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Characterization of generalised Petersen graphs that are Kronecker covers*

Matjaž Krnc[†] and Tomaž Pisanski[‡]

FAMNIT, University of Primorska, Slovenia

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Abstract

The family of generalised Petersen graphs $G(n, k)$, introduced by Coxeter et al. [4] and named by Mark Watkins (1969), is a family of cubic graphs formed by connecting the vertices of a regular polygon to the corresponding vertices of a star polygon. The Kronecker cover $KC(G)$ of a simple undirected graph G is a special type of bipartite covering graph of G , isomorphic to the direct (tensor) product of G and K_2 . We characterize all generalised Petersen graphs that are Kronecker covers, and describe the structure of their respective quotients. We observe that some of such quotients are again generalised Petersen graphs, and describe all such pairs.

Keywords— generalised Petersen graphs, Kronecker cover

MSC: 57M10, 05C10, 05C25

1 Introduction

The *generalised Petersen graphs*, introduced by Coxeter et al. [4] and named by Watkins [16], form a very interesting family of trivalent graphs that can be described by only two integer parameters. They include Hamiltonian and non-Hamiltonian graphs, bipartite and non-bipartite graphs, vertex-transitive and non-vertex-transitive graphs, Cayley and non-Cayley graphs, arc-transitive graphs and non-arc-transitive graphs, graphs of girth 3, 4, 5, 6, 7 or 8. Their generalization to I -graphs does not introduce any new vertex-transitive graphs but it contains also non-connected graphs and has in special cases unexpected symmetries [2]. For further properties of I -graphs also see [8, 12].

Following the notation of Watkins [16], for given integers n and $k < \frac{n}{2}$, we can define a generalised Petersen graph $G(n, k)$ as a graph on vertex-set $\{u_0, \dots, u_{n-1}, v_0, \dots, v_{n-1}\}$.

*Dedicated to Mark Watkins on the Occasion of his 80th Birthday.

[†]matjaz.krnc@upr.si

[‡]tomaz.pisanski@upr.si

The edge-set may be naturally partitioned into three equal parts (note that all subscripts are assumed modulo n): the edges $E_O(n, k) = \{u_i u_{i+1}\}_{i=0}^{n-1}$ form the *outer rim*, inducing a cycle of length n ; the edges $E_I(n, k) = \{v_i v_{i+k}\}_{i=0}^{n-1}$ form the *inner rims*, inducing $\gcd(n, k)$ cycles of length $\frac{n}{\gcd(n, k)}$; and the edges $E_S(n, k) = \{u_i v_i\}_{i=0}^{n-1}$, also called *spokes*, which induce a perfect matching in $G(n, k)$. Hence the edge-set may be defined as $E(G(n, k)) = E_O(n, k) \cup E_I(n, k) \cup E_S(n, k)$.

Various structural aspects of the mentioned family have been pointed out. Examples include identifying generalised Petersen graphs that are Hamiltonian [1] or Cayley [11, 14], or isomorphic [8, 12, 15], or finding their automorphism group [5]. Also, a related generalization to I -graphs has been introduced in the Foster census [3], and further studied by Boben et al. [2].

The theory of covering graphs became one of the most important and successful tools of algebraic graph theory. It is a discrete analog of the well known theory of covering spaces in algebraic topology. In general, covers depend on the values called voltages assigned to the edges of the graphs. Only in some cases the covering is determined by the graph itself. One of such cases is the recently studied *clone cover* [10]. The other, more widely known case is the Kronecker cover.

The *Kronecker cover* $KC(G)$ (also called bipartite or canonical double cover) of a simple undirected graph G is a bipartite covering graph with twice as many vertices as G . Formally, $KC(G)$ is defined as a tensor product $G \times K_2$, i.e. a graph on a vertex-set $V(KC(G)) = \{v', v''\}_{v \in V(G)}$, and the edge-set $E(KC(G)) = \{u'v'', u''v'\}_{uv \in E(G)}$. For $H = KC(G)$, we also say that G is a *quotient* of H . Some recent work on Kronecker covers includes Gévay and Pisanski [6] and Imrich and Pisanski [9].

In this paper, we study the family of generalised Petersen graphs in conjunction with the Kronecker cover operation. Namely, in the next section we state our main theorem characterizing all generalised Petersen graphs that are Kronecker covers, and describing the structure of their corresponding quotient graphs. In Section 3 we focus on the necessary and sufficient conditions for a generalised Petersen graph to be a Kronecker cover while in Section 4 we complement the existence results with the description of the structure of the corresponding quotient graphs. We conclude the paper with some remarks and directions for possible future research.

2 Main result

In order to state the main result we need to introduce the graph \mathcal{H} and two 2-parametric families of cubic, connected graphs.

Let \mathcal{H} be the graph defined by the following procedure: Take the Cartesian product $K_3 \square P_3$, remove the edges of the middle triangle, add a new vertex and connect it to all three 2-vertices. Note that the graph \mathcal{H} is mentioned in [9] and is depicted in Figure 1.

As shown in [9], the Desargues graph $G(10, 3)$ is the Kronecker cover of both $G(5, 2)$ and \mathcal{H} . Note that in Figure 1 the edge-colored subgraphs of $G(5, 2)$ and \mathcal{H} lift to the edge-colored subgraphs of $G(10, 3)$, respectively.

