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(Co)Feynman transform and cohomological field theories.-I

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Introduction

I describe a construction of cohomology classes in $H^*(\bar{\mathcal{M}}_{g,n})$ from A_∞ -algebra with A_∞ -scalar product. The construction is analogous to the construction from [B3]. The constructed cohomology classes define a cohomological field theory.

0.1 Notations.

Throughout the paper A denotes an A_∞ -algebra over an algebraically closed field k , $\dim_k A < \infty$ unless specified otherwise. We denote via $C^*(A, B)$ the Hochschild cochain complex of the A_∞ -bimodule B , and by $A^\vee = \text{Hom}_k(A, k)$ the dual A_∞ -bimodule. For two A_∞ -bimodules B and B' we denote via $\text{Hom}_{A-A}(B, B')$ the chain complex of (pre)morphisms $\text{Hom}_k^n(T\bar{A}[1] \otimes B \otimes T\bar{A}[1], B')$ with the differential $\hat{f} \rightarrow d_{\mu_{B'}} \circ \hat{f} \pm \hat{f} \circ d_{\mu_B}$.

1 Propagator

1.1 Quasi-isomorphisms of $\text{Hom}_{A-A}(A, A^\vee)$ and the Hochschild cochains $C^*(A, A^\vee)$.

The two complexes $\text{Hom}_{A-A}(A, A^\vee)$ and $(C^*(A, A^\vee), b)$ are quasi-isomorphic. A quasi-isomorphism

$$\begin{aligned} \Phi : \text{Hom}_{A-A}(A, A^\vee) &\rightarrow C^*(A, A^\vee) \\ (\Phi f)(\mathbf{a}, a_1, \dots, a_s) &= \sum_{r=0}^s (-1)^\varepsilon f^{s-r|1|^r}(a_{r+1}, \dots, a_s, \mathbf{a}, a_1, \dots, a_r)(\mathbf{e}) \end{aligned}$$

is induced by the quasi-isomorphism of bimodules $A \rightarrow A \otimes_{A-A} A$ with components

$$\varphi^{0|1|^r} : (\mathbf{a}, a_1, \dots, a_r) \rightarrow (\mathbf{a}, a_1, \dots, a_r, \mathbf{e}).$$

An inverse quasi-isomorphism

$$\Theta : C^*(A, A^\vee) \rightarrow \text{Hom}_{A-A}(A, A^\vee)$$

$$(\Theta \gamma)^{s|1|^r}(a_s, \dots, a_1, \mathbf{a}, a'_1, \dots, a'_r)(\mathbf{a}') = \sum_{s \geq s' \geq \bar{s} \geq 0} (-1)^\varepsilon \gamma(\mu_A^{\bar{s}|1|^r+1+s-s'}(a_{\bar{s}}, \dots, a_1, \mathbf{a}, a'_1, \dots, a'_r, a', a_s, \dots, a_{s'+1}), a_{s'}, \dots, a_{\bar{s}+1})$$

is induced by the quasi-isomorphism of bimodules

$$\begin{aligned} A \otimes_{A-A} A &\rightarrow A \\ \theta^{r|1|^r} : (a_1, \dots, a_r, \mathbf{a}, a'_1, \dots, a'_r, \mathbf{a}', a''_1, \dots, a''_{r''}) &\rightarrow (-1)^\varepsilon \mu_A^{r|1|^r'+1+r''}(a_1, \dots, a_r, \mathbf{a}, a'_1, \dots, a'_r, a', a''_1, \dots, a''_{r''}) \end{aligned}$$

1.2 Scalar product.

Let

$$\begin{aligned}\Psi &\in \text{Hom}_k^n(T\bar{A}[1], A^\vee), \\ b\Psi &= 0, \\ B\Psi &= 0\end{aligned}$$

denotes a cochain inducing an isomorphism of the A_∞ -bimodules

$$\begin{aligned}\Theta(\Psi)^{*|1|^*} &\in \text{Hom}_k^n(T\bar{A}[1] \otimes \mathbf{A} \otimes T\bar{A}[1], \mathbf{A}^\vee) \\ \Theta(\Psi) &: A \xrightarrow{\cong} A^\vee\end{aligned}$$

