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# Decomposition techniques applied to the Clique-Stable set Separation problem

Nicolas Bousquet\*    Aurélie Lagoutte\*    Frédéric Maffray\*    Lucas Pastor\*

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## Abstract

In a graph, a Clique-Stable Set separator (CS-separator) is a family  $\mathcal{C}$  of cuts (bipartitions of the vertex set) such that for every clique  $K$  and every stable set  $S$  with  $K \cap S = \emptyset$ , there exists a cut  $(W, W')$  in  $\mathcal{C}$  such that  $K \subseteq W$  and  $S \subseteq W'$ . Starting from a question concerning extended formulations of the Stable Set polytope and a related complexity communication problem, Yannakakis [20] asked in 1991 the following questions: does every graph admit a polynomial-size CS-separator? If not, does every perfect graph do? Several positive and negative results related to this question were given recently. Here we show how graph decomposition can be used to prove that a class of graphs admits a polynomial CS-separator. We apply this method to apple-free graphs and cap-free graphs.

*Keywords:* clique-stable set separation, extended formulation, stable set polytope, graph decomposition, apple-free graphs.

## 1 Introduction

Let  $G$  be any simple finite undirected graph without loop. In 1991 Yannakakis [20] asked for the existence of a compact extended formulation of the Stable Set polytope of  $G$ , i.e., the existence of a simpler polytope in higher dimension whose projection is the Stable Set polytope of  $G$  (the Stable Set polytope  $STAB(G)$  is the convex hull in  $\mathbb{R}^{|V(G)|}$  of the characteristic vectors of all the stable sets of  $G$ ). Such a simpler polytope would correspond to a linear program with extra variables to solve the Maximum Weighted Stable Set problem in  $G$ . However, there is no good explicit description of the Stable Set polytope for general graphs. Consequently, Yannakakis considered this problem in restricted classes of graphs on which a description of the Stable Set polytope with equalities and inequalities is known. This is the case for perfect graphs: the stable set polytope is exactly the polytope described by the clique inequalities ( $\sum_{v \in K} x_v \leq 1$ , for every clique  $K \subseteq V(G)$ ) together with the non-negativity constraints. This case highlights a combinatorial object which provides a combinatorial lower bound (the so-called *rectangle covering bound*) for the size of any extended formulation for  $STAB(G)$ .

Yannakakis then raised the question below for both perfect graphs and general graphs. We state this question in its graph-theoretical version but it can also be stated as a communication complexity problem. In a graph  $G$ , a *cut* is a partition  $(W, W')$  of  $V(G)$  into two subsets. Given two disjoint subsets of vertices  $K$  and  $S$ , a cut  $(W, W')$  *separates*  $K$  and  $S$  if  $K \subseteq W$  and  $S \subseteq W'$ . A *Clique-Stable Set separator* (*CS-separator*) is a family  $\mathcal{C}$  of cuts such that, for every clique  $K$  and every stable set  $S$  with  $K \cap S = \emptyset$ , there is a member of  $\mathcal{C}$  that separates  $K$

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and  $S$ . The *size* of a CS-separator is the number of cuts contained in the family. Yannakakis's question is the following:

*Does there exist a Clique-Stable set separator consisting of polynomially many cuts?*

Yannakakis proved in [20] that every graph on  $n$  vertices admits a CS-Separator of size  $n^{\mathcal{O}(\log n)}$ . He also proved that the existence of a polynomial-size CS-separator is a necessary condition for the existence of a polynomial (more frequently called *compact*) extended formulation of the Stable Set polytope.

The existence or not, for every graph  $G$ , of a polynomial-size CS-separator and/or a compact extended formulation for  $STAB(G)$  has been a big open question since 1991; see e.g. Lovász's survey [16]. It has seen a renewed interest since 2012 with the following two results: on the one hand, Fiorini *et al.* [9] proved that some graphs do not admit a compact extended formulation for  $STAB(G)$ ; on the other hand, Huang and Sudakov [13] proved the first superlinear lower bound for the size of a CS-Separator in general. This lower bound was then improved in [2] and then in [19]. Eventually, Göös [10] proved in 2015 that some graphs do not admit a polynomial-size CS-Separator. Hence, Yannakakis's two questions have been finally given a negative answer in the general case. However, determining which graph classes admit a polynomial CS-Separator remains a widely open problem. A class  $\mathcal{C}$  of graphs is said to *have the polynomial CS-Separation property* if there exists a polynomial  $P$  such that every graph  $G \in \mathcal{C}$  admits a CS-Separator of size  $P(|V(G)|)$ . We further say that  $\mathcal{C}$  has the  $\mathcal{O}(n^c)$ -CS-Separation property if  $P(n)$  is a  $\mathcal{O}(n^c)$  in the previous definition.

Given a family of graphs  $\mathcal{F}$ , a graph  $G$  is  $\mathcal{F}$ -free if no induced subgraph of  $G$  is isomorphic to a member of  $\mathcal{F}$ . If  $\mathcal{F}$  is composed of only one element  $F$ , we say that  $G$  is  $F$ -free. As usual we let  $P_k$  and  $C_k$  denote respectively the chordless path and chordless cycle on  $k$  vertices. A *hole* is any chordless cycle on at least four vertices. An *antihole* is the complementary graph of a hole.

It is easy to see that when the number of maximal cliques, or the number of maximal stable sets, is a polynomial in the size of the graph, then there exists a polynomial-size CS-Separator. In particular, this happens when  $\omega(G)$  or  $\alpha(G)$  is bounded by a constant, and by [1], this also happens for  $C_4$ -free graphs.

Concerning subclasses of perfect graphs, Yannakakis [20] proved that comparability graphs and chordal graphs have the polynomial CS-separator property, and they even have compact extended formulations. Since a CS-Separator of the complement graph can be trivially obtained from a CS-Separator of the graph, the polynomial CS-Separation property also holds for the complements of comparability graphs and complements of chordal graphs. Moreover, Lagoutte *et al.* [15] proved that perfect graphs with no balanced skew-partition have the quadratic CS-Separation property. Another interesting subclass of perfect graphs is the class of *weakly chordal graphs*, which are the graphs that contain no hole of length at least 5 and no antihole of length at least 5 [12]. Weakly chordal graphs have the polynomial CS-separator property: this follows from the main result of [3] and Theorem 12 of [4].

Leaving the domain of perfect graphs, Yannakakis proved that  $t$ -perfect graphs have the polynomial CS-Separation property, and even that they have compact extended formulations for the Stable Set polytope. Bousquet *et al.* [4] showed that, asymptotically almost surely, there exists a polynomial size CS-separator for  $G$  if  $G$  is picked at random. Furthermore, they showed that  $H$ -free graphs, where  $H$  is any split graph (i.e., a graph whose vertex set can be partitioned into a clique and a stable set),  $P_5$ -free graphs, and  $(P_k, \overline{P_k})$ -free graphs all satisfy the polynomial CS-Separation property. Furthermore, for any  $k$  the class of graphs that contain no hole of length at least  $k$  and no antihole of length at least  $k$  have the "strong

Erdős-Hajnal property” [3] (note that weakly chordal graphs correspond to the case  $k = 5$ ), and every hereditary class of graphs that has the strong Erdős-Hajnal property has the polynomial CS-Separation property [4]. With the results from [11] in which the authors devise a polynomial-time algorithm for the Maximum Weight Independent Set problem in  $P_6$ -free graphs and by the same techniques as for the  $P_5$ -free graphs and Corollary 15 from [4], the class of  $P_6$ -free graphs also has the polynomial CS-Separation property.

**Our contribution.** The goal of this paper is to show that graph decomposition can be used to prove that a class of graphs has the polynomial CS-Separation property. In the past decades many decomposition theorems have been proposed for graph classes, and especially for subclasses of perfect graphs. The existence of such a decomposition was used for example in [15] to prove that a certain subclass of perfect graphs has the polynomial CS-Separation property. Conforti *et al.* [8] have recently followed a similar approach as we do in this paper: they study how graph decomposition can help to prove that the Stable Set polytope admits a compact extended formulation, when this property holds in “basic” graphs.

