HAL open science

(Co)Feynman transform and cohomological field theories-1

Serguei Barannikov

▶ To cite this version:

Serguei Barannikov. (Co)Feynman transform and cohomological field theories-1. 2018. hal-01804639

HAL Id: hal-01804639
https://hal.science/hal-01804639

Preprint submitted on 31 May 2018

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

(Co)Feynman transform and cohomological field theories.-I

Serguei Barannikov

January, 2018

UMR7586 CNRS (Paris), NRU HSE (Moscow)

Introduction

I describe a construction of cohomology classes in $H^*(\bar{\mathcal{M}}_{g,n})$ from $A_\infty-$ algebra with $A_\infty-$ scalar product. The construction is analogous to the construction from [B3]. The constructed cohomology classes define a cohomological field theory.

0.1 Notations.

Throughout the paper A denotes an $A_\infty-$ algebra over an algebraically closed field k, $\dim_k A < \infty$ unless specified otherwise. We denote via $C^*(A,B)$ the Hochshild cochain complex of the $A_\infty-$ bimodule B, and by $A^\vee = \mathrm{Hom}_k(A,k)$ the dual $A_\infty-$ bimodule. For two $A_\infty-$ bimodules B and B' we denote via $\mathrm{Hom}_{A-A}(B,B')$ the chain complex of (pre)morhisms $\mathrm{Hom}^n_k(T\bar{A}\,[1]\otimes \boldsymbol{B}\otimes T\bar{A}\,[1]\,,B')$ with the differential $\hat{f}\to d_{\mu_{B'}}\circ\hat{f}\pm\hat{f}\circ d_{\mu_B}$.

1 Propagator

1.1 Quasi-isomoprhisms of $\mathrm{Hom}_{A-A}(A,A^\vee)$ and the Hochschild cochains $C^*(A,A^\vee)$.

The two complexes $\mathrm{Hom}_{A-A}(A,A^\vee)$ and $(C^*(A,A^\vee,b)$ are quasi-isomorphic. A quasi-isomorphism

$$\Phi:\mathrm{Hom}_{A-A}(A,A^\vee)\to C^*(A,A^\vee)$$

$$(\Phi f)(\boldsymbol{a},a_1,\dots,a_s)=\sum_{r=0}^{s}(-1)^\varepsilon f^{s-r|1|r}(a_{r+1},\dots,a_s,\boldsymbol{a},a_1,\dots,a_r)(\boldsymbol{e})$$

is induced by the quasi-isomorpshim of bimodules $A\to A\otimes_{A-A}A$ with components

$$\varphi^{0|1|r}:(\boldsymbol{a},a_1,\dots,a_r)\to(\boldsymbol{a},a_1,\dots,a_r,\boldsymbol{e}).$$

An inverse quasi-isomorphism

$$\Theta:C^*(A,A^\vee)\to\mathrm{Hom}_{A-A}(A,A^\vee)$$

$$(\Theta\gamma)^{s|1|r}(a_s,\dots,a_1,\boldsymbol{a},a'_1,\dots,a'_r)(\boldsymbol{a'})=\sum_{s\geq s'\geq\bar{s}\geq 0}(-1)^\varepsilon\gamma(\boldsymbol{\mu}_A^{\bar{s}|1|r+1+s-s'}(a_{\tilde{s}},\dots,a_1,\boldsymbol{a},a'_1,\dots,a'_r,a',a_s,\dots,a_{s'+1}),a_{s'},\dots a_{\tilde{s}+1})$$

is induced by the quasi-isomorphism of bimodules

$$A\otimes_{A-A}A\to A$$

$$\theta^{r|1|r''}:(a_1,\dots,a_r,\boldsymbol{a},a'_1,\dots,a'_{r'},\boldsymbol{a'},a''_1,\dots,a''_{r''})\to(-1)^\varepsilon\mu_A^{r|1|r'+1+r''}(a_1,\dots,a_r,\boldsymbol{a},a'_1,\dots,a'_{r'},a',a''_1,\dots,a''_{r''})$$

1.2 Scalar product.

Let

$$\Psi \in \mathrm{Hom}_k^n(T\bar{A}\,[1]\,, A^\vee),$$
$$b\Psi = 0,$$
$$B\Psi = 0$$

denotes a cochain inducing an isomorphism of the $A_\infty-$ bimodules

$$\Theta(\Psi)^{*|1|*} \in \mathrm{Hom}_k^n(T\bar{A}\,[1] \otimes \boldsymbol{A} \otimes T\bar{A}\,[1]\,, \boldsymbol{A}^\vee)$$
$$\Theta(\Psi):\ A \xrightarrow{\simeq} A^\vee$$

