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FUJIKI CLASS \mathcal{C} AND HOLOMORPHIC GEOMETRIC STRUCTURES

INDRANIL BISWAS AND SORIN DUMITRESCU

ABSTRACT. For compact complex manifolds with vanishing first Chern class that are compact torus principal bundles over Kähler manifolds, we prove that all holomorphic geometric structures on them, of affine type, are locally homogeneous. For a compact simply connected complex manifold in Fujiki class \mathcal{C} , whose dimension is strictly larger than the algebraic dimension, we prove that it does not admit any holomorphic rigid geometric structure, and also it does not admit any holomorphic Cartan geometry of algebraic type. We prove that compact complex simply connected manifolds in Fujiki class \mathcal{C} and with vanishing first Chern class do not admit any holomorphic Cartan geometry of algebraic type.

1. INTRODUCTION

The article deals, in particular, with holomorphic geometric structures, in the sense of Gromov [Gr, DG], on compact complex manifolds. The definition is very general (see Section 2), and interesting classical examples of such structures to keep in mind are holomorphic tensors, holomorphic affine connections, holomorphic projective connections and holomorphic conformal structures. Compact complex manifolds bearing any of these kind of geometric structures are rather special. Based on results obtained in [Du3, BD] we formulate the following:

Question 1.1. *Is it true that any holomorphic geometric structure of affine type on any compact complex manifold with trivial canonical bundle is locally homogeneous?*

Question 1.1 is known to have a positive answer if the manifold is either Kähler (hence Calabi-Yau) [Du3], or if it is in Fujiki class \mathcal{C} with a polystable holomorphic tangent bundle (with respect to some Gauduchon metric) [BD]. The results of [Du3] show that the answer is also yes for compact parallelizable manifolds and for Ghys's deformations of the complex structure on the parallelizable manifolds $SL(2, \mathbb{C})/\Gamma$ with Γ being a uniform lattice [Gh2].

Here we prove that Question 1.1 has a positive answer for compact complex torus principal bundles over compact Kähler Calabi-Yau manifolds (Theorem 1.2).

Since compact complex surfaces with trivial canonical bundle are either complex tori, or K3 surfaces, or primary Kodaira surfaces (elliptic principal bundles over elliptic curves) [BHPV, Chapter 6], from Theorem 1.2 it follows that the answer to Question 1.1 is yes when the dimension of the manifold is two.

More precisely, our result in this direction is:

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Theorem 1.2. *Let X be a compact complex torus holomorphic principal bundle over a compact Kähler manifold with trivial first Chern class in $H^2(X, \mathbb{R})$. Then any holomorphic geometric structure of affine type ϕ on X is locally homogeneous.*

The previous result also stands for holomorphic projective connections and for holomorphic conformal structures even though these two geometric structures are not of affine type. This is because on manifolds with trivial canonical bundle these two geometric structures admit global representatives which are of affine type, namely, a holomorphic affine connection and a holomorphic Riemannian metric respectively.

On the other hand, the previous result does not work in general for non-affine geometric structures. To give an example, recall that compact complex tori $T^n = \mathbb{C}^n/\Lambda$ (with Λ a cocompact lattice) admit holomorphic foliations that are defined by nonconstant holomorphic maps into the complex projective space $\mathbb{C}P^{n-1}$ which are not translation invariant [Gh]. Together with the standard holomorphic parallelization of the holomorphic tangent bundle of T^n they form holomorphic *rigid* geometric structures of non-affine type in Gromov's sense (see Definition 2.1 and Definition 2.2) which are not locally homogeneous.

We expect the answer for Question 1.1 to be yes for all holomorphic geometric structures of affine type on compact complex manifolds in the Fujiki class \mathcal{C} that have trivial canonical bundle. We also expect holomorphic Riemannian metrics to be always locally homogeneous on compact complex manifolds (this was proved in complex dimension three [Du4]) and holomorphic affine connections to be always locally homogeneous on compact complex manifolds with trivial canonical bundle; it may be noted that contrary to the case of holomorphic Riemannian metrics, here the condition on the triviality of the canonical bundle is not automatically satisfied and is in fact even necessary: there exists non-locally homogeneous holomorphic affine connections on principal elliptic bundles with odd first Betti number (hence non-Kähler) over Riemann surfaces of genus $g \geq 2$ [Du5].

In the case where the holomorphic geometric structure ϕ is rigid, Theorem 1.2 enables us to gather information about the fundamental group of the manifold X :

Corollary 1.3. *Let X be a compact complex torus holomorphic principal bundle over a compact Kähler manifold with trivial first Chern class. If X is endowed with a holomorphic rigid geometric structure of affine type ϕ , then the fundamental group of X is infinite.*

Corollary 1.4. *Let X be a compact complex manifold in Fujiki class \mathcal{C} bearing a holomorphic affine connection in TX . Then the fundamental group of X is infinite.*

Notice that Theorem 1.2 applies to many complex torus principal bundles which are not Kähler. For example, one could consider the complex Heisenberg group of upper triangular (3×3) matrices with complex entries and the compact parallelizable manifold obtained by taking the quotient of this group by the lattice of matrices with Gaussian integers as entries. This quotient is biholomorphic to a (non-Kähler) principal elliptic bundle over a two-dimensional compact complex torus. Many other examples and results about complex torus principal bundles can be found in [Hof].

