A LOGARITHMICALLY COMPLETELY MONOTONIC FUNCTION INVOLVING THE q-GAMMA FUNCTION

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Abstract. In the paper, the author proves the logarithmically complete monotonicity of a function involving the q-gamma function and derives some inequalities for the q-multinomial coefficient and the q-multivariate beta function. These conclusions generalize corresponding ones for the gamma function, the multinomial coefficient, and the multivariate beta function respectively.

Contents

1. Preliminaries

Suppose that $f : I \to \infty$ is an infinite differentiable function on an interval I. If

$$0 \le (-1)^{n-1} f^{(n-1)}(x) < \infty, \quad n \in \mathbb{N},$$

then we call f a completely monotonic function; if $f : I \to (0, \infty)$ is positive and

$$(-1)^n [\ln f(x)]^{(n)} \ge 0, \quad n \in \mathbb{N},$$

then we call f a logarithmically completely monotonic function; if $f : (0, \infty) \to [0, \infty)$ can be written in the form

$$f(x) = \frac{a}{x} + b + \int_0^\infty \frac{1}{s+x} \mathrm{d}\, \mu(s),$$

where a, b are non-negative constants and μ is a measure on $(0, \infty)$ such that $\int_0^\infty \frac{1}{1+s} \mathrm{d}\, \mu(s) < \infty$, then we call f a Stieltjes transform. For more details on definitions and properties of these kinds of functions, please see [15, Chapter XIII], [32, Chapter 1], and [33, Chapter IV]. Among these three kinds of functions, there are the following relations:

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(1) a function f is completely monotonic on $(0,\infty)$ if and only if it is a Laplace transform, that is, there is a positive measure μ on $[0,\infty)$ such that

$$f(x)=\int_0^\infty e^{-xt}\,\mathrm{d}\,\mu(t);$$

(2) the set of all logarithmically completely monotonic functions is a strict subset of all completely monotonic functions;
(3) the set of all Stieltjes transforms is a strict subset of all logarithmically completely monotonic functions on $(0,\infty)$.

For details on these relations, please refer to [5, 8, 24, 25, 32] and closely related references therein. For information on new developments of these kinds of functions, please refer to [7, 9, 23, 26, 30, 31, 32] and closely related references therein.

The classical Euler's gamma function $\Gamma(z)$ can be defined [22] by

$$\Gamma(z)=\int_0^\infty t^{z-1}e^{-t}\,\mathrm{d}\,t,\quad \Re(z)>0$$

or by

$$\Gamma(z)=\lim_{n\to\infty}\frac{n!n^z}{\prod_{k=0}^n(z+k)},\quad z\in\mathbb{C}\setminus\{0,-1,-2,\dots\}.$$

The logarithmic derivative of $\Gamma(z)$, denoted by $\psi(z)=\frac{\Gamma'(z)}{\Gamma(z)}$, and the derivatives $\psi^{(i)}(x)$ for $i\in\mathbb{N}$ are respectively called the digamma (or psi) funciton and polygamma functions. See [1, Chapter 6] and [16, Chapter 5].

The q-analogue $\Gamma_q(x)$ of the gamma function $\Gamma(x)$ for $q>0$ and $x>0$, called the q-gamma function, can be defined [3, pp. 493–496] and [6, Section 1.10] by

$$\Gamma_q(x)=\begin{cases}(1-q)^{1-x}\prod\limits_{i=0}^\infty\dfrac{1-q^{i+1}}{1-q^{i+x}}, & 0<q<1;\\ (q-1)^{1-x}q^{\binom{x}{2}}\prod\limits_{i=0}^\infty\dfrac{1-q^{-(i+1)}}{1-q^{-(i+x)}}, & q>1;\\ \Gamma(x), & q=1.\end{cases}\tag{1.1}$$

They satisfy

$$\lim_{q\to1^+}\Gamma_q(z)=\lim_{q\to1^-}\Gamma_q(z)=\Gamma(z)\quad\text{and}\quad\Gamma_q(x)=q^{\binom{x-1}{2}}\Gamma_{1/q}(x).$$

The q-analogue $\psi_q(x)$ of the digamma function $\psi(x)$ for $q>0$ and $x>0$, called the q-digamma function, is defined by

$$\psi_q(x)=\begin{cases}\dfrac{\Gamma_q'(x)}{\Gamma_q(x)}; & q\ne1,\\ \psi(x), & q=1.\end{cases}$$