To describe the quotients of generalised Petersen graphs, we use the LCF notation, named by developers Lederberg, Coxeter and Frucht, for the representation of cubic hamiltonian

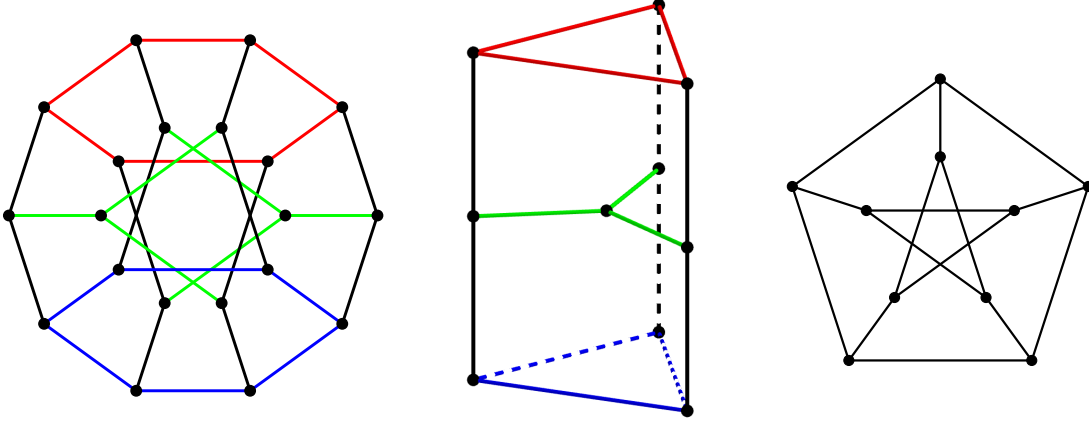


Figure 1: The Desargues graph and both its quotients; \mathcal{H} and the Petersen graph.

graphs (for extended description see [13]).

In a Hamiltonian cubic graph, the vertices can be arranged in a cycle, which accounts for two edges per vertex. The third edge from each vertex can then be described by how many positions clockwise (positive) or counter-clockwise (negative) it leads. The basic form of the LCF notation is just the sequence $[a_0, a_1, \dots, a_{n-1}]$ of numbers of positions, starting from an arbitrarily chosen vertex and written in square brackets.

To state our results, we only use a special type of such LCF-representable graphs, namely $C^+(n, k)$ and $C^-(n, k)$, which we define below.

Definition 1. Assuming all numbers are modulo n , define graphs

$$C^+(n, k) = \left[\frac{n}{2}, \frac{n}{2} + (k - 1), \frac{n}{2} + 2(k - 1), \dots, \frac{n}{2} + (n - 1)(k - 1) \right],$$

and similarly

$$C^-(n, k) = \left[\frac{n}{2}, \frac{n}{2} - (k + 1), \frac{n}{2} - 2(k + 1), \dots, \frac{n}{2} - (n - 1)(k + 1) \right].$$

Note that throughout the paper, any instance of the definition above is used with such values (n, k) , so that the corresponding graph is well defined. In [9] it was proven that $G(10, 3)$ is the Kronecker cover of two non-isomorphic graphs. Here we prove among other things that this is the only generalised Petersen graph that is a multiple Kronecker cover. Every other generalised Petersen graph is either a Kronecker cover of a single graph or it is not a Kronecker cover at all. More precisely;

Theorem 1. *Among the members of the family of generalised Petersen graphs, $G(10, 3)$ is the only graph that is the Kronecker cover of two non-isomorphic graphs, the Petersen graph and the graph \mathcal{H} . For any other $G \simeq G(n, k)$, the following holds:*

- a) *If $n \equiv 2 \pmod{4}$ and k is odd, G is a Kronecker cover. In particular*

- a₁) if $4k < n$, the corresponding quotient graph is $G\left(\frac{n}{2}, k\right)$, and
- a₂) if $n < 4k < 2n$ the quotient graph is $G\left(\frac{n}{2}, \frac{n}{2} - k\right)$.
- b) If $n \equiv 0 \pmod{4}$ and k is odd, G is a Kronecker cover if and only if $n \mid \frac{k^2-1}{2}$ and $k < \frac{n}{2}$. Moreover,
 - b₁) if $k = 4t + 1$ the corresponding quotient is $C^+(n, k)$ while
 - b₂) if $k = 4t + 3$ the quotient is $C^-(n, k)$.
- c) Any other generalised Petersen graph is not a Kronecker cover.

For $k = 1$ and even n each $G(n, 1)$ is a Kronecker cover. If $n = 4t$ case b₁) applies and the quotient graph is the Möbius ladder M_n (see [7]). For $G(4, 1)$ the quotient is $K_4 = M_4$. Similarly, the 8-sided prism $G(8, 1)$ is the Kronecker cover of M_8 . In case $n = 4t + 2$ the case a₁) applies and the quotient is $G(n/2, 1)$. For instance, the 6-sided prism is the Kronecker cover of the 3-sided prism. For $k > 1$ the smallest cases stated in Theorem 1 are presented in Table 1.

It is well-known that any automorphism of a connected bipartite graph either preserves the two sets of bipartition or interchanges the two sets of bipartition. In the former case we call the automorphism *colour preserving* and in the latter case *colour reversing*. Clearly, the product of two color-reversing automorphisms is a color preserving automorphism and the collection of all color preserving automorphisms determines a subgroup of the full automorphism group that is of index at most 2.

3 Identifying the Kronecker involutions

Before we state an important condition that classifies Kronecker covers we give the following definition.

Definition 2. A color-reversing involution ω from the automorphism group of a bipartite graph is called a *Kronecker involution*, if the vertices v and $\omega(v)$ are non-adjacent for every vertex v .

We proceed by a well-known proposition from [9], regarding the existence of Kronecker covers.

Theorem 2. For a bipartite graph G , there exists G' such that $\text{KC}(G') \simeq G$, if and only if $\text{Aut}(G)$ admits a Kronecker involution. Furthermore, the corresponding quotient graph may be obtained by contracting all pairs of vertices, naturally coupled by a given Kronecker involution.

The following result is well-known. One can find it, for instance in [8].

Theorem 3. A generalised Petersen graph $G(n, k)$ is bipartite if and only if n is even and k is odd.

We also include the classification concerning symmetries of generalised Petersen graphs, which follows from the work of Frucht et al. [5], Nedela and Skoviera [11], and Lovrečić-Saražin [14].

Theorem 4 ([5, 11, 14]). *Let $G(n, k)$ be a generalised Petersen graph. Then*

a) *it is symmetric if and only if*

$$(n, k) \in \{(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5), (24, 5)\},$$

b) *it is vertex-transitive if and only if $k^2 \equiv \pm 1 \pmod{n}$ or if $n = 10$ and $k = 2$,*

c) *it is a Cayley graph if and only if $k^2 \equiv 1 \pmod{n}$.*

In general the word *symmetric* means arc-transitive. For cubic graphs this is equivalent to saying vertex-transitive and edge-transitive. For a generalised Petersen graph symmetric is equivalent to edge-transitive.

