



**HAL**  
open science

## Octal Games on Graphs: The game 0.33 on subdivided stars and bistars. Full proof of Theorem 22

Laurent Beaudou, Pierre Coupechoux, Antoine Dailly, Sylvain Gravier, Julien Moncel, Aline Parreau, Eric Sopena

### ► To cite this version:

Laurent Beaudou, Pierre Coupechoux, Antoine Dailly, Sylvain Gravier, Julien Moncel, et al.. Octal Games on Graphs: The game 0.33 on subdivided stars and bistars. Full proof of Theorem 22. 2018. hal-01807116

**HAL Id: hal-01807116**

**<https://hal.science/hal-01807116>**

Preprint submitted on 4 Jun 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Octal Games on Graphs:

## The game 0.33 on subdivided stars and bistars

### Full proof of Theorem 22

Laurent Beaudou, Pierre Coupechoux, Antoine Dailly, Sylvain Gravier,  
Julien Moncel, Aline Parreau, Éric Sopena

This work has been supported by the ANR-14-CE25-0006 project of the French National Research Agency.

We define an equivalence relation on subdivided stars  $\sim_2$ . Let  $S$  and  $S'$  be two subdivided stars. We say that  $S$  and  $S'$  are  $\sim_2$ -equivalent, denoted  $S \sim_2 S'$ , if and only if for any subdivided star  $\hat{S}$ ,  $S^{\bullet\bullet}\hat{S} \equiv S'^{\bullet\bullet}\hat{S}$ . By Lemma 17 in [1], we already know that  $P_3 \sim_2 \emptyset$ , and thus  $S_2 \sim_2 \emptyset$  and  $S_{1,1} \sim_2 \emptyset$ .

We will prove that there are exactly ten equivalence classes for  $\sim_2$ :

- $\mathcal{D}_0^*$ : subdivided stars  $S$  such that  $\mathcal{G}(S) = 0$  and  $S$  contains zero or two paths of length 2, plus  $S_2$ ;
- $\mathcal{D}_1^* = \{P_1, S_{2,1}, S_{2,2,2}\}$  (these stars have Grundy value 1);
- $\mathcal{D}_1^\square$ : subdivided stars  $S$  such that  $\mathcal{G}(S) = 1$ ,  $S$  contains zero or two paths of length 2 and  $S \neq P_1$ ;
- $\mathcal{D}_2^* = \{P_2, S_{2,2}\}$  (these stars have Grundy value 2);
- $\mathcal{D}_2^\square$ : subdivided stars  $S$  such that  $\mathcal{G}(S) = 2$  and  $S$  contains one or three paths of length 2;
- $\mathcal{D}_3^\square$ : subdivided stars  $S$  such that  $\mathcal{G}(S) = 3$  and  $S$  contains one or three paths of length 2;
- For  $i \in \{0, 1, 2, 3\}$ ,  $\mathcal{D}_i$ : subdivided stars  $S$  with  $\mathcal{G}(S) = i$  and  $S$  is not in a previous class.

Figure 1 shows the equivalence classes of the subdivided stars.

**Theorem 22.** *The equivalence classes for  $\sim_2$  are exactly the sets  $\mathcal{D}_0, \mathcal{D}_0^*, \mathcal{D}_1, \mathcal{D}_1^*, \mathcal{D}_1^\square, \mathcal{D}_2, \mathcal{D}_2^*, \mathcal{D}_2^\square, \mathcal{D}_3$  and  $\mathcal{D}_3^\square$ . Moreover, Table 1 describes how the Grundy value of  $S^{\bullet\bullet}S'$  can be computed depending on the equivalence class of  $S$  and  $S'$ .*

*Proof.* We prove by induction on the total number of vertices in  $S$  and  $S'$  that the Grundy value of  $S^{\bullet\bullet}S'$  is given by Table 1.

One can check that the rows and columns for  $\mathcal{D}_1^*$  and  $\mathcal{D}_2^*$  in Table 1 are correct: it suffices to prove it for one representant for  $\mathcal{D}_1^*$  (say  $P_1$ ) and for  $\mathcal{D}_2^*$  (say  $P_2$ ). Indeed, Lemma 23 in [1] proves that  $S \sim_2 S'$  if  $S, S' \in \mathcal{D}_1^*$  or  $S, S' \in \mathcal{D}_2^*$ . For any subdivided star  $\hat{S}$ ,  $\hat{S}^{\bullet\bullet}P_1$  is  $\hat{S}$  with a path of length 2 attached to its central vertex. Thus, for every class, we only need to look at the value diagonally to the right and below in Figure 1. One can check that if  $\mathcal{G}(\hat{S}) = 0$ , then  $\mathcal{G}(\hat{S}^{\bullet\bullet}P_1) = 2$ , if  $\hat{S} \in \mathcal{D}_1^*, \mathcal{D}_2, \mathcal{D}_3^\square$ , then  $\mathcal{G}(\hat{S}^{\bullet\bullet}P_1) = 0$ , if  $\hat{S} \in \mathcal{D}_1, \mathcal{D}_1^\square$ , then  $\mathcal{G}(\hat{S}^{\bullet\bullet}P_1) = 3$ , if  $\hat{S} \in \mathcal{D}_2^*, \mathcal{D}_2^\square, \mathcal{D}_3$ , then  $\mathcal{G}(\hat{S}^{\bullet\bullet}P_1) = 1$ . For any subdivided star  $\hat{S}$ ,  $\hat{S}^{\bullet\bullet}P_2$  is  $\hat{S}$  with a path of length 3 attached to its central vertex. Thus,  $\mathcal{G}(\hat{S}^{\bullet\bullet}P_2) = \mathcal{G}(\hat{S})$ .

Now we study the Grundy value of  $S^{\bullet\bullet}S'$  depending on the class of  $S$  and  $S'$ . We can suppose that  $S, S' \notin \mathcal{D}_1^*, \mathcal{D}_2^*$ , and that neither  $S$  nor  $S'$  are  $S_{1,1}$  or  $P_3$  (since  $S_{1,2} \sim_2 \emptyset$ ; and  $P_3 \sim_2 \emptyset$ ). We can find the