The q-analogues $\psi_q^{(k)}(x)$ of the polygamma functions $\psi^{(k)}(x)$ for $k\in\mathbb{N}$ are called the q-polygamma functions. From (1.1), we obtain that,

(1) when $0<q<1$ and $x\in(0,\infty)$,

$$\psi_q(x)=-\ln(1-q)+(\ln q)\sum_{k=0}^\infty\frac{q^{k+x}}{1-q^{k+x}}=-\ln(1-q)+(\ln q)\sum_{k=1}^\infty\frac{q^{kx}}{1-q^k},$$

(2) when $q>1$ and $x\in(0,\infty)$,

$$\psi_q(x)=-\ln(q-1)+(\ln q)\left(x-\frac12-\sum_{i=1}^\infty\frac{q^{-ix}}{1-q^{-i}}\right).$$

The formula (1.11) in the paper [11] and its corrected version [12] reads that

$$\psi_q(x) = -\ln(1-q) - \int_0^\infty \frac{e^{-xt}}{1-e^{-t}}\,\mathrm{d}\,\gamma_q(t) \tag{1.2}$$

for $0<q<1$ and $x>0$, where

$$\gamma_q(t) = \begin{cases} -\ln q \sum\limits_{k=1}^{\infty} \delta(t+k\ln q), & 0<q<1 \\ t, & q=1 \end{cases} \tag{1.3}$$

and $\delta(t)$ represents the Dirac delta function, that is, $\mathrm{d}\,\gamma_q(t)$ is a discrete measure with positive masses $|\ln q|$ at the positive points $k|\ln q|$ for $k\in\mathbb{N}$. Accordingly, we obtain

$$\int_0^\infty e^{-xt}\,\mathrm{d}\,\gamma_q(t) = -\frac{q^x\ln q}{1-q^x}$$

and

$$\int_0^\infty te^{-xt}\,\mathrm{d}\,\gamma_q(t) = \frac{q^x(\ln q)^2}{(1-q^x)^2}$$

for $0<q<1$ and $x>0$. Differentiating with respect to x on both sides of (1.2) yields

$$\psi_q^{(k)}(x) = (-1)^{k+1}\int_0^\infty \frac{t^k e^{-xt}}{1-e^{-t}}\,\mathrm{d}\,\gamma_q(t), \quad 0<q<1, \quad k\in\mathbb{N}. \tag{1.4}$$

In [10, p. 1245, Theorem 4.4, (4.15)], [18, Lemma 2.3 and Remark 2.1], [19, Theorem 7.2, (7.5)], and [28, p. 152, Theorem 4.22, (4.20)], it was presented that, when $0<q<1$, the identity

$$\psi_q^{(k-1)}(x) - \psi_q^{(k-1)}(x+1) = (\ln q)\frac{\mathrm{d}^{k-1}}{\mathrm{d}\,x^{k-1}}\left(\frac{q^x}{1-q^x}\right)$$

is valid for $x\in(0,\infty)$ and $k\in\mathbb{N}$. One can also find these knowledge in [18, 19, 20, 21, 27] and closely related references.

2. Motivations

Let $\boldsymbol{a}=(a_1,a_2,\dots,a_m)$ with $a_i>0$ for $1\le i\le m$, let $\boldsymbol{p}=(p_1,p_2,\dots,p_m)$ with $\sum_{i=1}^m p_i=1$ and $p_i\in(0,1)$ for $1\le i\le m$, let

$$\binom{\sum_{i=1}^m a_i}{a_1,a_2,\dots,a_m} = \frac{\Gamma\bigl(1+\sum_{i=1}^m a_i\bigr)}{\Gamma(1+a_1)\Gamma(1+a_2)\cdots\Gamma(1+a_m)}$$

denote the multinomial coefficient, and let

$$\mathrm{B}(a_1,a_2,\dots,a_m) = \frac{\Gamma(a_1)\Gamma(a_2)\cdots\Gamma(a_m)}{\Gamma(a_1+a_2+\cdots+a_m)}$$

be the multivariate beta function. In [17] and [29], the authors considered the function