In order to understand which generalised Petersen graphs are Kronecker covers we have to identify all Kronecker involutions for each $G(n, k)$. In what follows, for a given pair (n, k) , our arguments rely on the structure of the automorphism group $A(n, k)$ of $G(n, k)$. We define three types of permutations on the vertex set of a generalised Petersen graph which are useful for describing elements of its automorphism group.

Definition 3. For $i \in [0, n - 1]$, define the permutations α, β and γ on $V(G(n, k))$ by

$$\begin{aligned} \alpha(u_i) &= u_{i+1}, & \alpha(v_i) &= v_{i+1}, \\ \beta(u_i) &= u_{-i}, & \beta(v_i) &= v_{-i}, \\ \gamma(u_i) &= v_{ki}, & \gamma(v_i) &= u_{ki}. \end{aligned}$$

Let us paraphrase Theorem 5 of Lovrečić-Saražin [14] that follows from Frucht et al. [5]

Theorem 5. *If (n, k) is not one of $(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5)$, or $(24, 5)$, then the following holds:*

- *if $k^2 \equiv 1 \pmod{n}$, then*

$$A(n, k) = \langle \alpha, \beta, \gamma \mid \alpha^n = \beta^2 = \gamma^2 = 1, \alpha\beta = \beta\alpha^{-1}, \alpha\gamma = \gamma\alpha^k, \beta\gamma = \gamma\beta \rangle$$

- *if $k^2 \equiv -1 \pmod{n}$, then*

$$A(n, k) = \langle \alpha, \beta, \gamma \mid \alpha^n = \beta^2 = \gamma^4 = 1, \alpha\beta = \beta\alpha^{-1}, \alpha\gamma = \gamma\alpha^k, \beta\gamma = \gamma\beta \rangle$$

In this case $\beta = \gamma^2$.

- *In all other cases the graph $G(n, k)$ is not vertex-transitive and*

$$A(n, k) = \langle \alpha, \beta \mid \alpha^n = \beta^2 = 1, \alpha\beta = \beta\alpha^{-1} \rangle$$

n	k	case	involution	quotient
4	1	b_1	$\alpha^2\gamma$	$C^+(4, 1)$
6	1	a_1	α^3	$G(3, 1)$
8	1	b_1	$\alpha^4\gamma$	$C^+(8, 1)$
10	1	a_1	α^5	$G(5, 1)$
10	3	\cdot	α^5, Δ	$G(5, 2), \mathcal{H}$
12	1	b_1	$\alpha^6\gamma$	$C^+(12, 1)$
12	5	b_1	$\alpha^6\gamma$	$C^+(12, 5)$
14	1	a_1	α^7	$G(7, 1)$
14	3	a_1	α^7	$G(7, 3)$
16	1	b_1	$\alpha^8\gamma$	$C^+(16, 1)$
18	1	a_1	α^9	$G(9, 1)$
18	3	a_1	α^9	$G(9, 3)$
18	5	a_2	α^9	$G(9, 4)$
20	1	b_1	$\alpha^{10}\gamma$	$C^+(20, 1)$
20	9	b_1	$\alpha^{10}\gamma$	$C^+(20, 9)$
22	1	a_1	α^{11}	$G(11, 1)$
22	3	a_1	α^{11}	$G(11, 3)$
22	5	a_1	α^{11}	$G(11, 5)$
24	1	b_1	$\alpha^{12}\gamma$	$C^+(24, 1)$
24	7	b_2	$\alpha^{12}\beta\gamma$	$C^-(24, 7)$
26	1	a_1	α^{13}	$G(13, 1)$
26	3	a_1	α^{13}	$G(13, 3)$
26	5	a_1	α^{13}	$G(13, 5)$
26	7	a_2	α^{13}	$G(13, 6)$
28	1	b_1	$\alpha^{14}\gamma$	$C^+(28, 1)$
28	13	b_1	$\alpha^{14}\gamma$	$C^+(28, 13)$

Table 1: The smallest generalised Petersen graphs that are Kronecker covers, together with their corresponding Kronecker involutions ω and the quotient graphs.

Observe that in the case when the underlying graph is symmetric, the automorphism group may not be described by α, β and γ . For illustration, consider the following permutation Δ of the vertex-set of Desargues graph $G(10, 3)$, as constructed on Figure 1, where the vertices touching the red or the blue hexagon are rotated around their ambient hexagon for 180° , while the mapping of the rest of the vertices is defined by a horizontal reflection. Since any member of $\langle \alpha, \beta, \gamma \rangle$ either fixes each rim set-wise or swaps them, it is easy to see that $\Delta \in \text{Aut}(G(10, 3))$ is not generated by α, β and γ . Hence the symmetric bipartite generalised Petersen graphs need to be checked separately. It turns out that the quotients of generalised Petersen graphs may be obtained by Kronecker involutions from $\langle \alpha, \beta, \gamma \rangle$, or by Δ (in the case $n = 10$ and $k = 3$).

However, for the non-symmetric members of generalised Petersen graphs, Theorem 5 implies that any element of the automorphism group (including any Kronecker involution) may be expressed in terms of α, β and γ . In fact, in the next lemma we show that any such element may be expressed in a canonical way.

Lemma 6. *For any automorphism ω from $A(n, k)$ we may associate a unique triple $(a, b, c) \in \mathbb{Z}_n \times \mathbb{Z}_2 \times \mathbb{Z}_2$ such that $\omega = \alpha^a \beta^b \gamma^c$, whenever (n, k) is not equal to one of the pairs $(4, 1), (5, 2), (8, 3), (10, 2), (10, 3), (12, 5)$, or $(24, 5)$.*

Proof. Let $G(n, k)$ be a generalised Petersen graph and let a, b, c be arbitrary integers. Then, by definition of the three generators α, β, γ (or the three permutations) it clearly holds

1. $\beta \alpha^a = \alpha^{-a} \beta$,
2. $\gamma \alpha^a = \alpha^{ak} \gamma$. If $k^2 \equiv 1 \pmod n$.
3. $\gamma \alpha^a = \alpha^{-ak} \gamma$. If $k^2 \equiv -1 \pmod n$.
4. $\gamma \beta = \beta \gamma$,

We omit the arguments for (1) and (4) as they are repeated from the definition. Property (2) follows from the facts $\alpha \gamma = \gamma \alpha^k$ and $k^2 \equiv 1 \pmod n$. Since $\alpha^a \gamma = \gamma \alpha^{ak}$ for any a , take $a = k$ and we get $\alpha^k \gamma = \gamma \alpha^{k^2} = \gamma \alpha$ and the result follows. In a similar way we prove (3).