Recall that complex projective spaces admit the standard flat holomorphic projective connection. Also recall that the smooth quadric $z_0^2 + z_1^2 + \dots + z_{n+1}^2 = 0$ in $\mathbb{C}P^{n+1}$ is endowed with a canonical (flat) holomorphic conformal structure given by the quadratic form (this standard holomorphic conformal structure is invariant by the subgroup $\mathrm{PO}(n+2, \mathbb{C})$ of the complex projective group $\mathrm{PGL}(n+2, \mathbb{C})$). More generally, if G is a complex semi-simple Lie group and $P \subset G$ a parabolic subgroup, then the rational manifold G/P is equipped with the standard flat Cartan geometry with model (G, P) .

For holomorphic rigid geometric structures and for holomorphic Cartan geometries of algebraic type (definitions are given in Section 2) we prove the following:

Theorem 4.2. *Let X be a compact complex manifold in the Fujiki class \mathcal{C} , of complex dimension n and of algebraic dimension $n - d$, with $d > 0$. If X admits a holomorphic rigid geometric structure ϕ , then the fundamental group of X is infinite.*

Theorem 4.3. *Let X be a compact complex simply connected manifold in the Fujiki class \mathcal{C} . If X bears a holomorphic Cartan geometry of algebraic type, then X is projective.*

Theorem 4.2 and Theorem 4.3 are in the same spirit as Borel-Remmert result asserting that compact simply connected homogeneous manifolds in Fujiki class \mathcal{C} are projective. Indeed, it should be mentioned that it was conjectured that compact simply connected complex manifolds bearing holomorphic Cartan geometries are homogeneous manifolds. Borel-Remmert theorem states precisely that compact homogeneous manifolds in class \mathcal{C} are biholomorphic to a product of a projective rational homogeneous manifold with a complex torus [Fu] (p. 255).

In the special case where the algebraic dimension of X is zero, Theorem 4.3 asserts that X does not bear holomorphic Cartan geometries of algebraic type. This was proved recently in [BDM] (Theorem 4.1) even for manifolds which are not necessarily in class \mathcal{C} .

Theorem 4.3 implies the following:

Corollary 1.5. *Let X be a compact complex manifold in Fujiki class \mathcal{C} with trivial first Chern class in $H^2(X, \mathbb{R})$. If X bears a holomorphic Cartan geometry of algebraic type, then the fundamental group of X is infinite.*

Notice that Corollary 1.5 should be seen as a natural generalization of Corollary 1.4. Indeed, manifolds X in Fujiki class \mathcal{C} bearing holomorphic affine connections on TX have vanishing Chern classes (see proof of Corollary 1.4 in Section 3).

2. GEOMETRIC STRUCTURES AND SYMMETRIES

Let X be a complex manifold of complex dimension n . For any integer $k \geq 1$, we associate the principal bundle of k -frames

$$R^k(X) \longrightarrow X,$$

which is the bundle of k -jets of local holomorphic coordinates on X . The corresponding structural group D^k is the group of k -jets of local biholomorphisms of \mathbb{C}^n fixing the origin. We note that D^k is a complex affine algebraic group.

Definition 2.1. A *holomorphic geometric structure* of order k on X is a holomorphic D^k -equivariant map ϕ from $R^k(X)$ to a complex algebraic manifold Z endowed with an algebraic action of D^k . The geometric structure ϕ is said to be of affine type if Z is a complex affine manifold.

Holomorphic tensors are holomorphic geometric structures of affine type of order one. More precisely, a holomorphic tensor on X is a holomorphic $\mathrm{GL}(n, \mathbb{C})$ -equivariant map from the frame bundle $R^1(X)$ to a linear complex algebraic representation W of $\mathrm{GL}(n, \mathbb{C})$. Holomorphic affine connections are holomorphic geometric structures of affine type of order two [Gr, DG]. Holomorphic foliations and holomorphic projective connections are holomorphic geometric structure of non-affine type.

The natural notion of symmetry of a holomorphic geometric structure is the following. A (local) biholomorphism of X preserves a holomorphic geometric structure ϕ if its canonical lift to $R^k(X)$ fixes each fiber of the map ϕ . Such a local biholomorphism is called a *local isometry* of ϕ .

A (local) holomorphic vector field on X is called a *Killing vector field* with respect to ϕ if its local flow acts on X by local isometries.

The connected component $\mathrm{Aut}_0(X, \phi)$, containing the identity element, of the automorphism group of (X, ϕ) is a complex Lie subgroup of the automorphism group of X . The corresponding Lie algebra is the vector space of globally defined (holomorphic) Killing vector fields for ϕ .

Definition 2.2. A holomorphic geometric structure ϕ is called *rigid* of order l in Gromov's sense if any local biholomorphism preserving ϕ is uniquely determined by its l -jet in any given point.