$$\begin{aligned}\mathcal{Q}(x) = \mathcal{Q}_{\boldsymbol{a},\boldsymbol{p};m}(x) &= \frac{\Gamma\bigl(1+x\sum_{i=1}^m a_i\bigr)}{\prod_{i=1}^m\Gamma(1+xa_i)}\prod_{i=1}^m p_i^{xa_i} = \binom{x\sum_{i=1}^m a_i}{xa_1,xa_2,\dots,xa_m}\prod_{i=1}^m p_i^{xa_i} \\ &= \frac{\sum_{i=1}^m a_i}{\prod_{i=1}^m a_i}\frac{\prod_{i=1}^m p_i^{xa_i}}{x^{m-1}\,\mathrm{B}(xa_1,xa_2,\dots,xa_m)}, \quad x\in(0,\infty), \quad m\in\mathbb{N}\end{aligned} \tag{2.1}$$

and, among other things, proved the following theorem.

Theorem 2.1 ([17, Theorem 2.1] and [29, Thereom 2.2])**.** *The function $\mathcal{Q}(x)=\mathcal{Q}_{\boldsymbol{a},\boldsymbol{p};m}(x)$ defined in* (2.1) *is logarithmically completely monotonic on $(0,\infty)$.*

For more information on the backgrounds, history, motivations, origins, and applications of the function $\mathcal{Q}(x)=\mathcal{Q}_{\boldsymbol{a},\boldsymbol{p};m}(x)$, please refer to [2, 13, 17, 29] and closely related references therein.

In this paper, we consider the q-analogue of the function $\mathcal{Q}(x)=\mathcal{Q}_{\boldsymbol{a},\boldsymbol{p};m}(x)$ and investigate its logarithmically complete monotonicity.

3. A logarithmically completely monotonic function

Let $\boldsymbol{a}=(a_1,a_2,\dots,a_m)$ with $a_i>0$ for $1\le i\le m$ and let $\boldsymbol{p}=(p_1,p_2,\dots,p_m)$ with $\sum_{i=1}^m p_i=1$ and $p_i\in(0,1)$ for $1\le i\le m$. Define

$$\mathcal{Q}_q(x)=\mathcal{Q}_{q;\boldsymbol{a},\boldsymbol{p};m}(x)=\frac{\Gamma_q\bigl(1+x\sum_{i=1}^m a_i\bigr)}{\prod_{i=1}^m\Gamma_q(1+xa_i)}\prod_{i=1}^m p_i^{xa_i} \tag{3.1}$$

for $q\in(0,1)$, $x\in(0,\infty)$, and $m\in\mathbb{N}$. It is clear that

$$\lim_{q\to1^-}\mathcal{Q}_q(x)=\lim_{q\to1^-}\mathcal{Q}_{q;\boldsymbol{a},\boldsymbol{p};m}(x)=\mathcal{Q}(x)=\mathcal{Q}_{\boldsymbol{a},\boldsymbol{p};m}(x).$$

Theorem 3.1. *The function $\mathcal{Q}_q(x)=\mathcal{Q}_{q;\boldsymbol{a},\boldsymbol{p};m}(x)$ defined in (3.1) is logarithmically completely monotonic on $(0,\infty)$.*

Proof. From (1.3) for $0<q<1$ and the well-known property $\delta(at)=\frac{1}{|a|}\delta(t)$, it follows that

$$\gamma_q\biggl(\frac{t}{\tau}\biggr)=-\ln q\sum_{k=1}^\infty\delta\biggl(\frac{t+k\tau\ln q}{\tau}\biggr)=-\tau\ln q\sum_{k=1}^\infty\delta(t+k\tau\ln q)$$

for $\tau>0$. This implies that

$$\mathrm{d}\,\gamma_q\biggl(\frac{t}{\tau}\biggr)=\tau\,\mathrm{d}\,\gamma_q(t). \tag{3.2}$$