By using the commuting rules (1–4) above we may transform any product of permutations α, β, γ to a form $\alpha^a \beta^b \gamma^c$ with $0 \leq b, c \leq 1$. In non-vertex-transitive case we have $c = 0$ while in vertex-transitive non-Cayley case, one could have $\gamma, \gamma^2, \gamma^3$. However, we may always use the fact that $\gamma^2 = \beta$ and the result follows readily.

A shorter proof was suggested by a referee: Namely, from Theorem 5 it clearly follows that both β and γ normalize α , and so the fact that the intersection of $\langle \alpha \rangle$ and $\langle \beta, \gamma \rangle$ (which is clearly isomorphic to one of $\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$ and \mathbb{Z}_4) is trivial implies that $A(n, k)$ is a semidirect product of \mathbb{Z}_n by $\mathbb{Z}_2, \mathbb{Z}_2 \times \mathbb{Z}_2$ or \mathbb{Z}_4 . \square

Note that in a bipartite $G(n, k)$ automorphisms α and γ are color reversing, while β is color preserving.

Proposition 7. *For a bipartite generalised Petersen graph $G(n, k)$, the following statements hold:*

1. α^a is a Kronecker involution if and only if $a = n/2$ and $n \equiv 2 \pmod 4$;

2. $\alpha^a\beta$ is not a Kronecker involution;
3. if $k^2 \equiv -1 \pmod{n}$, then neither $\alpha^a\gamma$ nor $\alpha^a\beta\gamma$ is a Kronecker involution, for any admissible a .

Proof. We prove the claims separately.

(1) Let $\omega = \alpha^a$ be a Kronecker involution. It is clear that ω does not fix any edge, and since ω is an involution we trivially have $a = \frac{n}{2}$. But since ω must be color-reversing, a must at the same time be odd, hence the conclusion.

(2) Let $\omega = \alpha^a\beta$ be a Kronecker involution. Since ω is color-reversing, a must be odd. Letting $i = \frac{a-1}{2}$, it is enough to observe that an edge u_iu_{i+1} is fixed by ω .

(3) In both cases, the resulting squared permutation can be written in form $\alpha^{a'}\beta$, which contradicts the fact that the original permutation is an involution. □

In every generalised Petersen graph $G(n, k)$ the permutations α and β are automorphisms. Moreover, they generate the dihedral group D_n of order $2n$ of automorphisms which is, in general, a subgroup of the full automorphism group $A(n, k)$. The two vertex orbits under D_n are exactly the outer rim and the union of the inner rims. The three edge orbits are outer-rim, inner-rim and the spokes. Clearly, Proposition 7 deals with Kronecker involutions from D_n and in particular implies the condition for $G(n, k)$ being Kronecker cover described in *a*) of Theorem 1. But additional Kronecker involutions may exist by the fact that the automorphism group of a generalised Petersen graph may be larger than D_n . In the next subsection we describe these additional Kronecker involutions, which may (see (3) of Proposition 7) only happen when $k^2 \equiv 1 \pmod{n}$.

3.1 Additional Kronecker involutions with $k^2 \equiv 1 \pmod{n}$

In what follows, we assume $k^2 \equiv 1 \pmod{n}$ and define Q , such that $k^2 - 1 = Qn$. The only two permissible types of involutions are $\alpha^a\gamma$ and $\alpha^a\beta\gamma$.

For an integer i let $b(i)$ be the maximal integer such that $2^{b(i)}$ divides i . In particular, we have

$$b(n) = b(k+1) + b(k-1) - b(Q). \tag{1}$$

In the following two subsections, we prove the condition for a generalised Petersen graph being a Kronecker cover, described in *(b₁)* and *(b₂)* of Theorem 1, respectively.

Involutions of type $\alpha^a\gamma$

We have $\omega_a = \alpha^a\gamma$ such that $\omega_a(v_i) = u_{ki+a}$ and $\omega_a(u_i) = v_{ki+a}$, so let us for easier notation define a function $\Omega_a : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ such that $\Omega_a(i) = ki + a$. By these definitions, we clearly have the following properties:

- P1. Permutation ω_a is color reversing if and only if $\Omega_a(i) \equiv i \pmod{2}$, in other words if a is even and k is odd.

P2. Permutation ω_a is an involution if and only if $\Omega_a(\Omega_a(i)) \equiv i \pmod{n}$, i.e. if $ak + a \equiv 0 \pmod{n}$.

P3. Permutation ω_a may only fix a spoke. In particular, α_a fixes some edge if and only if there exists an integer i , such that $\Omega_a(i) \equiv i \pmod{n}$.

Finally, let us define a constant $a_{\min} = \frac{n}{\gcd(n, k+1)}$. The next lemma describes necessary conditions for ω_a to be a Kronecker involution.

Lemma 8. *Let ω_a be a Kronecker involution. Then the following claims are true:*

(C₁) ω_{2a} is not a Kronecker involution;

(C₂) there exists an odd integer s , such that $a = sa_{\min}$;

(C₃) a_{\min} is even;

(C₄) Q is even;

(C₅) $k \equiv 1 \pmod{4}$.

Proof. We prove the claims consecutively.

(C₁) Since ω_a is an involution, by (P2.) we have $a + ka \equiv 0 \pmod{n}$ and

$$\Omega_{2a}(a) \equiv (ak + a) + a \equiv a \pmod{n},$$

hence a is a fixed point.

(C₂) From (P2.) it follows that $a(k+1)$ is a multiple of n . In other words, there exists a positive integer C , such that $a = \frac{Cn}{k+1}$. It is clear that a is minimized whenever $Cn = \text{lcm}(k+1, n)$, i.e.

$$a_{\min} = \frac{\text{lcm}(k+1, n)}{k+1} = \frac{n}{\gcd(n, k+1)}.$$

Note that in general C may be some s -th multiple of $\text{lcm}(k+1, n)$, however the value of s from our claim may by (C₁) only be odd.