Here we investigate further the relation between graph decomposition and CS-separators. To this end, we define in Section 2 the general setting of a decomposition tree and highlight in Section 3 some decomposition rules that behave well with respect to CS-Separators. We then apply this technique in Section 4 to two classes of graphs where existing decomposition theorems provide both “friendly” decompositions on one hand, and “basic” graphs already having the polynomial CS-Separation property on the other hand. These two classes are apple-free graphs and cap-free graphs (see Section 4 for definitions). Moreover, we briefly mention diamond-wheel-free graphs and  $k$ -windmill graphs. In Section 5, we show that the existence of a recursive star-cutset decomposition for a graph  $G$  cannot be enough to ensure a polynomial size CS-separator.

**Notations and definitions.** For any vertex  $v \in V(G)$ , we denote by  $N(v) = \{u \in V(G) \mid uv \in E(G)\}$  the set of vertices adjacent to  $v$ , called the *neighborhood* of  $v$ . The *closed neighborhood* of  $v$ , denoted by  $N[v]$  is the set  $\{v\} \cup N(v)$ . The set  $V(G) \setminus N[v]$  of vertices not adjacent to  $v$  is called the *anti-neighborhood* of  $v$ . For any  $S \subseteq V(G)$  we denote by  $G[S]$  the *induced subgraph* of  $G$  with vertex set  $S$ . Given a class of graphs  $\mathcal{C}'$ , a graph  $G$  is *nearly  $\mathcal{C}'$*  if for every  $v \in V(G)$ , we have  $G \setminus N[v] \in \mathcal{C}'$ .

For two sets  $A, B \subseteq V(G)$ , we say that  $A$  is *complete* to  $B$  if every vertex of  $A$  is adjacent to every vertex of  $B$ , and we say that  $A$  is *anticomplete* to  $B$  if no vertex of  $A$  is adjacent to any vertex of  $B$ . A *module* is a set  $M \subseteq V(G)$  such that every vertex in  $V(G) \setminus M$  is either complete to  $M$  or anticomplete to  $M$ . A module  $M$  is *trivial* if either  $|M| = 1$  or  $M = V(G)$ . A graph is *prime* if all its modules are trivial.

A *vertex cut*  $C$  is a subset of vertices such that  $G[V \setminus C]$  is disconnected. We say that  $C$  is a *minimal vertex cut* if  $C$  does not contain any other vertex cut. A *clique-cutset* is a vertex cut that induces a clique. A graph that does not contain any clique-cutset is called an *atom*.

A graph is *chordal* if it does not contain as an induced subgraph any  $C_k$  for  $k \geq 4$ . A class of graphs is *hereditary* if it is closed under taking induced subgraphs.

## 2 General setting of the method

Decomposition trees have been widely used to solve algorithmic problems on graphs. The idea consists in breaking the graph into smaller parts in a divide-and-conquer approach. The problem is solved on the smaller parts and then these partial solutions are combined in some way into a solution for the whole graph. To solve the problem on the smaller parts, one recursively breaks

them down into smaller and smaller graphs until one reaches graphs that are simple enough – they form the leaves of the decomposition tree. Then, from solutions on the leaves, one follows a bottom-up approach to build a solution for the parents of the leaves, and ultimately for the whole graph. The decomposition rules depend both on the class of graphs under study and on the problem under consideration. We try to present here the most general setting.

A *decomposition* of a graph  $G$  is a pair  $(G_1, G_2)$  where  $G_1$  and  $G_2$  are proper induced subgraphs of  $G$  ( $G_1$  and  $G_2$  are often called *blocks of the decomposition* in the literature). A decomposition  $(G_1, G_2)$  is *valid (with respect to the CS-Separation)* if, given a CS-Separator of size  $f_1$  of  $G_1$  and a CS-Separator of size  $f_2$  of  $G_2$ , there exists a CS-Separator of size  $f_1 + f_2$  of  $G$ . We discuss in detail in Section 3 which graph decompositions are valid.

Let  $\mathcal{C}'$  be a class of graphs. Given a graph  $G$ , a rooted binary tree  $T$  is a *valid decomposition tree for  $G$  with leaves in  $\mathcal{C}'$*  if the following conditions hold (an example is described in Figure 2):

- There exists a map  $\varphi : V(T) \rightarrow \mathcal{P}(V(G))$  that associate with every  $t \in V(T)$  a subset of vertices of  $G$ . We say that  $G[\varphi(t)]$  is the *subgraph at node  $t$* <sup>1</sup>.
- If  $r$  is the root of  $T$ , then  $\varphi(r) = V(G)$ , i.e.,  $G$  itself is the subgraph at node  $r$ .
- If  $f$  is a leaf of  $T$ , then  $G[\varphi(f)] \in \mathcal{C}'$ .
- If  $t$  is a node of  $V(T)$  with children  $s, s'$ , then  $(G[\varphi(s)], G[\varphi(s')])$  is a valid decomposition of  $G[\varphi(t)]$ .

We easily obtain the following:

**Lemma 2.1.** *Let  $\mathcal{C}'$  be a class of graphs having the  $\mathcal{O}(n^c)$ -CS-Separation property, for some constant  $c > 0$ . Let  $G$  be a graph on  $n$  vertices admitting a valid decomposition tree  $T$  with leaves in  $\mathcal{C}'$ , and  $L(T)$  be the number of leaves of  $T$ . Then there exists a CS-Separator for  $G$  of size  $\mathcal{O}(L(T) \cdot n^c)$ .*

*Proof.* Let  $\ell_1, \dots, \ell_{L(T)}$  be the leaves of  $T$ . Then for each  $i$ ,  $G[\varphi(\ell_i)]$  admits a CS-Separator of size  $f_i$  with  $f_i = \mathcal{O}(|\varphi(\ell_i)|^c) = \mathcal{O}(n^c)$  (since  $|\varphi(\ell_i)| \leq n$ ). By definition of a valid decomposition, we obtain that  $G$  admits a CS-Separator of size  $\sum_{i=1}^{L(T)} f_i = \mathcal{O}(L(T) \cdot n^c)$ .  $\square$

From Lemma 2.1, we can design a proof strategy. Let  $\mathcal{C}$  be a class of graphs in which we want to prove that the polynomial CS-Separation property holds. We need to reach the following intermediate goals:

- Find a suitable class  $\mathcal{C}'$  having the polynomial CS-Separation property,
- Prove that every  $G \in \mathcal{C}$  admits a valid decomposition tree  $T(G)$  with leaves in  $\mathcal{C}'$ , and
- Prove that  $T(G)$  has size polynomial in  $|V(G)|$ .

In practice, the last item is often the hardest to obtain. However, one way to get over this problem consists in labeling the nodes of the tree  $T(G)$ . Let  $\mathcal{S}$  be a polynomial set of subsets of  $V(G)$  (e.g.  $\mathcal{S}$  contains only subsets of at most  $k$  vertices for some fixed  $k$ ). A  $\mathcal{S}$ -labeling of  $T(G)$  is a map  $\ell : I(T) \rightarrow \mathcal{S}$  where  $I(T)$  is the set of internal nodes of  $T$ , with the condition that  $\ell(t) \subseteq \varphi(t)$  for every  $t \in T$ . In other words, the label of a node  $t$  must contain only vertices from the subgraph at node  $t$  (see Figure 2 for an illustration of the definition). The  $\mathcal{S}$ -labeling is *injective* if  $\ell(t) \neq \ell(t')$  whenever  $t \neq t'$ . The existence of an injective  $\mathcal{S}$ -labeling ensures that  $|I(T)| \leq |\mathcal{S}|$  and consequently  $|V(T)| \leq 2|I(T)| + 1 \leq 2|\mathcal{S}| + 1$  since the number of leaves of a binary tree is at most its number of internal vertices plus one. The following lemma provides sufficient conditions for a  $\mathcal{S}$ -labeling to be injective.

