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Approximation of variational problems with a convexity constraint by PDEs of Abreu type

Guillaume Carlier^{*} Teresa Radice[†]

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Abstract

Motivated by some variational problems subject to a convexity constraint, we consider an approximation using the logarithm of the Hessian determinant as a barrier for the constraint. We show that the minimizer of this penalization can be approached by solving a second boundary value problem for Abreu's equation which is a well-posed nonlinear fourth-order elliptic problem. More interestingly, a similar approximation result holds for the initial constrained variational problem.

Keywords: Abreu equation, Monge-Ampère operator, calculus of variations with a convexity constraint.

MS Classification: 35G30, 49K30.

1 Introduction

Given Ω , a bounded, open, convex subset of \mathbb{R}^d with $d \geq 2$, $F : \Omega \times \mathbb{R} \to \mathbb{R}$ strictly convex in its second argument, and φ a uniformly convex and smooth function defined in a neighbourhood of Ω , we are interested in the variational problem with a convexity constraint:

$$\inf_{u\in\overline{S}[\varphi,\Omega]}\mathcal{J}_0(u) := \int_{\Omega} F(x,u(x)) \mathrm{d}x \tag{1.1}$$

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where $\overline{S}[\varphi, \Omega]$ consists of all convex functions on Ω which admit a convex extension by φ in a neighbourhood of Ω . This is a way to express in some weak sense the boundary conditions

$$u = \varphi \text{ and } \partial_{\nu} u \le \partial_{\nu} \varphi \text{ on } \partial\Omega,$$
 (1.2)

where ν denotes the outward normal to $\partial\Omega$ and ∂_{ν} denotes the normal derivative.

Due to the convexity constraint, it is really difficult to write a tractable Euler-Lagrange equation for (1.1) (see [7], [2]). One may therefore wish to construct suitable penalizations for the convexity constraint which force the minimizers to somehow remain in the interior of the constraint and thus to be a critical point of the penalized functional. Since the seminal work of Trudinger and Wang [9, 10] on the prescribed affine mean curvature equation, the regularity of convex solutions of fourth-order nonlinear PDEs which are Euler-Lagrange equations of convex functionals involving the Hessian determinant have received a lot of attention. In particular, the Abreu equation which corresponds to the logarithm of the Hessian determinant has been studied by Zhou [11] in dimension 2 and more recently by Chau and Weinkove [3] and Le [5, 6] in higher dimensions. What the well-posedness and regularity results of these references in particular suggest is that a penalization involving the logarithm of the Hessian determinant should act as a good barrier for the convexity constraint in problems like (1.1). This was indeed confirmed numerically at a discretized level, see [1].

Our goal is precisely to show that one can indeed approximate (1.1) by a suitable boundary value problem for the Abreu equation. To do so, we first introduce a penalized version of (1.1) with a small parameter $\varepsilon > 0$:

$$\inf_{v\in\overline{S}[\varphi,\Omega]}\mathcal{J}_{\varepsilon}(v) := \mathcal{J}_{0}(v) - \varepsilon\mathfrak{F}_{\Omega}(v)$$
(1.3)

where, when $v \in \overline{S}[\varphi, \Omega]$ is smooth and strongly convex, (see section 2 for the definition for an arbitrary $v \in \overline{S}[\varphi, \Omega]$), $\mathfrak{F}_{\Omega}(v)$ is defined by

$$\mathfrak{F}_{\Omega}(v) := \int_{\Omega} \log(\det D^2 v).$$

Using the convexity of $\mathcal{J}_{\varepsilon}$ setting $f(x, u) := \partial_u F(x, u)$, one can easily see that if u is smooth and uniformly convex up to $\partial\Omega$, and solves the first-boundary problem for Abreu equation

$$\varepsilon U^{ij} w_{ij} = f(x, u) \text{ in } \Omega, \ u = \varphi \text{ and } \partial_{\nu} u = \partial_{\nu} \varphi \text{ on } \partial\Omega$$
 (1.4)

where $w := \det(D^2 u)^{-1}$ and U denotes the cofactor matrix of $D^2 u$ then it is indeed the solution of (1.3). It turns out however that the second-boundary value problem (where instead of prescribing both values of u and ∇u one rather prescribes u and $\det(D^2 u)$ on $\partial\Omega$) is much more well-behaved, see [3, 5, 6] and it was indeed used as an approximation for the affine Plateau problem in [9]. We shall also consider an extra approximation parameter and a second-boundary value problem on a larger domain and show that it approximates correctly not only (1.3) but also the initial problem (1.1) as the parameter converges to zero.