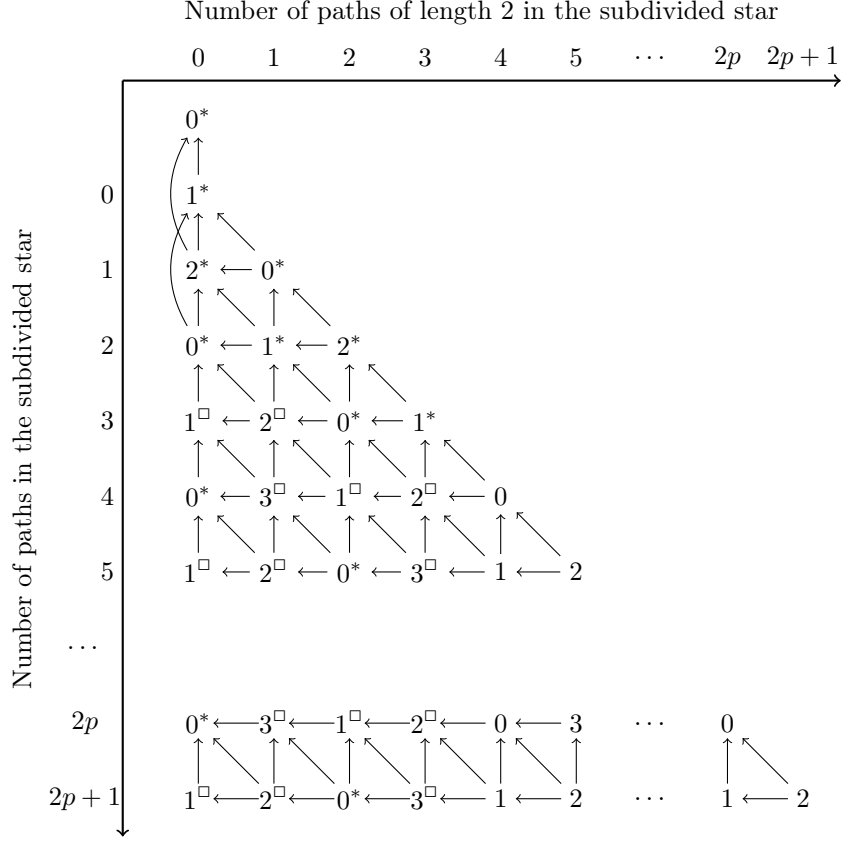


Figure 1: First six rows, and rows  $2p$  and  $2p+1$ , of the table of equivalence classes for  $\sim_2$  of the subdivided stars. Stars belonging to resp.  $\mathcal{D}_0^*$ ,  $\mathcal{D}_1^*$ ,  $\mathcal{D}_1^\square$ ,  $\mathcal{D}_2^*$ ,  $\mathcal{D}_2^\square$ ,  $\mathcal{D}_3^\square$  are depicted by resp.  $0^*$ ,  $1^*$ ,  $1^\square$ ,  $2^*$ ,  $2^\square$ ,  $3^\square$ , while the  $\mathcal{D}_i$ 's are depicted by  $i$ .

	$\mathcal{D}_0$	$\mathcal{D}_0^*$	$\mathcal{D}_1$	$\mathcal{D}_1^*$	$\mathcal{D}_1^\square$	$\mathcal{D}_2$	$\mathcal{D}_2^*$	$\mathcal{D}_2^\square$	$\mathcal{D}_3$	$\mathcal{D}_3^\square$
$\mathcal{D}_0$	$\oplus$	$\oplus_1$	$\oplus$	2	$\oplus_1$	$\oplus$	0	$\oplus_1$	$\oplus$	$\oplus_1$
$\mathcal{D}_0^*$	$\oplus_1$	$\oplus_1$	$\oplus_1$	2	$\oplus_1$	$\oplus_1$	0	$\oplus_1$	$\oplus_1$	$\oplus_1$
$\mathcal{D}_1$	$\oplus$	$\oplus_1$	$\oplus$	3	$\oplus_1$	$\oplus$	1	$\oplus_1$	$\oplus$	$\oplus_1$
$\mathcal{D}_1^*$	2	2	3	0	3	0	1	1	1	0
$\mathcal{D}_1^\square$	$\oplus_1$	$\oplus_1$	$\oplus_1$	3	$\oplus_1$	$\oplus_1$	1	$\oplus_1$	$\oplus_1$	$\oplus_1$
$\mathcal{D}_2$	$\oplus$	$\oplus_1$	$\oplus$	0	$\oplus_1$	$\oplus$	2	$\oplus_1$	$\oplus$	$\oplus_1$
$\mathcal{D}_2^*$	0	0	1	1	1	2	2	2	3	3
$\mathcal{D}_2^\square$	$\oplus_1$	$\oplus_1$	$\oplus_1$	1	$\oplus_1$	$\oplus_1$	2	0	$\oplus_1$	1
$\mathcal{D}_3$	$\oplus$	$\oplus_1$	$\oplus$	1	$\oplus_1$	$\oplus$	3	$\oplus_1$	$\oplus$	$\oplus_1$
$\mathcal{D}_3^\square$	$\oplus_1$	$\oplus_1$	$\oplus_1$	0	$\oplus_1$	$\oplus_1$	3	1	$\oplus_1$	0

Table 1: Computing the Grundy value of  $S^{\bullet-2} \bullet S'$  depending on the equivalence class of  $S$  and  $S'$ . Recall that  $\oplus$  denotes the nim-sum. Moreover,  $x \oplus_1 y$  stands for  $x \oplus y \oplus 1$ .

Grundy values of the options of  $S$  and  $S'$  thanks to Figure 1. None of the options of  $S$  and  $S'$  involves taking their central vertex. We can verify Table 1 by computing the Grundy value of  $S^{\bullet-2} \bullet S'$  depending on the Grundy values of their options, by using the induction hypothesis:

$$\mathcal{G}(S^{\bullet^2} \bullet S') = \text{mex}(\mathcal{G}(T^{\bullet^2} \bullet S'), \mathcal{G}(S^{\bullet^2} \bullet T') | T \text{ option of } S, T' \text{ option of } S')$$

In order to prove that the equivalence classes are correct, we need to check that the Grundy value of  $S^{\bullet^2} \bullet S'$  does not change with the classes of the options of  $S$  and  $S'$ . Indeed, two subdivided stars belonging to the same class can have different options.

There is a total of 36 cases to consider, which is why the full proof was not included in [1]. Indeed, all those cases work the exact same way: we verify that for each class of each possible option of  $S$  and  $S'$ , the mex value of those options is what is predicted by Table 1. The notation  $\mathcal{D}_i^{\bullet^2} \bullet \mathcal{D}_j$  expresses the fact that the two subdivided stars  $S$  and  $S'$  are in the classes  $\mathcal{D}_i$  and  $\mathcal{D}_j$  respectively. Furthermore, on figures, the original  $S^{\bullet^2} \bullet S'$  will be boxed while the options obtained when playing on  $S$  will be displayed on the left, and those obtained when playing on  $S'$  will be displayed on the right (if  $S$  and  $S'$  are in the same class, then we will display them on only one side). Below each option will be the Grundy value predicted by Table 1 (we can apply the induction hypothesis on the options since they have strictly less vertices than  $S^{\bullet^2} \bullet S'$ ). To compute the Grundy value of  $S^{\bullet^2} \bullet S'$ , one only needs to compute, for all sets on the left, the mex of this set and every set on the right. If all those mex computations lead to the same value, then this means that it is the Grundy value of  $S^{\bullet^2} \bullet S'$ .

**Case 1:**  $S, S' \in \mathcal{D}_0^*$

Figure 2 depicts all possible options, their Grundy values and show that the mex computation always leads to 1 for every set of options. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 1$ , which is what is predicted by Table 1.

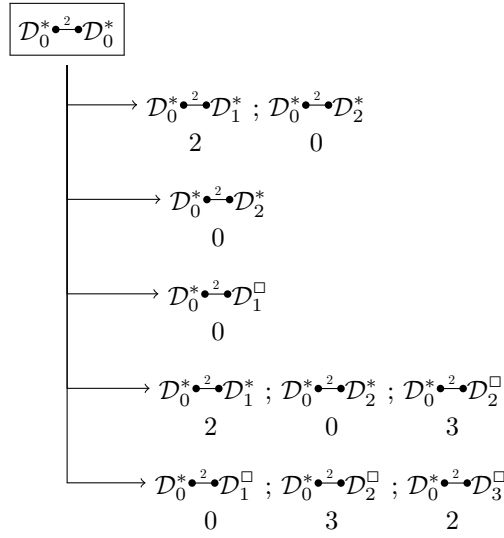


Figure 2: Possible options for the case  $S, S' \in \mathcal{D}_0^*$ .

**Case 2:**  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_1^\square$

Figure 3 depicts all possible options, their Grundy values and show that the mex computation always leads to 0 for every set of options. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 0$ , which is what is what is predicted by Table 1.

**Case 3:**  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_2^\square$

Figure 4 depicts all possible options, their Grundy values and show that the mex computations of the right options always leads to 3, while the left options are always at most 3 and as such the computation of any left option with every right option will always yield 3. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 3$ , which is what is predicted by Table 1.