(C₃) The claim follows from (P1.) and (C₂) by the fact that a is even if and only if a_{\min} is even.

(C₄) Suppose that Q is odd, which implies that $b(Q) = 0$ and $b(n) > b(k+1)$. By (C₂) we have that $a = sa_{\min}$, for some odd s . We will show that Ω_a has a fixed point in $r = \frac{k+a+1}{2}$. Indeed, we have

$$\begin{aligned} \Omega_a(r) - r &\equiv \left(\frac{k+1}{2} + \frac{a}{2} \right) (k-1) + a \\ &\equiv \frac{Qn}{2} + \frac{k+1}{2}a \equiv \frac{n}{2} + \frac{s \cdot \text{lcm}(n, k+1)}{2} \pmod{n}. \end{aligned}$$

To conclude the proof it is enough to notice that $\text{lcm}(n, k+1)$ is an odd multiple of n by the fact that $b(n) > b(k+1)$.

(C_5) By contradiction assume that $k + 1 \equiv 0 \pmod{4}$. We will show that in such case we have $b(n) = b(\gcd(n, k + 1))$, implying that a_{\min} is odd. Let $t = b(n)$ and $s = b(k + 1)$ and note that by (C_4) we have

$$t = s + b(k - 1) - b(Q) \leq s + b(k - 1) - 1 = s,$$

hence there exist odd integers o_1, o_2 such that $n = o_1 2^t$ and $k + 1 = o_2 2^s$, and

$$\begin{aligned} a_{\min} &= \frac{n}{\gcd(n, k + 1)} = \frac{o_1 2^t}{\gcd(o_1 2^t, o_2 2^s)} \\ &= \frac{o_1}{\gcd(o_1, o_2 2^{s-t})} \equiv 1 \pmod{2}. \end{aligned}$$

As desired, we are in contradiction with (C_3). \square

We conclude the proof of condition (b_1) of Theorem 1 by showing that the necessary conditions (C_4) and (C_5) mentioned in Lemma 8 are in fact sufficient.

Proposition 9. *Let Q be even and $k \equiv 1 \pmod{4}$. Then ω_a is a Kronecker involution for any $a = sa_{\min}$ and odd s .*

Proof. We consequently prove that ω_a is an involution, that it is color-reversing, and that it does not fix any edge. Observe that ω_a is an involution by the fact that

$$(k + 1)a \equiv (k + 1)s \cdot \frac{\text{lcm}(n, k + 1)}{k + 1} \equiv 0 \pmod{n}.$$

Notice that $k \equiv 1 \pmod{4}$ implies $\gcd(n, k + 1) \equiv 2 \pmod{4}$ hence a_{\min} and a are even which implies that ω_a is color-reversing. Using $(k + 1)(k - 1) = Qn$, it follows

$$a_{\min} = \frac{Qn}{Q\gcd(n, k + 1)} = \frac{(k + 1)(k - 1)}{(k + 1) \cdot \gcd(k - 1, Q)} = \frac{k - 1}{\gcd(k - 1, Q)}. \quad (2)$$

Now suppose that ω_a fixes a spoke $u_i v_i$. Then

$$\begin{aligned} 0 &\equiv i(k - 1) + a \pmod{n} \\ &\equiv i \cdot a_{\min} \cdot \gcd(k - 1, Q) + s \cdot a_{\min} \pmod{n} \\ &\equiv a_{\min}(i \cdot \gcd(k - 1, Q) + s) \pmod{n}, \end{aligned}$$

which is equivalent to

$$i \cdot \gcd(k - 1, Q) + s \equiv 0 \pmod{\gcd(n, k + 1)} \quad (3)$$

by the fact that $a_{\min} \mid n$. But (3) is a contradiction since clearly both $\gcd(n, k + 1)$ and $\gcd(Q, k - 1)$ are even while s is odd. \square

Involutions of type $\alpha^a\beta\gamma$

In this section we focus on Kronecker involutions that also include the reflection β . While this fact requires some adjustments by the fact that we are now considering involutions of type $\alpha^a\beta\gamma$, the subsection is mostly a compact transcript of the previous one.

Define $\omega'_a = \alpha^a\beta\gamma$ i.e. $\omega'_a(v_i) = u_{a-ki}$ and $\omega'_a(u_i) = v_{a-ki}$, and let $\Omega'_a : \mathbb{Z}_n \rightarrow \mathbb{Z}_n$ be a function defined as $\Omega'_a(i) = a - ki$. In this case, the requirements for the ω'_a being Kronecker involution imply:

1. Permutation ω'_a is color reversing if and only if a is even and k is odd.
2. Permutation ω'_a is an involution if and only if $a - ak \equiv 0 \pmod{n}$.
3. Permutation ω'_a may only fix an i -th spoke if and only if there exists an integer i , such that $\Omega'_a(i) \equiv i \pmod{n}$.

Define a constant $a'_{\min} = \frac{n}{\gcd(n, n-k+1)}$, and note the following necessary conditions for ω'_a to be a Kronecker involution.

Lemma 10. *Let ω'_a be a Kronecker involution. Then the following claims are true:*

- (C'_1) ω_{2a} is not a Kronecker involution;
- (C'_2) there exists an odd integer s , such that $a = sa_{\min}$;
- (C'_3) a_{\min} is even;
- (C'_4) Q is even;
- (C'_5) $k \equiv -1 \pmod{4}$.

Proof. We omit the proofs of (C'_1), (C'_3) and (C'_5), since they may be transcribed from the proofs of (C_1), (C_3) and (C_5) along the same lines.

(C'_2) Since $a(n-k+1)$ is a multiple of n , there exists a constant C such that $a = \frac{Cn}{n-k+1}$. It is clear that a is minimized whenever $Cn = \text{lcm}(n-k+1, n)$, i.e.

$$a_{\min} = \frac{\text{lcm}(n-k+1, n)}{n-k+1} = \frac{n}{\gcd(n, n-k+1)}.$$

Again, C may be some multiple of $\text{lcm}(k+1, n)$, however the value of s from our claim may by (C'_1) only be odd.

