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Center for Research in Economics and Management



University of Rennes 1

University of Caen Normandie

Prudence and preference for flexibility gain

Daniel DANAU

Normandie Univ, UNICAEN, CNRS, CREM, F-14000 Caen, France

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Working Paper



Prudence and preference for flexibility gain*

Daniel Danau[†]

Abstract

We investigate the properties of the preference of an individual for an unknown profit $\pi(x)$ over a certain profit $\pi(\mathbf{E}(x))$, where x is unknown at the time when the decision is made, whereas its expected value $\mathbf{E}(x)$ is known. The additional benefit to the individual of the former over the latter consists in a flexibility gain. For instance, a flexibility gain arises in investment decisions when the individual obtains a higher benefit if she postpones the investment until after some technology x is realized, rather than making it today with technology $\mathbf{E}(x)$. We show that prudence (positive third derivative of the utility function) is related to the size of the flexibility gain, in the same vein as it is related to the utility premium of an individual who prefers a certain profit π to a variable profit $\pi + \tilde{\varepsilon}$, where $\mathbf{E}(\tilde{\varepsilon}) = 0$. We further examine the way in which the concept of flexibility gain is linked to a variety of notions and problems, namely downside risk aversion, concave surplus of a risk neutral individual, stochastic dominance, optimal prevention, and principal-agent relationships with unknown distribution of some relevant variable. This permits to highlight the role of the degree of absolute prudence (or of the third derivative of the surplus function, when the individual is risk neutral) in decision making.

Keywords: Prudence; Flexibility gain; Utility premium; Downside risk aversion

J.E.L. Classification Numbers: D81

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[†]Université de Caen Normandie - Centre de Recherche en Economie et Management, Esplanade de la Paix, 14000 Caen (France). E-mail: daniel.danau@unicaen.fr

1 Introduction

The notion of prudence introduced by Kimball [14], was expressed both in terms of preferences and in terms of behaviour under uncertainty. In terms of preferences, prudence is the third derivative of the utility function being positive. In terms of behaviour, prudence is a precautionary motive for savings. A prudent individual saves more, whereas an imprudent individual saves less, if faced with an increased risk about her future resources.¹

Kimball's definition of prudence, whether in terms of preferences or in terms of behaviour, is related to another familiar concept in Decision theory, namely that of utility premium.² The utility premium measures the reduction induced in the expected utility of the individual, if a zero mean lottery is added to her initial wealth. Its size is inversely related to the initial wealth if and only if the third derivative of the utility function is positive (Hanson and Menezes [13]), hence if and only if the individual is prudent. In the consumption-savings model, an individual trades off current consumption against the expected second-period utility reduction, as measured by the utility premium. The utility premium is lower the less is consumed during the first period (and so the more is saved) if and only if the individual is prudent. Similar to Kimball [14], in the expected utility framework, the more recent literature tends to consider prudence as a measure of preference of a certain outcome over an uncertain outcome (see Eeckhoudt and Schlesinger [9], among others).

One may think of a prudent individual either as a consumer or as an investor. As a consumer, she derives some utility from consumption of the bundle of goods she can afford given her income, as measured by the indirect utility function. As an investor, she derives a utility from the monetary return on her investment.³ In the usual approach of the literature, one can say that prudence expresses the extent to which the individual dislikes uncertainty being added to a certain amount of money in the future, regardless of whether that money is income or return on investment.

What about an individual who prefers the uncertain outcome over the certain one? This is very often the case in investment problems. As widely demonstrated by the studies on investment under uncertainty (Dixit and Pindyck [6]), future investment opportunities might be preferable over current ones when investments are irreversible. By delaying an investment, the investor can acquire additional information, which will be useful to decide how much to invest. To the best of our knowledge, the literature on Decision theory has not

¹Prior to Kimball [14], Leland [16] and Sandmo [19] investigated savings decision under uncertainty and showed that savings increase as a zero mean risk is added to the future revenues if and only if the third derivative of the utility function is positive. Kimball calls an individual with this type of behaviour (or, equivalently, with a positive third derivative of the utility function) a prudent individual. One can say that, unlike risk aversion, prudence was first defined in terms of behaviour and then in terms of preferences (see Crainich and Eeckhoudt [3] for more details).

²Eeckhoudt and Schlesinger [9] point out that the notion of utility premium seems to have first been introduced by Friedman and Savage [11].

³In Bernoulli's examples, the investor derives a utility from money, consistent with Bernoulli's and Cramer's conjecture that the investor cares about the utility derived from money rather than about money *per se*. See, for instance, Levy [17] on page 25.

established what exact characteristics the utility function of an individual should display, in order to provide a measure of the benefit associated with the delay, when the concerned individual does prefer to delay the decision. In particular, one cannot rely on the notion of prudence, as defined by the literature, since the usual definition of prudence rests on the opposite hypothesis. However, in this paper we show that prudence is, indeed, related to the benefit ensuing from the investment delay, which we will refer to as a "flexibility gain," in line with Dixit and Pindyck [6]. Therefore, the definition of prudence is broader than usually considered by scholars.

As an illustration, take an individual who derives a net utility of $\widehat{u}(y) - \theta y$ from an immediate investment in a capacity of size y . Here, θ is the expected cost of acquiring a capacity unit, whereas the true cost is $\theta + \widetilde{\eta}$, where $\widetilde{\eta} \in \{-\eta, \eta\}$ with equal probabilities. This simple formulation, in which the gross utility ($\widehat{u}(y)$) - to be called "surplus" henceforth - is taken to be separable from the linear cost (θy), is yet indicative of our results. The optimal choice of y depends on the production technology, as expressed by the (expected or true) unit cost. Not surprisingly, assuming no discounting, the decision maker will prefer to delay the decision until after she will have observed the realization of $\theta + \widetilde{\eta}$. Hence, she has a flexibility gain. Applying Jensen's inequality, a greater flexibility gain is associated with a lower value of θ if and only if $\widehat{u}'''(\cdot)$ is positive.

To develop our study, we consider a general utility function $u(\pi(\cdot))$ and investigate the preference for $\pi(x)$ over $\pi(\theta)$, where θ is known and x is unknown. For instance, in the investment problem, $\pi(\cdot)$ may represent an optimized profit, namely the highest value that a profit function $\widehat{\pi}(y, \cdot)$ can take, such that $\pi(\cdot) = \widehat{\pi}(y(\cdot), \cdot)$. Exploring the preference for $\pi(x)$ over $\pi(\theta)$ requires developing a comparative statics analysis of the benefit $\mathbf{E}(u(\pi(x)))$ over the benefit $u(\pi(\mathbf{E}(x)))$, hence of the flexibility gain, similar to the analysis of the utility premium developed in Hanson and Menezes [13]. The link between flexibility gain and $u'''(\cdot)$ follows the same principle as in the previous example, except that now it refers to the indirect utility function, in line with Kimball's definition of prudence. We show that the flexibility gain to the individual increases with a favorable shift of the initial value θ if and only if the individual degree of absolute prudence is sufficiently high. How large the degree of absolute prudence must be for that result to be obtained will also depend on the properties of profit function and the distribution function of the unknown variable.

We further explore the link between flexibility gain and a few familiar concepts in Decision theory and problems in Contract theory. This enables us to clarify in which ways the definition of prudence extends to situations other than those already considered in previous studies. Economic applications are identified accordingly. Specifically, we show that there is a link between the concept of flexibility gain and the notions of downside risk aversion, concave surplus of a risk neutral individual, stochastic dominance and optimal prevention. Besides, we identify a connection with principal-agent relationships in which the distribution of some relevant variable is unknown. In each of those scenarios, we underline the role of

the degree of absolute prudence (or the third derivative of the utility function, when the individual is risk neutral) in decision making.

The paper is first related to studies in which individual's preferences or simply her behaviour are identified according to the third derivative of her utility function, like Hanson and Menezes [13], Kimball [14], as already explained. Within this bunch of studies, a specific trend is that of expressing prudence as aversion against downside risk (Menezes et al. [18], Bigelow and Menezes [2] and Eeckhoudt and Schlesinger [9]). In those studies, prudence is considered in terms of behaviour. The individual is asked to choose between two compounded lotteries, one of which is associated with less downside risk, but is also less valuable when a good outcome is realized. The lottery with a lower downside risk is more beneficial to the individual if and only if the utility premium decreases with the initial wealth, which is tantamount to prudence, according to Kimball's definition. The novelty of our study is that a lower downside risk may also emerge when an unknown outcome is preferred to a certain one and, hence, the individual enjoys a flexibility gain. We propose two specific lotteries and highlight that the preference for one over the other will depend on how the flexibility gain evolves with the stochastic variable. This link is consistent with that between lotteries and utility premium identified in the literature.

This paper is also related to the studies on Decision theory which focus on the link between prudence and irreversibility of decision making (Gollier et al. [12] and Eeckhoudt and Gollier [7]). However, that literature restrict attention to problems of preventive investment, particularly in situations in which no information acquisition can take place before deciding to invest. Our contribution is to show that the individual degree of prudence is relevant also when new information can be acquired over time, prior to investing.