**Case 4:**  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_3^\square$

Figure 5 depicts all possible options, their Grundy values and show that the mex computations of the right options always leads to 2, while the left options are always at most 2 and as such the computation of

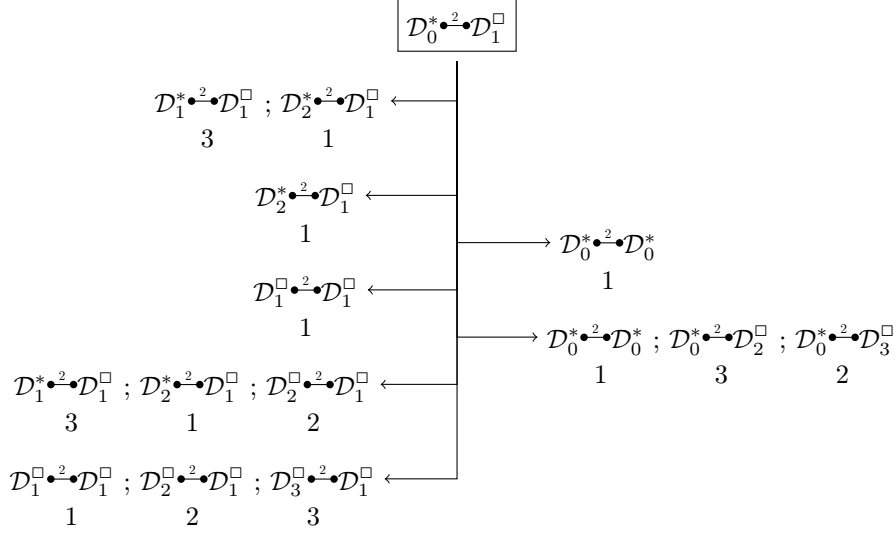


Figure 3: Possible options for the case  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_1^\square$ .

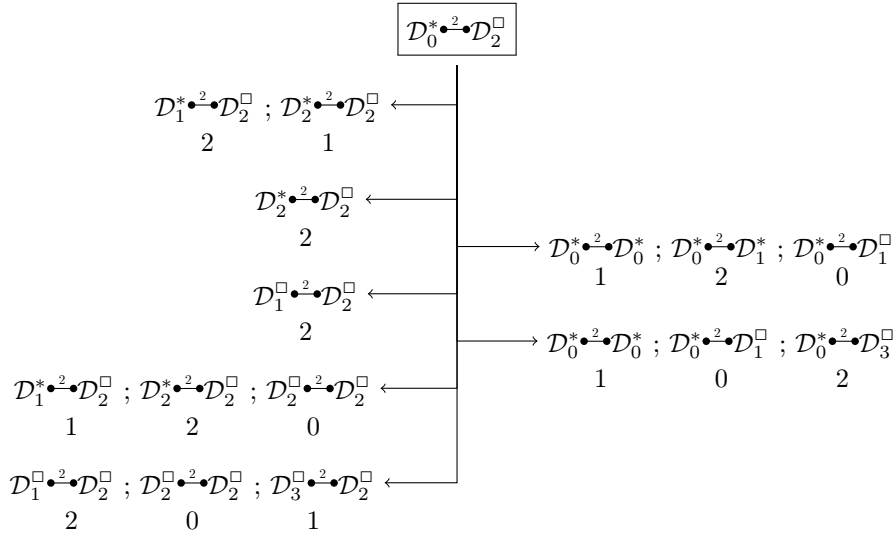


Figure 4: Possible options for the case  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_2^\square$ .

any left option with every right option will always yield 2. Thus,  $\mathcal{G}(S^{\bullet-2} \bullet S') = 2$ , which is what is predicted by Table 1.

**Case 5:**  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_0$

Figure 6 depicts all possible options, their Grundy values and show that the mex computations of the left options always leads to 1, while the right options are always at most 1 and as such the computation of any left option with every right option will always yield 1. Thus,  $\mathcal{G}(S^{\bullet-2} \bullet S') = 1$ , which is what is predicted by Table 1.

**Case 6:**  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_1$

Figure 7 depicts all possible options, their Grundy values and show that the mex computations of every set of options always leads to 0. Thus,  $\mathcal{G}(S^{\bullet-2} \bullet S') = 0$ , which is what is predicted by Table 1.

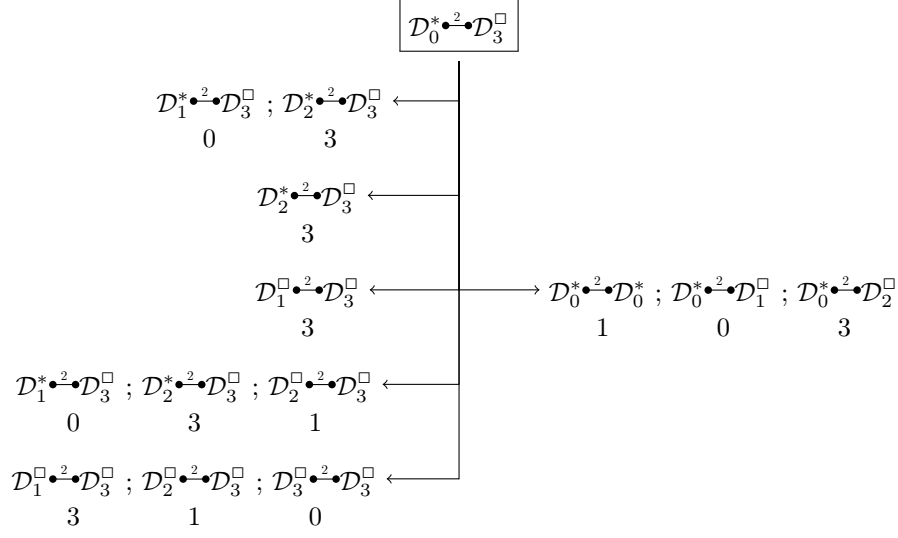


Figure 5: Possible options for the case  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_3^\square$ .

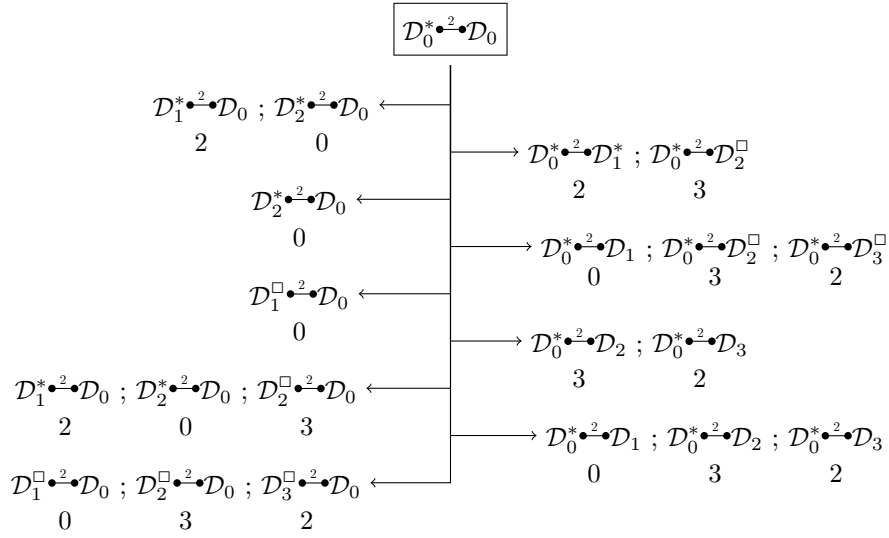


Figure 6: Possible options for the case  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_0$ .

**Case 7:**  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_2$

Figure 8 depicts all possible options, their Grundy values and show that the computation of any left option with every right option will always yield 3. Thus,  $\mathcal{G}(S^{\bullet-2} \bullet S')$  = 3, which is what is predicted by Table 1.