<sup>1</sup>As is usually the case to avoid confusion, vertices of the tree decomposition will be called nodes.

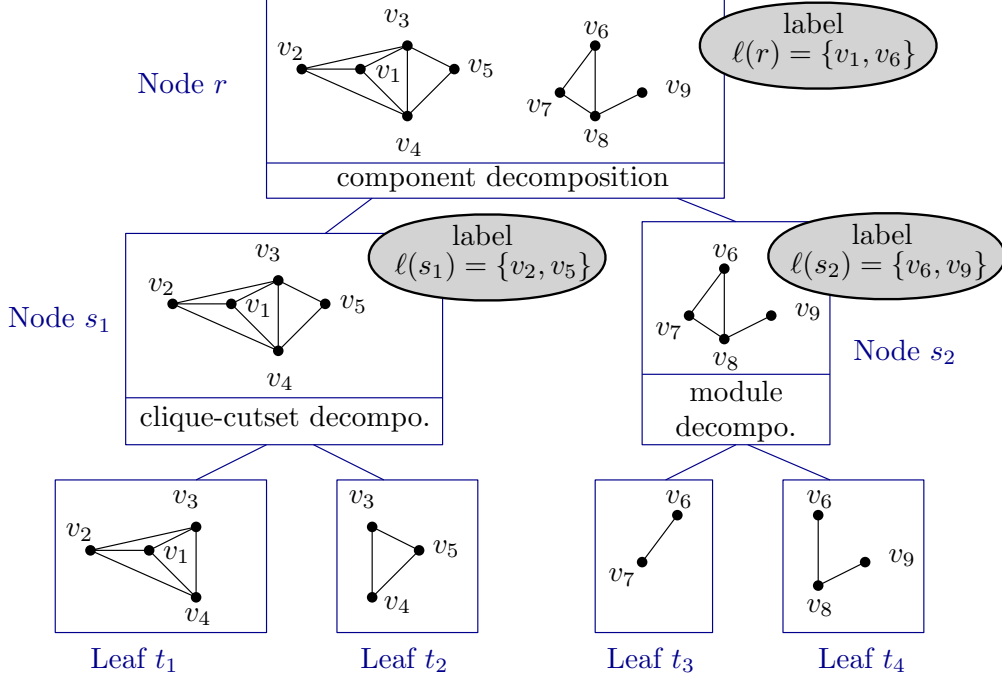


Figure 1: Example of a valid decomposition tree  $T$  for a graph  $G$  on 9 vertices (displayed in the upper box), with leaves in  $\mathcal{C}$ , where  $\mathcal{C}$  denote the union of triangle-free graphs and complete graphs. In this example,  $T$  has 3 internal nodes and 4 leaves, with  $\varphi(r) = \{v_1, \dots, v_9\}$ ,  $\varphi(s_1) = \{v_1, \dots, v_5\}$ ,  $\varphi(s_2) = \{v_6, \dots, v_9\}$ ,  $\varphi(t_1) = \{v_1, \dots, v_4\}$ ,  $\varphi(t_2) = \{v_3, \dots, v_5\}$ ,  $\varphi(t_3) = \{v_6, v_7\}$ , and  $\varphi(t_4) = \{v_6, \dots, v_9\}$ . Moreover, this tree is equipped with a  $\mathcal{S}$ -labeling  $\ell$ , where  $\mathcal{S}$  denotes the set of all pairs of non-adjacent vertices. Observe that  $\ell$  satisfies the condition of Lemma 2.2. See section 3 for details about the valid decompositions that are used.

**Lemma 2.2.** *Let  $G$  be a graph admitting a valid decomposition tree  $T(G)$  and  $\ell$  be a  $\mathcal{S}$ -labeling of  $T(G)$ . If for every node  $t$  with children  $s, s'$  we have:*

1.  $\ell(t) \not\subseteq \varphi(s)$  and  $\ell(t) \not\subseteq \varphi(s')$ , and
2. No member of  $\mathcal{S}$  is included in  $\varphi(s) \cap \varphi(s')$ , unless  $s$  or  $s'$  is a leaf,

then  $\ell$  is injective.

Informally, the first condition ensures that the label we choose for node  $t$  is "broken" at each decomposition (i.e. it does not appear in any children), and the second condition ensures that no potential label is duplicated, i.e. no potential label appears in both children.

*Proof.* Assume by contradiction that  $\ell$  is not injective and let  $t \neq t'$  be two internal nodes such that  $\ell(t) = \ell(t')$ . Suppose first that  $t$  is an ancestor of  $t'$ . Let  $s$  and  $s'$  be the children of  $t$ , where  $s'$  is an ancestor of  $t'$  (possibly  $s' = t'$ ). By definition,  $\ell(t') \subseteq \varphi(t') \subseteq \varphi(s')$ . But  $\ell(t)$  is not a subset of  $\varphi(s')$ , a contradiction with  $\ell(t) = \ell(t')$ . Now, suppose that none of  $t, t'$  is an ancestor of the other. Let  $t_0$  be the closest common ancestor of  $t$  and  $t'$ , and let  $s$  (resp.  $s'$ ) be the child of  $t_0$  which is an ancestor of  $t$  (resp. of  $t'$ ). Clearly none of  $s, s'$  is a leaf. Since  $\ell(t) \subseteq \varphi(t) \subseteq \varphi(s)$  and similarly  $\ell(t') \subseteq \varphi(t') \subseteq \varphi(s')$ , we obtain that  $\varphi(s) \cap \varphi(s')$  contains  $\ell(t) = \ell(t') \in \mathcal{S}$ : a contradiction to the second condition of the lemma.  $\square$

### 3 Valid decompositions

The core of the paper consists in showing that many usual graph decompositions are actually valid decompositions with respect to the CS-Separation.

- Suppose that  $G$  is not connected. So  $V(G)$  admits a partition  $(A, B)$  where  $A$  is anticomplete to  $B$  and both  $A$  and  $B$  are non-empty. Then  $(G[A], G[B])$  is a *component decomposition* of  $G$ . (Note that  $G[A]$  and  $G[B]$  may not be connected, but this does not matter.)
- Suppose that  $\overline{G}$  is not connected. So  $V(G)$  admits a partition  $(A, B)$  where  $A$  is complete to  $B$ , and both  $A$  and  $B$  are non-empty. Then  $(G[A], G[B])$  is an *anticomponent decomposition* of  $G$ .
- More generally, suppose that  $V(G)$  admits a partition  $(A, C, B)$  such that  $A$  and  $B$  are not empty and  $A$  is anticomplete to  $B$ . Then  $(G[A \cup C], G[B \cup C])$  is a *cutset decomposition* of  $G$ . If  $C$  is a clique, we say that it is a *clique-cutset decomposition*.
- Suppose that  $G$  admits a non-trivial module  $M$ . Pick any  $m \in M$ . Then  $(G[M], G[\{m\} \cup (V(G) \setminus M)])$  is a *module decomposition* of  $G$ .
- Let  $\mathcal{C}'$  be a class of graphs. Suppose that there is a vertex  $v \in V(G)$  such that  $G[N(v)] \in \mathcal{C}'$ . Then  $(G \setminus v, G[N(v)])$  is a  $\mathcal{C}'$ -*neighborhood decomposition* (at vertex  $v$ ).
- Similarly, suppose that there exists  $v \in V(G)$  such that  $G[V(G) \setminus N[v]] \in \mathcal{C}'$ . Then  $(G \setminus v, G \setminus N[v])$  is a  $\mathcal{C}'$ -*antineighborhood decomposition* (at vertex  $v$ ).
- An *amalgam split* of  $G$  (see [6]) is a partition of  $V(G)$  into five sets  $B_1, A_1, C, A_2, B_2$  with  $A_1, A_2$  both non-empty,  $|A_1 \cup B_1| \geq 2$ ,  $|A_2 \cup B_2| \geq 2$  satisfying the following conditions:  $C$  is a clique,  $C$  is complete to  $A_1 \cup A_2$ ,  $A_1$  is complete to  $A_2$ ,  $B_1$  is anticomplete to  $A_2 \cup B_2$ , and  $B_2$  is anticomplete to  $A_1 \cup B_1$ . Pick any  $a_1 \in A_1$  and  $a_2 \in A_2$ . Then  $(G[B_1 \cup A_1 \cup C \cup \{a_2\}], G[\{a_1\} \cup C \cup A_2 \cup B_2])$  is an *amalgam decomposition* of  $G$ .