Holomorphic affine connections are rigid of order one in Gromov's sense (see [Gr] and the nice expository survey [DG]). The rigidity comes from the fact that local biholomorphisms fixing a point and preserving a connection linearize in exponential coordinates, so they are indeed completely determined by their differential at the fixed point.

Holomorphic Riemannian metrics, holomorphic projective connections and holomorphic conformal structures in dimension ≥ 3 are all rigid holomorphic geometric structures, while holomorphic symplectic structures and holomorphic foliations are non-rigid geometric structures [DG].

The sheaf of local Killing fields of a holomorphic rigid geometric structure ϕ is locally constant. Its fiber is a finite dimensional Lie algebra called *the Killing algebra* of ϕ [DG, Gr].

Gromov's study of local symmetries of analytic rigid geometric structure [DG, Gr] led, in the particular case of simply connected manifolds X , to the following description of

$\text{Aut}_0(X, \phi)$ (see Section 3.5 in [Gr]). The action of $\text{Aut}_0(X, \phi)$ preserves each connected component of the fibers of a meromorphic map

$$\phi^r : X \longrightarrow W^r$$

into an algebraic manifold W^r (representing the r -jets of ϕ), and furthermore, the action of $\text{Aut}_0(X, \phi)$ on each of these connected components is transitive. The two main ingredients of the proof are

- (1) the integrability result showing that, for any r large enough, local isometries are exactly the class of local biholomorphisms that preserve the r -jet of ϕ , and
- (2) the extendibility result proving that local Killing fields on simply connected manifolds extend to the entire manifold [Am, DG, Gr, No].

It now follows that $\text{Aut}_0(X, \phi)$ -orbits in X are locally closed as they coincide with the connected components of the fibers of the meromorphic map ϕ^r .

Inspiration of these results led to Theorem 2.1 in [Du1] which says that the Killing Lie algebra of ϕ act transitively on the connected components of the *algebraic reduction* of X (see also Theorem 3 in [Du3]).

The algebraic dimension $a(X)$ of X is the transcendence degree over \mathbb{C} of the field of meromorphic functions $\mathcal{M}(X)$.

Let us recall the following classical result called the *algebraic reduction* theorem (see [Ue, pp. 25–26]).

Theorem 2.3 (Algebraic Reduction, [Ue, p. 25, Definition 3.3], [Ue, p. 26, Proposition 3.4]). *Let X be a compact connected complex manifold of dimension n and algebraic dimension $a(X) = n - d$. There exists a bi-meromorphic modification*

$$\Psi : \tilde{X} \longrightarrow X$$

and a holomorphic map

$$t : \tilde{X} \longrightarrow V$$

with connected fibers onto a $(n - d)$ -dimensional projective manifold V such that

$$t^*(\mathcal{M}(V)) = \Psi^*(\mathcal{M}(X)).$$

Let $\pi_{red} : X \longrightarrow V$ be the meromorphic fibration given by $t \circ \Psi^{-1}$; it is called the algebraic reduction of X .

This meromorphic fibration π_{red} is called *almost holomorphic* if the Ψ -exceptional locus does not intersect generic t -fibers.

Manifolds with maximal algebraic dimension are those for which the algebraic dimension coincides with the complex dimension. They are called Moishezon manifolds [Ue, p. 26, Definition 3.5]. The algebraic reduction of a Moishezon manifold is a bimeromorphism with a smooth complex projective manifold [Mo], [Ue, p. 26, Theorem 3.6].

More generally, a compact complex manifold is said to be in *the Fujiki class \mathcal{C}* if it is the image of a compact Kähler space under a holomorphic map. A result of Varouchas says

that a compact complex manifold belongs to Fujiki's class \mathcal{C} if and only if it is bimeromorphic to a compact Kähler manifold (in other words, admits compact Kähler modifications) [Va, Section IV.3]. Manifolds in Fujiki's class \mathcal{C} share many common features with Kähler manifolds.

We will investigate in Section 4 the holomorphic geometric structures on compact complex manifolds in Fujiki class \mathcal{C} .

The following theorem is proved using Theorem 2.1 in [Du1].

Theorem 2.4. *Let X be a compact complex simply connected manifold of dimension n endowed with a holomorphic rigid geometric structure. Then there exists a connected complex abelian Lie subgroup L in the group of automorphisms of (X, ϕ) which preserves each fiber of the algebraic reduction π_{red} , and which acts transitively on the generic fibers of π_{red} . Moreover the following hold.*

- (i) *If π_{red} is almost holomorphic, then the L -orbits are compact.*
- (ii) *If X is in the Fujiki class \mathcal{C} , then $a(X) > 0$.*

Proof. It follows from Theorem 2.1 in [Du1] that the orbits of $\text{Aut}_0(X, \phi)$ contain the generic orbits of π_{red} (see also Theorem 3 in [Du3]). Let $\{X_1, \dots, X_l\}$ be a basis of the Lie algebra of $\text{Aut}_0(X, \phi)$; these X_i are globally defined holomorphic vector fields on X .

