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# Spaces with No Infinite Discrete Subspace

Jean Goubault-Larrecq

LSV, ENS Paris-Saclay, CNRS, Université Paris-Saclay  
61 avenue du président Wilson, 94230 Cachan, France  
goubault@lsv.fr

## Abstract

We show that the spaces with no infinite discrete subspace are exactly those in which every closed set is a finite union of irreducibles. Call them FAC spaces: this generalizes a theorem by Erdős and Tarski (1943), according to which a preordered set has no infinite antichain—the finite antichain, or FAC, property—if and only if all its downwards-closed subsets are finite unions of ideals. All Noetherian spaces are FAC spaces, and we show that sober FAC spaces have a simple order-theoretic description.

## 1 Introduction

A preorder is *FAC* (for: has the *finite antichain* property) if and only if all its antichains are finite. An antichain is a subset of pairwise incomparable elements. A well-known result in the theory of preorders states that a preorder is FAC if and only if all its downwards-closed subsets are finite unions of ideals—an ideal is a downwards-closed, directed subset. This was discovered many times [1, 9, 10, 3, 7], and is credited to Erdős and Tarski [2].

The purpose of this paper is to generalize that result to the case of topological spaces. An antichain will simply be a discrete subspace, namely a subspace whose topology is discrete. Downwards-closed subsets will be replaced by closed subsets, and ideals by irreducible closed subsets. We shall retrieve the above preorder-theoretic result by looking at spaces with the Alexandroff topology of the preorder. This is in line with related results, such as the fact that Noetherian spaces are a topological generalization of well-quasi-orders [4].

## 2 Main Results

The main result of this paper is the following, and will be proved in Section 3.

**Theorem 2.1** *For a topological space  $X$ , the following are equivalent:*

1.  $X$  has no infinite discrete subspace;
2. every closed subset of  $X$  is a finite union of irreducible closed subsets.

We recall that a closed subset  $C$  of  $X$  is *irreducible* if and only if it is non-empty, and for all closed subsets  $C_1, C_2$  of  $X$ , if  $C$  is included in  $C_1 \cup C_2$  then  $C$  is included in  $C_1$  or in  $C_2$ . Equivalently,  $C$  is irreducible (closed) if and only if it is non-empty and, for any two open subsets  $U$  and  $V$  of  $X$ , if  $C$  intersects both  $U$  and  $V$  then it intersects  $U \cap V$ .

Perhaps the closest result in the literature, except for the preorder-theoretic results mentioned in the introduction, is due to A. H. Stone [11, Theorem 2]: a space  $X$  has no infinite discrete subspace if and only if every open cover of every subspace  $A$  of  $X$  has a finite subfamily whose union is dense in  $A$ , if and only if every continuous real-valued function on every subspace of  $X$  is bounded. Another one is due to Milner and Pouzet [8], and we shall return to it in Remark 3.4.

**Definition 2.2** *A space satisfying any of the equivalent properties of Theorem 2.1 is called a FAC space.*

The closures of points are always irreducible. A space is *sober* if and only if every irreducible closed subset is the closure of a unique point. The *sobrification*  $\mathcal{S}(X)$  of  $X$  can be defined as the set of all irreducible closed subsets of  $X$ , with the topology whose open subsets are  $\diamond U = \{C \in \mathcal{S}(X) \mid C \cap U \neq \emptyset\}$ , where  $U$  ranges over the open subsets of  $X$  (see [5, Section 8.2.3] for example).

The map  $U \mapsto \diamond U$  defines an isomorphism between the lattice of open subsets of  $X$  and of  $\mathcal{S}(X)$ . It follows that  $X$  and  $\mathcal{S}(X)$  have isomorphic lattices of closed subsets. Since Item 2 of Theorem 2.1 only depends on the lattice of closed subsets, we immediately obtain:

**Fact 2.3** *A space  $X$  is FAC if and only if its sobrification  $\mathcal{S}(X)$  is.*

Hence it is legitimate to study *sober* FAC spaces. This is best done by comparing the situation with Noetherian spaces (see Section 9.7 of [5] for an introduction to Noetherian spaces).