(C'_4) Suppose that Q is odd and let ω_a be such a Kronecker involution. By ((C'_2)) we have that $a = sa'_{\min}$, for some odd s . Note that $b(Q) = 0$ implies $b(k-1) < b(n)$ and

$$b(n - (k-1)) = \min(b(n), b(k-1)) = b(k-1),$$

hence $b(n) > b(n - k + 1)$. We will show that Ω'_a has a fixed point in $r = \frac{k-1+a}{2}$. Indeed, we have

$$\begin{aligned}\Omega'_a(r) - r &\equiv a - \left(\frac{k-1}{2} + \frac{a}{2}\right)(k+1) \equiv a - \frac{Qn}{2} - \frac{a}{2} \cdot (k+1) \pmod{n} \\ &\equiv -\frac{Qn}{2} + \frac{a}{2} \cdot (n-k+1) \equiv \frac{n}{2} - \frac{s \cdot \text{lcm}(n, n-k+1) \cdot (n-k+1)}{2 \cdot (n-k+1)} \pmod{n} \\ &\equiv \frac{n}{2} - \frac{s \cdot \text{lcm}(n, n-k+1)}{2} \pmod{n}.\end{aligned}$$

To conclude the proof it is enough to notice that $\text{lcm}(n, n-k+1)$ is an odd multiple of n by the fact that $b(n) \geq b(n-k+1)$. □

The proposition below shows that the necessary conditions mentioned in Lemma 10 are in fact sufficient, which proves the condition (b_2) of Theorem 1.

Proposition 11. *Let Q be even and $k \equiv 3 \pmod{4}$. Then ω'_a is a Kronecker involution for any $a = sa'_{\min}$ and odd s .*

Proof. We consequently prove that ω'_a is an involution, that it is color-reversing, and that it does not fix any edge. Observe that ω'_a is an involution by the fact that

$$(1-k)a \equiv (n+1-k)s \cdot \frac{\text{lcm}(n-k+1, n)}{n-k+1} \equiv 0 \pmod{n}.$$

Notice that $k \equiv -1 \pmod{4}$ implies $\text{gcd}(n, n-k+1) \equiv 2 \pmod{4}$ hence a_{\min} and a are even which implies that ω_a is color-reversing. Using $(k+1)(k-1) = Qn$, it follows

$$\begin{aligned}a'_{\min} &= \frac{Qn}{Q \cdot \text{gcd}(n, n-k+1)} = \frac{(k+1)(k-1)}{\text{gcd}(Qn, (k+1-Q)(k-1))} \\ &= \frac{(k+1)(k-1)}{(k-1) \cdot \text{gcd}(k+1, k+1-Q)} = \frac{k+1}{\text{gcd}(k+1, k+1-Q)}.\end{aligned} \tag{4}$$

Now suppose that ω'_a fixes a spoke $u_i v_i$ and, for easier notation, let $g = \text{gcd}(k+1, k+1-Q)$. Then

$$\begin{aligned}0 &\equiv a - i(k+1) \pmod{n} \\ &\equiv s \cdot a'_{\min} - i \cdot a'_{\min} \cdot g \pmod{n} \\ &\equiv a'_{\min}(s - i \cdot g) \pmod{n},\end{aligned}$$

which is equivalent to

$$s - i \cdot g \equiv 0 \pmod{\text{gcd}(n, n-k+1)} \tag{5}$$

by the fact that $a'_{\min} \mid n$. But (5) is a contradiction since clearly both $\text{gcd}(n, n-k+1)$ and $\text{gcd}(k+1, k+1-Q)$ are even while s is odd. □

We conclude with an important corollary that holds for Kronecker involutions of both types.

Corollary 12. *If Q is even then $\omega_{n/2}$ is a Kronecker involution.*

Proof. Suppose that $k \equiv 1 \pmod{4}$ and let $s = \frac{\gcd(n, k+1)}{2}$. Since $k+1 \equiv 2 \pmod{4}$, clearly also $\gcd(n, k+1) \equiv 2 \pmod{4}$ and s is an odd integer. But then $\omega_{s \cdot a_{\min}}$ is a Kronecker involution by Proposition 9, and clearly $s \cdot a_{\min} = n/2$.

Similarly, if $k \equiv -1 \pmod{4}$, set $s' = \frac{\gcd(n, n-k+1)}{2}$ and notice that it is an odd integer, while $s' \cdot a'_{\min} = n/2$. \square

In the next section we prove that for any generalised Petersen graph except $G(10, 3)$, all quotients are isomorphic, hence it will be convenient to always (when applicable) use the canonical value of $a = n/2$.

4 The quotients of generalised Petersen graphs

For a given generalised Petersen graph, so far we identified all its Kronecker involutions. In this section we determine the structure of the corresponding quotient graphs, for each of these involutions. Namely, the next two subsections deal with the structural part of the statements a) and b) of Theorem 1, respectively.

4.1 Involutions of D_n

We already know that the only Kronecker involution in the Dihedral group is the rotation $\alpha^{n/2}$, which is realized whenever $n \equiv 2 \pmod{4}$ and k is odd. In order to prove a) of Theorem 1, it is enough to show the following proposition, which describes the corresponding quotient graph explicitly.

Proposition 13. *For an odd n and an integer $k < \frac{n}{2}$, we have*

$$\text{KC}(G(n, k)) \simeq \begin{cases} G(2n, k); & k \text{ is odd;} \\ G(2n, n-k); & k \text{ is even.} \end{cases}$$

Proof of a) from Theorem 1. Let $G \simeq G(n, k)$ and $G' \simeq \text{KC}(G)$, for an odd integer n and $k < \frac{n}{2}$. The edges of $\text{KC}(G)$ are naturally partitioned into the following three groups:

(E1) $u'_i v''_i$ and $u''_i v'_i$;

(E2) $u'_i u''_{i+1}$ and $u''_i u'_{i+1}$;

(E3) $v'_i v''_{i+k}$ and $v''_i v'_{i+k}$.