Observe that some of these decompositions were introduced in the context of perfect graphs, as partial attempts to prove Berge's Strong Perfect Graph Conjecture by proving that minimal counter-examples to the conjecture do not admit such a decomposition. Since it is not known whether the polynomial CS-Separation property holds in perfect graphs, it makes sense to follow these steps and to carefully study the behavior of these decompositions with respect to CS-Separators.

We now prove that each of these decompositions is valid. It is worth noting that if  $G$  admits a so-called *2-join*, then a valid decomposition can be found too. We will not use it in our applications in Section 4, but we refer the interesting reader to [15] for more details.

**Lemma 3.1.** *Let  $G$  be a graph and let  $(G_1, G_2)$  be a decomposition of  $G$  which is either a component decomposition, or an anticomponent decomposition, or a cutset decomposition, or a module decomposition, or a  $\mathcal{C}'$ -neighborhood decomposition, or a  $\mathcal{C}'$ -antineighborhood decomposition, or an amalgam decomposition. Then this decomposition is valid.*

*Proof.* We follow the notations used in the definitions above. Let  $\mathcal{F}_1$  be a CS-separator of  $G_1$  and  $\mathcal{F}_2$  be a CS-separator of  $G_2$ , of respective sizes  $f_1$  and  $f_2$ . In each case, we will define a family  $\mathcal{F}$  of cuts of  $G$  of size  $f_1 + f_2$  and prove that  $\mathcal{F}$  is a CS-separator for  $G$ .

*Cutset decomposition:* Here  $G$  has a partition  $(A, C, B)$  where  $A$  and  $B$  are not empty and  $A$  is anticomplete to  $B$ , and  $G_1 = G[A \cup C]$ , and  $G_2 = G[B \cup C]$ . For each cut  $(W, W') \in \mathcal{F}_1$ ,

we put  $(W, W' \cup B)$  in  $\mathcal{F}$ ; and similarly for each cut  $(W, W') \in \mathcal{F}_2$ , we put  $(W, W' \cup A)$  in  $\mathcal{F}$ . Clearly  $\mathcal{F}$  is a set of cuts of  $G$ . Let us check that  $\mathcal{F}$  is a CS-Separator of  $G$ . Pick any clique  $K$  and any stable set  $S$  of  $G$ , such that  $K \cap S = \emptyset$ . Note that  $K$  cannot intersect both  $A$  and  $B$  since  $A$  is anticomplete to  $B$ . Up to symmetry, let us assume that  $K \cap B = \emptyset$ , so  $K \subseteq A \cup C$ . Then there is a cut  $(W, W') \in \mathcal{F}_1$  that separates  $K$  and  $S \cap (A \cup C)$  (possibly empty). Then the cut  $(W, W' \cup B)$  separates  $K$  and  $S$  in  $G$  and it is a member of  $\mathcal{F}$ .

*Component decomposition:* This is similar to a cutset decomposition with  $C$  being empty.

*Anticomponent decomposition:* Since a CS-separator of a graph  $G$  is also a CS-separator of its complement  $\overline{G}$  (up to swapping both sets in the cuts),  $\mathcal{F}_1$  and  $\mathcal{F}_2$  also are CS-separators of the complement of  $G_1$  and  $G_2$ , respectively. The first point ensures that the complement of  $G$  has a CS-separator of size  $f_1 + f_2$ , and then so does  $G$ .

*Module decomposition:* Here  $G$  has a partition  $(M, A, B)$  where  $A$  is complete to  $M$ , and  $B$  is anticomplete to  $M$ , and  $G_1 = G[M]$ , and  $G_2 = G[\{m\} \cup A \cup B]$  for some  $m \in M$ . For each cut  $(W, W') \in \mathcal{F}_1$ , we put the cut  $(W \cup A, W' \cup B)$  in  $\mathcal{F}$ . For each cut  $(W, W') \in \mathcal{F}_2$ , if  $m \in W$  then we put  $(W \cup M, W')$  in  $\mathcal{F}$ , otherwise we put  $(W, W' \cup M)$  in  $\mathcal{F}$ . Let us check that  $\mathcal{F}$  is a CS-Separator of  $G$ . Pick any clique  $K$  and any stable set  $S$  of  $G$ , such that  $K \cap S = \emptyset$ . First, suppose that  $K \subseteq M \cup A$  and  $S \subseteq M \cup B$ . Then there is a cut  $(W, W') \in \mathcal{F}_1$  separating  $K \cap M$  and  $S \cap M$ , and the corresponding cut  $(W \cup A, W' \cup B) \in \mathcal{F}$  separates  $K$  and  $S$ . So  $K \not\subseteq M \cup A$  or  $S \not\subseteq M \cup B$ . This implies that  $M$  cannot intersect both  $K$  and  $S$ . Let  $K_2 = K$  if  $K \cap M = \emptyset$  and  $(K \setminus M) \cup \{m\}$  otherwise. Similarly let  $S_2 = S$  if  $S \cap M = \emptyset$  and  $(S \setminus M) \cup \{m\}$  otherwise. Note that  $K_2$  and  $S_2$  are respectively a clique and a stable set in  $G_2$ . So there exists a cut  $(W, W') \in \mathcal{F}_2$  that separates  $K_2$  and  $S_2$ . If  $m \in W$ , then  $(W \cup M, W')$  is a member of  $\mathcal{F}$  that separates  $K \cup M$  and  $S$ , and consequently  $K$  and  $S$ . If  $m \in W'$ , then  $(W, W' \cup M)$  is a member of  $\mathcal{F}$  that separates  $K$  and  $S \cup M$ , and consequently  $K$  and  $S$ .

*$\mathcal{C}'$ -neighborhood decomposition:* Here  $G$  has a partition  $(\{v\}, A, B)$  where  $A = N(v)$ , and  $G[A] \in \mathcal{C}'$ , and  $G_1 = G[V(G) \setminus \{v\}]$ , and  $G_2 = G[N(v)]$ . For each cut  $(W, W') \in \mathcal{F}_1$ , we put  $(W, W' \cup \{v\})$  in  $\mathcal{F}$ . For each cut  $(W, W') \in \mathcal{F}_2$ , we put  $(W \cup \{v\}, W' \cup B)$  in  $\mathcal{F}$ . Let us check that  $\mathcal{F}$  is a CS-Separator of  $G$ . Pick any clique  $K$  and any stable set  $S$  of  $G$ , such that  $K \cap S = \emptyset$ . First, suppose that  $v \notin K$ , hence  $K \subseteq A \cup B$ . Then there is a cut  $(W, W') \in \mathcal{F}_1$  that separates  $K$  and  $S \cap (A \cup B)$ , and consequently the cut  $(W, W' \cup \{v\})$  is a member of  $\mathcal{F}$  that separates  $K$  and  $S$  in  $G$ . Now, assume that  $v \in K$ , hence  $K \subseteq N[v]$  and  $S \subseteq A \cup B$ . Then there is a cut  $(W, W') \in \mathcal{F}_2$  that separates  $K \cap A$  and  $S \cap A$ , and the cut  $F = (W \cup \{v\}, W' \cup B)$  is a member of  $\mathcal{F}$  that separates  $K$  and  $S$  in  $G$ .