Direct calculation gives

$$\ln\mathcal{Q}_q(x)=\ln\Gamma_q\biggl(1+x\sum_{i=1}^m a_i\biggr)-\sum_{i=1}^m\ln\Gamma_q(1+a_ix)+x\sum_{i=1}^m a_i\ln p_i,$$

$$[\ln\mathcal{Q}_q(x)]'=\biggl(\sum_{i=1}^m a_i\biggr)\psi_q\biggl(1+x\sum_{i=1}^m a_i\biggr)-\sum_{i=1}^m a_i\psi_q(1+a_ix)+\sum_{i=1}^m a_i\ln p_i,$$

and

$$[\ln\mathcal{Q}_q(x)]''=\biggl(\sum_{i=1}^m a_i\biggr)^2\psi_q'\biggl(1+x\sum_{i=1}^m a_i\biggr)-\sum_{i=1}^m a_i^2\psi_q'(1+a_ix).$$

Making use of (1.4) for $k=1$ and the equality (3.2), it follows that

$$\begin{aligned}\psi_q'(\tau x+1)&=\int_0^\infty\frac{t}{1-e^{-t}}e^{-(\tau x+1)t}\,\mathrm{d}\,\gamma_q(t)=\int_0^\infty\frac{t}{e^t-1}e^{-\tau xt}\,\mathrm{d}\,\gamma_q(t)\\&=\int_0^\infty h\biggl(\frac{v}{\tau}\biggr)e^{-vx}\,\mathrm{d}\,\gamma_q\biggl(\frac{v}{\tau}\biggr)=\tau\int_0^\infty h\biggl(\frac{v}{\tau}\biggr)e^{-vx}\,\mathrm{d}\,\gamma_q(v),\end{aligned}$$

where $\tau>0$ and $h(t)=\frac{t}{e^t-1}$. Hence, we have

$$[\ln\mathcal{Q}_q(x)]''=\int_0^\infty\biggl[\biggl(\sum_{i=1}^m a_i\biggr)^3h\biggl(\frac{v}{\sum_{i=1}^m a_i}\biggr)-\sum_{i=1}^m a_i^3h\biggl(\frac{v}{a_i}\biggr)\biggr]e^{-xv}\,\mathrm{d}\,\gamma_q(v). \tag{3.3}$$

By calculus, we have

$$\frac{\mathrm{d}}{\mathrm{d}\,t}\biggl[t^3h\biggl(\frac{1}{t}\biggr)\biggr]=\frac{\mathrm{d}}{\mathrm{d}\,t}\biggl(\frac{t^2}{e^{1/t}-1}\biggr)=\frac{e^{1/t}(2t+1)-2t}{(e^{1/t}-1)^2},$$

$$\frac{\mathrm{d}^2}{\mathrm{d}\,t^2}\left[t^3h\left(\frac{1}{t}\right)\right]=\frac{2t^2-e^{1/t}\left(4t^2+2t-1\right)+e^{2/t}\left(2t^2+2t+1\right)}{(e^{1/t}-1)^3t^2}\to0,\quad t\to0^+,$$

$$\frac{\mathrm{d}^3}{\mathrm{d}\,t^3}\left[t^3h\left(\frac{1}{t}\right)\right]=\frac{e^{1/t}\left(4e^{1/t}+e^{2/t}+1\right)}{(e^{1/t}-1)^4t^4}>0$$

on $(0,\infty)$. As a result, the function $t^3h\left(\frac{1}{t}\right)$ is strictly convex on $(0,\infty)$. Recall from [14, p. 650] that

(1) a function $\phi:[0,\infty)\to\mathbb{R}$ is said to be star-shaped if $\phi(\alpha x)\le\alpha\phi(x)$ for all $\alpha\in[0,1]$ and $x\ge0$;
(2) a real function ϕ defined on a set $S\subset\mathbb{R}^n$ is said to be super-additive if $x,y\in S$ implies $x+y\in S$ and $\phi(x+y)\ge\phi(x)+\phi(y)$;
(3) if ϕ is a real function defined on $[0,\infty)$, $\phi(0)\le0$, and ϕ is convex, then ϕ is star-shaped, but convexity is not a property of all star-shaped functions;
(4) if $\phi:[0,\infty)\to\mathbb{R}$ is star-shaped, then ϕ is super-additive.