**Case 8:**  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_3$

Figure 9 depicts all possible options, their Grundy values and show that the mex computations of the right options always leads to 2, while the left options are always at most 2 and as such the computation of any left option with every right option will always yield 2. Thus,  $\mathcal{G}(S^{\bullet-2} \bullet S')$  = 2, which is what is predicted by Table 1.

**Case 9:**  $S, S' \in \mathcal{D}_1^\square$

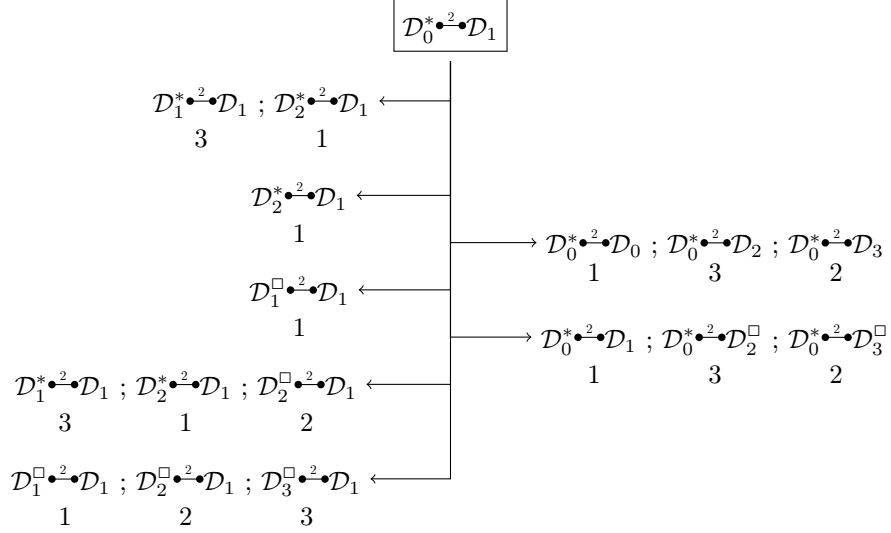


Figure 7: Possible options for the case  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_1$ .

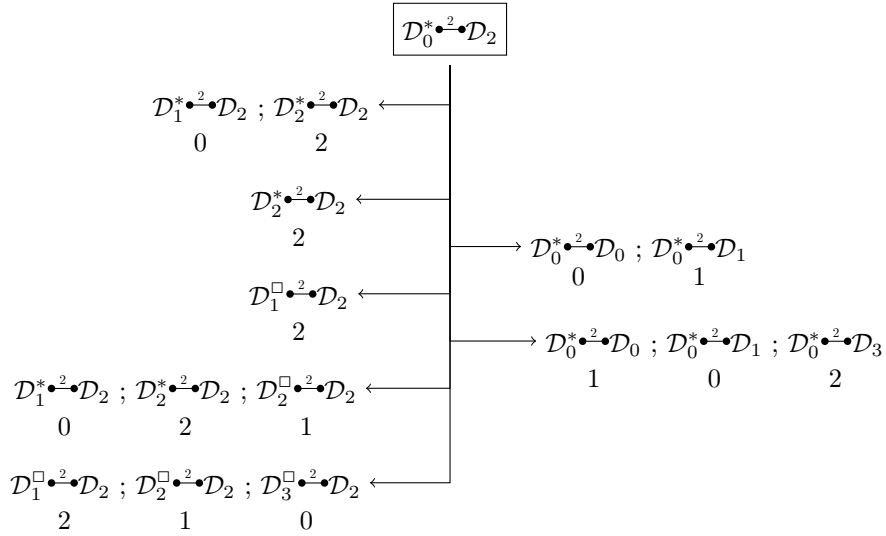


Figure 8: Possible options for the case  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_2$ .

Figure 10 depicts all possible options, their Grundy values and show that the mex computation always leads to 1 for every set of options. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 1$ , which is what is predicted by Table 1.

**Case 10:**  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_2^\square$

Figure 11 depicts all possible options, their Grundy values and show that the mex computation of the right options always leads to 2, while the left options are always at most 2 and as such the computation of any left option with every right option will always yield 2. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 2$ , which is what is predicted by Table 1.

**Case 11:**  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_3^\square$

Figure 12 depicts all possible options, their Grundy values and show that the mex computation of the right options always leads to 3, while the left options are always at most 3 and as such the computation of

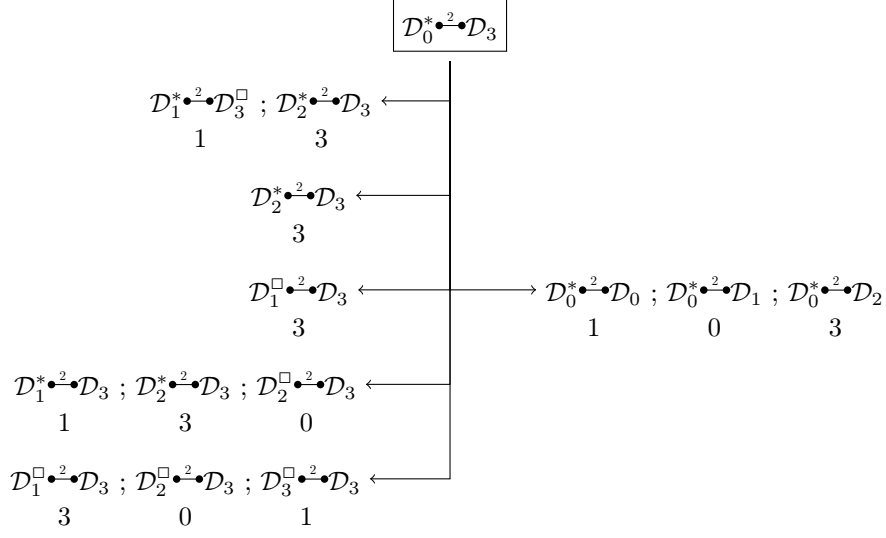


Figure 9: Possible options for the case  $S \in \mathcal{D}_0^*$  and  $S' \in \mathcal{D}_3$ .

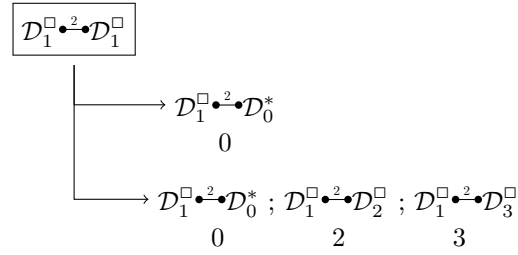


Figure 10: Possible options for the case  $S, S' \in \mathcal{D}_1^{\square}$ .

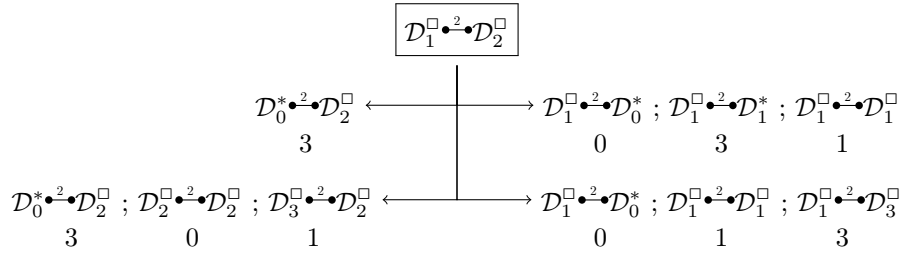


Figure 11: Possible options for the case  $S \in \mathcal{D}_1^{\square}$  and  $S' \in \mathcal{D}_2^{\square}$ .

any left option with every right option will always yield 3. Thus,  $\mathcal{G}(S \bullet^2 \bullet S') = 3$ , which is what is predicted by Table 1.