The paper is organized as follows. Section 2 gives some preliminaries. In section 3, we show a Γ -convergence result for $\mathcal{J}_{\varepsilon}$. In section 4, we consider an approximation by a second boundary value problem on a ball B containing $\overline{\Omega}$, with a further penalization $\frac{1}{\delta}(u-\varphi)$ on $B \setminus \Omega$, for which we prove existence and uniqueness of a smooth solution. In section 5, we show that when $\delta \to 0$, we recover the minimizer of the problem from section 3. Finally, we also show full convergence of the second boundary value problem to the initial constrained variational problem (1.1) when $\delta = \delta_{\varepsilon} \to 0$ as $\varepsilon \to 0$, provided F satisfies a suitable uniform convexity assumption.

2 Preliminaries

In the sequel, Ω will be an open, bounded and convex subset of \mathbb{R}^d , $d \geq 2$. We are also given an open ball *B* containing $\overline{\Omega}$ and assume that the boundary datum φ satisfies for some $\lambda > 0$:

$$\varphi \in C^{3,1}(\overline{B}), \ \varphi = 0 \text{ on } \partial B, \ D^2 \varphi \ge \lambda \text{ id on } B.$$
 (2.1)

We then define $\overline{S}[\varphi, \Omega]$ as the set of convex functions on Ω , which, once extended by φ on $B \setminus \Omega$, are convex on B. Note that elements of $\overline{S}[\varphi, \Omega]$ coincide with φ on $\partial\Omega$ and are Lipschitz continuous with Lipschitz constant at most $\|\nabla\varphi\|_{L^{\infty}(B)}$ so that $\overline{S}[\varphi, \Omega]$ is compact for the topology of uniform convergence.

Finally, we assume that the integrand $F: (x, u) \in \Omega \times \mathbb{R} \mapsto F(x, u)$ in the definition of \mathcal{J}_0 in (1.1) is measurable with respect to x, strictly convex and differentiable with respect to u and such that that $F(.,0) \in L^1(\Omega)$ and $f(x, u) := \partial_u F(x, u)$ satisfies $f(., u) \in L^{\infty}(\Omega)$ for every $u \in \mathbb{R}$. These assumptions in particular guarantee that the convex functional \mathcal{J}_0 is everywhere continuous and Gâteaux differentiable on $\overline{S}[\varphi, \Omega]$.

Following [9, 10, 11], let us recall how to define $\mathfrak{F}_{\Omega}(v)$ for an arbitrary convex function v on Ω , first recall that the subdifferential of v at $x \in \Omega$ is

given by

$$\partial v(x) := \{ p \in \mathbb{R}^d : v(y) - v(x) \ge p \cdot (y - x), \ \forall y \in \Omega \}.$$

The Monge-Ampère measure of v, denoted $\mu[v]$ is then defined by

$$\mu[v](A) := |\partial v(A)|$$

for every Borel subset A of Ω . From the seminal results of Alexandrov (see [4]), $\mu[v]$ is indeed a Radon measure and $v \mapsto \mu[v]$ is weakly continuous in the sense that whenever v_n are convex functions which locally uniformly converge to v then

$$\limsup_{n} \mu[v_n](F) \le \mu[v](F), \ \forall \ F \subset \Omega, \ \text{closed}$$

Decomposing the Monge-Ampère measure into its absolutely continuous part and its singular part (with respect to the Lebesgue measure \mathcal{L}^d) as

$$\mu[v] = \mu_r[v] + \mu_s[v], \ \mu_r[v] \ll \mathcal{L}^d, \ \mu_s[v] \perp \mathcal{L}^d$$