Lastly, in some respects, the paper is related to the study of Athey [1] on comparative statics predictions in stochastic optimization problems. Considering a decision maker who seeks to maximize the expected utility $\mathbf{E}[u(y, \hat{x})]$, the author investigates the conditions under which some choice variable y is monotonic with respect to the unknown variable \hat{x} . In our study, the variable \hat{x} is allowed to evolve stochastically over two periods, and the investor has the opportunity of deciding about y only in the second period.

The outline of the paper is as follows. In Section 2, we provide the benchmark of our analysis, in which we recall the link between utility premium and prudence identified in previous studies. In Section 3, we present our model and explore the link between prudence and flexibility gain. In Section 4, we discuss various concepts to which the flexibility gain is related, in order to highlight the broadness of the definition of prudence we propose and its relevance in economic applications. Section 5 briefly concludes.

2 Benchmark: utility premium and prudence

Before proceeding to the analysis, it is useful to briefly recall the link between utility premium and prudence. Kimball [14] defines prudence in terms of preferences as $u'''(m) > 0$, where m is the amount of money and $u(\cdot)$ is the increasing and concave utility function of the individual. In turn, the utility premium measures the decrease which is induced in the individual utility, as a zero mean risk $\tilde{\varepsilon}$ is added to the initial amount of money m . Therefore, the utility premium is given by $-\hat{w}(m) > 0$, where

$$\hat{w}(m) = \mathbf{E}[u(m + \tilde{\varepsilon})] - u(m). \quad (1)$$

Before the notion of prudence was introduced, Hanson and Menezes [13] showed that $\hat{w}'(m) > 0$ if and only if $u'''(\cdot) > 0$. This equivalence is obtained by applying Jensen's inequality to the marginal utility function. In words, the reduction in the expected utility, which results from the introduction of the lottery, is higher the lower the initial amount of money m is.

In terms of behaviour, Kimball [14] shows that saving decisions under uncertainty about future revenues depend on whether the individual is prudent or he is not. Following the studies of Leland [16] and Sandmo [19], which identified the link between the amount of savings and the sign of the third derivative of the utility function, he considers a setting where the individual lives two periods, her utility is additively separable between periods, and she must decide how much to save during the first period. When a zero mean lottery is added to the initial revenue during the second period, the individual saves more if and only if she is prudent. Formally, if m is the initial revenue and a lottery $\tilde{\varepsilon}$ is added during the second period, where $\tilde{\varepsilon}$ is as previously defined, then an individual who consumes c_1 during the first period will be left with $m - c_1 + \tilde{\varepsilon}$ in the second period. If there were no risk, then the optimal consumption would be such that $u'(c_1) = u'(m - c_1)$. Because of the risk, the individual decides how much to consume according to the rule $u'(c_1) = \mathbf{E}[u'(m - c_1 + \tilde{\varepsilon})]$, which mirrors a trade-off between the consumption in the first period and the expected utility in the second period. Now recall from above that $\mathbf{E}[u'(m - c_1 + \tilde{\varepsilon})] > u'(m - c_1)$ if and only if the individual is prudent. We see that the marginal utility is higher than in a risk-free situation. That is, the individual consumes less and saves more in the presence of risk. In substance, the reason why c_1 is decreased is that, by doing so, the utility premium $\hat{w}(m - c_1)$ is reduced in the second period. Hence, the trade-off faced by the individual is one between first-period consumption and the utility premium, which arises because of uncertainty about second period's revenues and which decreases with c_1 if and only if the individual is prudent.

Being based on these results, we can assess that prudence is related to how much a certain outcome is preferred over an uncertain outcome. Indeed, in terms of preferences, prudence is related to how much $u(m)$ is above $\mathbf{E}(u(m + \tilde{\varepsilon}))$. In terms of behaviour, prudence is related

to how much the individual dislikes facing $\mathbf{E}[u(m - c_1 + \tilde{\varepsilon})]$ rather than $u(m - c_1)$. This approach is systematically adopted in more recent studies, in that it is assumed that the individual prefers a certain outcome over an uncertain one, whenever prudence is expressed in terms of expected utility (see, for instance, Eeckhoudt and Schlesinger [9]).

3 Prudence and preference for flexibility gain

We now turn to present the basic model, which will be used for the purpose of the analysis. We consider an individual who has an increasing and concave utility function $u(\pi(\hat{x}))$, where $\pi(\hat{x})$ is an amount of money ("profit") and \hat{x} is a variable which $\pi(\hat{x})$ depends upon, such that $\pi'(\hat{x})$ is not a constant. For instance, in a case where the individual invests in a project, the profit function could be $\hat{\pi}(y, \hat{x}) = R(y) - C(y, \hat{x})$, where y is the size of the project, $R(y)$ is the return function and $C(y, \hat{x})$ is the cost function parameterized by the technology \hat{x} . Denoting $y(\hat{x})$ the unique level of y which maximizes $u(\hat{\pi}(y, \hat{x}))$, we can say that the optimized profit is $\pi(\hat{x}) = \hat{\pi}(y(\hat{x}), \hat{x})$ and grants a utility of $u(\pi(\hat{x}))$, $\forall \hat{x}$.

The interest of our study is to characterize the preference of the individual for obtaining $\mathbf{E}[u(\pi(x))]$ rather than $u[\pi(\mathbf{E}(x))]$, given that x is unknown. We take $\mathbf{E}(x) \equiv \theta$ and $x \in [\theta - \eta, \theta + \eta]$, for some known θ and η , and the distribution function to be $f(x)$. For instance, in the investment problem considered above, the individual may need to choose between investing today, when the technology is θ , and delaying the investment until after the technology x becomes known. Assuming no discounting, the individual will obviously prefer to wait and obtain $\mathbf{E}[u(\pi(x))]$ rather than $u[\pi(\mathbf{E}(x))]$. Henceforth, we use the notation

$$w(\theta) = \int_{\theta-\eta}^{\theta+\eta} u(\pi(x)) f(x) dx - u(\pi(\theta)) \quad (2)$$

to indicate the *flexibility gain*. As in the studies on investment under uncertainty (Dixit and Pindyck [6]), this is the benefit that can be obtained (when $w(\theta)$ is positive) by conditioning an investment decision on an unknown variable rather than on a known value. The reason why the flexibility gain depends on θ is that a change in θ induces a shift in the support of x , namely the interval $[\theta - \eta, \theta + \eta]$. Our goal is to show that the size of $w(\theta)$ is related to prudence, as was previously done for $\hat{w}(m)$, defined in (1). To do so, we investigate how the flexibility gain varies with θ . This boils down to looking at the properties of the following derivative:

$$w'(\theta) = \int_{\theta-\eta}^{\theta+\eta} \frac{d}{dx} [u(\pi(x)) f(x)] dx - \frac{d}{d\theta} u(\pi(\theta)). \quad (3)$$

The expression in (3) evidences that to assess the sign of $w'(\theta)$, it is first necessary to investigate the characteristics of the utility function. The reason is that, as θ changes, it is not clear whether the individual gains (loses) more on $\mathbf{E}[u(\pi(x))]$ or $u[\pi(\theta)]$. If $\pi(x)$ were independent of the individual decision, namely, $\pi(x) = \Pi(\theta) + \tilde{\varepsilon}$ and $\pi(\theta) = \Pi(\theta)$ for

some given $\Pi(\theta)$ and some variable $\tilde{\varepsilon}$ such that $\mathbf{E}(\tilde{\varepsilon}) = 0$, then the sign of $w'(\theta)$ could be ascertained by simply applying Jensen's inequality to the marginal utility function, according to the usual procedure, once the variations in $\Pi(\theta)$ with respect to θ were accounted for. Because $\pi(x)$ is not a linear function, this approach is not appropriate. In addition, as one would expect, $w'(\theta)$ also depends on the properties of the profit function $\pi(x)$, and on those of the distribution function $f(x)$. Before presenting our first result, we conveniently define $v(x) = u(\pi(x))$, $\forall x \in [\theta - \eta, \theta + \eta]$, as well as

$$\begin{aligned} \psi(x) = & -[3v''(x)f'(x) + 2v'(x)f''(x) + v(x)f'''(x)] \frac{1}{f(x)} \\ & - \frac{6}{\eta^3} \{[v(\theta)f(\theta) - v(\theta - \eta)f(\theta - \eta)] + \eta v(\theta)f'(\theta)\} \frac{1}{f(x)}, \end{aligned}$$

$\forall \theta$ and $\forall x \in [\theta - \eta, \theta + \eta]$.