The morphism $\Theta(\Psi)$ is an isomorphism when the component

$$\Theta(\Psi)^{0|1|0} \in \mathrm{Hom}_k^n(\boldsymbol{A}, \boldsymbol{A}^\vee)$$

is an isomorphism. The isomorphism $\eta = \Theta(\Psi)$ is invertible:

$$\eta^{-1}:\ A^\vee \xrightarrow{\simeq} A$$
$$(\eta^{-1})^{*|1|*} \in \mathrm{Hom}_k^{-n}(T\bar{A}\,[1] \otimes \boldsymbol{A}^\vee \otimes T\bar{A}\,[1]\,, \boldsymbol{A})$$

Without loss of generality the isomoprhism η can be assumed to be self-dual:

$$\eta = \eta^\vee.$$

Proposition 1. *Let* $\eta \in Hom_{A-A}(A, A^\vee) = Hom_{A-A}(A \otimes_{A-A} A, k)$ *, such that* $d_{DR}\eta = 0$ *. Then* $\eta = d_{DR}\psi_1$ *,* $\psi_1 \in C^*(A, A^\vee)$ *and*

$$\Phi(\eta) = B\psi_1.$$

Proof. The composition Φd_{DR} coincides with B □

1.3 Propagator.

Proposition 2. *The degeneration of the Hochshild to cyclic spectral sequence is equivalent to the existence of operators*

$$I_k : C^*(A, A^\vee) \to C^*(A, A^\vee)\,[-2k]\,,\ k \geq 1$$

such that

$$(1 + uI_1 + u^2 I_2 + \ldots)(b + uB) = b(1 + uI_1 + u^2 I_2 + \ldots)$$

$$\begin{aligned} B &= [b, I_1] \\ I_1 B &= [b, I_2] \\ I_2 B &= [b, I_3] \ldots \end{aligned} \tag{1.1}$$

Proof. The operators I_k can be constructed from the homotopy contraction operators for the differentials on the k-th page of the spectral sequence, see e.g. [BT]. □

Proposition 3. *If*

$$\Psi \in \ker b \cap ImB$$

then for some $\Psi_i \in Hom_k^{n+2i-1}(T\bar{A}\,[1]\,, A^\vee)$ *,* $i \geq 1$ *,*

$$\Psi = B\Psi_1 \tag{1.2}$$

and

$$b\Psi_i + B\Psi_{i+1} = 0 \tag{1.3}$$

for $i \geq 1$.

Proof. Since

$$\Psi \in \ker b \cap \mathrm{Im} B$$

it follows that

$$\Psi = B\Psi_1$$

for some $\Psi_1 \in \mathrm{Hom}_k^{n+1}(T\bar{A}\,[1]\,, A^\vee)$. Then

$$b\Psi_1 \in \ker b \cap \ker B$$

since $Bb\Psi_1 = -b\Psi = 0$. Therefore there exist $\psi_1' \in \ker B$ and Ψ_2 such that

$$b\Psi_1 + b\psi_1' = -B\Psi_2$$

for some $\Psi_2 \in \mathrm{Hom}_k^{n+3}(T\bar{A}\,[1]\,, A^\vee)$, see e.g. [C], lemma 36. Replacing Ψ_1 by $\Psi_1 + \psi_1'$ we get Ψ_1 , Ψ_2 satisfying (1.2), (1.3). Then

$$b\Psi_2 \in \ker(b) \cap \ker(B)$$

and there exist $\psi_2' \in \ker B$ and Ψ_3 such that

$$b\Psi_2 + b\psi_2' = -B\Psi_3$$

Replacing Ψ_2 by $\Psi_2 + \psi_2'$ we get Ψ_1 , Ψ_2 , Ψ_3 satisfying (1.2), (1.3) and so on. □

Proposition 4. *Let*

$$\rho = \eta^{-1} \circ \sum_{i=1}^{\infty} \Theta(I_i\Psi_i) \circ \eta^{-1}$$
$$\rho \in (pre)Hom_{A-A}(A^\vee, A). \tag{1.4}$$

Then

$$d_{\mu_A} \circ \rho + (-1)^n \rho \circ d_{\mu_{A^\vee}} = \eta^{-1}$$

Proof. It follows from eqs.(1.1),(1.2) and (1.3) that

$$\Theta(bI_1\Psi_1) = \Theta([b, I_1]\,\Psi_1) + \Theta(I_1 b\Psi_1) = \Theta(B\Psi_1) - \Theta(I_1 B\Psi_2) = \Theta(\Psi) - \Theta(I_1 B\Psi_2)$$

and

$$\Theta(bI_i\Psi_i) = \Theta([b, I_i]\,\Psi_i) + \Theta(I_i b\Psi_i) = \Theta(I_{i-1}B\Psi_i) - \Theta(I_i B\Psi_{i+1})$$

