linear Schrödinger potential from local Cauchy data
[arXiv:1702.04222, 2017](#)

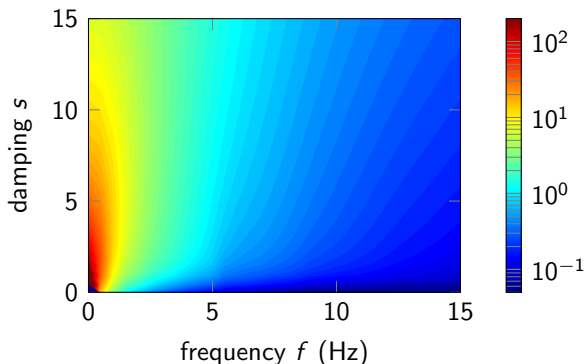
Stability with frequency

2/ Stability

- ▶ ensures reconstruction accuracy.

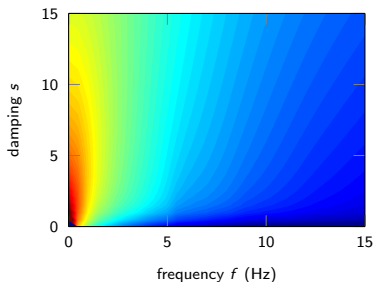
The stability indicates how minimizing the data recovers the model

$$\|m_1 - m_2\| \leq C \|\mathcal{F}(m_1) - \mathcal{F}(m_2)\|.$$

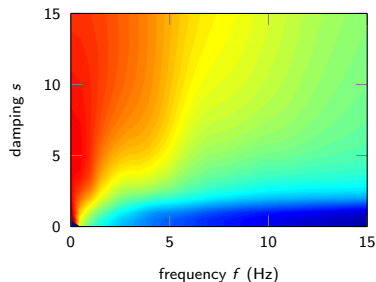


Stability & Convergence estimates

From our estimates, we see different conditions for stability and convergence.



(a) stability



(b) convergence

Quantitative estimates provide an initial, comprehensive, relation to guide the iterative procedure.

Plan

3 Numerical experiments

Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency



high stability constant.

Multi-level algorithm: stability and convergence

0./ No initial information on the subsurface

largest radius of convergence,

1./ Start with low frequency



high stability constant.

improve stability, *i.e.* resolution,

2./ Increase frequency

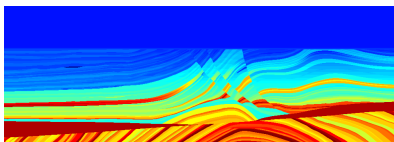


reduced radius of convergence.

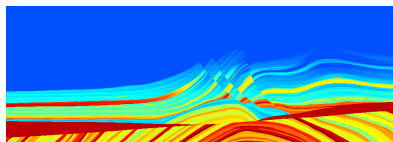
Elastic Marmousi model

Multi-parameters inversion $17 \times 3.5\text{km}$, true models.

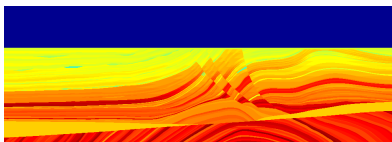
$$-\rho\omega^2\mathbf{u} - \nabla(\lambda\nabla\cdot\mathbf{u}) - \nabla\cdot(\mu[\nabla\mathbf{u} + (\nabla\mathbf{u})^T]) = 0.$$



(a) P-wave speed $\sqrt{\frac{\lambda+2\mu}{\rho}}$



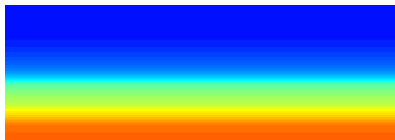
(b) S-wave speed $\sqrt{\frac{\mu}{\rho}}$



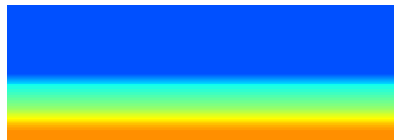
(c) Density

Elastic Marmousi model

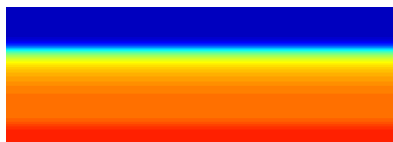
Multi-parameters inversion $17 \times 3.5\text{km}$, starting models.