Consider the holomorphic rigid geometric structure ϕ' which is the juxtaposition of ϕ with the family of holomorphic vector fields $\{X_1, \dots, X_l\}$. Denote by L' the automorphism group $\text{Aut}_0(X, \phi')$ of ϕ' . It can be shown that the Lie algebra of $L' = \text{Aut}_0(X, \phi')$ is in fact the center of the Lie algebra of $\text{Aut}_0(X, \phi)$. Indeed, an element of the family $\{X_1, \dots, X_l\}$ preserves ϕ' if and only if it commutes with all elements in this family (it belongs to the center of the Lie algebra). It follows that L' is the maximal abelian connected subgroup in $\text{Aut}_0(X, \phi)$.

Applying Theorem 2.1 in [Du1] to ϕ' it follows that at generic points in X , the Killing Lie algebra of ϕ' (and hence L') acts transitively on the fibers of the algebraic reduction π_{red} .

Let us consider the almost rigid meromorphic geometric structure (in the sense of Definition 1.2 in [Du1]) ϕ'' obtained by juxtaposing ϕ , the family of holomorphic vector fields $\{X_1, \dots, X_l\}$ and the meromorphic map π_{red} . When applied to ϕ'' , Theorem 2.1 in [Du1] says that, at generic points, the Killing Lie algebra of ϕ'' acts transitively on the fibers of the algebraic reduction π_{red} . Notice that Killing vector fields of ϕ'' are exactly those Killing vector fields of ϕ lying in the center of the Killing Lie algebra of ϕ and preserving each fiber of π_{red} . The connected component of identity in the automorphism group $\text{Aut}(X, \phi'')$ is the maximal connected complex abelian Lie subgroup $L = \text{Aut}_0(X, \phi'')$ of $\text{Aut}_0(X, \phi)$ (and hence of L') preserving each fiber of the algebraic reduction.

(i) If π_{red} is almost holomorphic, its generic fibers are compact. Therefore the generic (and hence all) L -orbits are compact.

(ii) If X is in the Fujiki class \mathcal{C} , the connected component of identity $\text{Aut}_0(X)$ of the automorphism group of the (simply connected) manifold X is a complex linear algebraic

group (see Corollary 5.8 in [Fu]). Then $\text{Aut}_0(X, \phi)$ is the connected component of identity of the subgroup of $\text{Aut}_0(X)$ preserving each fiber of the meromorphic fibration

$$\phi^r : X \longrightarrow W^r.$$

In particular, $\text{Aut}_0(X, \phi)$ is closed in $\text{Aut}_0(X)$. Moreover, $L = \text{Aut}_0(X, \phi'')$ coincides with the connected component of identity of the maximal abelian subgroup of $\text{Aut}_0(X, \phi)$ preserving each fiber of the algebraic reduction of X . It follows that L is closed in $\text{Aut}_0(X, \phi)$. A priori L is not Zariski closed with respect to the complex linear algebraic group structure of $\text{Aut}_0(X)$. Its Zariski closure L^* is an algebraic abelian subgroup of $\text{Aut}_0(X)$ in the terminology of Fujiki (see Definition 2.1 in [Fu]), so L^* is a meromorphic subgroup of $\text{Aut}_0(X)$ (equivalently, the L^* -action on X is compactifiable in Lieberman's terminology [Li, Section 3]).

Since L^* is a connected complex abelian algebraic group, a classical result of Rosenlicht [Ro] shows that L^* is isomorphic to $\mathbb{C}^p \times \mathbb{C}^{*q}$ for some nonnegative integers p and q .

Assume, by contradiction, that $a(X) = 0$. Then L acts with an open dense orbit on X . Consequently, L^* acts with an open dense orbit on X . Since L^* is abelian, this open dense orbit is biholomorphic to L^* and X is a compactification of L^* .

Since L^* is algebraic, it can be seen as a Zariski open dense set in a complex rational manifold L^{**} [Ro]. The action of L^* on X is meromorphic in Fujiki's sense, meaning the holomorphic action map

$$L^* \times X \longrightarrow X$$

extends to a meromorphic map $L^{**} \times X \longrightarrow X$ (see [Fu], Proposition 2.2 and Remark 2.3). Therefore, X is a bi-meromorphic image of L^{**} and hence it is a projective rational manifold (see Lemma 3.8 and Remark 4.1 in [Fu]). It follows that $a(X) = n > 0$: a contradiction. \square

Notice that statement (ii) in Theorem 2.4 will be improved in Theorem 4.2 which asserts that simply connected manifolds in Fujiki class \mathcal{C} bearing holomorphic rigid geometric structures are Moishezon.

The following classical result will be useful in the proof of Proposition 3.3.