**Case 12:**  $S \in \mathcal{D}_1^{\square}$  and  $S' \in \mathcal{D}_0$

Figure 13 depicts all possible options, their Grundy values and show that the mex computation always leads to 0. Thus,  $\mathcal{G}(S \bullet^2 \bullet S') = 0$ , which is what is predicted by Table 1.

**Case 13:**  $S \in \mathcal{D}_1^{\square}$  and  $S' \in \mathcal{D}_1$

Figure 14 depicts all possible options, their Grundy values and show that the mex computation always leads to 1. Thus,  $\mathcal{G}(S \bullet^2 \bullet S') = 1$ , which is what is predicted by Table 1.



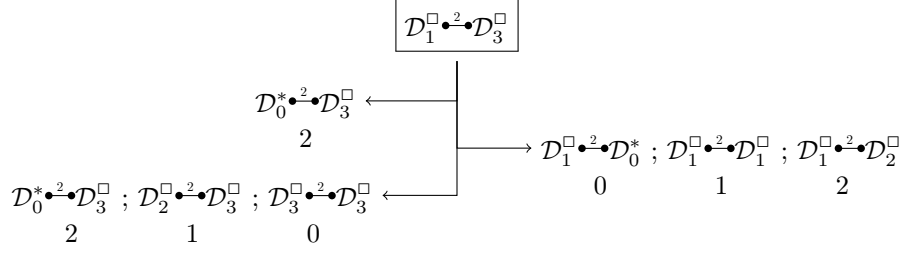


Figure 12: Possible options for the case  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_3^\square$ .

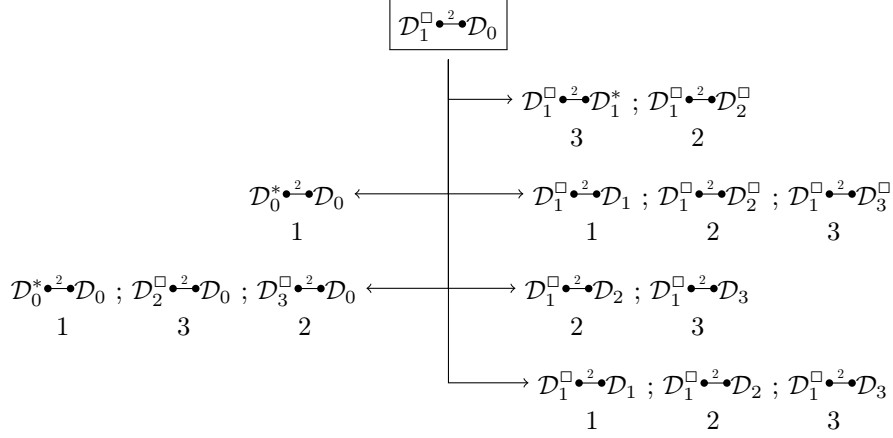


Figure 13: Possible options for the case  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_0$ .

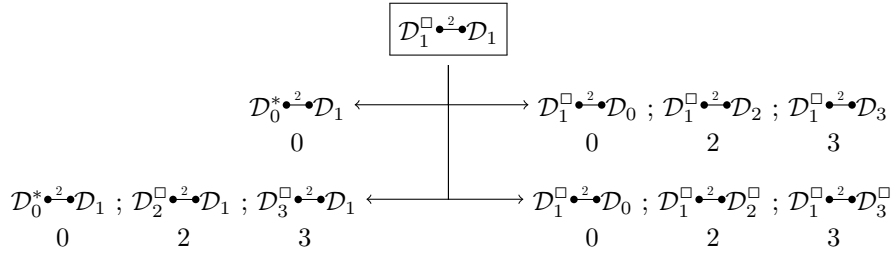


Figure 14: Possible options for the case  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_1$ .

**Case 14:**  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_2$

Figure 15 depicts all possible options, their Grundy values and show that the mex computation of the right options always leads to 2, while the left options are always at most 2, and as such the computations of any left option with every right option will always yield 2. Thus,  $\mathcal{G}(S \bullet^2 \bullet S') = 2$ , which is what is predicted by Table 1.

**Case 15:**  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_3$

Figure 16 depicts all possible options, their Grundy values and show that the mex computation of the right options always leads to 3, while the left options are always at most 3, and as such the computations of any left option with every right option will always yield 3. Thus,  $\mathcal{G}(S \bullet^2 \bullet S') = 3$ , which is what is predicted by Table 1.

**Case 16:**  $S, S' \in \mathcal{D}_2^\square$

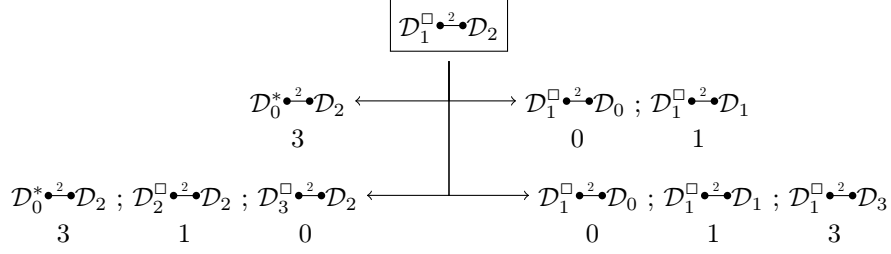


Figure 15: Possible options for the case  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_2$ .

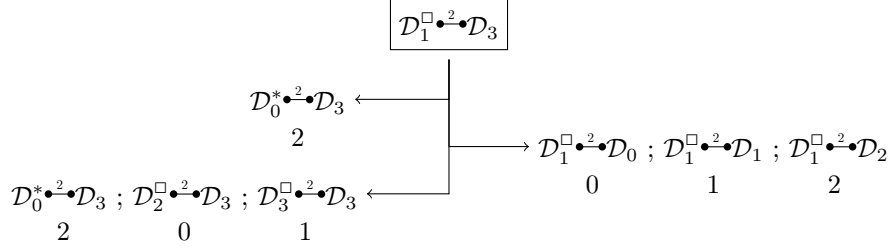


Figure 16: Possible options for the case  $S \in \mathcal{D}_1^\square$  and  $S' \in \mathcal{D}_3$ .

Figure 17 depicts all possible options, their Grundy values and show that the mex computation always leads to 0 for every set of options. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 0$ , which is what is predicted by Table 1.

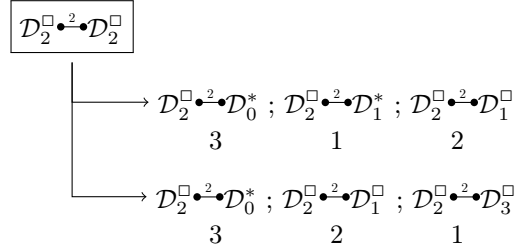


Figure 17: Possible options for the case  $S, S' \in \mathcal{D}_2^\square$ .

**Case 17:**  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_3^\square$

Figure 18 depicts all possible options, their Grundy values and show that the mex computation always leads to 1 for every set of options. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 1$ , which is what is predicted by Table 1.

**Case 18:**  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_0$

Figure 19 depicts all possible options, their Grundy values and show that the mex computation of the right options always leads to 3, while the left options are always at most 3, and as such the computation of any left option with every right options will always yield 3. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 3$ , which is what is predicted by Table 1.

**Case 19:**  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_1$

Figure 20 depicts all possible options, their Grundy values and show that the mex computation always leads to 2. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 2$ , which is what is predicted by Table 1.