*$\mathcal{C}'$ -anti-neighborhood decomposition:* the complement of a graph with a  $\mathcal{C}'$ -antineighborhood decomposition on  $v$  is a graph with a  $\overline{\mathcal{C}'}$ -neighborhood decomposition on  $v$ , so we apply the same construction as above in the complement graph.

*Amalgam decomposition:* Here  $G$  has a partition  $(B_1, A_1, C, A_2, B_2)$  where  $C$  is a clique,  $C$  is complete to  $A_1 \cup A_2$ ,  $A_1$  is complete to  $A_2$ ,  $B_1$  is anticomplete to  $A_2 \cup B_2$ , and  $B_2$  is anticomplete to  $A_1 \cup B_1$ , and  $G_1 = G[A_1 \cup B_1 \cup C \cup \{a_2\}]$ , and  $G_2 = G[A_2 \cup B_2 \cup C \cup \{a_1\}]$  for some  $a_1 \in A_1$  and  $a_2 \in A_2$ . For each cut  $(W, W') \in \mathcal{F}_1$ , if  $a_2 \in W$  we put the cut  $(W \cup A_2, W' \cup B_2)$  in  $\mathcal{F}$ , otherwise we put  $(W, W' \cup A_2 \cup B_2)$  in  $\mathcal{F}$ . Similarly, for each cut  $(W, W') \in \mathcal{F}_2$ , if  $a_1 \in W$  we put the cut  $(W \cup A_1, W' \cup B_1)$  in  $\mathcal{F}$ , otherwise we put  $(W, W' \cup A_1 \cup B_1)$  in  $\mathcal{F}$ . Let us check that  $\mathcal{F}$  is a CS-Separator of  $G$ . Pick any clique  $K$  and any stable set  $S$  of  $G$ , such that  $K \cap S = \emptyset$ . First, suppose that  $K \cap (B_1 \cup B_2) \neq \emptyset$ . Up to symmetry, we may assume that  $K \cap B_1 \neq \emptyset$  and let  $S_1 = S \setminus B_2$  if  $S \cap (A_1 \cup C) \neq \emptyset$  and  $S_1 = (S \setminus (A_2 \cup B_2)) \cup \{a_2\}$  otherwise. Note that  $K$  and  $S_1$  are respectively cliques and stable sets in  $G_1$ . So there exists a cut  $(W, W') \in \mathcal{F}_1$  that separates  $K$  and  $S_1$  in  $G_1$ . If  $S \cap (A_1 \cup C) = \emptyset$ , then  $a_2 \in S_1$ , hence  $a_2 \notin W$ , so the cut  $(W, W' \cup A_2 \cup B_2)$



is a member of  $\mathcal{F}$  that separates  $K$  and  $S$  in  $G$ . On the other hand, if  $S \cap (A_1 \cup C) \neq \emptyset$ , then  $S \cap A_2 = \emptyset$ , so whether or not  $a_2$  is in  $W$ , both possible cuts  $(W \cup A_2, W' \cup B_2)$  and  $(W, W' \cup A_2 \cup B_2)$  separate  $K$  and  $S$  in  $G$ , and one of them is a member of  $\mathcal{F}$ . We may assume now that  $K \cap (B_1 \cup B_2) = \emptyset$ . First, suppose that  $S \cap (A_1 \cup A_2) \neq \emptyset$ , so, up to symmetry, we may assume that  $S \cap A_1 \neq \emptyset$  and let  $K_1 = (K \setminus A_2) \cup \{a_2\}$ . Note that  $K_1$  is a clique in  $G_1$ . So there exists a cut  $(W, W') \in \mathcal{F}_1$  that separates  $K_1$  and  $S \cap V(G_1)$ . Hence, the cut  $(W \cup A_2, W' \cup B_2)$  is a member of  $\mathcal{F}$  that separates  $K$  and  $S$  in  $G$ . Finally, suppose that  $S \cap (A_1 \cup A_2) = \emptyset$  and let  $K_1 = (K \setminus A_2) \cup \{a_2\}$ . Note that  $K_1$  is a clique in  $G_1$ . So there exists a cut  $(W, W') \in \mathcal{F}_1$  that separates  $K_1$  and  $S \cap V(G_1)$ . Hence, the cut  $(W \cup A_2, W' \cup B_2)$  is a member of  $\mathcal{F}$  that separates  $K$  and  $S$  in  $G$ .  $\square$

Observe now that degeneracy can be seen as a kind of decomposition:

**Theorem 3.2.** *Let  $s > 0$  be a constant and let  $G$  be a graph on  $n$  vertices admitting an ordering  $v_1, \dots, v_n$  of its vertices such that  $|N(v_i) \cap \{v_{i+1}, \dots, v_n\}| \leq n^{s/\sqrt{\log n}}$  for every  $i \in \{1, \dots, n-1\}$ . Then  $G$  admits a CS-Separator of size  $n^{\mathcal{O}(1)}$ .*

*Proof.* Let  $\mathcal{C}'$  be the class of all graphs of size at most  $n^{s/\sqrt{\log n}}$ . By [20], every graph with  $m$  vertices admits a CS-Separator of size  $m^{\mathcal{O}(\log m)}$ , i.e. of size at most  $m^{c \log m}$  for some constant  $c$ . Hence every graph  $H \in \mathcal{C}'$  admits a CS-Separator of size

$$\left(n^{s/\sqrt{\log n}}\right)^{c \log(n^{s/\sqrt{\log n}})} = n^{cs^2} \quad \text{where } n \text{ still denotes } |V(G)|.$$

Now we recursively decompose  $G[v_i, \dots, v_n]$  with a  $\mathcal{C}'$ -neighborhood decomposition at vertex  $v_i$ , namely  $(G[\{v_{i+1}, \dots, v_n\}], N(v_i) \cap \{v_{i+1}, \dots, v_n\})$ , which is possible by degeneracy hypothesis. We obtain a valid decomposition tree  $T$  for  $G$  with leaves in  $\mathcal{C}'$ , such that the right son of every internal node of  $T$  is a leaf, and the height of  $T$  is  $n - 1$ . This proves that  $|V(T)|$  is linear in  $n$ . By Lemma 2.1,  $G$  admits a CS-Separator of size  $\mathcal{O}(n^{cs^2+1})$  which is a  $n^{\mathcal{O}(1)}$ .  $\square$

We finish this section by studying two types of situations that are not really, or not at all, decompositions but nevertheless are nice to encounter when one tries to obtain polynomial-size CS-separators.

**Theorem 3.3.** *Let  $\mathcal{C}$  be a hereditary class of graphs and  $c \geq 1$ . Assume that  $\mathcal{C}'$  is a class of graphs having the  $\mathcal{O}(n^c)$ -CS-Separation property. If every prime atom of  $\mathcal{C}$  is nearly- $\mathcal{C}'$ , then  $\mathcal{C}$  has the  $\mathcal{O}(n^{c+3})$ -CS-Separation property.*

*Proof.* We follow the method introduced in Section 2 by recursively decomposing every member  $G$  of  $\mathcal{C}$  along components, anticomponents, modules, clique-cutset, or  $\mathcal{C}'$ -antineighborhood until reaching graphs of  $\mathcal{C}'$  or cliques.