Proposition 1 *If $v'''(x) \geq \psi(x)$, $\forall x \in [\theta - \eta, \theta + \eta]$, then $w'(\theta) \geq 0$. If $v'''(x) \leq \psi(x)$, $\forall x \in [\theta - \eta, \theta + \eta]$, then $w'(\theta) \leq 0$.*

Being based on this result, we see that the flexibility gain is related to the third derivative of the utility function $u(\pi)$. Indeed, defining $P(x) \equiv -u'''(\pi(x))/u''(\pi(x))$ the coefficient of absolute prudence and $A(x) \equiv -u''(\pi(x))/u'(\pi(x))$ the coefficient of absolute risk aversion, the condition $v'''(x) \geq \psi(x)$ is rewritten as

$$P(x) \geq -\frac{\psi(x)}{u''(\pi(x))\pi'(x)} + 2\frac{\pi''(x)}{\pi'(x)} + \frac{1}{A(x)}\frac{\pi'''(x)}{\pi'(x)}, \quad (4)$$

if $\pi'(x) > 0$, and as

$$P(x) \leq -\frac{\psi(x)}{u''(\pi(x))\pi'(x)} + 2\frac{\pi''(x)}{\pi'(x)} + \frac{1}{A(x)}\frac{\pi'''(x)}{\pi'(x)},$$

if $\pi'(x) < 0$. This shows that if the interval $[\theta - \eta, \theta + \eta]$ shifts either to higher values or to lower values, in such a way as to have a positive impact on the profit obtained by the individual in each state x , then the flexibility gain increases if and only if the individual is sufficiently prudent. Indeed, if $\pi'(x) > 0$, then an upward shift in the set $[\theta - \eta, \theta + \eta]$ induces a higher flexibility gain if and only if $P(x)$ is sufficiently high. If $\pi'(x) < 0$, then it is a downward shift in the interval that induces a higher flexibility gain if and only if $P(x)$ is sufficiently high. Clearly, the benefit of delaying the decision to acquire the good is related to how prudent the individual is. Not surprisingly, as both the frequency and the profit function depend on the specific value of x , so does condition (4) as well. The two numerical examples presented hereafter illustrate the result.

Example 1 Take $u(\pi) = \frac{1}{1-\gamma}\pi^{1-\gamma}$, where $\gamma \in (0, 1)$, and $\hat{\pi}(y, x) = (a - by)y - xy$. Then,

$$\begin{aligned} y(x) &= \frac{a-x}{2b}, \\ \pi(x) &= \hat{\pi}(y(x), x) = \frac{(a-x)^2}{4b}, \\ v(x) &= u(\pi(x)) = g(\gamma)(a-x)^{2(1-\gamma)}, \\ v'''(x) &= g(\gamma)2(1-\gamma)(1-2\gamma)2\gamma(a-x)^{-2\gamma-1}, \end{aligned}$$

where $g(\gamma) = \frac{1}{1-\gamma} \left(\frac{1}{4b}\right)^{1-\gamma}$. Assume that x is uniformly distributed, for simplicity. Then, $\psi(x)$ is specified as

$$\begin{aligned} \psi(x) &= \frac{6}{\eta^3} [v(\theta - \eta) - v(\theta)] \\ &= \frac{6}{\eta^3} g(\gamma) \left[(a - \theta + \eta)^{2(1-\gamma)} - (a - \theta)^{2(1-\gamma)} \right] \equiv \bar{\psi}. \end{aligned}$$

It is easy to see that the condition $v'''(x) \geq \psi(x)$ is tightest for $x = \theta + \eta$. Hence, the condition $v'''(x) \geq \psi(x)$ holds $\forall x$ if and only if $v'''(\theta + \eta) \geq \bar{\psi}$. Taking $\gamma = 0.25$, $\theta = 10$, $\eta = 4$ and $a \geq 12$ (so as to ensure that $y(\cdot) \geq 0$), the condition is satisfied if and only if $a \leq 14.4$, all other values being unchanged.

Example 2 Take $u(\pi) = A\pi - \pi^\alpha$ and $\hat{\pi}(y, x)$ as defined in Example 1. Then, $y(x)$ and $\pi(x)$ are the same as in Example 1, whereas here

$$\begin{aligned} v(x) &= u(\pi(x)) = \frac{A}{4b}(a-x)^2 - \frac{1}{4b}(a-x)^{2\alpha}, \\ v'''(x) &= -\frac{\alpha}{2b}(2\alpha-1)(2\alpha-2)(2\alpha-3)(a-x)^{2\alpha-4}. \end{aligned}$$

Considering again a uniform distribution of x , one has

$$\psi(x) = \frac{6}{\eta^3} \frac{1}{2b} \left[A \left(\eta(a-\theta) + \frac{\eta^2}{2} \right) - \frac{1}{2}(a-\theta+\eta)^{2\alpha} + \frac{1}{2}(a-\theta)^{2\alpha} \right] \equiv \bar{\psi}.$$

Take $\theta = 10$, $\eta = 4$ and $a \geq 12$ as in Example 1, but also $a \leq 20$ (to account for the restrictions that the choice of a impose on A). Further take $\alpha = 1.7$ and $A = 35$. Then, the condition $v'''(x) \geq \psi(x)$ is satisfied if and only if $a \geq 16.5$, all other values being unchanged.

The result of the proposition accommodates a broader interpretation of prudence than is usually thought of, allowing for both the preference for a certain outcome over an uncertain one and the converse preference. Recall that the preference for a certain outcome is measured by the utility premium $-\hat{w}(\cdot)$. By comparing (2) with (1), we can say that $w(\cdot)$ is a generalization of $\hat{w}(\cdot)$, and that, under specific conditions, the relationship between flexibility gain and prudence is reminiscent to that between utility premium and prudence.

Corollary 1 Take $d^2\Pi(\hat{x})/d\hat{x}^2 = 0$ and $(d\Pi(\hat{x})/d\hat{x}) > 0 \forall \hat{x}$. Then, $w'(\theta) > 0$ if and only if $u'''(\cdot) > 0$.

The specificity of our setting is that $\pi(\hat{x})$ is not an initial amount of money to which an unknown outcome is possibly added. Instead, $\pi(\hat{x})$ is a profit which depends non linearly on \hat{x} , as in the investment problem, in which the individual decides the size of the project $y(\hat{x})$. To emphasize better why the definition of prudence is broader than usually considered so far, and why is it useful in applications, we will next examine a few other concepts related to the notion of flexibility gain.

4 Related concepts

4.1 Preference for lower downside risk

The concept of downside risk was introduced by Menezes et al. [18] to show that an individual, whose utility function is such that $u'''(\cdot) > 0$, prefers to have some risk added to a good outcome of a lottery rather than to a bad outcome. Hence, that preference can be used as a definition of prudence. In what follows, we will refer to the lotteries proposed by Bigelow and Menezes [2] and Eeckhoudt and Schlesinger [9], who show as well that having a preference for one such lotteries is equivalent to being averse to downside risk. The two lotteries are $\mathbf{A} = (0; \tilde{\varepsilon} - k)$ and $\mathbf{B} = (-k; \tilde{\varepsilon})$; in each lottery the two outcomes are equally likely. Further suppose that k is known and $\tilde{\varepsilon}$ is unknown and such that $\mathbf{E}(\tilde{\varepsilon}) = 0$. The individual is prudent if and only if lottery \mathbf{B} is preferred to lottery \mathbf{A} . In the words of Eeckhoudt and Schlesinger [9], lottery \mathbf{B} allows for a disaggregation of the two harms $\tilde{\varepsilon}$ and $-k$, which explains why an individual with $u'''(\cdot) > 0$ prefers \mathbf{B} to \mathbf{A} . The preference for \mathbf{B} over \mathbf{A} is proved to be equivalent to $u'''(\cdot) \geq 0$ by assuming that the individual has some initial wealth m , which adds up to each of the potential outcomes of the selected lottery. Then, \mathbf{B} is preferred to \mathbf{A} if and only if

$$\mathbf{E}(u(m + \tilde{\varepsilon})) - \mathbf{E}(u(m + \tilde{\varepsilon} - k)) \geq u(m + 0) - u(m - k).$$

By taking k arbitrarily small, this reduces to $\mathbf{E}(u'(m)) \geq u'(m)$, which is equivalent to $u'''(\cdot) \geq 0$ by Jensen's inequality.

Remarkably, the preference for \mathbf{B} over \mathbf{A} , as previously defined, can be reinterpreted by making specific considerations about the range of values from which $\tilde{\varepsilon}$ is drawn. Suppose that $\tilde{\varepsilon} \in [-\hat{\varepsilon}, \hat{\varepsilon}]$, for some $\hat{\varepsilon} > 0$. If $k < \hat{\varepsilon}$ and lottery \mathbf{A} is used, then the outcome is included in the interval $[-\hat{\varepsilon} - k, \hat{\varepsilon} - k]$; if lottery \mathbf{B} is used, then the outcome lies in $[-k, \hat{\varepsilon}]$. If $k > \hat{\varepsilon}$, then the outcomes of lottery \mathbf{A} are drawn from the interval $[-\hat{\varepsilon} - k, 0]$; those of lottery \mathbf{B} are drawn again from the interval $[-k, \hat{\varepsilon}]$. In either case, we observe an *upward shift in the support* of the final outcomes when switching from lottery \mathbf{A} to lottery \mathbf{B} , in

spite of earning less with lottery \mathbf{B} when $\tilde{\varepsilon}$ turns out to be high. Hence, we can say that a prudent individual has a specific preference for a shift in that support.

In the literature, the preference for \mathbf{B} over \mathbf{A} is considered to be a transposition in terms of behaviour of the preference for a certain outcome over an uncertain one, as described in the benchmark (see Eeckhoudt and Schlesinger [9]). We hereafter show that also the preference for an unknown outcome over a certain one can be represented in terms of individual behaviour, particularly, as a choice between two specific lotteries.