(a) P-wave speed



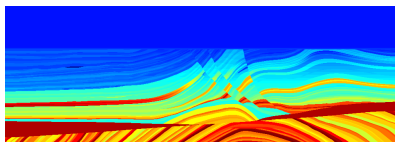
(b) S-wave speed



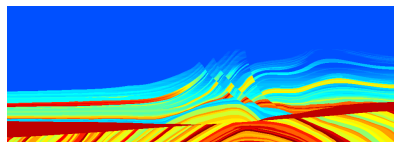
(c) Density

Elastic Marmousi reconstruction $17 \times 3.5\text{km}$

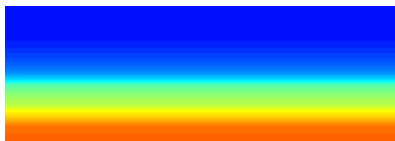
Frequency from 1 to 10Hz, unknown density, multi-level algorithm.



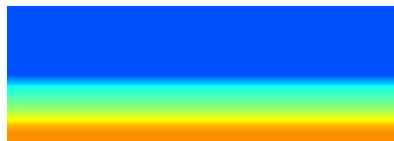
(a) True P-wave speed



(b) True S-wave speed



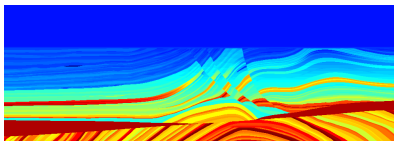
(c) Initial P-wave speed



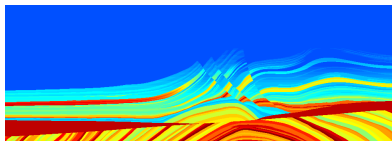
(d) Initial S-wave speed

Elastic Marmousi reconstruction $17 \times 3.5\text{km}$

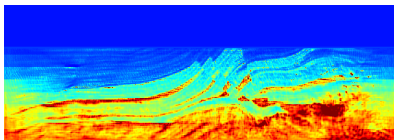
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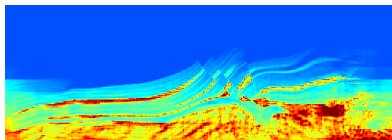
(a) True P-wave speed



(b) True S-wave speed



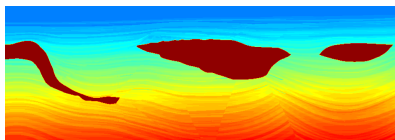
(c) 10Hz P-wave speed



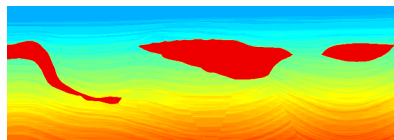
(d) 10Hz S-wave speed

Elastic Pluto models

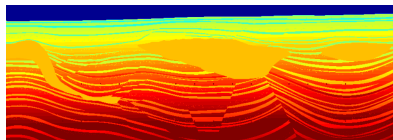
Multi-parameters inversion $31 \times 7\text{km}$, true models.



(a) P-wave speed



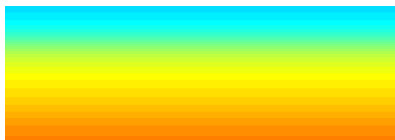
(b) S-wave speed



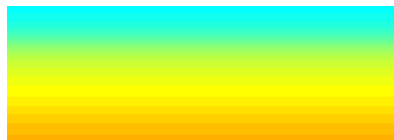
(c) Density

Elastic Pluto models

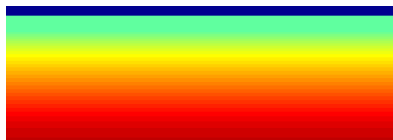
Multi-parameters inversion $31 \times 7\text{km}$, starting models.



(a) P-wave speed



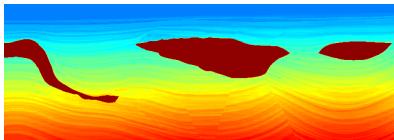
(b) S-wave speed



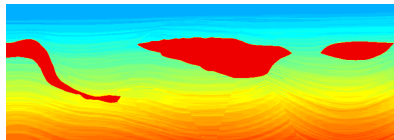
(c) Density

Elastic Pluto reconstruction 31×7 km

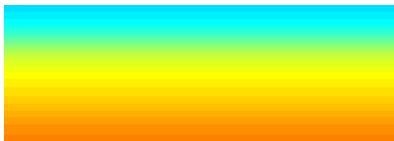
Frequency from 1 to 10Hz, unknown density, multi-level algorithm.