Thanks to Alexandrov's theorem, v is differentiable twice a.e., at such points of twice differentiability, we denote by $\partial^2 v$ its Hessian matrix, Trudinger and Wang proved in [9] that $\det(\partial^2 v)$ is the density of $\mu_r[v]$ with respect to \mathcal{L}^d , and following their approach, one can define the functional \mathfrak{F}_{Ω} by

$$\mathfrak{F}_{\Omega}(v) := \int_{\Omega} \log(\det \partial^2 v(x)) \mathrm{d}x, \ \forall v \in \overline{S}[\varphi, \Omega].$$
(2.2)

It is well-known that \mathfrak{F}_{Ω} is a concave functional and we refer to [8, 9, 11] for a proof of the useful properties of \mathfrak{F}_{Ω} recalled below in Lemmas 2.1 and 2.2

Lemma 2.1. The functional $v \in \overline{S}[\varphi, \Omega] \mapsto \mathfrak{F}_{\Omega}(v)$ defined in (2.2) is concave, upper semi-continuous for the topology of local uniform convergence and bounded from above on $\overline{S}[\varphi, \Omega]$ with the explicit bound (where c_d denotes the measure of the unit ball of \mathbb{R}^d)

$$\mathfrak{F}_{\Omega}(v) \le C_{\Omega,\varphi} := |\Omega| \log\left(\frac{c_d \|\nabla\varphi\|_{L^{\infty}}^d}{|\Omega|}\right), \ \forall v \in \overline{S}[\varphi,\Omega].$$
(2.3)

As we shall also work on the larger domain B, it will be also convenient to consider for every open subset ω of B and every convex function u on Bthe concave functional

$$\mathfrak{F}_{\omega}(v) := \int_{\omega} \log(\det \partial^2 v(x)) \mathrm{d}x.$$
(2.4)

Following the same lines as Lemma 6.4 in Trudinger-Wang [8], we also have:

Lemma 2.2. If ω is an open subset of B with $\omega \subset B$, then for every sequence of convex functions u_n converging locally uniformly on B to u, one has

$$\limsup_{n} \mathfrak{F}_{\omega}(u_n) \leq \mathfrak{F}_{\omega}(u).$$

3 Logarithmic penalization

Given $\varepsilon > 0$, we consider

$$\inf_{v\in\overline{S}[\varphi,\Omega]}\mathcal{J}_{\varepsilon}(v) := \mathcal{J}_0(v) - \varepsilon\mathfrak{F}_{\Omega}(v).$$
(3.1)

Since $\mathcal{J}_{\varepsilon}$ is strictly convex and lsc on the convex compact set $\overline{S}[\varphi, \Omega]$, we immediately have:

Proposition 3.1. Problem (3.1) admits a unique solution v_{ε} .

Arguing exactly as in [9, 11] by using Alexandrov's maximum principle, one can show:

Lemma 3.2. Let $\varepsilon > 0$ and v_{ε} be the solution of (3.1) then $\mu_s[v_{\varepsilon}] = 0$ i.e. $\mu[v_{\varepsilon}]$ has no singular part.

Remark 3.3. Let us remark that Lemma 3.2 enables one to express $-\mathfrak{F}_{\Omega}(v_{\varepsilon})$ in an alternative way as the entropy of the push-forward of the Lebesgue measure on Ω by ∇v_{ε} . Also, thanks to Lemma 3.2, one can prove uniqueness of the solution of (3.1) when \mathcal{J}_0 is convex but not necessarily strictly convex.

In dimension 2, we actually even have a uniform local bound on $det(\partial^2 v_{\varepsilon})$:

Proposition 3.4. Let d = 2, $\varepsilon > 0$ and v_{ε} be the solution of (3.1), then $\mu[v_{\varepsilon}] = \det(\partial^2 v_{\varepsilon}) \in L^{\infty}_{loc}(\Omega).$

Proof. It follows from Theorem 5.1 and Proposition 4.3 that v_{ε} is the uniform limit as $\delta \to 0$ of a sequence of smooth functions $(v_{\varepsilon}^{\delta})$ in $\overline{S}[\varphi, \Omega]$ such that, for every open subset ω with $\omega \subset \subset \Omega$, $\|\det(D^2 v_{\varepsilon}^{\delta})\|_{L^{\infty}(\omega)} \leq C$ where C is a constant that depends on ε and ω but not on δ . By weak convergence of Monge-Ampère measures we deduce that $\det(\partial^2 v_{\varepsilon}) \in L^{\infty}_{\text{loc}}(\Omega)$.