Lemma 2.5. *Let X be a compact complex manifold with trivial canonical line bundle K_X , and let L be a connected complex Lie group acting holomorphically on X . Then L preserves any nonzero holomorphic section ω of K_X and, consequently, the smooth finite measure on X defined by the section $\omega \wedge \bar{\omega}$.*

Proof. Let K be a holomorphic fundamental vector field of the L -action on X . Consider the corresponding 1-parameter family of automorphisms of X . The automorphism for any $t \in \mathbb{C}$ will be denoted by Φ^t . The Lie derivative $L_K \omega$ of ω with respect to K is another holomorphic section of K_X which must be of the form $c\omega$, for some $c \in \mathbb{C}$ (also called the divergence of K with respect to ω), because K_X is trivial. This implies that $(\Phi^t)^* \omega = \exp(ct) \cdot \omega$, for all $t \in \mathbb{C}$. Since the total volume $\int_X \omega \wedge \bar{\omega}$ of X is invariant by automorphisms, it follows that the modulus of $\exp(ct)$ equals one. By Liouville Theorem the entire holomorphic map

$t \rightarrow \exp(ct)$ must be constant, equal to 1 (the value in $t = 0$). Consequently, we have $c = 0$ and ω is K -invariant. Since L is connected, it is generated by the flows of its fundamental vector fields. Since all of them preserve ω , we get that ω is L -invariant. \square

In the last section we will deal also with another concept of geometric structure: *the Cartan geometry*. Cartan geometries are geometric structures which are infinitesimally modeled on homogeneous spaces G/H , where G is a complex Lie group and $H \subset G$ is a closed subgroup.

Denote by \mathfrak{g} and \mathfrak{h} the Lie algebras of the Lie groups G and H respectively. Then we have the following definition (see [Sh]).

Definition 2.6. A *holomorphic Cartan geometry* (P, ω) on X with model (G, H) is a holomorphic principal (right) H -bundle

$$\pi : P \longrightarrow X$$

endowed with a holomorphic \mathfrak{g} -valued 1-form ω satisfying:

- (1) $\omega_p : T_p P \rightarrow \mathfrak{g}$ is a complex linear isomorphism for all $p \in P$,
- (2) the restriction of ω to every fiber of π coincides with the left invariant Maurer-Cartan form of H , and
- (3) $(R_h)^* \omega = \text{Ad}(h)^{-1} \omega$, for all $h \in H$, where R_h is the right action of h on P and Ad is the adjoint representation of G on \mathfrak{g} .

The Cartan geometry is said to be *of algebraic type* if the image of H through the adjoint representation of G is a complex algebraic subgroup of $\text{GL}(\mathfrak{g})$.

Usual geometric structures such as holomorphic affine connections, holomorphic projective connections, or holomorphic conformal are examples of holomorphic Cartan geometry of algebraic type [Sh, Pe].

A (local) holomorphic vector field on X is a *Killing field* for the Cartan geometry (P, ω) if it admits a lift to P whose (local) flow commutes with the action of H and also preserves ω . The sheaf of local Killing fields of a holomorphic Cartan geometry is locally constant with fiber a finite dimensional Lie algebra [Me, Pe]: the Killing algebra of (P, ω) .

Gromov's results describing the decomposition in orbits of the Killing algebra of a rigid geometric structure [Gr] was adapted to the context of Cartan geometries of algebraic type by Melnick and Pecastaing [Me, Pe]. Using their results, Theorem 1.2 in [Du2] adapts Theorem 2.1 in [Du1] to Cartan geometries of algebraic type. Therefore, Theorem 2.4 stands also for holomorphic Cartan geometries of algebraic type.

3. TRIVIAL CANONICAL BUNDLE AND GEOMETRIC STRUCTURES

In this section we prove Theorem 1.2, and we deduce Corollary 1.3 and Corollary 1.4. We also prove Proposition 3.2 which is a particular case of Theorem 3.3 (the case where the canonical bundle is trivial).

Let us first prove Theorem 1.2.

Proof of Theorem 1.2. Let T be a complex torus of complex dimension $d > 0$, and let

$$\pi : X \longrightarrow Y$$

be a principal T -fibration over a compact Kähler manifold Y . Since the torus action trivializes the relative tangent bundle of the fibration, the exact sequence induced by $d\pi$ yields $K_X \simeq \pi^*K_Y$.

We also assume that

- the real first Chern class of K_X vanishes (or equivalently, the real first Chern class of K_Y vanishes), and
- X is endowed with a holomorphic geometric structure ϕ of order k which is of affine type.

We need to consider only the case where the complex dimension $n - d$ of Y is positive. Otherwise, X coincides with the complex torus T and, since $R^k(T)$ is isomorphic to $D^k \times T$, the geometric structure ϕ is completely determined by a holomorphic map from T to a complex affine manifold. This map must be constant, and consequently, ϕ is invariant under the action of T on X .

Since Y is Kähler with vanishing first Chern class, its canonical bundle K_Y is of finite order [Be]. Replacing Y by a suitable finite unramified cover of it we will assume that K_Y is trivial (equivalently, K_X is trivial).

Denote by X_1, \dots, X_d a basis of the fundamental vector fields of the T -action on X ; they are holomorphic global vector fields on X which span (and trivialize) the kernel of $d\pi$ (the vertical subbundle $V \subset TX$).

Assume, by contradiction, that ϕ is not locally homogeneous on X . Then Lemma 3.2 in [Du3] proves that, for some integers $a, b \geq 0$, there exists a nontrivial holomorphic section of $(TX)^{\otimes a} \otimes ((TX)^*)^{\otimes b}$ which vanishes at some point in X . Since K_X is trivial, there is a canonical contraction isomorphism between TX and $\Lambda^{n-1}(T^*X)$. Using this isomorphism, we get a nontrivial holomorphic section t of $(T^*X)^{\otimes m}$, with $m = (n-1)a + b$, such that t vanishes at some point in $x_0 \in X$.