Consequently, since $\lim_{t\to0^+}\frac{t^2}{e^{1/t}-1}=0$, the function $t^3h\left(\frac{1}{t}\right)$ is star-shaped, and then super-additive, on $(0,\infty)$. As a result, it follows inductively that

$$\left(\sum_{i=1}^m\frac{a_i}{v}\right)^3h\left(\frac{1}{\sum_{i=1}^m(a_i/v)}\right)\ge\sum_{i=1}^m\left(\frac{a_i}{v}\right)^3h\left(\frac{1}{a_i/v}\right)$$

which can be rearranged as

$$\left(\sum_{i=1}^m a_i\right)^3h\left(\frac{v}{\sum_{i=1}^m a_i}\right)\ge\sum_{i=1}^m a_i^3h\left(\frac{v}{a_i}\right).$$

Combining this with (3.3) yields that the second derivative $[\ln\mathcal{Q}_q(x)]''$ is completely monotonic on $(0,\infty)$.

Complete monotonicity of $[\ln\mathcal{Q}_q(x)]''$ implies that the first derivative $[\ln\mathcal{Q}_q(x)]'$ is strictly increasing on $(0,\infty)$. Hence, from (1.2), it follows that

$$\begin{aligned}[\ln\mathcal{Q}_q(x)]'&\le\lim_{x\to\infty}\left[\left(\sum_{i=1}^m a_i\right)\psi_q\left(1+x\sum_{i=1}^m a_i\right)-\sum_{i=1}^m a_i\psi_q(1+a_ix)\right]+\sum_{i=1}^m a_i\ln p_i\\&=\left[\left(\sum_{i=1}^m a_i\right)\lim_{x\to\infty}\psi_q\left(1+x\sum_{i=1}^m a_i\right)-\sum_{i=1}^m a_i\lim_{x\to\infty}\psi_q(1+a_ix)\right]+\sum_{i=1}^m a_i\ln p_i\\&=\left[-\left(\sum_{i=1}^m a_i\right)\ln(1-q)+\sum_{i=1}^m a_i\ln(1-q)\right]+\sum_{i=1}^m a_i\ln p_i=\sum_{i=1}^m a_i\ln p_i<0.\end{aligned}$$

By definition, the function $\mathcal{Q}_q(x)$ is logarithmically completely monotonic on $(0,\infty)$. The proof of Theorem 3.1 is complete. □

4. Inequalities for the q-multinomial coefficient

Let $\boldsymbol{a}=(a_1,a_2,\dots,a_m)$ with $a_i>0$ for $1\le i\le m$. Then the q-multinomial coefficient for $q\in(0,1)$ is

$$\binom{\sum_{i=1}^m a_i}{a_1,a_2,\dots,a_m}_q=\frac{\Gamma_q\left(1+\sum_{i=1}^m a_i\right)}{\Gamma_q(1+a_1)\Gamma_q(1+a_2)\dots\Gamma_q(1+a_m)}.\tag{4.1}$$

With the aid of logarithmically complete monotonicity of $\mathcal{Q}_q(x)=\mathcal{Q}_{q;\boldsymbol{a},\boldsymbol{p};m}(x)$, we now present some inequalities for the q-multinomial coefficient.

Theorem 4.1. *Let $q\in(0,1)$, $\ell,m\in\mathbb{N}$, and $\boldsymbol{a}=(a_1,a_2,\dots,a_m)$ with $a_i>0$ for $1\le i\le m$. If $x_j>0$ for $1\le j\le\ell$ and $\lambda_j\in(0,1)$ with $\sum_{j=1}^\ell\lambda_j=1$, then*

$$\binom{\sum_{j=1}^{\ell}\lambda_j x_j \sum_{i=1}^{m} a_i}{a_1\sum_{j=1}^{\ell}\lambda_j x_j, a_2\sum_{j=1}^{\ell}\lambda_j x_j, \dots, a_m\sum_{j=1}^{\ell}\lambda_j x_j}_q \le \prod_{j=1}^{\ell}\binom{x_j\sum_{i=1}^{m} a_i}{x_j a_1, x_j a_2, \dots, x_j a_m}_q^{\lambda_j} \tag{4.2}$$

and the equality in (4.2) *holds if and only if* $x_1 = x_2 = \cdots = x_\ell$.