Let us now state a Γ -convergence result for $\mathcal{J}_{\varepsilon}$:

Proposition 3.5. The family of functionals $\mathcal{J}_{\varepsilon}$ defined on $\overline{S}[\varphi, \Omega]$ equipped with the topology of uniform convergence Γ -converges to \mathcal{J}_0 in particular v_{ε} converges uniformly to the solution of (1.1).

Proof. Assume u_{ε} is a family in $\overline{S}[\varphi, \Omega]$ that converges uniformly as $\varepsilon \to 0$ to u, thanks to (2.3) and Fatou's Lemma, we have

$$\liminf_{\varepsilon} \mathcal{J}_{\varepsilon}(u_{\varepsilon}) \geq \liminf_{\varepsilon} (\mathcal{J}_{0}(u_{\varepsilon}) - \varepsilon C_{\Omega,\varphi}) \geq \mathcal{J}_{0}(u).$$

Given $u \in \overline{S}[\varphi, \Omega]$, we now look for a recovery sequence $u_{\varepsilon} \in \overline{S}[\varphi, \Omega]$ converging to u and such that $\limsup_{\varepsilon} \mathcal{J}_{\varepsilon}(u_{\varepsilon}) \leq \mathcal{J}_{0}(u)$, we simply take

$$u_{\varepsilon} := (1 - \varepsilon)u + \varepsilon\varphi$$

since $\partial^2 u_{\varepsilon} \geq \varepsilon D^2 \varphi$ we have

$$\mathfrak{F}_{\Omega}(u_{\varepsilon}) \ge d|\Omega|\log(\varepsilon) + \int_{\Omega}\log(\det(D^{2}\varphi))$$

with the convexity of \mathcal{J}_0 , we then have

$$\limsup_{\varepsilon} \mathcal{J}_{\varepsilon}(u_{\varepsilon}) \leq \limsup_{\varepsilon} ((1-\varepsilon)\mathcal{J}_{0}(u) + \varepsilon \mathcal{J}_{0}(\varphi)) + O(\varepsilon \log(\varepsilon))) \leq \mathcal{J}_{0}(u).$$

4 Second boundary value approximation

Having Proposition 3.5 in mind, we now fix the value of ε . Throughout this section, to simplify notations, we therefore take $\varepsilon = 1$ and we are interested in approximating the solution of

$$\inf_{v\in\overline{S}[\varphi,\Omega]}\mathcal{J}_1(v) := \int_{\Omega} F(x,v(x)) \mathrm{d}x - \mathfrak{F}_{\Omega}(v), \tag{4.1}$$

by a second-boundary value problem for Abreu equation. More precisely given $\delta > 0$, we consider

$$U^{ij}w_{ij} = f_{\delta}(x, u), \quad \text{in } B, \ u = \varphi, \ w = \psi \text{ on } \partial B$$
(4.2)

where $\psi := \det((D^2 \varphi)^{-1})$ and

$$f_{\delta}(x,u) := \begin{cases} f(x,u) \text{ if } x \in \Omega\\ \frac{1}{\delta}(u - \varphi(x)) \text{ if } x \in B \setminus \Omega \end{cases}$$

and as before $w = \det(D^2 u)^{-1}$ and U is the cofactor matrix of $D^2 u$. In view of (4.2) and the definition of f_{δ} , it is natural to introduce the functional defined over convex functions on B by

$$\mathcal{J}_1^{\delta}(v) := \int_{\Omega} F(x, v(x)) \mathrm{d}x + \frac{1}{2\delta} \int_{B \setminus \Omega} (v - \varphi)^2 - \mathfrak{F}_B(v)$$

where

$$\mathfrak{F}_B(v) := \int_B \log(\det(\partial^2 v))$$

so that

$$\mathcal{J}_1^{\delta}(v) = \mathcal{J}_1(v) + \frac{1}{2\delta} \int_{B \setminus \Omega} (v - \varphi)^2 - \int_{B \setminus \Omega} \log(\det(\partial^2 v)).$$
(4.3)

4.1 A priori estimates for the second boundary value problem

Following a similar convexity argument as in Lemma 2.2 in Chau and Weinkove [3], we first have

Proposition 4.1. Let u be a smooth and uniformly convex solution of (4.2), then

$$\max_{B} |u| + \int_{\partial B} |\partial_{\nu} u|^{d} + \frac{1}{\delta} \int_{B \setminus \Omega} |u - \varphi|^{2} \le C$$
(4.4)

for some constant C only depending on B, $\|\varphi\|_{C^{3,1}(\overline{B})}$ and the constant λ in (2.1).