Let  $\mathcal{K}$  be the class of complete graphs. For every  $G \in \mathcal{C}$ , we recursively define a valid decomposition tree  $T(G)$  with leaves in  $\mathcal{C}' \cup \mathcal{K}$  as follows. If  $G \in \mathcal{C}' \cup \mathcal{K}$ , then the root  $r$  is the only node of  $T(G)$ . Otherwise, we use a valid decomposition  $(G_1, G_2)$  of  $G$  as described below and  $T(G)$  is obtained from  $T(G_1)$  and  $T(G_2)$  by connecting the root  $r$  to the respective roots of  $T(G_1)$  and  $T(G_2)$ . Moreover the map  $\varphi : V(T) \rightarrow \mathcal{P}(V(G))$  is naturally obtained from the maps  $\varphi_1$  and  $\varphi_2$  associated to  $T(G_1)$  and  $T(G_2)$  respectively, by setting  $\varphi(t) = \varphi_i(t)$  if  $t \in V(T(G_i))$  for  $i = 1, 2$ , and  $\varphi(r) = V(G)$ . We proceed as follows:

- If  $G$  is not connected or not anticonnected, we use a component or anticomponent decomposition.
- Otherwise, if  $G$  has a non-trivial module  $M$  we use the module decomposition.

- Otherwise, if  $G$  has a clique-cutset, we use a clique-cutset decomposition.
- Finally if  $G$  is nearly  $\mathcal{C}'$ , then we use the  $\mathcal{C}'$ -antineighborhood decomposition.

Observe that, since prime atoms of  $\mathcal{C}$  are nearly  $\mathcal{C}'$ , then every  $G \in \mathcal{C}$  satisfies at least one condition, hence  $T(G)$  is well-defined (but maybe not unique - this does not matter).

We now want to define an injective  $\mathcal{S}$ -labeling of  $T(G)$ , for some well-chosen  $\mathcal{S}$ . A *trio* of  $G$  is a subset  $X \subseteq V(G)$  of at most three vertices, containing a non-edge. Let  $\mathcal{S}$  be the set of trios of  $G$ . Clearly  $|\mathcal{S}| \leq |V(G)|^3$ .

Once again, we distinguish cases depending on the rule that was used to decompose each internal node  $t$ .

- Rule 1:  $G[\varphi(t)]$  is not connected, i.e.  $\varphi(t) = V_1 \uplus V_2$  with  $V_1$  anticomplete to  $V_2$ . Let  $v_i$  be any vertex of  $V_i$ , for  $i = 1, 2$  and define  $\ell(t) = \{v_1, v_2\}$ .
- Rule 2:  $G[\varphi(t)]$  is not anticonnected, i.e.,  $\varphi(t) = V_1 \uplus V_2$  with  $V_1$  complete to  $V_2$ . Since  $t$  is not a clique (otherwise,  $t$  would be a leaf), there exists a non-edge  $uv$  in, say,  $V_1$ . Let  $v_2$  be any vertex of  $V_2$  and define  $\ell(t) = \{u, v, v_2\}$ .
- Rule 3:  $G[\varphi(t)]$  has a non-trivial module  $M$ . Let  $u, v$  be two vertices of  $M$  and  $x$  be a common non-neighbor of  $u$  and  $v$  (which exists otherwise Rule 2 applies). We define  $\ell(t) = \{u, v, x\}$ .
- Rule 4:  $G[\varphi(t)]$  has a partition  $(V_1, K, V_2)$  where  $K$  is a clique-cutset that separates  $V_1$  from  $V_2$ . Let  $v_i$  be any vertex of  $V_i$ , for  $i = 1, 2$  and define  $\ell(t) = \{v_1, v_2\}$ .
- Rule 5:  $G[\varphi(t)]$  has a  $\mathcal{C}'$ -antineighborhood decomposition at vertex  $x$ : let  $v$  be any non-neighbor of  $x$  (which exists, otherwise Rule 2 applies) and define  $\ell(t) = \{x, v\}$ .

For Rules 1 to 5, we have that  $\ell(t) \not\subseteq \varphi(s)$  and  $\ell(t) \not\subseteq \varphi(s')$  where  $s, s'$  are the children of  $t$ . Moreover, for Rules 1 up to 3,  $\varphi(s)$  and  $\varphi(s')$  intersect on at most one vertex, and for Rule 4,  $\varphi(s) \cap \varphi(s')$  is a clique. Finally for Rule 5, the node  $s'$  is a leaf. Hence  $\varphi(s) \cap \varphi(s')$  does not contain any trio, unless  $s'$  is a leaf. Hence  $\ell$  satisfies both conditions of Lemma 2.2, consequently  $\ell$  is injective. This implies that  $|V(T(G))|$  is a  $\mathcal{O}(|V(G)|^3)$ . By Lemma 3.1, every decomposition used above is valid so  $T(G)$  is a valid decomposition tree with leaves in  $\mathcal{C}' \cup \mathcal{K}$ . Every graph  $H$  of  $\mathcal{C}'$  admits a CS-Separator of size  $\mathcal{O}(|V(H)|^c)$  by assumption, and every graph  $H$  of  $\mathcal{K}$  is a clique so admits a linear CS-Separator. According to Lemma 2.1,  $G \in \mathcal{C}$  admits a CS-Separator of size  $\mathcal{O}(|V(G)|^{c+3})$ . This concludes the proof.  $\square$

We say that a class of graph  $\mathcal{C}$  is at distance  $f(n)$  of a class  $\mathcal{C}'$  if for every member  $G$  of  $\mathcal{C}$  with  $|V(G)| = n$  there exists a set  $D \subseteq V(G)$  such that  $|D| \leq f(n)$  and  $G \setminus D \in \mathcal{C}'$ .

**Lemma 3.4.** *Let  $\mathcal{C}'$  be a class of graphs having the  $\mathcal{O}(n^{c'})$ -CS-Separation property, and let  $\mathcal{C}$  be a class of graphs at distance  $c \cdot \log(n)$  of  $\mathcal{C}'$ , for some positive constants  $c, c'$ . Then,  $\mathcal{C}$  has the  $\mathcal{O}(n^{c'+c})$ -CS-Separation property.*

*Proof.* Let  $G$  be a graph in  $\mathcal{C}$  with  $|V(G)| = n$  and  $D \subseteq V(G)$  be a set of at most  $c \log(n)$  vertices such that  $G \setminus D$  is in  $\mathcal{C}'$ . Let  $G' = G \setminus D$  be the subgraph induced by  $V(G) \setminus D$ . Since  $G'$  is in  $\mathcal{C}'$ , it admits a CS-Separator of size  $\mathcal{O}(n^{c'})$  that separates any disjoint clique and stable set in  $G'$ . We build the family  $\mathcal{F}$  of cuts of  $G$  as follows. For each cut  $F' = (W, W')$  in  $\mathcal{F}'$  and for each subset of vertices  $X \subseteq D$ , add the cut  $F = (W \cup X, W' \cup (D \setminus X))$  to  $\mathcal{F}$ . Let us check that  $\mathcal{F}$  is a CS-separator of  $G$ : for every clique  $K$  and every stable set  $S$  disjoint from  $K$  in  $G$ , there is a cut in  $\mathcal{F}' = (W, W')$  that separates  $K \cap V(G')$  and  $S \cap V(G')$ , furthermore, by choosing  $X = K \cap D$ , the corresponding cut  $F = (W \cup X, W' \cup (D \setminus X))$  of  $\mathcal{F}$  separates  $K$  and  $S$  in  $G$ . The family  $\mathcal{F}$  of cuts is of size  $\mathcal{O}(n^{c'} 2^{c \log n}) = \mathcal{O}(n^{c'+c})$ .  $\square$

## 4 Applications

Let us now apply the method exposed above to prove that the polynomial CS-Separation property holds in various classes of graphs.

### 4.1 Apple-free graphs

An *apple*  $A_k$  is the graph obtained from a chordless cycle  $C_k$  of length at least 4 by adding a vertex having exactly one neighbor on the cycle (see Figure 2). A graph is *apple-free* if it does not contain any  $A_k$  for  $k \geq 4$  as an induced subgraph. This class of graphs was introduced in [17] under the name *pan-free graphs* as a generalization of claw-free graphs. Brandstädt et al. [5] provide an in-depth structural study of apple-free graphs in order to design a polynomial-time algorithm for solving the maximum weighted stable set problem in this class.

