(Co)Feynman transform and cohomological field theories.-I

Serguei Barannikov

January, 2018

UMR7586 CNRS (Paris), NRU HSE (Moscow)

Introduction

I describe a construction of cohomology classes in $H^*(\bar{\mathcal{M}}_{g,n})$ from $A_\infty-$ algebra with $A_\infty-$ scalar product. The construction is analogous to the construction from [B3]. The constructed cohomology classes define a cohomological field theory.

0.1 Notations.

Throughout the paper $A$ denotes an $A_\infty-$ algebra over an algebraically closed field $k$, $\dim_k A < \infty$ unless specified otherwise. We denote via $C^*(A,B)$ the Hochshild cochain complex of the $A_\infty-$ bimodule $B$, and by $A^\vee = \text{Hom}_k(A,k)$ the dual $A_\infty-$ bimodule. For two $A_\infty-$ bimodules $B$ and $B'$ we denote via $\text{Hom}_{A\otimes A}(B,B')$ the chain complex of (pre)morhisms $\mu_k(\bar{T}A[1] \otimes B \otimes \bar{T}A[1],B')$ with the differential $\hat{f} \to d_{\mu_B} \circ \hat{f} \pm \hat{f} \circ d_{\mu_B}$.

1 Propagator

1.1 Quasi-isomorphisms of $\text{Hom}_{A\otimes A}(A,A^\vee)$ and the Hochschild cochains $C^*(A,A^\vee)$.

The two complexes $\text{Hom}_{A\otimes A}(A,A^\vee)$ and $(C^*(A,A^\vee),b)$ are quasi-isomorphic. A quasi-isomorphism

$$\Phi : \text{Hom}_{A\otimes A}(A,A^\vee) \to C^*(A,A^\vee)$$

$$(\Phi f)(a,a_1,\ldots,a_s) = \sum_{r=0}^s (-1)^{s-r+1} f^{s-r+1}(a_0,a_1,\ldots,a_s,a_1,\ldots,a_r)$$

is induced by the quasi-isomorphism of bimodules $A \to A \otimes_{A\otimes A} A$ with components

$$\varphi^{0,0}_{s,0} : (a_1,\ldots,a_s) \to (a_1,\ldots,a_s,e).$$

An inverse quasi-isomorphism

$$\Theta : C^*(A,A^\vee) \to \text{Hom}_{A\otimes A}(A,A^\vee)$$

$$(\Theta \gamma)^{s,0}(a_1,\ldots,a_s,a_1',\ldots,a_s') = \sum_{s' \geq s \geq 0} (-1)^{s'} \mu_A^{s,s'}(a_1,\ldots,a_s,a_1',\ldots,a_s')$$

is induced by the quasi-isomorphism of bimodules

$$A \otimes_{A\otimes A} A \to A$$

$$\theta^{s,0}_{s,s} : (a_1,\ldots,a_s,a_1',\ldots,a_s') \to (-1)^{s} \mu_A^{s,s}(a_1,\ldots,a_s,a_1',\ldots,a_s')$$
1.2 Scalar product.

Let

\[ \Psi \in \text{Hom}_k^n(T\tilde{A}[1], A^\vee), \]
\[ b\Psi = 0, \]
\[ B\Psi = 0 \]

denotes a cochain inducing an isomorphism of the \( A_\infty \)-bimodules

\[ \Theta(\Psi)^{[1]} : \text{Hom}_k^n(T\tilde{A}[1] \otimes A \otimes T\tilde{A}[1], A^\vee) \]
\[ \Theta(\Psi) : A \xrightarrow{\sim} A^\vee \]

The morphism \( \Theta(\Psi) \) is an isomorphism when the component

\[ \Theta(\Psi)^{[0]} \in \text{Hom}_k^0(A, A^\vee) \]

is an isomorphism. The isomorphism \( \eta = \Theta(\Psi) \) is invertible:

\[ \eta^{-1} : A^\vee \xrightarrow{\sim} A \]
\[ (\eta^{-1})^{[1]} \in \text{Hom}_k^n(T\tilde{A}[1] \otimes A^\vee \otimes T\tilde{A}[1], A) \]