Proof. First observe that by convexity and (2.1), u < 0 in B and $\partial_{\nu} u > 0$ on ∂B . Define $\tilde{u} := \varphi$, \tilde{U} as the cofactor matrix of $D^2 \varphi$, $\tilde{w} := \det(D^2 \varphi)^{-1}$ and $\tilde{f} := \tilde{U}^{ij} \tilde{w}_{ij}$ (whose L^{∞} norm only depends on $\|\varphi\|_{C^{3,1}(\overline{B})}$ and the constant λ in (2.1)) we have by the concavity of \mathfrak{F}_B , (4.2) and the monotonicity of

f(x, .):

$$\begin{split} 0 &\geq (\mathfrak{F}'_B(u) - \mathfrak{F}'_B(\widetilde{u}))(u - \widetilde{u}) \\ &= \int_B (U^{ij} w_{ij} - \widetilde{U}^{ij} \widetilde{w}_{ij})(u - \varphi) + \int_{\partial B} \psi (U^{ij} - \widetilde{U}^{ij}) \partial_i (u - \varphi) \nu_j \\ &= \int_\Omega f(x, u)(u - \varphi) - \int_B \widetilde{f}(u - \varphi) + \frac{1}{\delta} \int_{B \setminus \Omega} (u - \varphi)^2 \\ &+ \int_{\partial B} \psi (U^{\nu\nu} - \widetilde{U}^{\nu\nu}) \partial_\nu (u - \varphi) \\ &\geq \int_\Omega f(x, \varphi)(u - \varphi) - \int_B \widetilde{f}(u - \varphi) + \frac{1}{\delta} \int_{B \setminus \Omega} (u - \varphi)^2 \\ &+ \int_{\partial B} \psi (U^{\nu\nu} - \widetilde{U}^{\nu\nu}) \partial_\nu (u - \varphi) \end{split}$$

where, in the last line, we have used the fact that $\nabla u - \nabla \varphi = \partial_{\nu}(u - \varphi)\nu$ on ∂B and set $U^{\nu\nu} = U\nu \cdot \nu$, $\widetilde{U}^{\nu\nu} = \widetilde{U}\nu \cdot \nu$. Using the fact that $f(x, \varphi)$, \widetilde{f} , φ , $\nabla \varphi$ and \widetilde{U} are bounded, we thus get

$$\frac{1}{\delta} \int_{B \setminus \Omega} (u - \varphi)^2 + \int_{\partial B} \psi U^{\nu\nu} \partial_{\nu} u \le C \left(1 + \int_B |u| + \int_{\partial B} \partial_{\nu} u + \int_{\partial B} U^{\nu\nu} \right).$$
(4.5)

Denoting by R the radius of B and by the same argument as in Lemma 2.2 in [3], one has

$$U^{\nu\nu} = \frac{1}{R^{d-1}} \partial_{\nu} u^{d-1} + E \text{ with } |E| \le C(1 + \partial_{\nu} u^{d-2}) \text{ on } \partial B.$$

$$(4.6)$$

Moreover since u is convex and $u = \varphi = 0$ on ∂B , one has

$$\max_{B} |u| = -\min_{B} u \le 2R\partial_{\nu}u(x) \text{ for all } x \in \partial B.$$
(4.7)

Putting together (4.5), (4.6), (4.7) and the fact that $\inf_{\partial B} \psi > 0$, we obtain

$$\int_{\partial B} (\partial_{\nu} u)^d \le C(1 + \int_{\partial B} (\partial_{\nu} u)^{d-1})$$

which gives a bound on $\|\partial_{\nu}u\|_{L^{d}(\partial B)}$ hence also on $\max_{B}|u|$ by (4.7) so that finally the bound on $\delta^{-1}\int_{B\setminus\Omega}(u-\varphi)^{2}$ follows from the latter bounds and (4.5).