**Case 20:**  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_2$

Figure 21 depicts all possible options, their Grundy values and show that the mex computation of the left options always leads to 1, while the right options are always at most 1, and as such the computation of

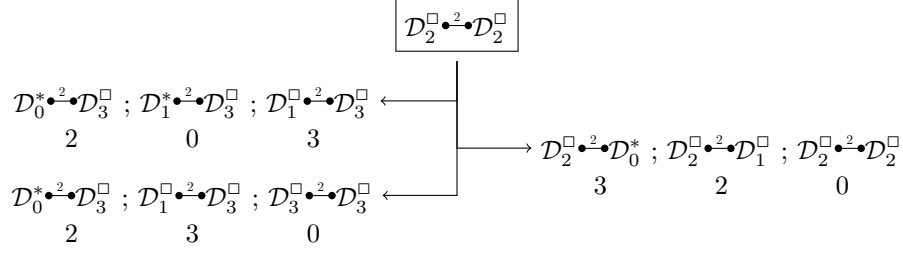


Figure 18: Possible options for the case  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_3^\square$ .

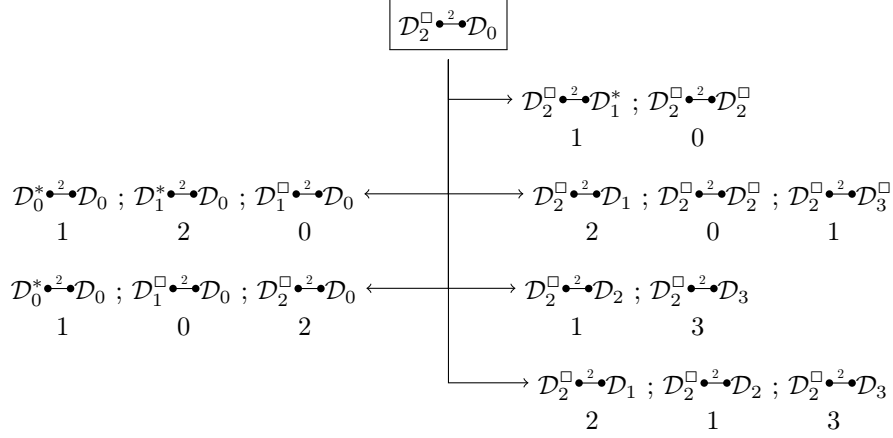


Figure 19: Possible options for the case  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_0$ .

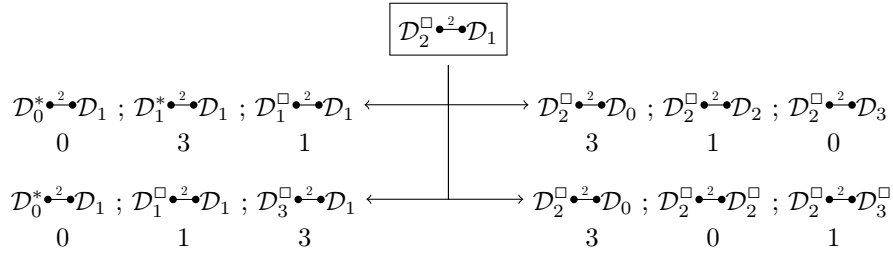


Figure 20: Possible options for the case  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_1$ .

any left option with every right options will always yield 1. Thus,  $\mathcal{G}(S^{\bullet^2} \cdot S') = 1$ , which is what is predicted by Table 1.

**Case 21:**  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_3$

Figure 22 depicts all possible options, their Grundy values and show that the mex computation always leads to 0. Thus,  $\mathcal{G}(S^{\bullet^2} \cdot S') = 0$ , which is what is predicted by Table 1.

**Case 22:**  $S, S' \in \mathcal{D}_3^\square$

Figure 23 depicts all possible options, their Grundy values and show that the mex computation always leads to 1 for every set of options. Thus,  $\mathcal{G}(S^{\bullet^2} \cdot S') = 1$ , which is what is predicted by Table 1.

**Case 23:**  $S \in \mathcal{D}_3^\square$  and  $S' \in \mathcal{D}_0$

Figure 24 depicts all possible options, their Grundy values and show that the mex computation always leads to 2. Thus,  $\mathcal{G}(S^{\bullet^2} \cdot S') = 2$ , which is what is predicted by Table 1.

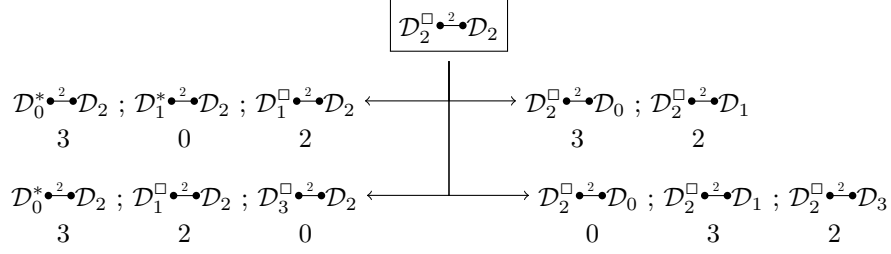


Figure 21: Possible options for the case  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_2$ .

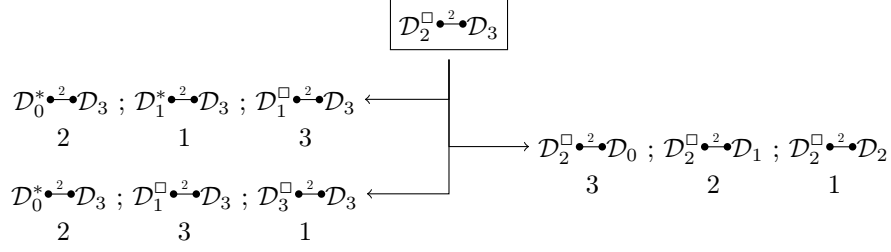


Figure 22: Possible options for the case  $S \in \mathcal{D}_2^\square$  and  $S' \in \mathcal{D}_3$ .

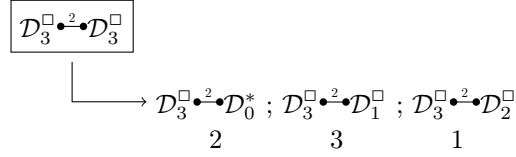


Figure 23: Possible options for the case  $S, S' \in \mathcal{D}_3^\square$ .

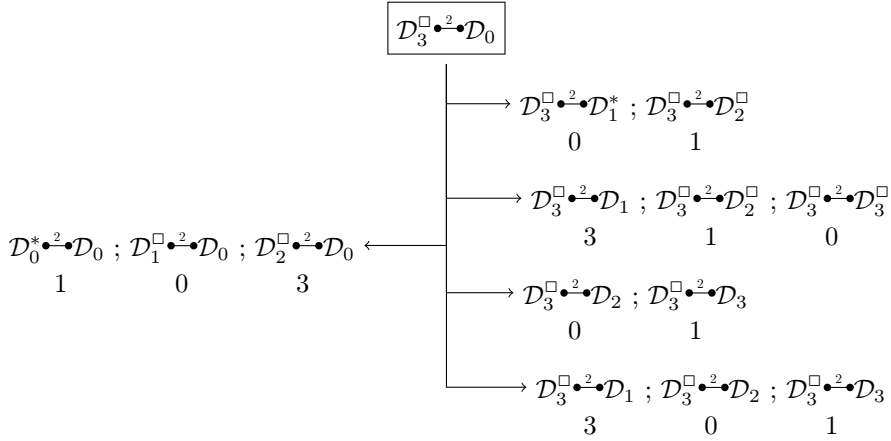


Figure 24: Possible options for the case  $S \in \mathcal{D}_3^\square$  and  $S' \in \mathcal{D}_0$ .