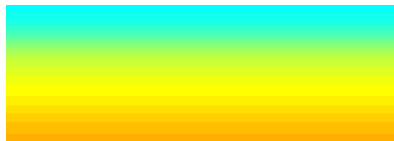
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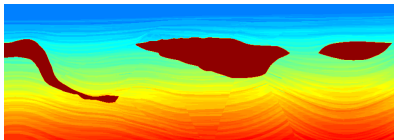
(c) Initial P-wave speed



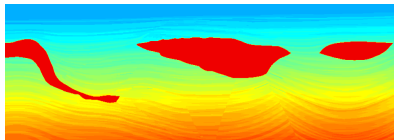
(d) Initial S-wave speed

Elastic Pluto reconstruction 31×7 km

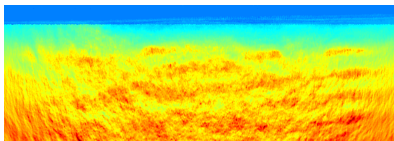
Frequency from 1 to 10Hz, unknown density, multi-level algorithm.



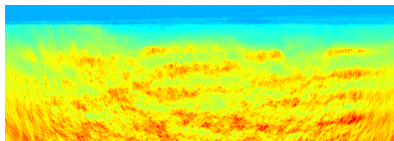
(a) True P-wave speed



(b) True S-wave speed



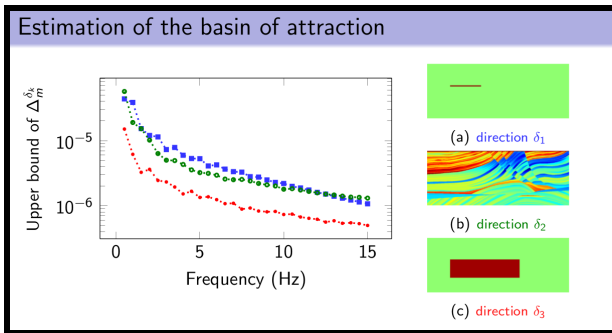
(c) 10Hz P-wave speed



(d) 10Hz S-wave speed

Elastic Pluto reconstruction $31 \times 7\text{km}$

Reminder from the convergence analysis.



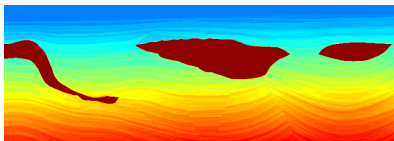
- ▶ Salt domes reduce the size of the radius of convergence.

Solution

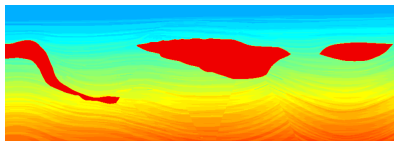
- ▶ Start with low or complex frequency.

Elastic Pluto reconstruction $31 \times 7\text{km}$

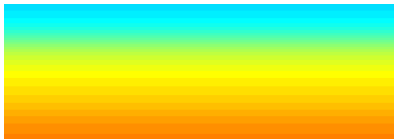
Using complex frequencies, unknown density, multi-level algorithm.



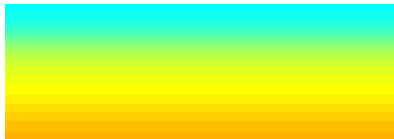
(a) True P-wave speed



(b) True S-wave speed



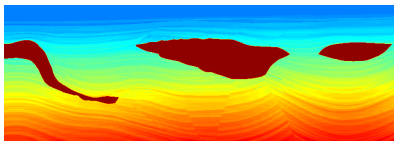
(c) Initial P-wave speed



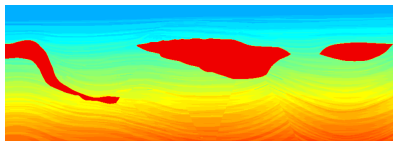
(d) Initial S-wave speed

Elastic Pluto reconstruction 31×7 km

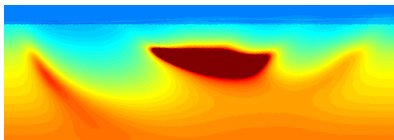
Using complex frequencies, unknown density, multi-level algorithm.