4.2 Existence and uniqueness of a smooth uniformly convex solution

Thanks to Theorem 1.1 in [3], a Leray-Schauder degree argument and the a priori estimate (4.4), one easily deduces the following:

Theorem 4.2. For every $\delta > 0$, the second boundary value problem (4.2) admits a unique uniformly convex solution which is $W^{4,p}(B)$ for every $p \in [1, +\infty)$.

Proof. Let $D := \{u \in C(\overline{B}), \|u\|_{C(\overline{B})} \leq C+1\}$ where C is the constant from (4.4). For $t \in [0, 1]$ and $u \in D$, it follows from Theorem 1.1 in [3] that there exists a unique $W^{4,p}$ for every $p \in [1, \infty)$ and uniformly convex solution of

$$V^{ij}w_{ij} = tf_{\delta}(x, u), w = \det(D^2 v)^{-1} \text{ in } B, \ v = \varphi, w = \psi \text{ on } \partial B$$
(4.8)

where V denotes the cofactor matrix of D^2v . We denote by $v = T_t(u)$ the solution of (4.8). Moreover, by Theorem 2.1 of [3], for every $\alpha \in (0, 1)$ there are a priori bounds on $\|v\|_{C^{3,\alpha}}$ and on $\sup_B(\det(D^2v) + \det(D^2v)^{-1})$ that only depend on C, α , δ , $\|\varphi\|_{C^{3,1}}$ and the constant λ in (2.1). Therefore $(t, u) \in [0, 1] \times D \mapsto T_t(u)$ is continuous on $[0, 1] \times D$ and T_t is compact in $C(\overline{B})$ for every $t \in [0, 1]$. Since T_0 is constant and by (4.4) it has a unique fixed point in D, again by (4.4), T_t has no fixed point on ∂D , it thus follows from the Leray-Schauder Theorem that T_1 has a fixed point in D, this proves the existence claim for (4.2).

Finally, uniqueness follows from the same argument as in Lemma 7.1 from [10] where it is proven that two smooth solutions actually have the same gradient on ∂B and then are the minimizers of the same strictly convex minimization problem hence coincide.

In dimension d = 2, following the argument of Remark 4.2 of Trudinger and Wang [9] and taking advantage of the fact that the right-hand side of the Abreu equation (4.2) does not depend on δ on Ω , we have the following local bound (which we have used in the proof of Proposition 3.4):

Proposition 4.3. Let d = 2 and u be the solution of (4.2) then for every open set $\omega \subset \subset \Omega$, $\|\det(D^2u)\|_{L^{\infty}_{loc}(\omega)}$ is bounded independently of δ .

Proof. Let $B_r := B_r(0) \subset \Omega$, and observe that thanks to (4.4) both $\|f_{\delta}(., u(.))\|_{L^{\infty}(\Omega)} = \|f(., u(.))\|_{L^{\infty}(\Omega)}$ and $\|\nabla u\|_{L^{\infty}(\Omega)}$ are bounded independently of δ . Define then $\eta(x) := \frac{1}{2}(r^2 - |x|^2)$ and consider $z := \log(w) - \log(w)$

 $2\log(\eta) - \frac{1}{2}|\nabla u|^2$, by construction z achieves its minimum at an interior point x_0 of B_r at such a point, we have

$$\frac{\nabla w}{w} = 2\frac{\nabla \eta}{\eta} + D^2 u \nabla u. \tag{4.9}$$

We also have

$$z_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} + 2\frac{\delta_{ij}}{\eta} + 2\frac{\eta_i \eta_j}{\eta^2} - u_{ijk} u_k - u_{ik} u_{jk}, \qquad (4.10)$$

multiplying by $wU = [D^2u]^{-1}$, using $wU^{ij}z_{ij} \ge 0$ at $x_0, U^{ij}w_{ij} = f(x, u) \le C$ and the identities

$$wU^{ij}u_{ik}u_{jk} = u_{ii} = \Delta u, \ wU^{ij}u_{ijk}u_k = -\frac{w_k}{w}u_k = -\frac{\nabla w}{w} \cdot \nabla u, \qquad (4.11)$$