Without loss of generality the isomorphism \( \eta \) can be assumed to be self-dual:

\[ \eta = \eta^\vee. \]

Proposition 1. Let \( \eta \in \text{Hom}_{A^\vee}(A, A^\vee) = \text{Hom}_{A^\vee}(A \otimes_{A^\vee} A, k) \), such that \( d_{DR}\eta = 0 \). Then \( \eta = d_{DR}\psi_1 \), \( \psi_1 \in C^*(A, A^\vee) \) and

\[ \Phi(\eta) = B\psi_1. \]

Proof. The composition \( \Phi d_{DR} \) coincides with \( B \)

1.3 Propagator.

Proposition 2. The degeneration of the Hochschild to cyclic spectral sequence is equivalent to the existence of operators

\[ I_k : C^*(A, A^\vee) \rightarrow C^*(A, A^\vee)[-2k], k \geq 1 \]

such that

\[ (1 + uI_1 + u^2I_2 + \ldots)(b + uB) = b(1 + uI_1 + u^2I_2 + \ldots) \]
\[ B = [b, I_1] \]
\[ I_1B = [b, I_2] \]
\[ I_2B = [b, I_3] \ldots \]

(1.1)

Proof. The operators \( I_k \) can be constructed from the homotopy contraction operators for the differentials on the \( k \)-th page of the spectral sequence, see e.g. [BT].

Proposition 3. If

\[ \Psi \in \ker b \cap \text{Im}B \]

then for some \( \Psi_i \in \text{Hom}_k^{n+2i-1}(T\tilde{A}[1], A^\vee), i \geq 1 \),

\[ \Psi = B\Psi_1 \]

(1.2)
and
\[ b\Psi_i + B\Psi_{i+1} = 0 \] (1.3)
for \( i \geq 1 \).

**Proof.** Since
\[ \Psi = B\Psi_1 \]
it follows that
\[ \Psi \in \ker b \cap \text{Im} B \]
for some \( \Psi_1 \in \text{Hom}^{n+1}(T\bar{A} [1], A^\vee) \). Then
\[ b\Psi_1 \in \ker b \cap \ker B \]
since \( Bb\Psi_1 = -b\Psi = 0 \). Therefore there exist \( \psi'_1 \in \ker B \) and \( \Psi_2 \) such that
\[ b\Psi_1 + b\psi'_1 = -B\Psi_2 \]
for some \( \Psi_2 \in \text{Hom}^{n+3}(T\bar{A} [1], A^\vee) \), see e.g. [C], lemma 36. Replacing \( \Psi_1 \) by \( \Psi_1 + \psi'_1 \) we get \( \Psi_1, \Psi_2 \) satisfying (1.2), (1.3). Then
\[ b\Psi_2 \in \ker(b) \cap \ker(B) \]
and there exist \( \psi'_2 \in \ker B \) and \( \Psi_3 \) such that
\[ b\Psi_2 + b\psi'_2 = -B\Psi_3 \]
Replacing \( \Psi_2 \) by \( \Psi_2 + \psi'_2 \) we get \( \Psi_1, \Psi_2, \Psi_3 \) satisfying (1.2), (1.3) and so on. \( \square \)

**Proposition 4.** Let
\[ \rho = \eta^{-1} \circ \sum_{i=1}^{\infty} \Theta(I_i\Psi_i) \circ \eta^{-1} \]
\[ \rho \in (\text{pre})\text{Hom}_{A-A}(A^\vee, A). \] (1.4)

Then
\[ d_{\mu_A} \circ \rho + (-1)^n \rho \circ d_{\mu_A^\vee} = \eta^{-1} \]