for $i \geq 2$. Since

$$\eta = \Theta(\Psi)$$

and

$$d_{\mu_{A^\vee}} \circ \Theta(x) - (-1)^{\bar{x}}\Theta(x) \circ d_{\mu_A} = \Theta(bx)$$

and also

$$d_{\mu_A} \circ \eta^{-1} + (-1)^{n+1}\eta^{-1} \circ d_{\mu_{A^\vee}} = 0,$$

it follows that

$$d_{\mu_A} \circ \rho + (-1)^n \rho \circ d_{\mu_{A^\vee}} = \eta^{-1}$$

□

2 Vertices

2.1 Tensors from $\mathbf{Hom}_{A-A}(\underbrace{A\otimes_{A-A}A\otimes\ldots\otimes_{A-A}A}_{r},A)$

The tensors from the chain complexes $\mathrm{Hom}^n_k(T\bar{A}\,[1]\otimes\boldsymbol{B}\otimes T\bar{A}\,[1]\,,B')$ representing the (pre)morhisms

Definition 5. $A_\infty-$ bimodule (pre-)morphisms

$$M_r\in\mathrm{Hom}_{A-A}(\underbrace{A\otimes_{A-A}A\otimes\ldots\otimes_{A-A}A}_{r},A)\ r\geq 1$$

$$M_r^{k_1|1|k_{r+1}}:(a_1^1,\ldots,a_{k_1}^1,\boldsymbol{a^1},\ldots,a_{k_r}^r,\boldsymbol{a^r},a_1^{r+1},\ldots,a_{k_{r+1}}^{r+1})\to(-1)^\varepsilon\mu_A^{r+\sum k_j}(a_1^1,\ldots,a_{k_1}^1,a^1,\ldots,a_{k_r}^r,a^r,a_1^{r+1},\ldots,a_{k_{r+1}}^{r+1})$$

for $r\geq 2$ and M_1 act as the differential of these complexes.

Proposition 6. *The operations M_r satisfy the $A_\infty-$ relations*

$$\sum_{r'+r''=r}M_{r'}\circ M_{r''}=0$$

2.2 Cyclically invariant morphisms

Definition 7. The bimodule morphisms

$$\tilde{M}_r\in\mathrm{Hom}_{A-A}(\underbrace{A\otimes_{A-A}A\otimes\ldots\otimes_{A-A}A}_{r+1},k)$$

are defined as compositions

$$\eta\circ M_r\in\mathrm{Hom}_{A-A}(\underbrace{A\otimes_{A-A}A\otimes\ldots\otimes_{A-A}A}_{r+1},A^\vee)$$

It follows from the compatibility of η with the $A_\infty-$ algebra that these tensors are invariant under natural $\mathbb{Z}/(r+1)\mathbb{Z}$ action

Definition 8. Using the tensor $\eta^{-1}_{\mathbb{Z}/r\mathbb{Z}}$ the cyclically invariant tensors from the various spaces $\left(\mathrm{Hom}_{A-A}(\underbrace{A\otimes_{A-A}A\otimes\ldots\otimes_{A-A}A}_{r},k)\right)$ can be naturally composed. This defines the (twisted) modular operad $\mathrm{End}^{\text{multi-cyclic}}_{A-A}$.

This operad is an analogue of the twisted modular operad of endomorphisms considered in [B1].

Proposition 9. *The operations $\tilde{M}_r$ define the "algebra" structure over the Feynman transform $\mathcal{F}(Ass)$ on $End^{multi\text{-}cyclic}_{A-A}$.*

3 Cohomology classes.

Theorem 10. *Assigning compositions of tensors to vertices, the Hochshild chains to external legs and the propagator (1.4) to edges gives naturally a cocycle in generalized stable ribbon graph complex. These cocycles define naturally a cohomological field theory.*

References

[B1] S.Barannikov, *Modular operads and Batalin-Vilkovisky geometry.* International Mathematics Research Notices, 2007, art. no. rnm075, doi:10.1093/imrn/rnm075

[B2] S.Barannikov, *Matrix De Rham complex and quantum A-infinity algebras.* Letters in Math. Phys., 2014, Volume 104, Issue 4, pp 373-395

[B3] S.Barannikov, *Supersymmetry and cohomology of graph complexes.* 2009 HAL

[BT] R.Bott, L.Tu, *Differential forms in algebraic topology.* 1982

[C] A.Connes, *Non-commutative differential geometry,* Publ. IHES

[T] Thomas Tradler, *Infinity-inner-products on A-infinity-algebras,* J. Homotopy Related Structures 3 (2008), no. 1, 245–271. MR2426181 (2010g:16016)