We will get a contradiction using an induction on the power m .

First consider the case of $m = 1$. In this case t is a holomorphic one form. Note that $t(X_i)$ is a holomorphic function on X vanishing at x_0 and therefore, it vanishes identically. This proves that V is in the kernel of t . The fibers of π being compact and connected, the form t is the pull-back of a holomorphic one form on Y . This leads to a holomorphic one form on Y vanishing at $\pi(x_0) \in Y$. Since Y is a Kähler Calabi-Yau manifold, holomorphic tensors on Y are parallel with respect to any Ricci flat Kähler metric on it [Be, p. 760, Principe de Bochner]. Hence they cannot vanish at a point without being trivial: a contradiction.

Let us consider now the case of $m > 1$. The induction hypothesis is that the vanishing holds if degree of the tensor is less than m .

We will show that for all $0 < k \leq m$, the contraction of t with any ordered family of k vector fields (they need not be distinct), chosen from the vector fields $\{X_1, \dots, X_d\}$, vanishes identically.

Let us consider first the case of $k = m$. When contracted with any ordered family of m vector fields (need not be distinct) chosen from $\{X_1, \dots, X_d\}$, the tensor t produces a holomorphic function on X which vanishes at x_0 . Therefore all those contractions vanish identically on X .

Now assume that $0 < k < m$. When contracted with any k vector fields among $\{X_1, \dots, X_d\}$, the tensor t produces a holomorphic section of $(T^*X)^{\otimes m-k}$ vanishing at x_0 . Since $m - k < m$, the induction hypothesis on m holds. So this section of $(T^*X)^{\otimes m-k}$ vanishes identically.

Since all those contractions vanishing identically, our tensor t is a holomorphic section of $((TX/V)^*)^{\otimes m}$.

The fibers of π being compact and connected, the tensor t is a pull-back from Y . We get a nontrivial holomorphic section of $((TY)^*)^{\otimes m}$ which vanishes at $\pi(x_0) \in Y$. But, as before, this holomorphic tensor should be parallel with respect to any Ricci flat metric on Y [Be]: a contradiction. \square

Remark 3.1. Assume that Y is a compact complex manifold such that any holomorphic geometric structure of affine type on it is locally homogeneous. The proof of Theorem 1.2 shows that any compact complex torus holomorphic principal bundle X over Y shares the same property (i.e. any holomorphic geometric structure of affine type on X is locally homogeneous as well).

Corollary 1.3 is a direct consequence of Theorem 1.2 because of the following proposition.

Proposition 3.2. *Let X be a compact complex manifold whose canonical bundle is of finite order such that X admits a holomorphic rigid geometric structure ϕ . If ϕ is locally homogeneous, then the fundamental group of X is infinite.*

Proof. Assume by contradiction that the fundamental group of X is finite. Replacing X by its universal cover and ϕ by its pull-back on the universal cover we assume that X is simply connected. Notice that K_X has now become trivial.

Since ϕ is locally homogeneous, the Killing Lie algebra of ϕ is transitive on X . The extendibility result of local Killing fields on simply connected manifolds [Am, No, DG, Gr] implies that the connected component of identity of $\text{Aut}(X, \phi)$ acts transitively on X . It now follows that X is a compact complex homogeneous manifold. Since K_X is trivial, this implies that X is a parallelizable manifold biholomorphic to a quotient of a connected complex Lie group by a co-compact lattice in it [Wa]. In particular, the fundamental group of X is infinite: a contradiction. \square

Proposition 3.3. *Let X be a compact complex manifold in the Fujiki class \mathcal{C} , of complex dimension n and of algebraic dimension $n - d$, with $d > 0$. Assume that the canonical bundle*

K_X of X is trivial and that X admits a holomorphic rigid geometric structure ϕ . Then the fundamental group of X is infinite.

Proof. We assume that X have complex dimension n and algebraic dimension $n - d$, with $d > 0$. Assume, by contradiction, that X have finite fundamental group. Replacing X by its universal cover and pulling-back ϕ , we can assume that X is compact simply connected. We have seen in the proof of Theorem 2.4 that the maximal connected abelian complex Lie subgroup L' of the group of automorphisms of (X, ϕ) acts transitively on the generic fibers of the algebraic reduction (in particular L' has positive dimension) and coincides with the automorphism group of the new holomorphic rigid geometric structure ϕ' constructed by juxtaposing ϕ with a basis $\{X_1, \dots, X_l\}$ of the Lie algebra of $\text{Aut}(X, \phi)$ (this construction was used earlier in the proof of Theorem 2.4). Moreover, since L' also preserves the smooth finite volume defined by a nontrivial holomorphic section of K_X (Lemma 2.5), Section 3.7 in [Gr] shows that the orbits of L' are all compact (see also Section 3.5.4 in [DG]). The orbits must be compact complex tori covered by L' .