Proof. The logarithmically complete monotonicity in Theorem 3.1 indicates that the function $\mathcal{Q}_q(x)$ is logarithmically convex on $(0,\infty)$. Hence, one acquires

$$\mathcal{Q}_q\left(\sum_{j=1}^{\ell}\lambda_j x_j\right) \le \prod_{j=1}^{\ell}\mathcal{Q}_q^{\lambda_j}(x_j).$$

Using the expression

$$\mathcal{Q}_q(x) = \binom{x\sum_{i=1}^{m} a_i}{xa_1, xa_2, \dots, xa_m}_q \prod_{i=1}^{m} p_i^{xa_i}$$

leads to

$$\binom{\sum_{j=1}^{\ell}\lambda_j x_j \sum_{i=1}^{m} a_i}{a_1\sum_{j=1}^{\ell}\lambda_j x_j, a_2\sum_{j=1}^{\ell}\lambda_j x_j, \dots, a_m\sum_{j=1}^{\ell}\lambda_j x_j}_q \prod_{i=1}^{m} p_i^{a_i\sum_{j=1}^{\ell}\lambda_j x_j} \le \prod_{j=1}^{\ell}\left[\binom{x_j\sum_{i=1}^{m} a_i}{x_j a_1, x_j a_2, \dots, x_j a_m}_q \prod_{i=1}^{m} p_i^{a_i x_j}\right]^{\lambda_j}$$

which can be rearranged as (4.2). The proof of Theorem 4.1 is complete. □

Theorem 4.2. *Let* $q \in (0,1)$, $\ell, m \in \mathbb{N}$, *and* $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ *with* $a_i > 0$ *for* $1 \le i \le m$. *If* $x_j > 0$ *for* $1 \le j \le \ell$, *then*

$$\prod_{j=1}^{\ell}\binom{x_j\sum_{i=1}^{m} a_i}{x_j a_1, x_j a_2, \dots, x_j a_m}_q < \binom{\sum_{j=1}^{\ell} x_j \sum_{i=1}^{m} a_i}{a_1\sum_{j=1}^{\ell} x_j, a_2\sum_{j=1}^{\ell} x_j, \dots, a_m\sum_{j=1}^{\ell} x_j}_q. \tag{4.3}$$

Proof. In [2, Lemma 3], it was proved that, if $g : [0,\infty) \to (0,1]$ is differentiable and $\frac{g'}{g}$ is strictly increasing on $(0,\infty)$, then the inequality $g(x)g(y) < g(x+y)$ is valid for $x, y \in (0,\infty)$. As a result, we can inductively deduce

$$\prod_{j=1}^{\ell} g(x_j) < g\left(\sum_{j=1}^{\ell} x_j\right).$$

Applying this inequality to the logarithmic convexity of the function $\mathcal{Q}_q(x)$ reveals

$$\prod_{j=1}^{\ell}\left[\binom{x_j\sum_{i=1}^{m} a_i}{x_j a_1, x_j a_2, \dots, x_j a_m}_q \prod_{i=1}^{m} p_i^{a_i x_j}\right] < \binom{\sum_{j=1}^{\ell} x_j \sum_{i=1}^{m} a_i}{a_1\sum_{j=1}^{\ell} x_j, a_2\sum_{j=1}^{\ell} x_j, \dots, a_m\sum_{j=1}^{\ell} x_j}_q \prod_{i=1}^{m} p_i^{a_i\sum_{j=1}^{\ell} x_j}$$

which can be rewritten as (4.3). The proof of Theorem 4.2 is complete. □

Theorem 4.3. *Let $q \in (0,1)$, $m \in \mathbb{N}$, and $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$. If $0 < a \le c$ and $x > 0$, then*

$$\begin{aligned}\begin{pmatrix}(a+x)\sum_{i=1}^m a_i\\(a+x)a_1,(a+x)a_2,\dots,(a+x)a_m\end{pmatrix}_q &\begin{pmatrix}c\sum_{i=1}^m a_i\\ca_1,ca_2,\dots,ca_m\end{pmatrix}_q\\ &\le \begin{pmatrix}a\sum_{i=1}^m a_i\\aa_1,aa_2,\dots,aa_m\end{pmatrix}_q \begin{pmatrix}(c+x)\sum_{i=1}^m a_i\\(c+x)a_1,(c+x)a_2,\dots,(c+x)a_m\end{pmatrix}_q \end{aligned} \tag{4.4}$$

and the equality in (4.4) *holds if and only if $a = c$.*

Proof. For $0 < a < c$, set

$$V(x) = \ln \mathcal{Q}_q(a+x) + \ln \mathcal{Q}_q(c) - \ln \mathcal{Q}_q(a) - \ln \mathcal{Q}_q(c+x).$$