The morphism $\Theta(\Psi)$ is an isomorphism when the component

$$\Theta(\Psi)^{0|1|^0} \in \text{Hom}_k^n(\mathbf{A}, \mathbf{A}^\vee)$$

is an isomorphism. The isomorphism $\eta = \Theta(\Psi)$ is invertible:

$$\begin{aligned}\eta^{-1} &: A^\vee \xrightarrow{\cong} A \\ (\eta^{-1})^{*|1|^*} &\in \text{Hom}_k^{-n}(T\bar{A}[1] \otimes \mathbf{A}^\vee \otimes T\bar{A}[1], \mathbf{A})\end{aligned}$$

Without loss of generality the isomorphism η can be assumed to be self-dual:

$$\eta = \eta^\vee.$$

Proposition 1. *Let $\eta \in \text{Hom}_{A-A}(A, A^\vee) = \text{Hom}_{A-A}(A \otimes_{A-A} A, k)$, such that $d_{DR}\eta = 0$. Then $\eta = d_{DR}\psi_1$, $\psi_1 \in C^*(A, A^\vee)$ and*

$$\Phi(\eta) = B\psi_1.$$

Proof. The composition Φd_{DR} coincides with B □

1.3 Propagator.

Proposition 2. *The degeneration of the Hochschild to cyclic spectral sequence is equivalent to the existence of operators*

$$I_k : C^*(A, A^\vee) \rightarrow C^*(A, A^\vee)[-2k], \quad k \geq 1$$

such that

$$(1 + uI_1 + u^2I_2 + \dots)(b + uB) = b(1 + uI_1 + u^2I_2 + \dots)$$

$$B = [b, I_1]$$

$$I_1B = [b, I_2] \tag{1.1}$$

$$I_2B = [b, I_3] \dots$$

Proof. The operators I_k can be constructed from the homotopy contraction operators for the differentials on the k -th page of the spectral sequence, see e.g. [BT]. □

Proposition 3. *If*

$$\Psi \in \ker b \cap \text{Im} B$$

then for some $\Psi_i \in \text{Hom}_k^{n+2i-1}(T\bar{A}[1], A^\vee)$, $i \geq 1$,

$$\Psi = B\Psi_1 \tag{1.2}$$

and

$$b\Psi_i + B\Psi_{i+1} = 0 \quad (1.3)$$

for $i \geq 1$.

Proof. Since

$$\Psi \in \ker b \cap \text{Im} B$$

it follows that

$$\Psi = B\Psi_1$$

for some $\Psi_1 \in \text{Hom}_k^{n+1}(T\bar{A}[1], A^\vee)$. Then

$$b\Psi_1 \in \ker b \cap \ker B$$

since $Bb\Psi_1 = -b\Psi = 0$. Therefore there exist $\psi'_1 \in \ker B$ and Ψ_2 such that

$$b\Psi_1 + b\psi'_1 = -B\Psi_2$$

for some $\Psi_2 \in \text{Hom}_k^{n+3}(T\bar{A}[1], A^\vee)$, see e.g. [C], lemma 36. Replacing Ψ_1 by $\Psi_1 + \psi'_1$ we get Ψ_1, Ψ_2 satisfying (1.2), (1.3). Then

$$b\Psi_2 \in \ker(b) \cap \ker(B)$$

and there exist $\psi'_2 \in \ker B$ and Ψ_3 such that

$$b\Psi_2 + b\psi'_2 = -B\Psi_3$$

Replacing Ψ_2 by $\Psi_2 + \psi'_2$ we get Ψ_1, Ψ_2, Ψ_3 satisfying (1.2), (1.3) and so on. \square

Proposition 4. *Let*

$$\begin{aligned} \rho &= \eta^{-1} \circ \sum_{i=1}^{\infty} \Theta(I_i \Psi_i) \circ \eta^{-1} \\ \rho &\in (\text{pre})\text{Hom}_{A-A}(A^\vee, A). \end{aligned} \quad (1.4)$$