A Noetherian space is a space where every open subset is compact. (A subset is *compact* if and only if all its open covers have a finite subcover; we do not require Hausdorffness.) Equivalently, a space is Noetherian if and only if every monotone chain of open subsets stabilizes, if and only if the inclusion ordering on the lattice of closed subsets is well-founded. It is well-known that, in a Noetherian space, all closed subsets are finite unions of irreducibles. A simple proof goes as follows. Imagine there were a closed subset  $C$  that one cannot write as a finite union of irreducibles. Since inclusion is well-founded on closed sets, we can require  $C$  to be minimal.  $C$  is not a finite union of irreducibles, hence is not empty and not irreducible itself. Hence there are two closed sets  $C_1$  and  $C_2$  such that  $C \subseteq C_1 \cup C_2$  but  $C$  is included neither in  $C_1$  nor in  $C_2$ . Then  $C \cap C_1$  and  $C \cap C_2$ , being strictly smaller than  $C$ , are finite unions of irreducibles. It follows that  $C = (C \cap C_1) \cup (C \cap C_2)$  is also a finite union of irreducibles—contradiction. That means that Item 2 of Theorem 2.1 is true of all Noetherian spaces, whence:

**Fact 2.4** *Every Noetherian space is FAC.*

That can also be obtained by using Theorem 1 of [11], which says that a space is Noetherian if and only if it has no weakly discrete infinite subspace, and using Item 1 of Theorem 2.1. (A space is *weakly discrete* if and only if every point has a finite neighborhood. Clearly, every discrete space is weakly discrete.)

The converse of Fact 2.4 fails. For example, consider the real line  $\mathbb{R}$ , with the Scott topology, whose non-trivial opens are the half-open intervals  $]r, +\infty[$ ,  $r \in \mathbb{R}$ . That is FAC, because all non-empty closed sets are of the form  $] - \infty, r]$  with  $r \in \mathbb{R}$ , or the whole of  $\mathbb{R}$ , and all of them are irreducible. However,  $\mathbb{R}$  with the Scott topology is not Noetherian, since no open set  $]r, +\infty[$  is compact.

Another example comes from the theory of preorders. Every topological space  $X$  comes with a so-called *specialization* preorder  $\leq$ , defined by  $x \leq y$  if and only if every open neighborhood of  $x$  contains  $y$ , if and only if  $x$  is in the closure of  $y$ .  $X$  is *Alexandroff* if and only if the open subsets of  $X$  are the upwards-closed subsets with respect to some preordering (which must be  $\leq$ ). The full subcategory of the category of topological spaces consisting of Alexandroff spaces is equivalent to the category of preorders. Hence, equating the two categories, a Noetherian Alexandroff space is the same thing as a well-quasi-order [5, Proposition 9.7.17]. The discrete subspaces of an Alexandroff space are exactly its antichains, hence:

**Fact 2.5** *The FAC Alexandroff spaces are exactly the preorders with no infinite antichain.*

Fact 2.4, once specialized to Alexandroff spaces, yields the following trivial fact: in a well-quasi-ordered set, every antichain is finite. The converse fails, take  $\mathbb{Z}$  with its usual ordering for example.

In an Alexandroff space, a closed subspace is the same as a downwards-closed subset, and an irreducible closed subset is the same thing as an ideal, namely a downwards-closed directed subset (see Fact 8.2.49 in [5], or [6].) Hence, specialized to Alexandroff spaces, Theorem 2.1 is a restatement of the Erdős-Tarski result mentioned in the introduction.