To that end, we proceed in two steps. First, instead of considering an individual who has an initial wealth of m and plays a lottery, as in the benchmark, we suppose that the individual has to choose between the two lotteries $\mathbf{A}' = (\Pi(\theta); \Pi(\theta) + \tilde{\varepsilon} - k)$ and $\mathbf{B}' = (\Pi(\theta) - k; \Pi(\theta) + \tilde{\varepsilon})$, where $\Pi(\theta)$ is again an exogenously given profit. This profit includes the initial wealth and is net of any cost which must be incurred to obtain an outcome. As is the case of \mathbf{A} and \mathbf{B} , the outcomes of \mathbf{A}' and those of \mathbf{B}' are both equally likely. Observe now that each of the lotteries \mathbf{A}' and \mathbf{B}' include, as outcomes, some given amount (say, a "known" outcome) and a lottery (say, an "unknown" outcome). \mathbf{B}' is preferable to \mathbf{A}' when $u'''(\cdot) > 0$ because the net gain on the unknown outcome in the former lottery exceeds the net gain on the known outcome in the latter lottery:

$$\mathbf{E}(u(\Pi(\theta) + \tilde{\varepsilon})) - \mathbf{E}(u(\Pi(\theta) + \tilde{\varepsilon} - k)) \geq u(\Pi(\theta)) - u(\Pi(\theta) - k).$$

As a second step, we return to our setting and let the profit be endogenous, namely $\pi(\cdot) = \hat{\pi}(y(\cdot), \cdot)$, as previously defined in the investment problem. The individual must choose between the lotteries \mathbf{A}'' and \mathbf{B}'' , instead of \mathbf{A}' and \mathbf{B}' , where $\mathbf{A}'' = (\pi(\theta); \pi(x - \hat{k}))$ and $\mathbf{B}'' = (\pi(\theta - \hat{k}); \pi(x))$, θ and \hat{k} are known, x is unknown, $\mathbf{E}(x) = \theta$ and $x \in [\theta - \eta, \theta + \eta]$. Essentially, as in the previous cases, the individual gains more on the certain outcome with lottery \mathbf{A}'' than \mathbf{B}'' , but has a higher downside risk that lottery \mathbf{B}'' on the unknown outcome. Again, the outcomes of \mathbf{A}'' and those of \mathbf{B}'' are equally likely. We also take the distribution of x to be uniform and refer, for simplicity, to the case where $\pi'(x) > 0$.

Proposition 2 *Lottery \mathbf{B}'' is preferred to lottery \mathbf{A}'' if*

$$P(x) \geq 2 \frac{\pi''(x)}{\pi'(x)} + \frac{1}{A(x)} \frac{\pi'''(x)}{\pi'(x)}, \quad \forall x. \quad (5)$$

4.2 Concave surplus function of a consumption good

In this subsection, we consider a risk neutral individual whose surplus from consumption is a concave function of the amount of the consumption good. Let the utility function be

$$\hat{u}(y) - \hat{x}y, \quad (6)$$

where y is the amount of the good, $\hat{x}y$ is the cost of acquisition, and $\hat{u}(y)$ is the surplus derived from consumption of y units, such that $\hat{u}'(\cdot) > 0$ and $\hat{u}''(\cdot) < 0$. Whereas it is unusual in Decision theory to refer to risk neutral individuals with concave surplus function, this is indeed the case of many applications, like delegation of tasks (*à la* Laffont and Martimort [15]), for instance. The flexibility gain is now written as

$$w(\theta) = \mathbf{E} [\hat{u}(y(x)) - xy(x)] - [\hat{u}(y(\theta)) - \theta y(\theta)].$$

In light of Proposition 1, one expects the benefit of the delay to depend on $\hat{u}'''(\cdot)$. The reason for this conjecture is that the surplus function $\hat{u}(\cdot)$ of the risk neutral individual has similar characteristics to the utility function $u(\cdot)$ of a risk averse individual, except that the argument of the function is the amount of the consumption good rather than an amount of money. We hereafter show that this simple conjecture is correct, indeed.

Take the distribution of x to be uniform, for simplicity. Because $y(\hat{x})$ solves $\hat{u}'(y(\hat{x})) = \hat{x}$, $\forall \hat{x}$, one can rewrite the flexibility gain as follows:

$$w(\theta) = \frac{1}{2\eta} \int_{\theta}^{\theta+\eta} \frac{\partial \hat{u}(y(x))}{\partial x} dx - \int_{\theta-\eta}^{\theta} \frac{\partial \hat{u}(y(x))}{\partial x} dx = \frac{1}{2\eta} \int_{\theta-\eta}^{\theta} [y(x+\eta) - y(x)] dx.$$

Define now the function

$$\zeta(a, b, c) = [f(a) - f(a+c)] - [f(b) - f(b+c)], \quad (7)$$

where $f(\cdot)$ is the inverse function of $\hat{u}'(\cdot)$. Being based on the lemma stated here below, we will next use $\zeta(a, b, c)$ as a measure of the concavity/convexity of the marginal surplus function.

Lemma 1 $\zeta(a, b, c) > 0$ if and only if $u(\cdot)''' > 0$, $\forall a, b, c$ such that $a < c$ and $b > 0$.

Observing that $w'(\theta) = -\zeta(\theta - \eta, \theta, \eta)$ and using Lemma 7, the following result is derived as a particular case of Proposition 1.

Proposition 3 $w(\theta) < 0$ if and only if $\hat{u}'''(y) > 0$.

Therefore, the flexibility gain increases as the support of unit cost values $[\theta - \eta, \theta + \eta]$ shifts to lower values if and only if the marginal surplus is convex. In the same vein as in the general model, the sign of the marginal flexibility gain depends on the third derivative of the utility function, except that here we refer to the utility of the consumption good rather than to the utility of money, as in the definition of prudence provided by Kimball. We will now show that this result is helpful to understand the preference over distributions that differ in terms of either mean value, or spread between values, or both.

4.3 Stochastic ordering and choice between distributions

As is well known, the preference of an individual, as derived with the expected utility approach, can be reformulated as a choice between distributions, instead of a choice between a certain outcome today and an uncertain outcome tomorrow. For that, the distributions must respect some stochastic ordering. Indeed, an individual whose utility $u(a)$ increases with a dislikes facing the distribution $F_2(a)$, which is first-order stochastically dominated (FSD) by $F_1(a)$, since low values of a are more likely to be drawn with $F_2(a)$ than they are with $F_1(a)$, whereas the converse is true for high values. If the individual is also risk averse, then she dislikes facing a distribution $F_2(a)$ which is second-order stochastically dominated (SSD) by $F_1(a)$, provided $F_2(a)$ is riskier than $F_1(a)$. Moreover, a prudent individual dislikes $F_2(a)$, if it is third-order stochastically dominated (TSD) by $F_1(a)$, provided $F_2(a)$ embodies a higher downside risk than $F_1(a)$. The literature has shown that this reasoning applies to the comparison between higher order derivatives of the utility function and higher orders of stochastic dominance (see Eeckhoudt and Schlesinger [9]).

Let us now come to the notion of flexibility gain and identify its link with stochastic dominance. To do so, we refer to the utility function $\hat{u}(y) - x_{ij}y$, and assume that $x_{ij} = \theta_i + \tilde{\sigma}_j$, where θ_i is given and $\tilde{\sigma}_j \in [-\eta_j, \eta_j]$, $i \in \{1, 2\}$, $j \in \{1, 2\}$, $\theta_2 > \theta_1$, $\eta_2 > \eta_1$ and $\mathbf{E}(\tilde{\sigma}_j) = 0$, $\forall j$. Moreover, we take the four distributions to be all uniform. Instead of choosing whether to invest in period zero or one, the individual must select one of the four cost distributions indexed by ij . As the distributions are defined, two of them have the same mean and two have the same spread. Hence, one can associate the former pair with first-order stochastic dominance and the latter pair with mean-preserving spread, which is a particular case of second-order stochastic dominance. In this sense, it is easy to check that the individual will prefer $1j$ to $2j$, and $i2$ to $i1$. However, neither first- nor second-order stochastic dominance applies, if the four distributions are considered altogether. Because of this, one cannot apply the usual reasoning to identify the preference order of the four distributions. We will prove that the third derivative of the function $\hat{u}(\cdot)$ measures the extent to which the individual prefers to be faced with $1j$ to $2j$, and $i2$ to $i1$. To that end, we define

$$D_{ij/i'j'} = \mathbf{E}(U_{ij} - U_{i'j'}),$$

where

$$U_{ij} = \hat{u}(y(x_{ij})) - x_{ij}y(x_{ij}).$$

For instance, $D_{1j/2j}$ measures the additional gain that a technology associated with an expected unit cost of θ_1 grants, relative to one associated with an expected unit cost of θ_2 , for any given value η_j of the spread.

Proposition 4 (i) $D_{1j/2j'}$ increases with $\hat{u}'''(\cdot)$, $\forall j \geq j'$.

(ii) $D_{i2/i'1}$ increases with $\hat{u}'''(\cdot)$, $\forall i \leq i'$.

(iii) $D_{11/22}$ increases with $\hat{u}'''(\cdot)$ if and only if $\theta_2 - \theta_1 > \eta_2 - \eta_1$.