The manifold X being in class \mathcal{C} , the main theorem in [GW] asserts that the action of L' factors through a holomorphic action of a compact complex torus T . In particular, generic orbits of the T -action have trivial stabilizer, while all orbits have discrete stabilizers [GW] (Lemma 2.1), and X is a holomorphic principal T -Seifert bundle.

Now notice that X being simply connected, its Albanese map is trivial [Ue], and the proof of Proposition 6.9 in [Fu] asserts that any solvable Lie subalgebra of the Lie algebra of holomorphic vector fields on X have a common zero on any invariant closed analytic set (see also Proposition 1.3 in [GW]). In particular, (holomorphic) fundamental vector fields of the T -action should vanish identically on any compact complex torus embedded in X . This means that they should vanish identically on all T -orbits: a contradiction. \square

We now deduce Corollary 1.4 from Proposition 3.3.

Proof of Corollary 1.4. Let us consider a holomorphic affine connection in TX . We apply a method of [At] (see also [IKO]) to show that all Chern classes of TX of positive degree must vanish. By Chern-Weil theory we compute a representative of the Chern class $c_k(X, \mathbb{R})$ using a Hermitian metric on TX and the associated Levi-Civita connection. We get a representative of $c_k(X, \mathbb{R})$ which is a closed form on X of type (k, k) . We perform the same computations using the holomorphic affine connection in TX and get another representative of $c_k(X, \mathbb{R})$ which is a holomorphic form on X . But on manifolds of type \mathcal{C} , forms of different types (here (k, k) and $(2k, 0)$) are never cohomologous, unless they represent zero in cohomology. This implies the vanishing of $c_k(X, \mathbb{R})$, for all $k > 0$.

In particular, $c_1(X, \mathbb{R}) = 0$ and Theorem 1.5 in [To] implies that there exists a finite integer l such that K_X^l is holomorphically trivial.

Assume, by contradiction, that the fundamental group of X is finite. We replace X by its universal cover which is a compact complex manifold in class \mathcal{C} with trivial canonical bundle bearing a holomorphic affine connection. Proposition 3.3 implies that X is a Moishezon

manifold. By Corollary 2 in [BM2], a Moishezon manifold X admitting a holomorphic Cartan geometry (in particular a holomorphic affine connection) must be a smooth complex projective manifold. But a compact complex projective manifold bearing a holomorphic affine connection (and hence having trivial real Chern classes of positive degree) is covered by a compact complex torus [IKO]: a contradiction. \square

4. FUJIKI CLASS \mathcal{C} AND GEOMETRIC STRUCTURES

This section deals with holomorphic geometric structures on Fujiki class \mathcal{C} manifolds with finite fundamental group. The main results are Theorem 4.2, Theorem 4.3 and Corollary 1.5.

Let us start by proving the following weak version of Theorem 4.2:

Proposition 4.1. *Let X be a compact simply connected Kähler manifold, of complex dimension n and of algebraic dimension $n - 1$. Then X does not admit any holomorphic rigid geometric structure.*

Proof. Assume, by contradiction, that X as in the proposition bears a holomorphic rigid geometric structure ϕ . Then Theorem 2.4 implies that there exists a holomorphic vector field K on X preserving the fibers of the algebraic reduction π_{red} of X . The algebraic reduction π_{red} of any compact Kähler manifold of algebraic dimension $n - 1$ is known to be an almost holomorphic fibration. Then Theorem 2.4 implies that all K -orbits in X are compact.

By a Theorem of Holman, [Ho], the K -action factors through the action of a compact complex torus T of dimension one, and X is a holomorphic Seifert T -principal bundle. It can be shown that the T -action must have trivial stabilizers on all of X . Indeed, if an element $g \in T$ fixes $x \in X$, then its differential at x preserves K and also acts trivially on the quotient $TX/\mathbb{R} \cdot K$ (since the action fixes all fibers of the algebraic reduction). On the other hand, since T is compact, the action of g must be linearizable in the neighborhood of x . This implies that the action of g is trivial in the neighborhood of x and hence on X . It follows that g is the identity element in T , and hence the T -action is free.

This implies that X is a T -principal bundle over a simply connected projective manifold (the basis V of the algebraic reduction). By a result of Blanchard [Bl] (see also Theorem 1.6 in [Hof]) those manifolds are Kähler if and only if the principal bundle is trivial. In particular, they are not simply connected if they are not Kähler: a contradiction. \square

Notice that the above proof of Proposition 4.1 works for Kähler manifolds X for which the algebraic reduction π_{red} is a nontrivial almost holomorphic map. Another point regarding the previous proof is that it adapts to holomorphic Cartan geometries of algebraic type. Indeed, one needs to apply Theorem 1.2 in [Du2] in order to show that there exists a nontrivial holomorphic vector field K preserving the Cartan geometry as well as the fibers of the algebraic reduction. Therefore, the proof of Proposition 4.1 remains valid for holomorphic Cartan geometries of algebraic type.