There is a purely order-theoretic characterization of sober Noetherian spaces (see Theorem 9.7.12 of [5]): they are exactly the posets  $X$  whose ordering  $\leq$  is well-founded and satisfies properties T and W, in the upper topology. Property T states that  $X$  is *finitary*, where a finitary subset is defined as the downwards-closure  $\downarrow E$  of a finite set  $E$ . Property W states that, for any two points  $x$  and  $y$ ,  $\downarrow x \cap \downarrow y$  is finitary. (We write  $\downarrow x$  for  $\downarrow\{x\}$ . This coincides with the closure of  $x$ .) The upper topology is the coarsest topology that has  $\leq$  as specialization ordering. Alternatively, this is the topology whose closed sets are the intersections of finitary subsets.

**Theorem 2.6** *The sober FAC spaces are exactly the posets  $X$  in which every intersection of finitary subsets is finitary, in the upper topology. Every closed set is finitary.*

Properties T and W are equivalent to the statement that *finite* intersections of finitary subsets are finitary. Together with well-foundedness, they imply

Noetherianness, hence FACness, but as we have seen, the fact that intersections of finitary subsets are finitary is strictly weaker than requiring that  $\leq$  is well-founded.

*Proof.* Let  $X$  be a sober FAC space. Since  $X$  is sober, its only irreducible closed subsets are the closures of points, and those are exactly the sets of the form  $\downarrow x$ ,  $x \in X$ , where downward closure is taken with respect to the specialization ordering  $\leq$  of  $X$ . Item 2 of Theorem 2.1 then implies that every closed set is finitary. Since intersections of closed sets are closed, we conclude.

Conversely, let  $X$  be a set with an ordering  $\leq$  in which intersections of finitary subsets are finitary. Since finite unions of finitary subsets are finitary, the finitary subsets form the closed sets of a topology, and that must be the upper topology. By Item 2 of Theorem 2.1, and since every set  $\downarrow x$  is irreducible closed,  $X$  is FAC in that topology.

Finally, we show that  $X$  is sober. Let  $C = \downarrow E$  be closed, where  $E$  is finite. We may assume that  $E$  is an antichain, by removing elements below another one, one by one. If  $C$  is irreducible, then it is non-empty, so  $E$  contains some element  $x$ . Then  $C$  is included in the union of the closed sets  $\downarrow x$  and  $\downarrow(E \setminus \{x\})$ , hence must be contained in one of them. It cannot be contained in the second one, since  $x \in C$  and  $E$  is an antichain. It follows that  $C$  is included in  $\downarrow x$ , hence is equal to it. Recall that  $\downarrow x$  is the closure of  $x$ . We have shown that every irreducible closed set is the closure of some point. This point must be unique because  $\leq$  is an ordering, not just a preordering.  $\square$

We finish with the following observation. That generalizes the fact that the Hausdorff Noetherian spaces are the finite discrete spaces.

**Proposition 2.7** *Given a FAC space  $X$ , the following are equivalent:*

1.  $X$  is sober and  $T_1$ ;
2.  $X$  is Hausdorff;
3.  $X$  is a KC-space, namely one where each compact subset is closed;
4.  $X$  is finite and discrete.

*Proof.* (1)  $\Rightarrow$  (4). In a sober space, every irreducible closed set is the closure of a point, and in a  $T_1$  space, all points are closed. Hence the finite unions of irreducible closed subsets are the finite subsets. That applies to every closed subset, in particular to  $X$  itself. Finally, every finite  $T_1$  space is discrete.

(4)  $\Rightarrow$  (2)  $\Rightarrow$  (1), (4)  $\Rightarrow$  (3): obvious.

(3)  $\Rightarrow$  (4). Every KC-space is  $T_1$ , since one-element sets are compact, hence closed. Then (3)  $\Rightarrow$  (4) follows from the well-known fact that every infinite KC-space contains an infinite discrete subspace. We give a proof below, for completeness. Hence, assuming that  $X$  is a FAC KC-space,  $X$  must be finite, and since every KC-space is  $T_1$ , its topology is discrete.