Therefore, how much an individual prefers being faced with a set $[\theta_i - \eta_j, \theta_i + \eta_j]$, instead of a set $[\theta_{i'} - \eta_{j'}, \theta_{i'} + \eta_{j'}]$, is related to $\hat{u}'''(\cdot)$. Remarkably, if two of the four distributions were to share the same support and could be ordered in the sense of either first-order stochastic dominance or mean preserving spread, then the preference for one distribution would be unrelated to $\hat{u}'''(\cdot)$, in line with the findings of the literature.

4.4 Optimal prevention

Eeckhoudt and Gollier [7] define prevention as an effort to be undertaken in order to reduce the probability of occurrence of an adverse effect. These authors find the counter-intuitive result that a prudent individual makes less preventive effort than an imprudent individual. In their setting, the individual has no current consumption and the probability of future consumption depends on the preventive effort. That effort is "inflexible" in the sense that there is no benefit to acquiring new information by postponing the decision. In what follows, we show that when the investment has the nature of a preventive effort, a prudent individual does not consider information acquisition as an opportunity and, hence, there is no flexibility gain.

We begin by considering the model of Eeckhoudt and Gollier [7]. The individual may or may not incur a loss of $L > 0$ in the future. The probability of being exposed to the loss, denoted $p(e_0)$, depends on the exerted effort e_0 , where 0 denotes period zero. The expected utility is written as

$$p(e_0) u(m - L - e_0) + [1 - p(e_0)] u(m - e_0).$$

We take the cost of effort to change between periods zero and one. We let it be $c_0(e) = e$ in period zero and $c_1(e) = xe$ in period one, where x is unknown and $x \in [0, 2]$. If the decision is delayed from period zero to period one, then the expected utility is given by

$$p(e_1) u(m - L - e_1 - x) + [1 - p(e_1)] u(m - e_1 - x),$$

for some realized x . The flexibility gain from delaying the decision is written as

$$\begin{aligned} w_p &= \mathbf{E} [p(e_1(x)) u(m - L - e_1(x) - x) + (1 - p(e_1(x))) u(m - e_1(x) - x)] \\ &\quad - [p(e_0) u(m - L - e_0) + (1 - p(e_0)) u(m - e_0)], \end{aligned}$$

where e_0 is the optimal effort in period zero, and $e_1(x)$ is the optimal effort in period one, depending on the realization of x . To state the next result, it is convenient to denote

$$k(x) \equiv m - e_1(x) - x, \quad \forall x.$$

Proposition 5 $w_p > 0$ if

$$P(k(t) - s) < -\frac{1}{k'(t)} \left\{ -\frac{k''(t)}{k'(t)} + 2\frac{\frac{dp(e_1(t))}{dt}}{p(e_1(t))} + \frac{1}{k'(t)} \frac{\frac{d^2p(e_1(t))}{dt^2}}{p(e_1(t))} \frac{1}{A(k(t) - s)} \right. \\ \left. + \frac{1}{Lp(e_1(t))} \frac{u''(k(t))}{u''(k(t) - s)} \left(1 - \frac{1}{A(k(t))} \frac{k''(t)}{k'(t)} \right) \right\}, \quad (8)$$

$\forall t, \forall s \in [0, L]$, and $w_p < 0$ if the converse is true.

To interpret this result, first consider a risk neutral individual. It is easy to check that she chooses the same level of effort regardless of the period and of the cost $c(e)$. The reason is that the marginal cost of effort is the same ($c'(e) = 1, \forall e$), regardless of how the cost changes over time. Indeed, because $e'(x) = 0, \forall x$, and $k'(x) = -1, \forall x$, a unitary change in the cost of effort simply involves that the amount of money to be spent in preventive effort will vary by one unit. Then, no flexibility gain is to be foreseen by delaying the effort decision. It follows that an individual is concerned with the timing of the preventive effort only if she is not risk neutral. Next consider a risk averse individual. According to (8), the timing of the preventive effort depends on the individual degree of absolute prudence. To illustrate, in the particular case where the optimal effort varies linearly with the cost of effort, namely, $e'(x) = -1/2$, and $p''(e) = 0, \forall e$, (8) reduces to

$$P(k(t) - s) < \frac{2}{p(e_1(t))} \left(1 + \frac{1}{L} \frac{u''(k(t))}{u''(k(t) - s)} \right),$$

where the right-hand side is strictly positive. This tells that the individual prefers to delay the effort decision from period zero to period one if and only if her degree of absolute prudence is sufficiently low. The reason is that delaying effort raises her exposure to risk, instead of granting her better opportunities. This result points to the conclusion that the crucial reason why prudence is related to the flexibility gain in our model is that the future brings more favourable investment opportunities rather than more risk.

4.5 Principal-agent relationships with unknown distribution

We now apply the result previously obtained to principal-agent models. For the sake of illustration, we refer to the utility function in (6), further taking

$$\hat{u}(y) = \frac{1}{1 - \gamma} y^{1 - \gamma},$$

for some $\gamma \in (0, 1)$. This is similar to a constant relative risk aversion function, except that here the argument of the function is the amount of a consumption good rather than an amount of money. Accordingly, we have $\hat{u}'(y) = y^{-\gamma} > 0$, $\hat{u}''(y) = -\gamma y^{-\gamma-1} < 0$ and $\hat{u}'''(y) = \gamma(\gamma + 1) y^{-\gamma-2} > 0$. Considering a function with these properties is convenient in

that it permits to look at variations in $\widehat{u}'''(\cdot)$ through variations in γ . Indeed, one has

$$\frac{d(\widehat{u}(y)''')}{d\gamma} = (2\gamma + 1)y^{-\gamma-2} + \gamma(\gamma + 1)y^{-\gamma-2} \ln y,$$

which is strictly positive if y is above one. The second and the third derivatives of $\widehat{u}(\cdot)$ are also the second and the third derivatives of $\widehat{u}(y) - \widehat{x}y$.

4.5.1 Delegation with privately known cost distribution

Take the decision maker to be a principal who delegates the production of y units of some good to an agent and derives a gross utility (or surplus) of $\widehat{u}(y)$ from consumption. By granting a profit of $\pi = t(\widehat{x}, y) - \widehat{x}y$ to the agent, where $t(\widehat{x}, y)$ is a transfer and \widehat{x} is the unit cost of production, the principal obtains a net utility of $\widehat{u}(y) - \widehat{x}y - \pi$.

Take $\widehat{x} = \theta$, which is known, in period zero; and $\widehat{x} \in \{\theta - \eta, \theta + \eta\}$, with equal probabilities, in period one. From the previous analysis, we know that, if $\pi = 0$, then the principal strictly prefers to condition the production level on the cost realization $\theta + \tilde{\sigma}$, where $\tilde{\sigma} \in \{-\eta, +\eta\}$, rather than on the expected cost θ . Indeed, by doing so, she enjoys a flexibility gain. Moreover, according to Proposition 3, higher values of θ are associated with a lower flexibility gain because $\widehat{u}'''(\cdot) > 0$.

In agency models, the characteristics of the distribution of some unknown variable, which matters in the principal-agent relationship, are usually expressed by making reference to the monotonic likelihood ratio. Remarkably, this property is more restrictive than that of stochastic dominance (see, for instance, Eeckhoudt et al [8], page 39). We hereafter show that, in the framework we consider, it is essential to refer to the third derivative of the surplus function to characterize the optimal contractual design.

Suppose that the principal can choose between an agent producing at a cost of $\theta_1 + \tilde{\sigma}_1$ and an agent producing at a cost of $\theta_2 + \tilde{\sigma}_2$, where $\theta_1 < \theta_2$, $\sigma_1 \in \{-\eta_1, \eta_1\}$, $\sigma_2 \in \{-\eta_2, \eta_2\}$ and $\eta_2 > \eta_1$. We saw that, if the principal can leave zero profit to the agent regardless of his cost, then the principal prefers a cost of $\theta_1 + \tilde{\sigma}_1$ if and only if $\theta_2 - \theta_1 > \eta_2 - \eta_1$ (Proposition 4). Moreover, the gain associated with the former choice relative to the latter increases with $\widehat{u}'''(\cdot)$. Indeed, using $\widehat{u}'(y) = \theta + \tilde{\sigma}$ and $\widehat{u}'(y) = y^{-\gamma}$, we see that $y(\theta + \tilde{\sigma}) = (\theta + \tilde{\sigma})^{-\frac{1}{\gamma}}$ and that

$$D_{11/22} = \mathbf{E} \int_{\theta_1 + \tilde{\sigma}_1}^{\theta_2 + \tilde{\sigma}_2} x^{-\frac{1}{\gamma}} dx$$

together with

$$\frac{dD_{11/22}}{d\gamma} = \frac{1}{\gamma^2} \mathbf{E} \int_{\theta_1 + \tilde{\sigma}_1}^{\theta_2 + \tilde{\sigma}_2} x^{-\frac{1}{\gamma}} \ln(x) dx,$$

which is positive if the unit cost is above one in all states. Since $d\widehat{u}'''/d\gamma > 0$, we can say that a greater value of \widehat{u}''' is associated with a greater value of $D_{11/22}$, as in Proposition 4.