For easier notation, define k' to be equal k or $n-k$, depending on whether k is odd or even, respectively. Furthermore, let $H := G(2n, k')$ and denote its vertex set with

$$V(H) = \{a_0, \dots, a_{2n-1}, b_0, \dots, b_{2n-1}\},$$

while its edge set consists of edges of form $a_i a_{i+1}$, $a_i b_i$ and $b_i, b_{i+k'}$. To show the left implication of Proposition 13, it is enough to show that $G' \simeq H$. Throughout the proof all subscripts for vertices of H (on the left-hand side) are assumed to be modulo $2n$, while all subscripts for vertices of G' (on the right-hand side) are assumed to be modulo n . To show an equivalence, we introduce a bijection $f : V(H) \rightarrow V(G')$, such that

$$a_i \mapsto \begin{cases} u'_i & \text{if } i \text{ is even,} \\ u''_i & \text{if } i \text{ is odd,} \end{cases} \quad \text{and} \quad b_i \mapsto \begin{cases} v''_i & \text{if } i \text{ is even,} \\ v'_i & \text{if } i \text{ is odd.} \end{cases}$$

for any $0 \leq i < 2n$. Since n is odd, f is clearly a bijection and it is enough to show that f is a homomorphism between H and G' . We now check that all edges of H map to edges in G' . First observe that in H , edges of types $a_i a_{i+1}$ and $a_i b_i$ map to these in (E2) and (E1), respectively. Indeed, by definition we have

$$f(a_i a_{i+1}) = \begin{cases} u'_i u''_{i+1} & \text{if } i \text{ is even,} \\ u''_i u'_{i+1} & \text{if } i \text{ is odd,} \end{cases} \quad \text{and} \quad f(a_i b_i) = \begin{cases} u'_i v''_i & \text{if } i \text{ is even,} \\ u''_i v'_i & \text{if } i \text{ is odd.} \end{cases}$$

Finally, for edges of type $b_i, b_{i+k'}$, we now observe that

$$f(b_i b_{i+k'}) = \begin{cases} v''_i v'_{i+k'} & \text{if } i \text{ is even,} \\ v'_i v''_{i+k'} & \text{if } i \text{ is odd.} \end{cases} \quad (6)$$

Indeed, if k is odd or even, we have

$$b_{i+k} \mapsto \begin{cases} v''_{i+k} & \text{if } i+k \text{ is even,} \\ v'_{i+k} & \text{if } i+k \text{ is odd.} \end{cases} \quad \text{and} \quad b_{i-k+n} \mapsto \begin{cases} v''_{i+n-k} & \text{if } i-k+n \text{ is even,} \\ v'_{i+n-k} & \text{if } i-k+n \text{ is odd,} \end{cases}$$

respectively. Keep in mind that all subscripts on the right hand side are modulo n . By (6) we conclude that edges of type $b_i b_{i+k'}$ correspond to the edges of type (E3) in G' . Since both G' and H are by definition cubic and of the same cardinality, the isomorphism follows. \square

It remains to describe the behavior of the rest of Kronecker involutions satisfying the conditions $n \equiv 0 \pmod{4}$ and $n \mid \frac{k^2-1}{2}$, while $k < \frac{n}{2}$. In the next subsection we describe their equivalence (for fixed n, k), and also the corresponding quotient structure.

4.2 The rim-switching Kronecker involutions

Let us now turn to the Kronecker involutions containing permutation α , which are described by item *b*) in Theorem 1, so we assume that $k^2 \equiv 1 \pmod{n}$ and $Q = \frac{k^2-1}{n}$ is even.

In such case, using Propositions 9 and 11 one may find a Kronecker involution of $G(n, k)$, depending on whether $k \equiv 1 \pmod{4}$ or $k \equiv 3 \pmod{4}$, respectively. In order to prove that several instances of Kronecker involutions are equivalent, we will need the following extension of the LCF notion.

Definition 4. For an involution g without fixed points of type $[n] \rightarrow [n]$, we define $f(i) = g(i) - i$ and write, for short, $[f]$ instead of $[f(0), f(1), \dots, f(n-1)]$.

It is easy to see that both graphs $C^+(n, k)$ and $C^-(n, k)$ from Definition 1 correspond to $[\frac{n}{2} + (k-1)x]$ and $[\frac{n}{2} - (k+1)x]$, respectively. In order to complete the proof of the main theorem, it remains to show that for all possible Kronecker involutions, the corresponding quotient is unique. We split the further analysis into two cases, depending on the value of $k \pmod{4}$.

Case 1: $k \equiv 1 \pmod{4}$

In this case, any odd s defines $a = sa_{\min}$ and subsequently a Kronecker involution of the form $\omega_a = \alpha^a \gamma$, with $\Omega_a(i) = a + ki$. By Definition 1 and Theorem 2 the corresponding quotient graph G' is isomorphic to an outer-rim, augmented by a matching edges of type $i \sim \Omega_a^{-1}(i)$, which implies $G' \simeq [f_a]$, where

$$f_a(i) = \Omega_a^{-1}(i) - i = ik + a - i.$$

To show that for any odd s , all instances of corresponding Kronecker involutions are equivalent, we first prove the following lemma.

Lemma 14. *Let $a' = a + \gcd(k-1, Q) \cdot a_{\min}$. Then $[f_a] \simeq [f_{a'}]$.*

Proof. To prove the claim it is enough to observe that the LCF sequence of graph $[f_{a'}]$ is equivalent to the LCF sequence of $[f_a]$, cyclically shifted by one, i.e. $f_{a'}(i) = f_a(i+1)$. Indeed, by definition we have

$$\begin{aligned} f_{a'}(i) &= a + \gcd(k-1, Q)a_{\min} + i(k-1) \\ &= a + (i+1)(k-1) = f_a(i+1), \end{aligned}$$

where the second line follows from (2). □

We are now ready to show item b_1) of Theorem 1.

Proposition 15. *Let $k^2 \equiv 1 \pmod{n}$ with $n \mid \frac{k^2-1}{2}$ and $k \equiv 1 \pmod{4}$. Then $G(n, k)$ have unique quotient $[f]$.*

Proof of b_1) of Theorem 1. Let S be the set of all $\frac{\gcd(n, k+1)}{2}$ Kronecker involutions. The Lemma 14 partitions S into equivalence classes with respect to the relation of having an isomorphic corresponding quotients. We will show that S consists of only one such equivalence class.