Then

$$d_{\mu_A} \circ \rho + (-1)^n \rho \circ d_{\mu_{A^\vee}} = \eta^{-1}$$

Proof. It follows from eqs.(1.1),(1.2) and (1.3) that

$$\Theta(bI_1\Psi_1) = \Theta([b, I_1]\Psi_1) + \Theta(I_1 b\Psi_1) = \Theta(B\Psi_1) - \Theta(I_1 B\Psi_2) = \Theta(\Psi) - \Theta(I_1 B\Psi_2)$$

and

$$\Theta(bI_i\Psi_i) = \Theta([b, I_i]\Psi_i) + \Theta(I_i b\Psi_i) = \Theta(I_{i-1} B\Psi_i) - \Theta(I_i B\Psi_{i+1})$$

for $i \geq 2$. Since

$$\eta = \Theta(\Psi)$$

and

$$d_{\mu_{A^\vee}} \circ \Theta(x) - (-1)^{\bar{x}} \Theta(x) \circ d_{\mu_A} = \Theta(bx)$$

and also

$$d_{\mu_A} \circ \eta^{-1} + (-1)^{n+1} \eta^{-1} \circ d_{\mu_{A^\vee}} = 0,$$

it follows that

$$d_{\mu_A} \circ \rho + (-1)^n \rho \circ d_{\mu_{A^\vee}} = \eta^{-1}$$

\square

2 Vertices

2.1 Tensors from $\text{Hom}_{A-A}(\underbrace{A \otimes_{A-A} A \otimes \dots \otimes_{A-A} A}_r, A)$

The tensors from the chain complexes $\text{Hom}_k^n(T\bar{A}[1] \otimes \mathbf{B} \otimes T\bar{A}[1], B')$ representing the (pre)morphisms

Definition 5. A_∞ -bimodule (pre-)morphisms

$$M_r \in \text{Hom}_{A-A}(\underbrace{A \otimes_{A-A} A \otimes \dots \otimes_{A-A} A}_r, A) \quad r \geq 1$$

$$M_r^{k_1|1|k_{r+1}} : (a_1^1, \dots, a_{k_1}^1, \mathbf{a}^1, \dots, a_{k_r}^r, \mathbf{a}^r, a_1^{r+1}, \dots, a_{k_{r+1}}^{r+1}) \rightarrow (-1)^\varepsilon \mu_A^{r+\sum k_j} (a_1^1, \dots, a_{k_1}^1, a^1, \dots, a_{k_r}^r, a^r, a_1^{r+1}, \dots, a_{k_{r+1}}^{r+1})$$

for $r \geq 2$ and M_1 act as the differential of these complexes.

Proposition 6. The operations M_r satisfy the A_∞ -relations

$$\sum_{r'+r''=r} M_{r'} \circ M_{r''} = 0$$

2.2 Cyclically invariant morphisms

Definition 7. The bimodule morphisms

$$\tilde{M}_r \in \text{Hom}_{A-A}(\underbrace{A \otimes_{A-A} A \otimes \dots \otimes_{A-A} A}_{r+1}, k)$$

are defined as compositions

$$\eta \circ M_r \in \text{Hom}_{A-A}(\underbrace{A \otimes_{A-A} A \otimes \dots \otimes_{A-A} A}_{r+1}, A^\vee)$$

It follows from the compatibility of η with the A_∞ -algebra that these tensors are invariant under natural $\mathbb{Z}/(r+1)\mathbb{Z}$ action

Definition 8. Using the tensor η^{-1} the cyclically invariant tensors from the various spaces $\left(\text{Hom}_{A-A}(\underbrace{A \otimes_{A-A} A \otimes \dots \otimes_{A-A} A}_r, k) \right)_{\mathbb{Z}/r\mathbb{Z}}$ can be naturally composed. This defines the (twisted) modular operad $\text{End}_{A-A}^{\text{multi-cyclic}}$.

This operad is an analogue of the twisted modular operad of endomorphisms considered in [B1].

Proposition 9. The operations \tilde{M}_r define the “algebra” structure over the Feynman transform $\mathcal{F}(\text{Ass})$ on $\text{End}_{A-A}^{\text{multi-cyclic}}$.

3 Cohomology classes.

Theorem 10. Assigning compositions of tensors to vertices, the Hochschild chains to external legs and the propagator (1.4) to edges gives naturally a cocycle in generalized stable ribbon graph complex. These cocycles define naturally a cohomological field theory.

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