To see why this is relevant, consider a situation in which the principal does not know

the type of agent she is facing, where the type is defined by the distribution indexed by $ij \in \{11, 12, 21, 22\}$. The agent holds this information, instead. In problems of this kind, the principal attempts to figure out a contractual allocation such that the agent is induced to reveal information. Indeed, information release allows the principal to minimize the agency cost (namely, the cost of the agent's private information). Applying the Revelation Principle, the agent is required to report his type to the principal. He will be motivated to reveal his true type (rather than delivering a fake report) if some incentive constraints are satisfied. For instance, an agent of type 11 will tell the truth if, by so doing, he obtains at least as high a profit as he would obtain by announcing any other type. Usually, one such constraint is implied by some other, hence only one of the constraints is potentially binding; what exact constraint is binding is unrelated to the properties of the principal's surplus function. However, this is true in agency problems where distributions respect either first- or second-order stochastic dominance (see, for instance, Courty and Li [4]). Here, this is not the case, instead. In what follows, we show that it depends on $\hat{u}'''(\cdot)$ whether the principal should be more concerned with an agent of type 11 announcing, say, 21 or 22, hence which of the associated incentive constraints is tighter (for a complete analysis, see Danau and Vinella [5]). Let us begin with the incentive of type 11 to announce 22. Denoting Π_1 and Π_2 the profits respectively designed for the two types, it is easy to verify that this incentive is eliminated if and only if $\Pi_1 - \Pi_2 \geq \Delta\Pi_{12}$, where

$$\Delta\Pi_{12} = (\theta_2 - \theta_1) \mathbf{E} [y(\theta_2 + \tilde{\sigma}_2)] - \frac{1}{2} (\eta_2 - \eta_1) [y(\theta_2 - \eta_2) - y(\theta_2 + \eta_2)]$$

is the additional profit that type 11 can obtain, if it announces 22 rather than telling the truth. Replacing $y(\theta_2 + \tilde{\sigma}_2) = (\theta_2 + \tilde{\sigma}_2)^{-\frac{1}{\gamma}}$, this becomes

$$\Delta\Pi_{12} = (\theta_2 - \theta_1) \mathbf{E} \left[(\theta_2 + \tilde{\sigma}_2)^{-\frac{1}{\gamma}} \right] - \frac{1}{2} (\eta_2 - \eta_1) \left[(\theta_2 - \eta_2)^{-\frac{1}{\gamma}} - (\theta_2 + \eta_2)^{-\frac{1}{\gamma}} \right].$$

Further denote Π_3 the profit assigned to type 21. This profit should be designed in such a way that an agent of type 11 is unwilling to announce 21. This lie is unattractive to type 11 if and only if $\Pi_1 - \Pi_3 \geq \Delta\Pi_{13}$, where

$$\Delta\Pi_{13} = (\theta_2 - \theta_1) \mathbf{E} [y(\theta_2 + \tilde{\sigma}_1)].$$

Replacing $y(\theta_2 + \tilde{\sigma}_1) = (\theta_2 + \tilde{\sigma}_1)^{-\frac{1}{\gamma}}$, this becomes

$$\Delta\Pi_{13} = (\theta_2 - \theta_1) \mathbf{E} \left[(\theta_2 + \tilde{\sigma}_1)^{-\frac{1}{\gamma}} \right]$$

To establish which of the two lies is more attractive, hence which of the two incentive constraints implies the other, it is necessary to consider $\hat{u}'''(\cdot)$, according to the following proposition.

Proposition 6 $\frac{d\Delta\Pi_{12}}{d\gamma} > 0$, $\frac{d\Delta\Pi_{13}}{d\gamma} > 0$ and $\frac{d\Delta\Pi_{12}}{d\gamma} - \frac{d\Delta\Pi_{13}}{d\gamma} > 0$.

The fact that the differences $\Pi_1 - \Pi_2$ and $\Pi_1 - \Pi_3$ increase with γ , as the first two results in the proposition highlight, involve that they are increasing with $\hat{u}'''(\cdot)$ as well. However, this does not necessarily have a qualitative impact on the optimal contractual design. Suppose that $\Pi_1 - \Pi_3 > \Pi_1 - \Pi_2$, regardless of the value of γ , and that $\Pi_1 - \Pi_3$ is sufficiently high, relative to $\Pi_1 - \Pi_2$, for an agent of type 11 to be more prone to claim 12 than 22, regardless of how Π_2 compares with Π_3 . Then, as usual in agency problems of task delegation, the only concern of the principal with an agent of type 11 is to prevent him from exaggerating the expected cost and claiming 12. The properties of the surplus function $\hat{u}(\cdot)$ only matter in the identification of the exact size of the information rent Π_{11} , which must be given up to prevent that lie. However, as evidenced by the second result of the proposition, one might also have $\Pi_1 - \Pi_2 > \Pi_1 - \Pi_3$ when γ is sufficiently large. In that case, type 11 might want to claim 12 for $\hat{u}'''(\cdot)$ low, and 22 for $\hat{u}'''(\cdot)$ high, and the third derivative of the surplus function comes to matter in the optimal contractual design.⁴

4.5.2 Price discrimination with unknown preferences

The previous application is an example of agency problems with privately known distributions, as explored by the recent literature on contract theoretic models. Most of the studies on the subject are about price discrimination in the relationship between a monopolist and a consumer, none of whom knows the consumer valuation for the good in the contracting stage, whereas the consumer has private information on the distribution of his valuation. The pioneering paper is Courty and Li [4]. The authors assume that the monopolist receives a fixed payment of a when the consumer is still uninformed of his valuation. This might be followed by a reimbursement of k , which the consumer can require in a later stage, after learning his true valuation. Of course, the consumer will want to be reimbursed, thus renouncing to consume, if and only if k exceeds his valuation. If the consumer does not renounce, then the monopolist will bear a cost of c to provide the service.

Essentially, in Courty and Li [4], the economic issue is how to choose the future disbursement k and the current revenue a , which is more in line with the classical savings-consumption model than with the issue of our interest. However, because this problem belongs to the kind of principal-agent models considered in the previous example, for the sake of completeness, we show that u''' plays a role in the solution adopted by the principal also in this case. To that end, we restrict attention to the case of symmetric information between players.

Whereas Courty and Li [4] and more recent studies assume that the monopolist is risk neutral, we consider a risk averse monopolist, whose utility $u(\cdot)$ is expressed as a function

⁴In Danau and Vinella [5], it is shown that the study of the optimal contractual design with privately known distributions that respect neither first- nor second-order stochastic dominance, as in this example, is a lengthy and complicated exercise, unless $\hat{u}'''(\cdot)$ is considered.

of money, as defined above. Accordingly, the total benefit of the monopolist is

$$u(a) - \nu u(k) - (1 - \nu) u(c),$$

where ν is the probability of a high valuation, namely $\nu > 0$, $(1 - \nu)$ is the probability of a low valuation, namely 0, and the reimbursement k is supposed to take values in $(0, \nu)$ at optimum. Under symmetric information, if the risk neutral consumer has zero outside opportunity, then the monopolist sets the fixed payment to

$$a(k) = (1 - \nu)k + \nu v.$$

The first-order condition of the maximization problem of the monopolist is given by

$$u'(k + \nu(v - k)) \geq \frac{\nu}{1 - \nu} u'(k).$$

For a positive solution to exist, it is necessary and sufficient that the high valuation is less likely than the low valuation: $\nu < 1/2$. Otherwise, the monopolist will choose $a = \nu v$ and concede no reimbursement. Accordingly, we take $\nu < 1/2$. Then, $k > 0$ involving that $a(k) > \nu v$. Replacing $u'(y) = y^{-\gamma}$, we pin down the following solution:

$$[k + \nu(v - k)]^{-\gamma} = \frac{\nu}{1 - \nu} k^{-\gamma} \Leftrightarrow k^* = \frac{\nu v}{\left(\frac{1-\nu}{\nu}\right)^{\frac{1}{\gamma}} - (1 - \nu)},$$

which is lower than ν and confirms our previous hypothesis. We see that $dk^*/d\gamma > 0$. Hence, the greater $u'''(\cdot)$ is the higher the value k^* that the solution takes. The intuition is that a more prudent monopolist is more prone to grant a reimbursement to the consumer in a later stage to be able to appropriate a higher certain payment $a(k^*)$ today.

5 Conclusion

We showed that the notion of prudence extends to situations the literature has not considered so far. Specifically, provided that an individual prefers future outcomes to current ones, the third derivative of her utility function (and, implicitly, her degree of prudence) is a measure of that preference. This is explained by the fact that when new information can be acquired over time, downside risk is lower as decisions are delayed.

The results we obtained suggest two directions along which the extended definition of prudence is potentially useful in applications. First, it applies to decisional processes in which a delay in the decision allows the individual to take advantage of newly available information. Second, it applies to situations in which the individual must choose between distributions that differ in terms of expected value and/or spread. We showed that, in the presence of distributions with these characteristics, one cannot make reference to the usual

link between expected utility and stochastic ordering. Therefore, our findings are essential for this kind of applications.