**Proof.** It follows from eqs.(1.1),(1.2) and (1.3) that
\[ \Theta(bI_1\Psi_1) = \Theta([b, I_1]\Psi_1) + \Theta(I_1b\Psi_1) = \Theta(B\Psi_1) - \Theta(I_1B\Psi_2) = \Theta(\Psi) - \Theta(I_1B\Psi_2) \]
and
\[ \Theta(bI_i\Psi_i) = \Theta([b, I_i]\Psi_i) + \Theta(I_ib\Psi_i) = \Theta(I_{i-1}B\Psi_i) - \Theta(I_iB\Psi_{i+1}) \]
for \( i \geq 2 \). Since
\[ \eta = \Theta(\Psi) \]
and
\[ d_{\mu_A^\vee} \circ \Theta(x) - (-1)^{\overline{i}} \Theta(x) \circ d_{\mu_A} = \Theta(bx) \]
and also
\[ d_{\mu_A} \circ \eta^{-1} + (-1)^n \eta^{-1} \circ d_{\mu_A^\vee} = 0, \]
it follows that
\[ d_{\mu_A} \circ \rho + (-1)^n \rho \circ d_{\mu_A^\vee} = \eta^{-1} \]
\[ \square \]
2 Vertices

2.1 Tensors from $\text{Hom}_{\mathcal{A}}(A \otimes_{\mathcal{A}} A \otimes \ldots \otimes_{\mathcal{A}} A, A)$

The tensors from the chain complexes $\text{Hom}^r_{\mathcal{A}}(T \mathcal{A}[1] \otimes B \otimes T \mathcal{A}[1], B')$ representing the (pre)morphisms

Definition 5. $A_{\infty}$– bimodule (pre-)morphisms

$$M_r \in \text{Hom}_{\mathcal{A}}(A \otimes_{\mathcal{A}} A \otimes \ldots \otimes_{\mathcal{A}} A, A) \quad r \geq 1$$

$$M_r^{k_1 \ldots k_r} : (a_1^1, \ldots, a_1^r, a_1^{r+1}, \ldots, a_{k_r+1}^r) \rightarrow (-1)^r \mu_A^{r + \sum k_j}(a_1^1, \ldots, a_1^r, a_1^{r+1}, \ldots, a_{k_r+1}^r)$$

for $r \geq 2$ and $M_1$ act as the differential of these complexes.

Proposition 6. The operations $M_r$ satisfy the $A_{\infty}$– relations

$$\sum_{r' + r'' = r} M_{r'} \circ M_{r''} = 0$$

2.2 Cyclically invariant morphisms

Definition 7. The bimodule morphisms

$$\tilde{M}_r \in \text{Hom}_{\mathcal{A}}(A \otimes_{\mathcal{A}} A \otimes \ldots \otimes_{\mathcal{A}} A, k)$$

are defined as compositions

$$\eta \circ \tilde{M}_r \in \text{Hom}_{\mathcal{A}}(A \otimes_{\mathcal{A}} A \otimes \ldots \otimes_{\mathcal{A}} A, \mathcal{A})$$

It follows from the compatibility of $\eta$ with the $A_{\infty}$– algebra that these tensors are invariant under natural $\mathbb{Z}/(r+1)\mathbb{Z}$ action

Definition 8. Using the tensor $\eta^{-1}$ the cyclically invariant tensors from the various spaces

$$\left(\text{Hom}_{\mathcal{A}}(A \otimes_{\mathcal{A}} A \otimes \ldots \otimes_{\mathcal{A}} A, k)\right)$$

can be naturally composed. This defines the (twisted) modular operad $\text{End}^\text{multi-cyclic}_{\mathcal{A}}$.

This operad is an analogue of the twisted modular operad of endomorphisms considered in [B1].

Proposition 9. The operations $\tilde{M}_r$ define the “algebra” structure over the Feynman transform $\mathcal{F}(\text{Ass})$ on $\text{End}^\text{multi-cyclic}_{\mathcal{A}}$.

3 Cohomology classes.

Theorem 10. Assigning compositions of tensors to vertices, the Hochshild chains to external legs and the propagator (1.4) to edges gives naturally a cocycle in generalized stable ribbon graph complex. These cocycles define naturally a cohomological field theory.

References


S. Barannikov, *Supersymmetry and cohomology of graph complexes*. 2009 HAL.


A. Connes, *Non-commutative differential geometry*, Publ. IHES.