Because

$$V'(x) = \frac{\mathcal{Q}_q'(a+x)}{\mathcal{Q}_q(a+x)} - \frac{\mathcal{Q}_q'(c+x)}{\mathcal{Q}_q(c+x)}$$

and the logarithmically complete monotonicity of $\mathcal{Q}_q(x)$ hints that the function $\frac{\mathcal{Q}_q'(x)}{\mathcal{Q}_q(x)}$ is strictly increasing on $(0,\infty)$, we see that $V'(x) < 0$ and $V(x) < V(0) = 0$. Consequently, it follows that

$$\ln \mathcal{Q}_q(a+x) + \ln \mathcal{Q}_q(c) \le \ln \mathcal{Q}_q(a) + \ln \mathcal{Q}_q(c+x)$$

which is equivalent to

$$\begin{aligned}\ln&\left[\begin{pmatrix}(a+x)\sum_{i=1}^m a_i\\(a+x)a_1,(a+x)a_2,\dots,(a+x)a_m\end{pmatrix}_q \prod_{i=1}^m p_i^{a_i(a+x)}\right]\\ &+\ln\left[\begin{pmatrix}c\sum_{i=1}^m a_i\\ca_1,ca_2,\dots,ca_m\end{pmatrix}_q \prod_{i=1}^m p_i^{a_i c}\right] \le \ln\left[\begin{pmatrix}a\sum_{i=1}^m a_i\\aa_1,aa_2,\dots,aa_m\end{pmatrix}_q \prod_{i=1}^m p_i^{a_i a}\right]\\ &+\ln\left[\begin{pmatrix}(c+x)\sum_{i=1}^m a_i\\(c+x)a_1,(c+x)a_2,\dots,(c+x)a_m\end{pmatrix}_q \prod_{i=1}^m p_i^{a_i(c+x)}\right].\end{aligned}$$

This can be simplified as (4.4). The proof of Theorem 4.3 is complete. □

5. Inequalities for the q-multivariate beta function

Let $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$. Then the q-multivariate beta function for $q \in (0,1)$ is

$$\mathrm{B}_q(a_1, a_2, \dots, a_m) = \frac{\Gamma_q(a_1)\Gamma_q(a_2)\cdots\Gamma_q(a_m)}{\Gamma_q(a_1 + a_2 + \cdots + a_m)}.$$

In [3, p. 544], it was proved that

$$\Gamma_q(x+1) = \frac{1-q^x}{1-q}\Gamma_q(x)$$

with $\Gamma_q(1) = 1$ for $q \in (0,1)$. Applying this recurrence relation to (4.1) yields

$$\begin{pmatrix}\sum_{i=1}^m a_i\\a_1,a_2,\dots,a_m\end{pmatrix}_q = \frac{(1-q)^{m-1}\left(1-q^{\sum_{i=1}^m a_i}\right)}{\prod_{i=1}^m (1-q^{a_i})}\frac{1}{\mathrm{B}_q(a_1,a_2,\dots,a_m)}. \tag{5.1}$$

Substituting (5.1) into those inequalities in Theorems 4.1 to 4.3, we can conclude several inequalities for the q-multivariate beta function $\mathrm{B}_q(a_1, a_2, \dots, a_m)$. Equivalently speaking, Theorems (4.1) to (4.3) can be respectively reformulated as follows.

Let $q \in (0,1)$, $\ell, m \in \mathbb{N}$, and $\boldsymbol{a} = (a_1, a_2, \dots, a_m)$ with $a_i > 0$ for $1 \le i \le m$. If $x_j > 0$ for $1 \le j \le \ell$ and $\lambda_j \in (0,1)$ with $\sum_{j=1}^{\ell} \lambda_j = 1$, then

$$\frac{\prod_{i=1}^m\left(1-q^{a_i\sum_{j=1}^\ell\lambda_jx_j}\right)}{1-q^{\sum_{j=1}^\ell\lambda_jx_j\sum_{i=1}^m a_i}}\mathrm{B}_q\left(a_1\sum_{j=1}^\ell\lambda_jx_j,a_2\sum_{j=1}^\ell\lambda_jx_j,\dots,a_m\sum_{j=1}^\ell\lambda_jx_j\right)$$
$$\ge\prod_{j=1}^\ell\left[\frac{\prod_{i=1}^m(1-q^{x_ja_i})}{1-q^{x_j\sum_{i=1}^m a_i}}\mathrm{B}_q(a_1x_j,a_2x_j,\dots,a_mx_j)\right]^{\lambda_j}\tag{5.2}$$

and the equality in (5.2) holds if and only if $x_1=x_2=\dots=x_\ell$.