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A Proof of Proposition 1

Because $v(x) = u[\pi(x)]$ we can rewrite (3) as

$$w'(\theta) = \int_{\theta-\eta}^{\theta+\eta} [v(x) f(x)]' dx - v'(\theta).$$

Adding and subtracting $\int_{\theta-\eta}^{\theta+\eta} [v(\theta) f(\theta)]' dx$, we obtain

$$\begin{aligned} w'(\theta) &= \int_{\theta}^{\theta+\eta} \{[v(x) f(x)]' - [v(\theta) f(\theta)]'\} dx - \int_{\theta-\eta}^{\theta} \{[v(\theta) f(\theta)]' - [v(x) f(x)]'\} dx \\ &\quad + \int_{\theta-\eta}^{\theta+\eta} [v(\theta) f(\theta)]' dx - v'(\theta). \end{aligned}$$

Changing the margins of the integrals, we can rewrite

$$\begin{aligned} w'(\theta) &= \int_0^{\eta} \{[v(\theta+x) f(\theta+x)]' - [v(\theta) f(\theta)]'\} dx \\ &\quad - \int_0^{\eta} \{[v(\theta) f(\theta)]' - [v(x+\theta-\eta) f(x+\theta-\eta)]'\} dx \\ &\quad + \int_{\theta-\eta}^{\theta+\eta} [v(\theta) f(\theta)]' dx - v'(\theta). \end{aligned}$$

Adding and subtracting $\int_0^{\eta} [v(\theta-x) f(\theta-x)]' dx$, we can further rewrite

$$\begin{aligned} w'(\theta) &= \int_0^{\eta} \int_{\theta}^{\theta+x} [v(z) f(z)]'' dz dx - \int_0^{\eta} \int_{\theta-x}^{\theta} dz dx + g(\theta), \\ &= \int_0^{\eta} \int_{\theta-x}^{\theta} \int_z^{\theta+x} [v(t) f(t)]''' dt dz dx + g(\theta), \end{aligned}$$

where

$$\begin{aligned} g(\theta) &= - \int_0^{\eta} \{[v(\theta-x) f(\theta-x)]' - [v(x+\theta-\eta) f(x+\theta-\eta)]'\} dx \\ &\quad + \int_{\theta-\eta}^{\theta+\eta} [v(\theta) f(\theta)]' dx - v'(\theta), \end{aligned}$$

which is rewritten as

$$g(\theta) \equiv 2[v(\theta) f(\theta) - v(\theta-\eta) f(\theta-\eta)] + 2\eta v(\theta) f'(\theta).$$

Further using

$$\int_0^\eta \int_{\theta-x}^\theta \int_z^{z+x} dt dz dx = \frac{\eta^3}{3},$$

together with the expression of $g(\theta)$, one finally obtains $w'(\theta) = \int_0^\eta \int_{\theta-x}^\theta \int_z^{z+x} \phi(t) dt dz dx$, where

$$\begin{aligned} \phi(t) &= [v(t) f(t)]''' \\ &+ \frac{6}{\eta^3} \{[v(\theta) f(\theta) - v(\theta - \eta) f(\theta - \eta)] + \eta v(\theta) f'(\theta)\}. \end{aligned}$$

Calculating derivatives we obtain

$$\begin{aligned} \phi(t) &= v'''(t) f(t) + 3v''(t) f'(t) + 2v'(t) f''(t) + v(t) f'''(t) \\ &+ \frac{6}{\eta^3} \{[v(\theta) f(\theta) - v(\theta - \eta) f(\theta - \eta)] + \eta v(\theta) f'(\theta)\}. \end{aligned} \quad (9)$$

We see that if $\phi(t) \geq 0, \forall t$, then $w'(\theta) \geq 0$. From (9), $\phi(t) \geq 0$ is rewritten as $v'''(t) \geq \psi(t)$. If $\phi(t) \leq 0, \forall t$, then $w'(\theta) \leq 0 \Leftrightarrow v'''(t) \leq \psi(t)$.

B Proof of Proposition 2

The difference between the expected value of lottery **B**' and that of lottery **A**' is written as

$$\begin{aligned} &\frac{1}{2} \{\mathbb{E}[u(\pi(x))] - u(\theta)\} - \frac{1}{2} \{\mathbb{E}[u(\pi(y(x-k), x-k))] - u(\pi(\theta-k))\} \\ &= \frac{1}{2} \frac{1}{2\eta} \int_0^\eta \int_{\theta-x}^\theta \int_{z-k}^{z+x-k} \int_t^{t+k} [u(\pi(v))]''' dv dt dz dx. \end{aligned}$$

Hence, if $[u(\pi(x))]''' \geq 0, \forall x$, then the difference is non-negative. If $[u(\pi(x))]''' < 0, \forall x$, then the difference is negative. Furthermore, $[u(\pi(x))]''' \geq 0$ is rewritten as (5), which completes the proof.

C Proof of Proposition 5

If the investment is made in period one, when x is known, then the optimal effort $e_1(x)$ solves

$$\begin{aligned} &u(m - e_1(x) - x) - u(m - L - e_1(x) - x) \\ &= -\frac{p(e_1(x))}{p'(e_1(x))} u'(m - L - e_1(x) - x) - \frac{1 - p(e_1(x))}{p'(e_1(x))} u'(m - e_1(x) - x). \end{aligned} \quad (10)$$

Denoting

$$\begin{aligned} k(x) &\equiv m - e_1(x) - x, \\ \Delta u(k(x)) &\equiv u(k(x)) - u(k(x) - L), \\ g(e, x) &\equiv -\frac{p(e)}{p'(e)}u'(m - L - e - x) - \frac{1 - p(e)}{p'(e)}u'(m - e - x), \end{aligned}$$

(10) is rewritten as

$$\Delta u(k(x)) = g(e_1(x), x). \quad (11)$$

Because

$$\frac{\partial \Delta u(k(x))}{\partial x} = u'(m - L - e - x) - u'(m - e - x) < 0,$$

and

$$\frac{\partial g(e, x)}{\partial x} = \frac{p(e)}{p'(e)}u''(m - L - e - x) + \frac{1 - p(e)}{p'(e)}u''(m - e - x) > 0,$$

one has $e_1'(x) < 0$.

If the investment is made in period zero, then the optimal effort e_0 is the solution to $\Delta u(k(0)) = g(e_0, 0)$. Hence, $e_0 = e_1(0)$.

Denoting further $h(x) \equiv g(e_1(x), x)$, we rewrite

$$w_p = \mathbf{E}[p(e_1(x))h(x)] - p(e_1(0))h(0) + \mathbf{E}[u(k(x))] - u(k(0)).$$

In integral form, this becomes

$$w_p = \int_{-\eta}^0 \int_x^0 \int_z^{z+\eta} \left(\frac{d^2[p(e_1(t))h(t)]}{dt^2} + \frac{d^2[u(k(t))]}{dt^2} \right) dt dz dx.$$

We see that $w_p > 0$ if

$$\frac{d^2[p(e_1(t))h(t)]}{dt^2} > -\frac{d^2[u(k(t))]}{dt^2}, \quad \forall t.$$

The left-hand side is developed as

$$\frac{d^2[p(e_1(t))h(t)]}{dt^2} = \frac{d^2 p(e_1(t))}{dt^2} h(t) + 2 \frac{dp(e_1(t))}{dt} h'(t) + p(e_1(t)) h''(t).$$

Hence, $w_p > 0$ if

$$h''(t) > -\frac{1}{p(e_1(t))} \left\{ \frac{d^2[u(k(t))]}{dt^2} + \frac{d^2 p(e_1(t))}{dt^2} h(t) + 2 \frac{dp(e_1(t))}{dt} h'(t) \right\}, \quad \forall t. \quad (12)$$

Using (11) and rearranging, (12) becomes

$$\begin{aligned} \frac{d^2 \Delta u(k(t))}{dt^2} &> -\frac{1}{p(e_1(t))} \left\{ \frac{d^2 p(e_1(t))}{dt^2} \Delta u(k(t)) + 2 \frac{dp(e_1(t))}{dt} \frac{d \Delta u(k(t))}{dt} \right\} \\ &\quad - \frac{1}{p(e_1(t))} \frac{d^2[u(k(t))]}{dt^2}. \end{aligned}$$

Using further

$$\begin{aligned}\frac{d\Delta u(k(t))}{dt} &= k'(t) \Delta u'(k(t)) \\ \frac{d^2\Delta u(k(t))}{dt^2} &= k''(t) \Delta u'(k(t)) + [k'(t)]^2 \Delta u''(k(t)),\end{aligned}$$

the condition becomes

$$\begin{aligned}\Delta u''(k(t)) &> -\frac{k''(t)}{[k'(t)]^2} \Delta u'(k(t)) \\ &- \frac{1}{p(e_1(t)) [k'(t)]^2} \left\{ \frac{d^2 p(e_1(t))}{dt^2} \Delta u(k(t)) + 2 \frac{dp(e_1(t))}{dt} k'(t) \Delta u'(k(t)) \right\} \\ &- \frac{1}{p(e_1(t)) [k'(t)]^2} \left\{ k''(t) u'(k(t)) + [k'(t)]^2 u''(k(t)) \right\},\end{aligned}$$

and, after regrouping,

$$\begin{aligned}\Delta u''(k(t)) &> \frac{1}{k'(t)} \left(-\frac{k''(t)}{k'(t)} + 2 \frac{dp(e_1(t))}{p(e_1(t)) dt} \right) \Delta u'(k(t)) - \frac{\frac{d^2 p(e_1(t))}{dt^2}}{p(e_1(t)) [k'(t)]^2} \Delta u(k(t)) \\ &- \frac{1}{p(e_1(t)) [k'(t)]^2} \left\{ k''(t) u'(k(t)) + [k'(t)]^2 u''(k(t)) \right\}.\end{aligned}$$