The polynomial CS-Separation property holds both for claw-free graphs and chordal graphs (and even for any  $C_4$ -free graphs), so it seems interesting to study the CS-Separation in the class of apple-free graphs, which generalizes both these classes. Moreover, it is not known whether the polynomial CS-Separation property holds in perfect graphs. Hence apple-free graphs appear as a successful attempt to forbid cycle-like structures, especially if the method seems to provide some insight on how to tackle the problem in further classes. We can prove the following:

**Theorem 4.1.** *The class of apple-free graphs has the polynomial CS-Separation property.*

Let  $A_4$ ,  $A_5$ ,  $D_6$  and  $E_6$  be the graphs described in Figure 2. For fixed  $p \geq 3$ , let  $C_{k \geq p}$  be the set of chordless cycles on at least  $p$  vertices. Here are properties from [5] that we will use.

**Theorem 4.2** ([5]).

- (i) *Every apple-free graph is either  $(C_{k \geq 7})$ -free or claw-free.*
- (ii) *Every prime  $(A_4, A_5, A_6, C_{k \geq 7})$ -free atom is nearly  $D_6$ - and  $E_6$ -free.*
- (iii) *Every  $(A_4, A_5, A_6, D_6, E_6, C_{k \geq 7})$ -free graph is either  $C_6$ -free or claw-free.*
- (iv) *Every prime  $(A_4, A_5, C_{k \geq 6})$ -free atom is nearly  $C_5$ -free.*
- (v) *Every prime  $(A_4, C_5, C_{k \geq 6})$ -free atom is nearly chordal.*

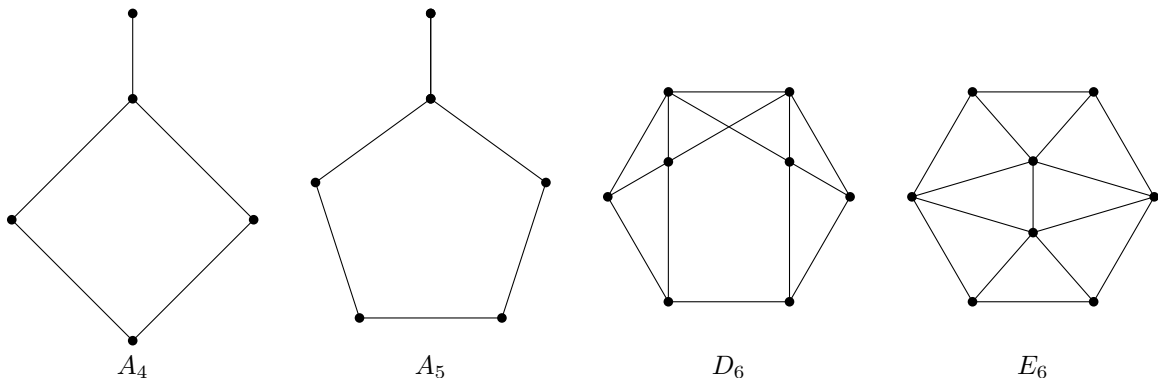


Figure 2: From left to right:  $A_4$ ,  $A_5$ ,  $D_6$  and  $E_6$ .

*Proof of Theorem 4.1.* We repeatedly use Theorem 3.3 for each item of Theorem 4.2.

By (v), prime  $(A_4, C_{k \geq 5})$ -free atoms are nearly chordal. Moreover chordal graphs have the linear-CS-Separation property (because a chordal graph has linearly many maximal cliques), hence by Theorem 3.3, every  $(A_4, C_{k \geq 5})$ -free graph on  $n$  vertices admits a CS-Separator of size  $\mathcal{O}(n^4)$ .

Now, by (iv), prime  $(A_4, A_5, C_{k \geq 6})$ -free atoms are nearly  $C_5$ -free, hence nearly  $(A_4, C_{k \geq 5})$ -free. By Theorem 3.3, we obtain that every  $(A_4, A_5, C_{k \geq 6})$ -free graph on  $n$  vertices admits a CS-separator of size  $\mathcal{O}(n^7)$ .

By (iii), every  $(A_4, A_5, A_6, D_6, E_6, C_{k \geq 7})$ -free graph  $G$  is either (a)  $C_6$ -free, and in that case  $G$  is  $(A_4, A_5, C_{k \geq 6})$ -free and admits a CS-separator of size  $\mathcal{O}(n^7)$ , according to the previous point; or (b) claw-free, in which case it also admits a CS-separator of size  $\mathcal{O}(n^4)$  (the polynomial bound in [4] is much worse but stated in a more general context; it can be improved in the specific case of claw-free graphs).

Now, by (ii), prime  $(A_4, A_5, A_6, C_{k \geq 7})$ -free atoms are nearly  $D_6$ - and  $E_6$ -free, so by Theorem 3.3, every  $(A_4, A_5, A_6, C_{k \geq 7})$ -free graph on  $n$  vertices admits a CS-separator of size  $\mathcal{O}(n^{10})$ .

Finally, by (i), any apple-free graph  $G$  on  $n$  vertices is either claw-free or  $(A_4, A_5, A_6, C_{k \geq 7})$ -free, so in any case it admits a CS-separator of size  $\mathcal{O}(n^{10})$ .  $\square$

When the polynomial CS-Separation property holds in a class  $\mathcal{C}$  of graphs, it is natural to wonder whether the following stronger condition also holds: does the Stable Set polytope admit a compact extended formulation in  $\mathcal{C}$ ? We can give a negative answer to this question in apple-free graphs: indeed, Rothvoss [18] proved the existence of line graphs with no compact extended formulation. Since apple-free graphs contain all claw-free graphs and in particular all line graphs, it follows that there is no compact extended formulation for apple-free graphs. In other words, the extension complexity of the Stable Set polytope in this class is not polynomial.

## 4.2 Cap-free graphs

A *hole* is a chordless cycle of length at least 4. A *cap* is a hole with an additional vertex adjacent to exactly two consecutive vertices in the hole. A graph is *cap-free* if it does not contain any cap graph as an induced subgraph. In the context of the Strong Perfect Graph Conjecture, Conforti et al. [7] studied various classes of graphs, including the class of cap-free graphs, for which they provide a decomposition theorem and a polynomial-time recognition algorithm. A *basic cap-free graph* is either a chordal graph or a 2-connected triangle-free graph with at most one additional universal vertex, which we denote by (2-connected) *almost triangle-free graph*. They proved the following:

**Theorem 4.3** ([7]). *Every connected cap-free graph either has an amalgam or is a basic cap-free graph.*

Recall that chordal graphs have the linear CS-separator property. Moreover, almost triangle-free graphs have a quadratic number of maximal cliques (each edge plus the universal vertex). So connected basic cap-free graphs have the quadratic-CS-Separation property.

We can finally prove the following:

**Theorem 4.4.** *Cap-free graphs have the  $\mathcal{O}(n^5)$ -CS-Separation property.*

*Proof.* We follow the method introduced in Section 2 by recursively decomposing every member  $G$  of  $\mathcal{C}$  along components, anticomponents and amalgams until reaching basic cap-free graphs.

For every  $G \in \mathcal{C}$ , we recursively define a valid decomposition tree  $T(G)$  with leaves in basic cap-free graphs as follows. If  $G$  is a basic cap-free graph, then the root  $r$  is the only node of

$T(G)$ . Otherwise, we use a valid decomposition  $(G_1, G_2)$  of  $G$  as described below and  $T(G)$  is obtained from  $T(G_1)$  and  $T(G_2)$  by connecting the root  $r$  to the respective roots of  $T(G_1)$  and  $T(G_2)$ . Moreover the map  $\varphi : V(T) \rightarrow \mathcal{P}(V(G))$  is naturally obtained from the maps  $\varphi_1$  and  $\varphi_2$  associated to  $T(G_1)$  and  $T(G_2)$  respectively, by setting  $\varphi(t) = \varphi_i(t)$  if  $t \in V(T(G_i))$  for  $i = 1, 2$ , and  $\varphi(r) = V(G)$ . We proceed as follows:

- If  $G$  is not connected, we use a component decomposition.
- If  $G$  is not anticonnected, we use an anticomponent decomposition.
- Otherwise, if  $G$  has an amalgam, we use an amalgam decomposition.