As promised, let us show that every infinite KC-space  $X$  contains an infinite discrete subspace. We first observe that  $X$  cannot be Noetherian. Otherwise all

its subsets would be compact [5, Proposition 9.7.7], hence closed since  $X$  is KC. Then the topology of  $X$  would be discrete. The finite subsets of  $X$  would then form an open cover of  $X$ . Since  $X$  is Noetherian, hence compact, that would have a finite subcover, contradicting the fact that  $X$  is infinite.

Since  $X$  is not Noetherian, the inclusion ordering is not well-founded on the lattice of closed subsets of  $X$ . We can therefore find a strictly decreasing sequence  $C_0 \supset C_1 \supset \dots \supset C_n \supset \dots$  of closed subsets of  $X$ . For each  $n \in \mathbb{N}$ , pick a point  $x_n$  in  $C_n \setminus C_{n+1}$ , and let  $A = \{x_n \mid n \in \mathbb{N}\}$ . Clearly,  $A$  is infinite. Since  $C_{n+1}$  is closed in  $X$ , and  $X$  is  $T_1$ ,  $C'_n = \{x_1, \dots, x_{n-1}\} \cup C_{n+1}$  is closed, and its complement  $U_n$  is open. However,  $U_n \cap A$  contains just  $x_n$ . This shows that the subspace topology on  $A$  is discrete.  $\square$

### 3 Proof of the Main Theorem

The proof of Theorem 2.1 is inspired from Lawson, Mislove and Priestley's proof of the Erdős-Tarski theorem [7, Proposition 4]. In fact, this *is* their proof, with all preorder-theoretic notions replaced by suitable topological notions.

We first need the following lemma. We write  $cl(A)$  for the closure of  $A$  in  $X$ .

**Lemma 3.1** *Let  $A$  be a subset of a topological space  $X$ . The subspace topology on  $A$  is discrete if and only if for every  $x \in A$ ,  $x$  is not in the closure  $cl(A \setminus \{x\})$ .*

*Proof.* If  $A$  is discrete, then for every  $x \in A$ ,  $\{x\}$  is open in  $A$ , so  $A \setminus \{x\}$  is closed in  $A$ . That means that  $A \setminus \{x\}$  occurs as the intersection of some closed subset of  $X$  with  $A$ , and the smallest one is  $cl(A \setminus \{x\})$ . That cannot contain  $x$  since its intersection with  $A$  is  $A \setminus \{x\}$ .

Conversely, for every  $x \in A$ , if  $x$  is not in the closure  $cl(A \setminus \{x\})$ , then it lies in some open subset  $U$  of  $X$  that does not intersect  $A \setminus \{x\}$ . Then  $U \cap A$  is open in  $A$ , but only contains the point  $x$  from  $A$ . It follows that every one-element subset of  $A$  is open in  $A$ , whence  $A$  is discrete.  $\square$

Given a closure operator  $\varphi$  on a set  $X$  (namely,  $\varphi$  is monotonic,  $A \subseteq \varphi(A)$  and  $\varphi(\varphi(A)) \subseteq \varphi(A)$  for every  $A \subseteq X$ ), a subset  $A$  is called  $\varphi$ -independent by Milner and Pouzet [8] if and only if, for every  $x \in A$ ,  $x$  is not in  $\varphi(A \setminus \{x\})$ . One may therefore restate Lemma 3.1 by saying that  $A$  is discrete if and only if it is  $cl$ -independent. Note that  $cl$  is not just a closure operator, but a *topological* closure operator, namely one which commutes with finite unions. This will be important in two places in the proof below.

Next, we prove a useful lemma, generalizing Lemma 3 of [7] to the topological case. For a family of sets  $M$ , we write  $\bigcup M$  for the union of all elements of  $M$ .