Because $\Delta u(k(t)) = -\int_0^L u'(k(t)-s) ds$, we can rewrite the condition in integral form as follows:

$$\begin{aligned}-\int_0^L u'''(k(t)-s) ds &> -\frac{1}{k'(t)} \left(-\frac{k''(t)}{k'(t)} + 2 \frac{dp(e_1(t))}{p(e_1(t)) dt} \right) \int_0^L u''(k(t)-s) ds \\ &+ \frac{\frac{d^2 p(e_1(t))}{dt^2}}{p(e_1(t)) [k'(t)]^2} \int_0^L u'(k(t)-s) ds \\ &- \frac{1}{p(e_1(t)) [k'(t)]^2} \left\{ k''(t) u'(k(t)) + [k'(t)]^2 u''(k(t)) \right\}.\end{aligned}$$

Dividing by $-u''(k(t)-s)$ we get

$$\begin{aligned}\int_0^L P(k(t)-s) ds &< -\int_0^L \frac{1}{[k'(t)]^2} \left\{ -k''(t) + \frac{\frac{d^2 p(e_1(t))}{dt^2}}{p(e_1(t)) A(k(t)-s)} \right\} ds \quad (13) \\ &- \int_0^L \frac{1}{k'(t)} \left\{ 2 \frac{dp(e_1(t))}{p(e_1(t)) dt} \right\} ds \\ &+ \frac{1}{L p(e_1(t))} \frac{u''(k(t))}{u''(k(t)-s)} \left(1 - \frac{1}{A(k(t))} \frac{k''(t)}{k'(t)} \right) ds\end{aligned}$$

This is satisfied if (8) is satisfied, where (8) is obtained by taking the integrands in the left- and in the right-hand side of (13).

D Proof of Lemma 1

One has $\zeta(a, b, c) > 0$ if and only if

$$\begin{aligned} f(a) - f(a+c) &> f(b) - f(b+c) \\ \Leftrightarrow \int_a^{a+c} [f'(x) - f'(x+b-a)] dx &< 0. \end{aligned}$$

Provided u''' has a constant sign, this is also the case of $f'(\cdot)$, and the above condition holds if and only if $f'(x) < f'(x+b-a)$, for any given $x \in [a, a+c]$. This is equivalent to $u''(y(x)) > u''(y(x+b-a))$. Because $b > a$ and $u'' < 0$, $y(x) > y(x+b-a)$. Hence, $u''(y(x)) > u''(y(x+b-a))$ is equivalent to $u''' > 0$.

E Proof of Proposition 4

Being based on the definition of U_{ij} , we compute

$$\begin{aligned} D_{ij/i'j'} &= \mathbf{E}[U_{ij} - U_{i'j'}] \\ &= -\mathbf{E}\left[\int_{x_{ij}}^{x_{i'j'}} u'(y(z)) y'(z) dz\right] + \mathbf{E}\left[\int_{x_{ij}}^{x_{i'j'}} (y(z) + zy'(z)) dz\right]. \end{aligned}$$

Because $u'(y(z)) = z, \forall z$, we have

$$D_{ij/i'j'} = \mathbf{E}\left[\int_{x_{ij}}^{x_{i'j'}} y(z) dz\right]. \quad (14)$$

Proof of (i) and (ii). Using (14) and (7), we further develop

$$\begin{aligned} D_{1j/2j} &= \mathbf{E}\int_{x_{1j}}^{x_{2j}} y(z) dz \\ &= \mathbf{E}\left[\int_{x_{1j}}^{\theta_1} y(z) dz + \int_{\theta_1}^{\theta_2} y(z) dz + \int_{\theta_2}^{x_{2j}} y(z) dz\right] \\ &= \int_{\theta_1-\eta_j}^{\theta_1} \zeta(x, x + \Delta\theta, n_j) dx + \int_{\theta_1}^{\theta_2} y(x) dx, \end{aligned}$$

where $\Delta\theta = \theta_2 - \theta_1$. In the expression of $D_{1j/2j}$, the term that depends on $u'''(\cdot)$ is $\zeta(\cdot, \cdot, \cdot)$, as defined in (7), which increases with $u'''(\cdot)$. Also,

$$\begin{aligned} D_{i2/i1} &= -\mathbf{E}\left[\int_{x_{i1}}^{x_{i2}} y(x) dx\right] \\ &= \left[\int_{\theta_i-\eta_2}^{\theta_i} y(x) dx - \int_{\theta_i}^{\theta_i+\eta_2} y(x) dx\right] - \left[\int_{\theta_i-\eta_1}^{\theta_i} y(x) dx - \int_{\theta_i}^{\theta_i+\eta_1} y(x) dx\right] \\ &= \int_{\theta_i-\eta_2}^{\theta_i} \zeta(x, x + \Delta\eta, \eta_2) dx + \int_{\theta_i}^{\theta_i+\Delta\eta} y(x) dx - \int_{\theta_i+\eta_1}^{\theta_i+\eta_1+\Delta\eta} y(x) dx, \end{aligned}$$

where $\Delta\eta = \eta_2 - \eta_1$. Again, the term that depends on $u(\cdot)'''$ is $\zeta(\cdot, \cdot, \cdot)$, which increases with $u(\cdot)'''$.

Rewriting

$$D_{1j/2j'} = D_{1j/2j} + D_{2j/2j'} = \begin{cases} D_{1j/2j}, & \text{if } j' = j, \\ D_{12/22} + D_{22/21}, & \text{if } j' = 1 < j = 2 \end{cases}$$

and

$$D_{i2/i'1} = D_{i2/i1} + D_{i1/i'1} = \begin{cases} D_{i2/i1}, & \text{if } i = i', \\ D_{12/11} + D_{11/21}, & \text{if } i = 1 < i' = 2 \end{cases},$$

and considering that $D_{1j/2j}$, $\forall j$, and $D_{i2/i1}$, $\forall i$, increase with $u'''(\cdot)$, we deduce that this is the case of $D_{1j/2j'}$ and $D_{i2/i'1}$ as well.

Proof of (iii). Using (14) and (7), we compute

$$\begin{aligned} D_{11/22} &= \mathbf{E} \left[\int_{x_{11}}^{x_{22}} y(x) dx \right] \\ &= \int_{\theta_1 - \eta_1}^{\theta_2 - \eta_2} \zeta(x, x + \eta_1, x + 2\eta_1) dx + \int_{\theta_1}^{\theta_1 + \Delta\theta - \Delta\eta} y(x) dx. \end{aligned}$$

If $\Delta\theta > \Delta\eta$, then $D_{11/22}$ is positive and increases with $\zeta(\cdot, \cdot, \cdot)$; if $\Delta\theta < \Delta\eta$, then it is negative and decreases with $\zeta(\cdot, \cdot, \cdot)$. Provided that $\zeta(\cdot, \cdot, \cdot)$ increases with $u'''(\cdot)$, the result follows.

F Proof of Proposition 6

Using the definition of $\Delta\Pi_{12}$ and $\Delta\Pi_{13}$, we calculate:

$$\begin{aligned} \frac{d\Delta\Pi_{12}}{d\gamma} &= \frac{1}{2\gamma^2} [(\theta_2 - \theta_1) - (\eta_2 - \eta_1)] (\theta_2 - \eta_2)^{-\frac{1}{\gamma}} \ln(\theta_2 - \eta_2) \\ &\quad + \frac{1}{2\gamma^2} [(\theta_2 - \theta_1) + (\eta_2 - \eta_1)] (\theta_2 + \eta_2)^{-\frac{1}{\gamma}} \ln(\theta_2 + \eta_2), \\ &> 0, \end{aligned}$$

$$\begin{aligned} \frac{d\Delta\Pi_{13}}{d\gamma} &= \frac{1}{2\gamma^2} (\theta_2 - \theta_1) \left[(\theta_2 - \eta_2)^{-\frac{1}{\gamma}} \ln(\theta_2 - \eta_2) + (\theta_2 + \eta_2)^{-\frac{1}{\gamma}} \ln(\theta_2 + \eta_2) \right] \\ &> 0 \end{aligned}$$

and

$$\begin{aligned} \frac{d\Delta\Pi_{12}}{d\gamma} - \frac{d\Delta\Pi_{13}}{d\gamma} &= \frac{1}{2\gamma^2} [-(\eta_2 - \eta_1)] (\theta_2 - \eta_2)^{-\frac{1}{\gamma}} \ln(\theta_2 - \eta_2) \\ &\quad + \frac{1}{2\gamma^2} (\eta_2 - \eta_1) (\theta_2 + \eta_2)^{-\frac{1}{\gamma}} \ln(\theta_2 + \eta_2) \\ &> 0 \end{aligned}$$