**Case 24:**  $S \in \mathcal{D}_3^\square$  and  $S' \in \mathcal{D}_1$

Figure 25 depicts all possible options, their Grundy values and show that the mex computation always leads to 3. Thus,  $\mathcal{G}(S \bullet^2 S') = 3$ , which is what is predicted by Table 1.

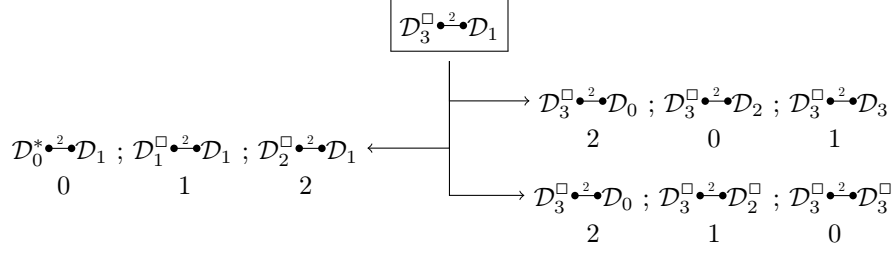


Figure 25: Possible options for the case  $S \in \mathcal{D}_3^\square$  and  $S' \in \mathcal{D}_1$ .

**Case 25:**  $S \in \mathcal{D}_3^\square$  and  $S' \in \mathcal{D}_2$

Figure 26 depicts all possible options, their Grundy values and show that the mex computation always leads to 0. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 0$ , which is what is predicted by Table 1.

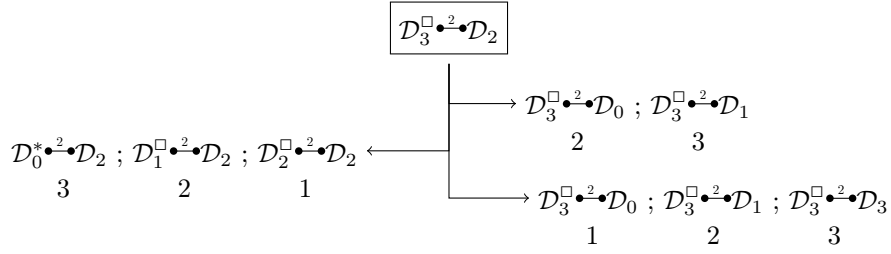


Figure 26: Possible options for the case  $S \in \mathcal{D}_3^\square$  and  $S' \in \mathcal{D}_2$ .

**Case 26:**  $S \in \mathcal{D}_3^\square$  and  $S' \in \mathcal{D}_3$

Figure 27 depicts all possible options, their Grundy values and show that the mex computation always leads to 1. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 1$ , which is what is predicted by Table 1.

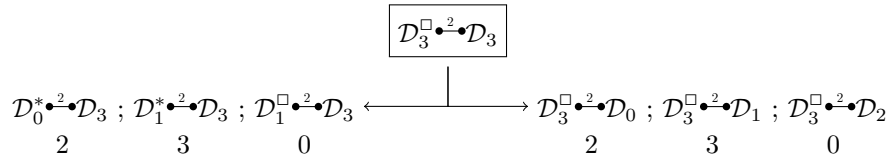


Figure 27: Possible options for the case  $S \in \mathcal{D}_3^\square$  and  $S' \in \mathcal{D}_3$ .

**Case 27:**  $S, S' \in \mathcal{D}_0$

Figure 28 depicts all possible options, their Grundy values and show that the mex computation always leads to 0. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 0$ , which is what is predicted by Table 1.

**Case 28:**  $S \in \mathcal{D}_0$  and  $S' \in \mathcal{D}_1$

Figure 29 depicts all possible options, their Grundy values and show that the mex computation of the right options always leads to 1, while the left options are always at most 1, and as such the computation of any left option with every right option will always yield 1. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 1$ , which is what is predicted by Table 1.

**Case 29:**  $S \in \mathcal{D}_0$  and  $S' \in \mathcal{D}_2$

Figure 30 depicts all possible options, their Grundy values and show that the mex computation always leads to 2. Thus,  $\mathcal{G}(S^{\bullet 2} \bullet S') = 2$ , which is what is predicted by Table 1.

**Case 30:**  $S \in \mathcal{D}_0$  and  $S' \in \mathcal{D}_3$

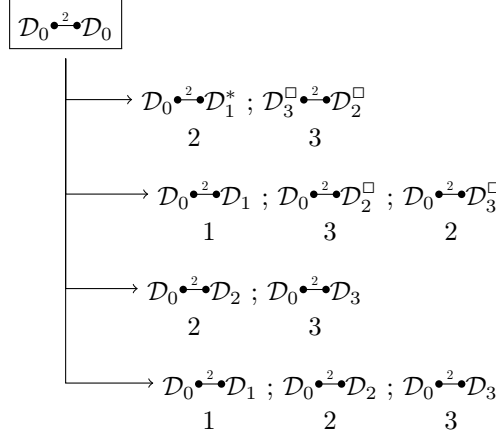


Figure 28: Possible options for the case  $S, S' \in \mathcal{D}_0$ .

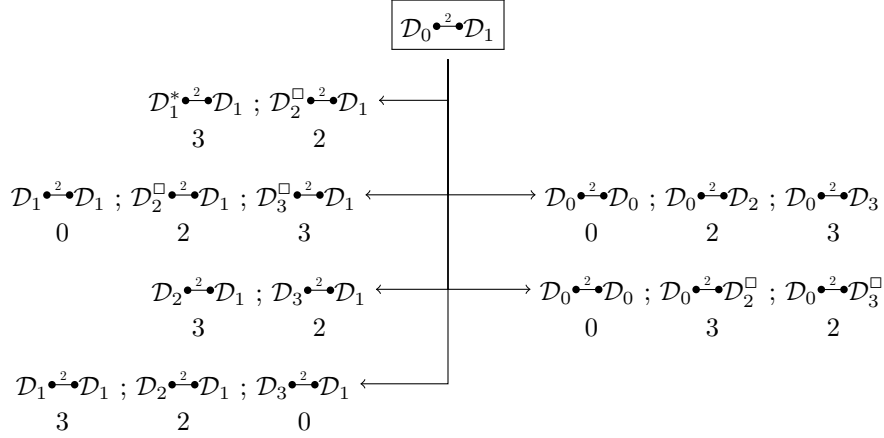


Figure 29: Possible options for the case  $S \in \mathcal{D}_0$  and  $S' \in \mathcal{D}_1$ .

Figure 31 depicts all possible options, their Grundy values and show that the mex computation of the right options always leads to 3, while the left options are always at most 3, and as such the computation of any left option with every right option will always yield 3. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 3$ , which is what is predicted by Table 1.

**Case 31:**  $S, S' \in \mathcal{D}_1$

Figure 32 depicts all possible options, their Grundy values and show that the mex computation always leads to 0. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 0$ , which is what is predicted by Table 1.

**Case 32:**  $S \in \mathcal{D}_1$  and  $S' \in \mathcal{D}_2$

Figure 33 depicts all possible options, their Grundy values and show that the mex computation of the left options always leads to 3, while the right options are always at most 3, and as such the computation of any left option with every right option will always yield 3. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 3$ , which is what is predicted by Table 1.

**Case 33:**  $S \in \mathcal{D}_1$  and  $S' \in \mathcal{D}_3$

Figure 34 depicts all possible options, their Grundy values and show that the mex computation always leads to 2. Thus,  $\mathcal{G}(S^{\bullet^2} \bullet S') = 2$ , which is what is predicted by Table 1.