Theorem 4.2. *Let X be a compact complex manifold in the Fujiki class \mathcal{C} , of complex dimension n and of algebraic dimension $n - d$, with $d > 0$. If X admits a holomorphic rigid geometric structure ϕ , then the fundamental group of X is infinite.*

Proof. Assume, by contradiction, that X has finite fundamental group. We replace X by its universal cover and the rigid geometric structure by the pull-back of ϕ on the universal cover. In this way we may assume that X is simply connected. By Theorem 2.4, the action of the connected complex abelian Lie group L preserves the algebraic reduction $\pi_{red} : X \rightarrow V$. Since X is in class \mathcal{C} , this L is a subgroup of the complex linear algebraic group $\text{Aut}_0(X)$ (Corollary 5.8 in [Fu]).

It follows that in Theorem 2.3 the model \tilde{X} may be chosen, using Hironaka's equivariant resolution with respect to $\text{Aut}_0(X)$ (see Lemma 2.4 point (4), Remark 2.4 point (2) and Lemma 2.5 in [Fu]), such that $\text{Aut}_0(\tilde{X})$ acts transitively, at the generic point in \tilde{X} , on the fibers of the map

$$t : \tilde{X} \rightarrow V.$$

Equivalently, for any generic point $x_0 \in \tilde{X}$, there exist holomorphic vector fields $\{X_1, \dots, X_d\}$ on \tilde{X} such that $\{X_1(x_0), \dots, X_d(x_0)\}$ span the tangent space at x_0 to the corresponding fiber $t^{-1}(t(x_0))$. Since the fiber $t^{-1}(t(x_0))$ is compact and connected, by Blanchard Theorem, $\{X_1, \dots, X_d\}$ span the tangent space to $t^{-1}(t(x_0))$ on an open dense set in $t^{-1}(t(x_0))$. It now follows that the Lie algebra of the stabilizer of $t^{-1}(t(x_0))$ in $\text{Aut}_0(\tilde{X})$ contains $\{X_1, \dots, X_d\}$. Therefore, this stabilizer of $t^{-1}(t(x_0))$ acts with an open dense orbit in $t^{-1}(t(x_0))$. Consequently, $t^{-1}(t(x_0))$ is an almost homogeneous space (the group is the stabilizer in $\text{Aut}_0(\tilde{X})$).

The closure of any $\text{Aut}_0(\tilde{X})$ -orbit is a compact analytic subset in \tilde{X} ([Fu], Lemma 2.4 point (2)). Recall that \tilde{X} being simply connected and in Fujiki class \mathcal{C} , this $\text{Aut}_0(\tilde{X})$ is a complex linear algebraic group acting meromorphically on X [Fu] (Lemma 1 and Corollary 5.8). This means — as in the proof of Theorem 2.3 (ii) — that the action of $\text{Aut}_0(\tilde{X})$ on \tilde{X} extends to a meromorphic map defined on a rational manifold. In particular, the closure of any $\text{Aut}_0(\tilde{X})$ -orbit is the meromorphic image of a rational manifold and hence it is Moishezon.

The smooth fiber $t^{-1}(t(x_0))$ of t is a closed analytic subset in the closure of the $\text{Aut}_0(\tilde{X})$ -orbit of x_0 . Consequently, $t^{-1}(t(x_0))$ is also Moishezon.

It can be shown that the smooth fiber $t^{-1}(t(x_0))$ does not admit any nontrivial holomorphic one-form. Indeed, any holomorphic one-form β on it would furnish a constant function when specialized on any of the vector fields $\{X_1, \dots, X_d\}$. Since all those vector fields vanish at some point in $t^{-1}(t(x_0))$ (Proposition 1.3 in [GW] and the proof of Proposition 6.9 in [Fu]), it follows that $\beta = 0$.

Since $t^{-1}(t(x_0))$ does not admit any nontrivial holomorphic one-form, the first Betti number of the smooth fiber of t is trivial. Then the results of Campana [Ca, Corollary 2] and Fujiki (Proposition 2.5 in [Fu2], see also [Fu3]) imply that \tilde{X} is Moishezon, and hence X is Moishezon: a contradiction. \square

Theorem 4.3. *Let X be a compact complex simply connected manifold in the Fujiki class \mathcal{C} . If X bears a holomorphic Cartan geometry of algebraic type, then X is a complex projective variety.*

Proof. By Corollary 2 in [BM2], a Moishezon manifold X admitting a holomorphic Cartan geometry must be a smooth complex projective manifolds.

We deal now with the case where the algebraic dimension of X is strictly less than the complex dimension of X . In view of Theorem 1.2 in [Du2], the method used in the proof of Theorem 4.2 works for holomorphic Cartan geometries of algebraic type, showing that compact simply connected manifolds in class \mathcal{C} which are not of maximal algebraic dimension (meaning not Moishezon) do not admit any holomorphic Cartan geometry of algebraic type. \square

Let us deduce Corollary 1.5 from Theorem 4.3.

Proof of Corollary 1.5. Assume, by contradiction, that the fundamental group of X is finite. Replacing X by its universal cover endowed with the pull-back of the Cartan geometry, enables one to assume that X is simply connected. Theorem 4.3 implies then that X is a projective manifold. With our assumption on the first Chern class, X is a projective Calabi-Yau manifold. But a projective Calabi-Yau manifold bearing a holomorphic Cartan geometry of algebraic type is covered by a compact complex torus [BM1, Du2]: a contradiction. \square

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