**Lemma 3.2** *Let  $X$  be a topological space, and  $M$  be an infinite family of irreducible closed subsets of  $X$ , with the property that for every infinite subfamily  $N$  of  $M$ ,  $cl(\bigcup N) = cl(\bigcup M)$ . Then  $cl(\bigcup M)$  is irreducible closed.*

*Proof.* We shall repeatedly use the fact that if an open set intersects the closure of a set  $E$ , then it intersects  $E$ . Let  $U$  and  $V$  be two open subsets of  $X$  that intersect  $cl(\bigcup M)$ . It suffices to show that  $cl(\bigcup M)$  intersects  $U \cap V$ . Let  $M'$  be the subfamily of those  $I \in M$  such that  $U$  intersects  $I$ .  $M'$  is non-empty: since  $U$  intersects  $cl(\bigcup M)$ , it intersects  $\bigcup M$ , hence some  $I \in M$ .

If  $M'$  were finite, then  $M \setminus M'$  would be infinite, so by assumption  $cl(\bigcup(M \setminus M'))$  would be equal to  $cl(\bigcup M)$ . However,  $U$  intersects the latter, so it would intersect the former, hence also  $\bigcup(M \setminus M')$ , hence also some  $I \in M \setminus M'$ . That would contradict the definition of  $M'$ . Therefore  $M'$  is infinite.

Since  $M'$  is infinite, we use the assumption again, and so  $cl(\bigcup M') = cl(\bigcup M)$ . We now use the fact that  $V$ , not just  $U$ , intersects  $cl(\bigcup M)$ . Then  $V$  must also intersect  $cl(\bigcup M')$ , hence  $\bigcup M'$ , and therefore it must intersect some  $I \in M'$ . By definition of  $M'$ ,  $U$  also intersects  $I$ . Since  $I$  is irreducible closed, it must therefore intersect  $U \cap V$ . Since  $I$  is included in  $cl(\bigcup M)$ , the latter must also intersect  $U \cap V$ .  $\square$

### 3.1 Proving the hard implication (2) $\Rightarrow$ (1)

We prove the contrapositive, namely: we assume that there is a closed subset  $C$  of  $X$  that cannot be written as a finite union of irreducible closed subsets, and we shall build an infinite discrete subspace of  $X$ .

**Lemma 3.3** (*Assuming that  $C$  is closed in  $X$ .*) *Every irreducible closed subset of  $X$  included in  $C$  is contained in some maximal irreducible closed subset of  $X$  included in  $C$ .*

*Proof.* Recall that  $\mathcal{S}(X)$  is sober, and that every sober space is a dcpo in its specialization ordering [5, Proposition 8.2.34]. A dcpo is a poset in which every directed subset has a supremum. The supremum of a directed family  $(I_k)_{k \in K}$  of irreducible closed subsets is the closure of their union. To show that, it suffices to show that  $cl(\bigcup_{k \in K} I_k)$  is irreducible: if it intersects two open subsets  $U$  and  $V$ , then  $U$  intersects some  $I_k$ ,  $V$  intersects some  $I_{k'}$ , and by directedness we may assume  $k = k'$ ; then  $I_k$  intersects both  $U$  and  $V$ , hence  $U \cap V$  since  $I_k$  is irreducible; this implies that the larger set  $cl(\bigcup_{k \in K} I_k)$  also intersects  $U \cap V$ .

In particular, the supremum of a directed family of irreducible closed subsets included in  $C$  is again included in  $C$ . It follows that the poset of all irreducible closed subsets of  $X$  included in  $C$  is a dcpo, in particular an inductive poset. We apply Zorn's Lemma and conclude.  $\square$

Alternatively, one may show that the irreducible closed subsets of  $X$  included in  $C$  are exactly the irreducible closed subsets of the subspace  $C$ . (For that, the fact that  $C$  is closed is important.) We obtain a slightly different proof by reasoning in  $\mathcal{S}(C)$ , also a dcpo, hence also an inductive poset, instead of  $\mathcal{S}(X)$ .

We now have everything to produce the argument, imitating the proof of [7, Proposition 4].