The inequality (5.2) implies that the function

$$\frac{\prod_{i=1}^m(1-q^{a_ix})}{1-q^{x\sum_{i=1}^m a_i}}\mathrm{B}_q(a_1x,a_2x,\dots,a_mx)$$

for $m\in\mathbb{N}$ is logarithmically concave with respect to $x\in(0,\infty)$. Generally, we claim that the function

$$\frac{1-q^{x\sum_{i=1}^m a_i}}{\prod_{i=1}^m(1-q^{a_ix})}\frac{1}{\mathrm{B}_q(a_1x,a_2x,\dots,a_mx)}$$

for $m\in\mathbb{N}$ is logarithmically completely monotonic with respect to $x\in(0,\infty)$.

Let $q\in(0,1)$, $\ell,m\in\mathbb{N}$, and $\boldsymbol{a}=(a_1,a_2,\dots,a_m)$ with $a_i>0$ for $1\le i\le m$. If $x_j>0$ for $1\le j\le\ell$, then

$$\frac{1}{(1-q)^{(\ell-1)(m-1)}}\prod_{j=1}^\ell\left[\frac{\prod_{i=1}^m(1-q^{a_ix_j})}{1-q^{x_j\sum_{i=1}^m a_i}}\mathrm{B}_q(a_1x_j,a_2x_j,\dots,a_mx_j)\right]$$
$$>\frac{\prod_{i=1}^m\left(1-q^{a_i\sum_{j=1}^\ell x_j}\right)}{1-q^{\sum_{j=1}^\ell x_j\sum_{i=1}^m a_i}}\mathrm{B}_q\left(a_1\sum_{j=1}^\ell x_j,a_2\sum_{j=1}^\ell x_j,\dots,a_m\sum_{j=1}^\ell x_j\right).$$

Let $q\in(0,1)$, $m\in\mathbb{N}$, and $\boldsymbol{a}=(a_1,a_2,\dots,a_m)$ with $a_i>0$ for $1\le i\le m$. If $0<a\le c$ and $x>0$, then

$$\frac{1-q^{(a+x)\sum_{i=1}^m a_i}}{1-q^{(c+x)\sum_{i=1}^m a_i}}\frac{\prod_{i=1}^m(1-q^{(c+x)a_i})}{\prod_{i=1}^m(1-q^{(a+x)a_i})}\frac{\mathrm{B}_q((c+x)a_1,(c+x)a_2,\dots,(c+x)a_m)}{\mathrm{B}_q((a+x)a_1,(a+x)a_2,\dots,(a+x)a_m)}$$
$$\le\frac{1-q^{c\sum_{i=1}^m a_i}}{1-q^{a\sum_{i=1}^m a_i}}\frac{\prod_{i=1}^m(1-q^{aa_i})}{\prod_{i=1}^m(1-q^{ca_i})}\frac{\mathrm{B}_q(ca_1,ca_2,\dots,ca_m)}{\mathrm{B}_q(aa_1,aa_2,\dots,aa_m)}\tag{5.3}$$

and the equality in (5.3) holds if and only if $a=c$.

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Institute of Mathematics, Henan Polytechnic University, Jiaozuo, Henan, 454010, China; College of Mathematics, Inner Mongolia University for Nationalities, Tongliao, Inner Mongolia, 028043, China; Department of Mathematics, College of Science, Tianjin Polytechnic University, Tianjin, 300387, China; Institute of Fundamental and Frontier Sciences, University of Electronic Science and Technology of China, Chengdu, Sichuan, 610054, China

Email address: qifeng618@gmail.com, qifeng618@hotmail.com, qifeng618@qq.com
URL: https://qifeng618.wordpress.com