**Case 34:**  $S, S' \in \mathcal{D}_2$

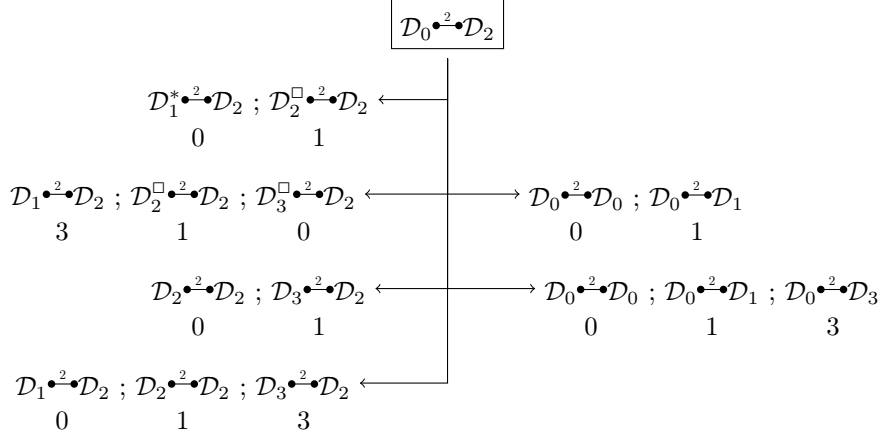


Figure 30: Possible options for the case  $S \in \mathcal{D}_0$  and  $S' \in \mathcal{D}_2$ .

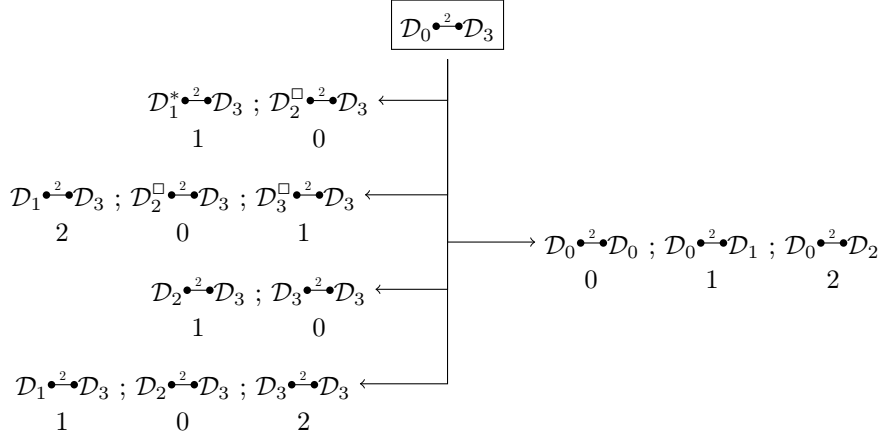


Figure 31: Possible options for the case  $S \in \mathcal{D}_0$  and  $S' \in \mathcal{D}_3$ .

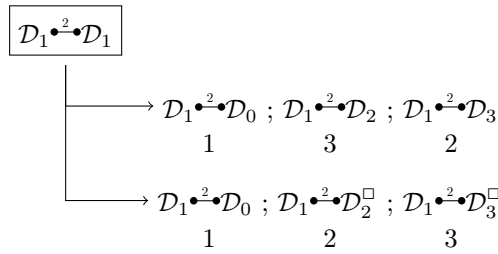


Figure 32: Possible options for the case  $S, S' \in \mathcal{D}_1$ .

Figure 35 depicts all possible options, their Grundy values and show that the mex computation always leads to 0. Thus,  $\mathcal{G}(S \bullet^2 S') = 0$ , which is what is predicted by Table 1.

**Case 35:**  $S \in \mathcal{D}_2$  and  $S' \in \mathcal{D}_3$

Figure 36 depicts all possible options, their Grundy values and show that the mex computation of the right options always leads to 1, while the left options are always at most 1, and as such the computation of

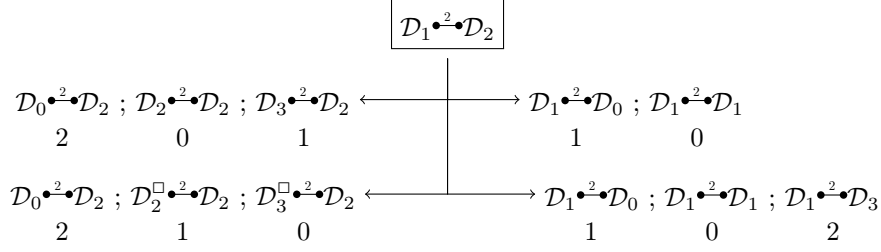


Figure 33: Possible options for the case  $S \in \mathcal{D}_1$  and  $S' \in \mathcal{D}_2$ .

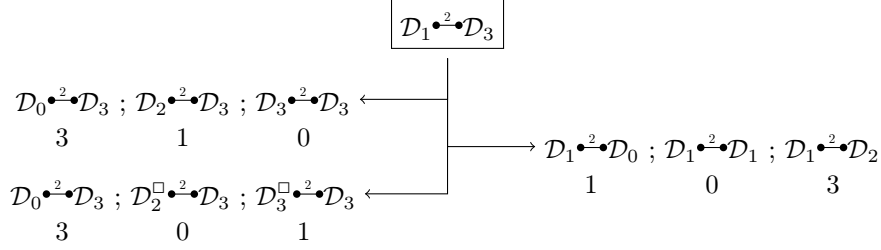


Figure 34: Possible options for the case  $S \in \mathcal{D}_1$  and  $S' \in \mathcal{D}_3$ .

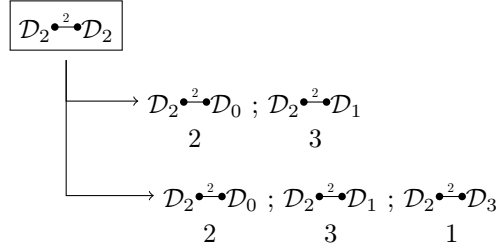


Figure 35: Possible options for the case  $S, S' \in \mathcal{D}_2$ .

any left option with every right option will always yield 1. Thus,  $\mathcal{G}(S \bullet^2 S') = 1$ , which is what is predicted by Table 1.

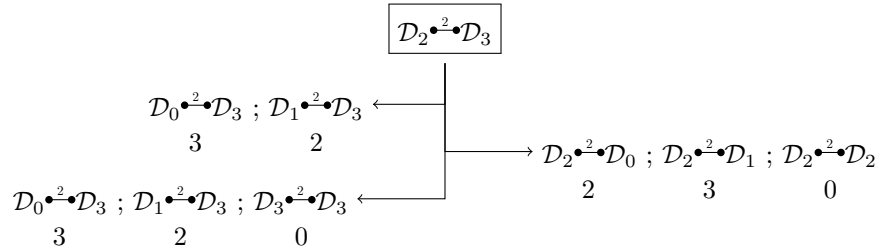


Figure 36: Possible options for the case  $S \in \mathcal{D}_2$  and  $S' \in \mathcal{D}_3$ .

**Case 36:**  $S, S' \in \mathcal{D}_3$

Figure 37 depicts all possible options, their Grundy values and show that the mex computation always leads to 0. Thus,  $\mathcal{G}(S \bullet^2 S') = 0$ , which is what is predicted by Table 1.



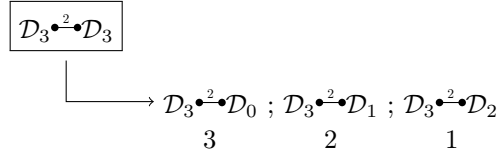


Figure 37: Possible options for the case  $S, S' \in \mathcal{D}_3$ .

□

## References

- [1] L. Beaudou, P. Coupechoux, A. Dailly, S. Gravier, J. Moncel, A. Parreau, É. Sopena, *Octal Games on Graphs: The game 0.33 on subdivided stars and bistars*, arXiv preprint arXiv:1612.05772.