Let  $M_0$  be the collection of all maximal irreducible closed subsets of  $X$  included in  $C$ . By Lemma 3.3, since  $\downarrow x$  is irreducible closed in  $X$  for every

$x \in C$ , and also included in  $C$ , we obtain that  $x$  is a member of some maximal irreducible closed subset of  $X$  included in  $C$ . Therefore  $C = \bigcup M_0$ , in particular,  $C = cl(\bigcup M_0)$ .

Since  $C$  is not a finite union of irreducible closed subsets, it is in particular not irreducible. The contrapositive of Lemma 3.2 then implies the existence of an infinite subset  $M_1$  of  $M_0$  such that  $cl(\bigcup M_1)$  is strictly included in  $cl(\bigcup M_0)$ .

If  $cl(\bigcup M_1)$  were an irreducible closed subset, and recalling that it is included in  $C$ , it would be included in some maximal irreducible closed subset of  $X$  included in  $C$  by Lemma 3.3, namely in some member of  $M_0$ . In particular,  $\bigcup M_1$  would be included in some member of  $M_0$ . Since all elements of  $M_0$  are maximal, hence pairwise incomparable, that would imply that  $M_1$  contains only one element. That is impossible, since  $M_1$  is infinite.

Hence we can reapply Lemma 3.2: there is an infinite subset  $M_2$  of  $M_1$  such that  $cl(\bigcup M_2)$  is strictly included in  $cl(\bigcup M_1)$ .

Again  $cl(\bigcup M_2)$  cannot be irreducible, hence we can find an infinite subset  $M_3$  of  $M_2$  such that  $cl(\bigcup M_3)$  is strictly included in  $cl(\bigcup M_2)$ . Let us proceed infinitely that way. To make that clear, we have an infinite, antitone sequence of infinite subsets  $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$  such that  $cl(\bigcup M_0) \supset cl(\bigcup M_1) \supset cl(\bigcup M_2) \supset \dots \supset cl(\bigcup M_n) \supset \dots$  (all inclusions here are strict; from which we can conclude that the inclusions  $M_0 \supseteq M_1 \supseteq M_2 \supseteq \dots \supseteq M_n \supseteq \dots$  are strict as well).

We can therefore pick an irreducible closed subset  $I_n$  in  $M_n$  that is not contained in  $cl(\bigcup M_{n+1})$ , for each natural number  $n$ . Indeed, if every element  $I$  of  $M_n$  were in  $cl(\bigcup M_{n+1})$ , then  $\bigcup M_n$  would be included in  $cl(\bigcup M_{n+1})$ , so  $cl(\bigcup M_n)$  would be included in  $cl(\bigcup M_{n+1})$  as well, contradicting  $cl(\bigcup M_n) \supset cl(\bigcup M_{n+1})$ .

We now claim that  $I_n$  cannot be included in  $cl(\bigcup \{I_m \mid m \neq n\})$ . Assume the contrary. Since for every  $m > n$ ,  $I_m$  is in  $M_m \subseteq M_{n+1}$ ,  $I_n$  would be included in  $cl(I_0 \cup I_1 \cup I_2 \cup \dots \cup I_{n-1} \cup (\bigcup M_{n+1}))$ . However, closure commutes with finite unions—this is where we require a *topological* closure operator, and where a mere closure operator  $\varphi$  as in [8] will not suffice. Therefore  $I_n$  would be included in  $cl(I_0) \cup cl(I_1) \cup cl(I_2) \cup \dots \cup cl(I_{n-1}) \cup cl(\bigcup M_{n+1})$ , that is, in  $I_0 \cup I_1 \cup I_2 \cup \dots \cup I_{n-1} \cup cl(\bigcup M_{n+1})$ . Since  $I_n$  is irreducible, it must be included in one term of the union. It cannot be included in  $cl(\bigcup M_{n+1})$ , by definition. And it cannot be included in any  $I_k$  with  $k < n$ , since all those irreducible closed subsets, being maximal, are pairwise incomparable.