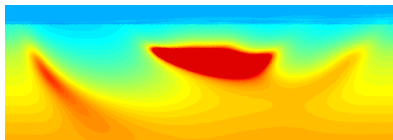
(a) True P-wave speed



(b) True S-wave speed



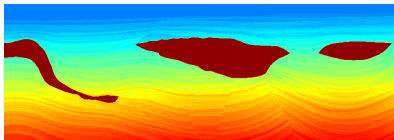
(c) (0Hz,10) P-wave speed



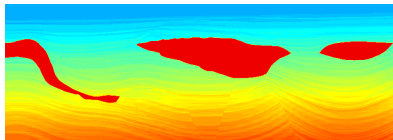
(d) (0Hz,10) S-wave speed

Elastic Pluto reconstruction $31 \times 7\text{km}$

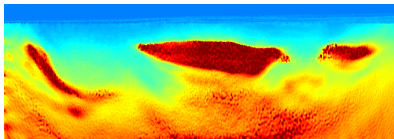
Using complex frequencies, unknown density, multi-level algorithm.



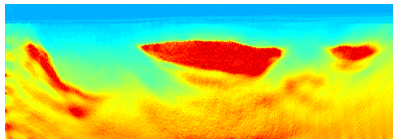
(a) True P-wave speed



(b) True S-wave speed



(c) 10Hz P-wave speed



(d) 10Hz S-wave speed

Plan

4 Conclusion

Conclusion

Comprehensive FWI

- ▶ Quantitative stability and convergence analysis:
 - ▶ depending on the frequency and geometry,
 - ▶ **depending on the methods.**
- ▶ We can also quantify the noise robustness.

Perspectives

- ▶ Analytical relation to develop between the two components,
- ▶ obtain conditional progression in frequency from stability and convergence estimates

Quantitative reconstruction method for inverse problem

- ▶ Discontinuous Galerkin discretization in HPC framework,
- ▶ acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

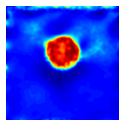
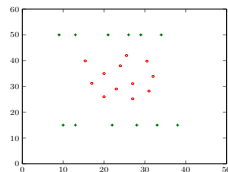
Applications and ongoing investigations:

Quantitative reconstruction method for inverse problem

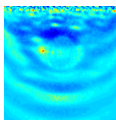
- ▶ Discontinuous Galerkin discretization in HPC framework,
- ▶ acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:

- ▶ **Inverse scattering** using obstacles positions,
- ▶ **visco-elastic** reconstruction: five unknowns,



(a) λ



(b) Q_λ

- ▶ $(\lambda, \mu, \rho, Q_\lambda, Q_\mu)$,
- ▶ unknown Q does not prevent λ recovery,
- ▶ procedure to recover attenuation parameter?

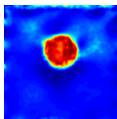
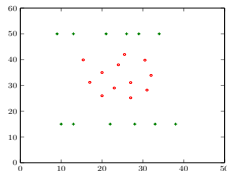
- ▶ **helioseismology**; **anisotropy**; model **parametrization**.

Quantitative reconstruction method for inverse problem

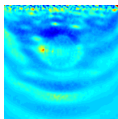
- ▶ Discontinuous Galerkin discretization in HPC framework,
- ▶ acoustic, elastic, anisotropy, viscosity; 2D, 3D, dual-sensors data.

Applications and ongoing investigations:

- ▶ **Inverse scattering** using obstacles positions,
- ▶ **visco-elastic** reconstruction: five unknowns,



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- ▶ $(\lambda, \mu, \rho, Q_\lambda, Q_\mu)$,
- ▶ unknown Q does not prevent λ recovery,
- ▶ procedure to recover attenuation parameter?

- ▶ **helioseismology**; **anisotropy**; model **parametrization**.

THANK YOU

APPENDIX

Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

- 1 the frequency ω ,
- 2 with direction δ_k (the geometry of the unknown),
- 3 with the parametrization.

The estimates allow a comparison of methods, in particular we want to compare the dependency of the optimization on the low frequencies.

Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

- 1 the frequency ω ,
- 2 with direction δ_k (the geometry of the unknown),
- 3 with the parametrization.

the MBTT method decomposes the model with a smooth part (propagator) and the reflectors:

$$c^{-2} = p + r = p + D\mathcal{F}^*(p)s.$$



F. Clément and G. Chavent

Waveform inversion through MBTT formulation – 1992



F. Clément, G. Chavent and S. Gómez

Migration-based traveltime waveform inversion of 2-d simple structures: A synthetic example
Geophysics 2001



G. Chavent, K. Gadylshin and V. Tcheverda

Reflection fwi in mbtt formulation
EAGE 2015

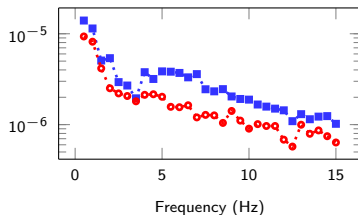
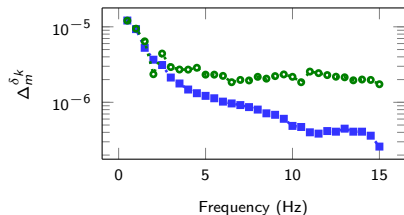
Quantitative convergence for MBTT

Numerical estimate of the size $\Delta_m^{\delta_k}$ the problem components

- 1 the frequency ω ,
- 2 with direction δ_k (the geometry of the unknown),
- 3 with the parametrization.

Comparison of convergence with parametrization,

$$c^{-2} = p + D\mathcal{F}^*(p)s.$$



Quantitative convergence estimates with frequency

Frequency progression for iterative algorithm, $-\omega^2 = (s + 2i\pi f)^2$.

Other application

- ▶ Single frequency gives high radius than range of frequencies,
- ▶ can be apply for other minimization problem,
- ▶ ...

Finite curvature indicates robustness to noise

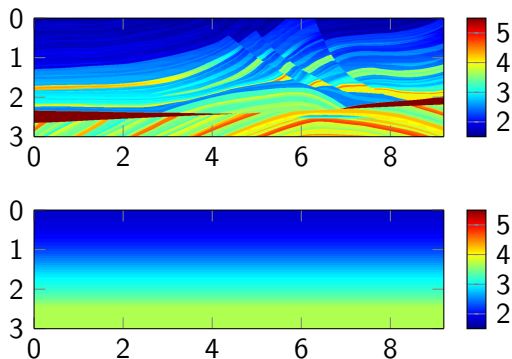
- ▶ Low and complex frequencies more affected.

Stability

- ▶ Indicates the accuracy of the reconstruction.

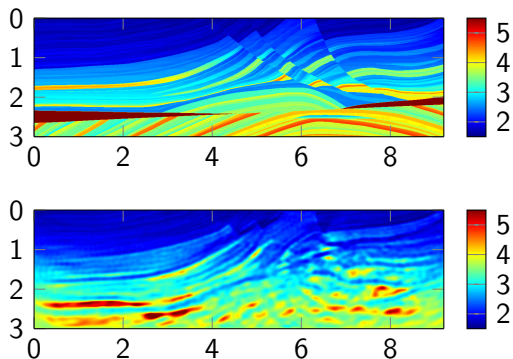
Acoustic Marmousi reconstruction $9.2 \times 3\text{km}$

Starting available frequency is 4Hz.



Acoustic Marmousi reconstruction $9.2 \times 3\text{km}$

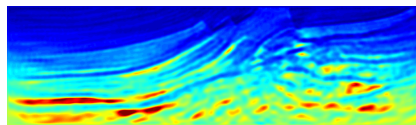
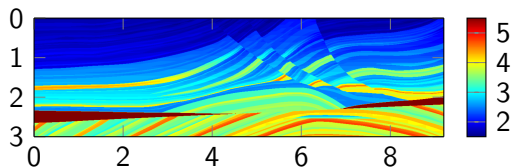
Starting available frequency is 4Hz.



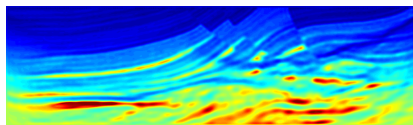
Using Fourier frequencies only: from 4 to 10Hz

Acoustic Marmousi reconstruction $9.2 \times 3\text{km}$

Starting available frequency is 4Hz.



(a) Using Fourier frequencies only



(b) Using Complex frequencies from (4Hz, 7)