By Theorem 4.3, every connected cap-free graph either has an amalgam or is a basic cap-free graph. Hence  $T(G)$  is well-defined (but maybe not unique - this does not matter). We define an injective  $\mathcal{S}$ -labeling of  $T(G)$ , for some well-chosen  $\mathcal{S}$ . Let  $\mathcal{S}$  be the set of trios of  $G$  (as in the proof of Theorem 3.3, a *trio* is a set of at most three vertices containing a non-edge). Clearly  $|\mathcal{S}| \leq |V(G)|^3$ .

Once again, we distinguish cases depending on the rule that was used to decompose each internal node  $t$ .

- Rule 1:  $G[\varphi(t)]$  is not connected, i.e.  $\varphi(t) = V_1 \uplus V_2$  with  $V_1$  anticomplete to  $V_2$ . Let  $v_i$  be any vertex of  $V_i$ , for  $i = 1, 2$  and define  $\ell(t) = \{v_1, v_2\}$ .
- Rule 2:  $G[\varphi(t)]$  is not anticonnected, i.e.,  $\varphi(t) = V_1 \uplus V_2$  with  $V_1$  complete to  $V_2$ . Since  $\varphi(t)$  is not a clique (otherwise,  $t$  would be a leaf), there exists a non-edge  $uv$  in, say,  $V_1$ . Let  $v_2$  be any vertex of  $V_2$  and define  $\ell(t) = \{u, v, v_2\}$ .
- Rule 3:  $G[\varphi(t)]$  contains an amalgam, i.e.,  $\varphi(t) = C \uplus A_1 \uplus A_2 \uplus B_1 \uplus B_2$  with  $C$  a clique complete to  $A_1 \cup A_2$ ,  $A_1$  complete to  $A_2$ ,  $B_1$  anticomplete to  $A_2 \cup B_2$  and symmetrically  $B_2$  anticomplete to  $A_1 \cup B_1$ . Note that  $B_1 \cup B_2 \neq \emptyset$  for otherwise,  $G[\varphi(t)]$  would not be anticonnected and we would have applied Rule 2 instead. So we may assume that  $B_1$  contains at least one vertex  $b_1$ . By definition of an amalgam decomposition, there exists at least two vertices  $u_2, v_2 \in A_2 \cup B_2$ . Then we define  $\ell(t) = \{b_1, u_2, v_2\}$ .

For Rules 1 to 3, we have that  $\ell(t) \not\subseteq \varphi(s)$  and  $\ell(t) \not\subseteq \varphi(s')$  where  $s, s'$  are the children of  $t$ . Moreover, for Rules 1 and 2,  $\varphi(s)$  and  $\varphi(s')$  do not intersect, and for Rule 3,  $\varphi(s) \cap \varphi(s')$  is a clique. Hence  $\varphi(s) \cap \varphi(s')$  does not contain any trio. Hence  $\ell$  satisfies both conditions of Lemma 2.2, consequently  $\ell$  is injective. This implies that  $|V(T(G))|$  is a  $\mathcal{O}(n^3)$ . Moreover, for every leaf  $f$ ,  $G[\varphi(f)]$  is a basic cap-free graph, hence it admits a CS-Separator of size  $\mathcal{O}(n^2)$ . According to Lemma 2.1,  $G \in \mathcal{C}$  admits a CS-Separator of size  $\mathcal{O}(n^5)$ . This concludes the proof.  $\square$

### 4.3 Diamond-wheel-free and $k$ -windmill-free graphs

Although it was already known that both diamond-wheel-free-graphs and  $k$ -windmill-free graphs have the polynomial CS-Separation property, as proved in [14, sections 2.3.2 and 3.4.3], it is worth noting that the general framework discussed in this paper works just fine with those classes of graphs, using  $\mathcal{C}'$ -neighborhood decomposition with some well-chosen class  $\mathcal{C}'$ .

## 5 Limits of the graph decompositions

In this section, contrary to previous sections, we present a negative result. More precisely, we highlight that some decomposition theorems have no chance to be useful for proving the

polynomial CS-Separation property. This happens when the decomposition theorem is based only on one classical decomposition called star-cutset decomposition.

A *star-cutset* in a graph  $G$  is a vertex cut  $X$  such that there is a vertex  $x \in X$  that is adjacent to every vertex in  $X \setminus \{x\}$ . Let  $V_1, V_2$  be a partition of  $G[V \setminus X]$  such that  $V_1$  is anticomplete to  $V_2$ . Then (as usual)  $(G[V_1 \cup X], G[V_2 \cup X])$  is a cutset decomposition of  $G$ , and in this case we call it a *star-cutset* decomposition.

**Theorem 5.1.** *There exists a class  $\mathcal{D}$  of graphs such that:*

1. *every graph of  $\mathcal{D}$  is either a clique or admits a star-cutset decomposition  $(G_1, G_2)$  with  $G_1, G_2 \in \mathcal{D}$ , and*
2.  *$\mathcal{D}$  does not have the polynomial CS-Separator property.*

*Proof.* Let  $\mathcal{C}$  be a class of graphs that does not have the polynomial CS-Separation property. Such a class of graphs exist by [10] (it may be the class of all graphs). Let us now denote by  $\mathcal{C}'$  the closure of  $\mathcal{C}$  by taking induced subgraphs, then we define

$$\mathcal{D} = \{H = (V, E) \mid \exists x \in V(H) \text{ such that } x \text{ is a universal vertex and } H[V \setminus x] \in \mathcal{C}'\}$$

Let us prove that  $\mathcal{D}$  satisfies the first item. Let  $G$  be a graph of  $\mathcal{D}$  that is not a clique. Let  $u, v$  be two non-adjacent vertices. Let  $x$  be a universal vertex of  $G$ , which exists by definition of  $\mathcal{D}$ . Note that  $(x \cup N(x)) \setminus \{u, v\} = V \setminus \{u, v\}$  is a star-cutset. So one can decompose  $G$  into  $G_1 = G[V \setminus u]$  and  $G_2 = G[V \setminus v]$ . By definition of  $\mathcal{D}$ , both graphs  $G_1$  and  $G_2$  are in  $\mathcal{D}$  because  $x$  remains a universal vertex in  $G_1, G_2$ , and  $G_1 \setminus x, G_2 \setminus x$  are induced subgraphs of  $G \setminus x \in \mathcal{C}'$ .

Moreover, we prove by contradiction that  $\mathcal{D}$  satisfies the second item. Assume that  $\mathcal{D}$  has the polynomial CS-Separation property, for some polynomial  $P$ . Let  $G \in \mathcal{C}$ , then  $G \in \mathcal{C}'$  so the graph  $H$  obtained from  $G$  by adding a universal vertex  $x$  is in  $\mathcal{D}$ . In particular, it admits a CS-Separator  $\mathcal{F}$  of size  $P(|V(H)|)$ , and replacing every cut  $(B, W)$  of  $\mathcal{F}$  by  $(B \setminus x, W \setminus x)$  provides a CS-Separator for  $G$  of the same size. Hence  $\mathcal{C}$  has the polynomial CS-Separation property, a contradiction.  $\square$

A drawback of this theorem is that  $\mathcal{D}$  is not hereditary. It would be nice to know whether there exists a hereditary class of graphs satisfying the condition of Theorem 5.1.

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