Finally, since  $I_n$  is not included in  $cl(\bigcup \{I_m \mid m \neq n\})$ , we can pick a point  $x_n$  of  $I_n$  that is not in  $cl(\bigcup \{I_m \mid m \neq n\})$ , for each  $n \in \mathbb{N}$ . Since  $x_m$  is in  $I_m$ , for every  $m \in \mathbb{N}$ ,  $cl(\bigcup \{I_m \mid m \neq n\})$  contains  $cl\{x_m \mid m \neq n\}$ . It follows that  $x_n$  is not in  $cl\{x_m \mid m \neq n\}$ . Lemma 3.1 states that the family  $\{x_n \mid n \in \mathbb{N}\}$  is a (necessarily infinite) discrete subspace.

**Remark 3.4** *The implication (2)  $\Rightarrow$  (1) can also be proved by using a result of Milner and Pouzet [8]. Stating it bluntly would not be helpful. Instead, we draw a few consequences of (2), walking quietly towards their result. Let  $C$  be a closed subset of  $X$ . By (2),  $C$  is a union of finitely many irreducible closed*



subsets  $I_1, \dots, I_n$ , and choosing them maximal in  $C$ , one may assume that they are pairwise incomparable. Let  $A_i = I_i \setminus \bigcup_{j \neq i} I_j$  (a non-empty set), and  $N_i$  be the family of all subsets  $E$  of  $A_i$  whose closure is different from  $I_i$ .  $N_i$  contains the empty set, hence is non-empty, and is clearly downwards-closed. For every pair  $E, E' \in N_i$ ,  $E \cup E'$  is also in  $N_i$ :  $cl(E \cup E') = cl(E) \cup cl(E')$  cannot be equal to  $I_i$ , since  $I_i$  is irreducible. Finally,  $N_i$  is proper, meaning that  $A_i \notin N_i$ . Hence  $N_i$  is a proper ideal of  $\mathbb{P}(A_i)$ .

For every subset  $E$  of  $C$ , one can see that  $cl(E) = C$  if and only if no  $E \cap A_i$  belongs to  $N_i$ . In one direction, if no  $E \cap A_i$  is in  $N_i$ , then  $cl(E) = cl(\bigcup_{i=1}^n E \cap A_i) = \bigcup_{i=1}^n cl(E \cap A_i) = \bigcup_{i=1}^n I_i = C$ . In the other direction, assume  $cl(E) = C$  and  $E \cap A_i \in N_i$  for some  $i$ . Then  $cl(E \cap A_i) \neq I_i$ , hence  $cl(E \cap A_i)$  is a proper subset of  $I_i$ . Since  $I_i \subseteq C = cl(E) = cl(E \cap A_i) \cup cl(E \cap \bigcup_{j \neq i} A_j) \subseteq cl(E \cap A_i) \cup \bigcup_{j \neq i} I_j$ , by irreducibility we would obtain that either  $I_i \subseteq cl(E \cap A_i)$ , or  $I_i \subseteq \bigcup_{j \neq i} I_j$ . Both cases are impossible.

Hence, for  $\varphi$  equal to the closure operator  $cl$ , we have obtained that (2) implies: for each closed subset  $C$  of  $X$ , (\*) there are finitely many, pairwise disjoint subsets  $A_1, \dots, A_n$  and proper ideals  $N_i$  of  $\mathbb{P}(A_i)$ ,  $1 \leq i \leq n$ , such that for every subset  $E$  of  $\bigcup_{i=1}^n A_i$ ,  $\varphi(E) = C$  if and only if no  $E \cap A_i$  belongs to  $N_i$ . Theorem 1.2 of [8] states that, for a closure operator  $\varphi$  (not necessarily topological), (\*) is equivalent to the fact that  $C$  contains no infinite  $\varphi$ -independent set.  $\square$

### 3.2 Proving the easier implication (1) $\Rightarrow$ (2)

We assume an infinite discrete subspace  $A$  of  $X$ . We shall simply show that  $cl(A)$  cannot be written as a finite union of irreducible closed subsets of  $X$ . Assume one could write  $cl(A)$  as the finite union  $I_0 \cup I_1 \cup I_2 \cup \dots \cup I_{n-1}$ , where each  $I_k$  is irreducible closed in  $X$ .