In other words, this is equivalent to being in an additive group of order $\frac{\gcd(n, k+1)}{2}$ and calculating the order of the element $\frac{\gcd(Q, k-1)}{2}$. Clearly, all classes of such partition of S have the same cardinality, while the number of these classes is equal to

$$\begin{aligned} \gcd\left(\frac{\gcd(n, k+1)}{2}, \frac{\gcd(Q, k-1)}{2}\right) &= \gcd\left(\frac{n}{2}, \frac{k+1}{2}, \frac{k-1}{2}, \frac{Q}{2}\right) \\ &\leq \gcd\left(\frac{k+1}{2}, \frac{k-1}{2}\right) = 1. \end{aligned}$$

But then any Kronecker involution, in particular the one with $a = \frac{n}{2}$, corresponds to the unique quotient of $G(n, k)$. □

Case 2: $k \equiv 3 \pmod{4}$

Also in this case, any odd s defines $a = sa'_{\min}$ and subsequently a Kronecker involution of the form $\omega'_a = \alpha^a \beta \gamma$, with $\Omega'_a(i) = a - ki$. By Definition 1 and Theorem 2 the corresponding quotient graph G' is isomorphic to an outer-rim, augmented by matching edges of type $i \sim \Omega_a^{-1}(i)$, which implies $G' \simeq [f_a]$, where

$$f_a(i) = \Omega'_a(i) - i = a - ik - i.$$

We prove a similar lemma as in the previous case.

Lemma 16. *Let $a' = a + \gcd(k+1, k+1-Q) \cdot a'_{\min}$. Then $[f_a] \simeq [f_{a'}]$.*

Proof. We similarly prove the claim by observing $f_{a'}(i) = f_a(i-1)$. Indeed, we have

$$\begin{aligned} f_{a'}(i) &= a + \gcd(k+1, k+1-Q)a'_{\min} - i(k+1) \\ &= a - (i-1)(k+1) = f_a(i-1), \end{aligned}$$

where the second line follows from (4). □

We are now ready to show item b_2) of Theorem 1.

Proposition 17. *Let $k^2 \equiv 1 \pmod{n}$ with $n \mid \frac{k^2-1}{2}$ and $k \equiv -1 \pmod{4}$. Then $G(n, k)$ have unique quotient $C^-(n, k)$.*

Proof of b_2) of Theorem 1. Let S be the set of all $\frac{\gcd(n, k+1)}{2}$ Kronecker involutions. Again, we show that Lemma 16 eventually covers the whole set S .

In this case one may consider an additive group of order $\frac{\gcd(n, n-k+1)}{2}$ and calculate the order of the element $\frac{\gcd(k+1, k+1-Q)}{2}$. The number of such orbits is equal to

$$\begin{aligned} \gcd\left(\frac{\gcd(n, n-k+1)}{2}, \frac{\gcd(k+1, k+1-Q)}{2}\right) &= \gcd\left(\frac{n}{2}, \frac{k+1}{2}, \frac{k-1}{2}, \frac{Q}{2}\right) \\ &\leq \gcd\left(\frac{k+1}{2}, \frac{k-1}{2}\right) = 1. \end{aligned}$$

□

Since we described the quotients of all existing Kronecker involutions, this concludes the proof of Theorem 1.

5 Concluding remarks and future work

In this paper, we classified parameters (n, k) such that $G(n, k)$ is a Kronecker cover of some graph, and described the corresponding quotients. From our main result it easily follows:

Corollary 18. *KC($G(n, k)$) is itself a generalised Petersen graph if and only if n is odd.*

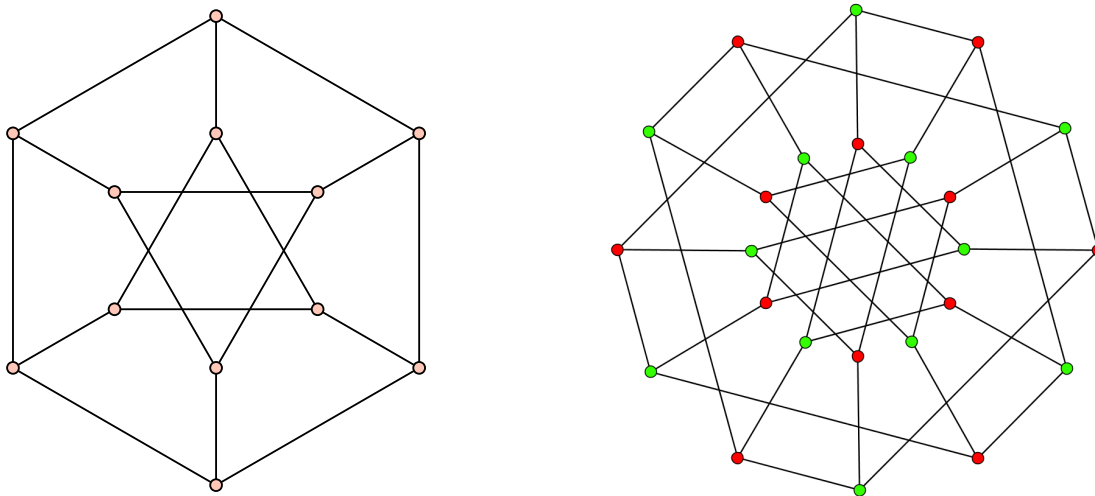


Figure 2: The Dürer graph $G(6, 2)$ and its Kronecker cover $KC(G(6, 2))$ with proper vertex two-coloring.

We analyzed the problem of Kronecker covers of the family of generalised Petersen graphs. It would be interesting to transfer this problem to the family of I -graphs [2, 3, 12, 8] or Rose-Window graphs [17], or some other families of cubic or quartic graphs.

Graphs $KC(G(n, k))$ that are not generalised Petersen graphs, in other words if n is odd, fall into two known classes, depending on the parity of k . If k is odd, we have $KC(G(n, k)) = 2G(n, k)$. It would be interesting to investigate the family of graphs $KC(G(n, k))$ with both n and k even. The smallest case is depicted in Figure 2. This is the Kronecker cover of the Dürer graph $G(6, 2)$.

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