For each  $k$ , if  $I_k$  contains some point  $x$  of  $A$ , then, writing  $cl(A)$  as  $cl(\{x\} \cup (A \setminus \{x\})) = \downarrow x \cup cl(A \setminus \{x\})$  (recall—for the second and final time—that closures commute with finite unions), and remembering that  $I_k$  is irreducible and included in  $cl(A)$ , we obtain that  $I_k$  is included in  $\downarrow x$  or in  $cl(A \setminus \{x\})$ . The latter is impossible if  $I_k$  contains  $x$ , since  $A$  is discrete: indeed, Lemma 3.1 states that  $x$  is not in  $cl(A \setminus \{x\})$ . Therefore  $I_k = \downarrow x$ .

That implies that each  $I_k$  can contain at most one point  $x$  from  $A$ . Explicitly, if it contained two distinct points  $x$  and  $y$  of  $A$ , then  $I_k$  would be equal to  $\downarrow x$ , and also to  $\downarrow y$ . If we had assumed  $X$  to be  $T_0$ , we could conclude  $x = y$ , but we haven't made that assumption, and we must therefore work slightly harder. Since  $A$  is discrete,  $x$  is not in  $cl(A \setminus \{x\})$ , hence neither in the smaller subset  $cl(\{y\}) = \downarrow y$ . That shows that  $x$  is not less than or equal to  $y$ . That contradicts  $\downarrow x = \downarrow y$ .

We now have infinitely many points in  $A$ , hence in  $cl(A) = I_0 \cup I_1 \cup I_2 \cup \dots \cup I_{n-1}$ , but each  $I_k$  can contain at most one point from  $A$ . That is impossible, by the pigeonhole principle.

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## References

- [1] Robert Bonnet. On the cardinality of the set of initial intervals of a partially ordered set. In *Infinite and finite sets: to Paul Erdős on his 60th birthday*, pages 189–198. North-Holland, 1975.
- [2] Paul Erdős and Alfred Tarski. On families of mutually exclusive sets. *Annals of Mathematics*, 2(44):315–329, 1943.
- [3] Roland Fraïssé. Theory of relations. *Studies in Logic and the Foundations of Mathematics*, 118:1–456, 1986.
- [4] Jean Goubault-Larrecq. On Noetherian spaces. In *Proceedings of the 22nd Annual IEEE Symposium on Logic in Computer Science (LICS'07)*, pages 453–462, Wrocław, Poland, July 2007. IEEE Computer Society Press.
- [5] Jean Goubault-Larrecq. *Non-Hausdorff Topology and Domain Theory, Selected Topics in Point-Set Topology*, volume 22 of *New Mathematical Monographs*. Cambridge University Press, 2013.
- [6] Rudolf-Eberhart Hoffmann. Sobrification of partially ordered sets. *Semigroup Forum*, 17:123–138, 1979.
- [7] Jimmie D. Lawson, Michael Mislove, and Hilary Priestley. Ordered sets with no infinite antichains. *Discrete Mathematics*, 63(2):225–230, 1987.
- [8] Eric Charles Milner and Maurice Pouzet. A decomposition theorem for closure systems having no infinite independent set. *Combinatorics, Paul Erdős is Eighty*, 1:277–299, 1993.
- [9] Maurice Pouzet. Relations non reconstructibles par leurs restrictions. *Journal of Combinatorial Theory, Series B*, 26(1):22–34, 1979.
- [10] Maurice Pouzet and Nejib Zaguia. Dimension de Krull des ensembles ordonnés. *Discrete Mathematics*, 53:173–192, 1985.
- [11] Arthur Harold Stone. Hereditarily compact spaces. *American Journal of Mathematics*, 82(4):900–916, 1960.