(the second identity is classically obtained by first differentiating the relation $-\log(w) = \log(\det D^2 u)$ and then taking the scalar product with ∇u) as well as the fact that $\operatorname{Tr}(U) = \Delta u$ in dimension d = 2, we get

$$0 \le C - wU\frac{\nabla w}{w} \cdot \frac{\nabla w}{w} + 2\frac{w}{\eta}\Delta u + 2wU\frac{\nabla\eta}{\eta} \cdot \frac{\nabla\eta}{\eta} + \frac{\nabla w}{w} \cdot \nabla u - \Delta u.$$
(4.12)

Using (4.9) and using again $wU = [D^2u]^{-1}$, we then obtain

$$wU\frac{\nabla w}{w} \cdot \frac{\nabla w}{w} = 4wU\frac{\nabla\eta}{\eta} \cdot \frac{\nabla\eta}{\eta} + D^2u\nabla u \cdot \nabla u + 4\frac{\nabla\eta}{\eta} \cdot \nabla u \qquad (4.13)$$

and

$$\frac{\nabla w}{w} \cdot \nabla u = 2\frac{\nabla \eta}{\eta} \cdot \nabla u + D^2 u \nabla u \cdot \nabla u.$$
(4.14)

Replacing (4.13), (4.14) in (4.12), multiplying by η and rearraging gives

$$\Delta u(\eta - 2w) \le C\eta - 2\frac{w}{\eta}U\nabla\eta \cdot \nabla\eta - 2\nabla\eta \cdot \nabla u \le C\eta + \|\nabla\eta\|_{L^{\infty}}\|\nabla u\|_{L^{\infty}(\Omega)} \le C'$$
(4.15)

If $\eta(x_0) \ge 4w(x_0)$, (4.15) gives $\eta(x_0)\Delta u(x_0) \le 2C'$ and since $\Delta u(x_0)w(x_0)^{1/2} \ge 2$ we get the desired lower bound on the minimum of $\eta^{-2}w$. In the remaining case $w(x_0) \ge \frac{1}{4}\eta(x_0) \ge \frac{\eta^2(x_0)}{2r^2}$ and we reach the same conclusion. This gives a local lower bound on w i.e. the desired local upper bound on det (D^2u) .

5 Convergence

5.1 Letting $\delta \to 0$ for fixed ε

In this paragraph, we fix ε (and thus normalize it to $\varepsilon = 1$ as we did in the whole of section 4).

Theorem 5.1. Let u_{δ} be the unique smooth strictly convex solution of (4.2), then u_{δ} converges uniformly on Ω to the unique minimizer of (4.1) as $\delta \to 0^+$.

Proof. We already know from (4.4) that (possibly up to an extraction) u_{δ} converges locally uniformly on B to some convex u and it also follows from (4.4) that $u \in \overline{S}[\varphi, \Omega]$. Let $v \in \overline{S}[\varphi, \Omega]$ (extended by φ on $B \setminus \Omega$), thanks to (4.2) and the convexity of \mathcal{J}_1^{δ} we first have

$$\mathcal{J}_1^{\delta}(v) - \mathcal{J}_1^{\delta}(u_{\delta}) \ge \int_{\partial B} U_{\delta}^{\nu\nu} \psi \partial_{\nu}(u_{\delta} - \varphi)$$

i.e.

$$\begin{aligned} \mathcal{J}_{1}(v) - \mathcal{J}_{1}(u_{\delta}) &\geq \frac{1}{2\delta} \int_{B \setminus \Omega} (u_{\delta} - \varphi)^{2} + \int_{B \setminus \Omega} (\log(\det(D^{2}\varphi)) - \log(\det(D^{2}u_{\delta}))) \\ &+ \int_{\partial B} U_{\delta}^{\nu\nu} \psi \partial_{\nu} (u_{\delta} - \varphi) \\ &\geq \int_{B \setminus \Omega} (\log(\det(D^{2}\varphi)) - \log(\det(D^{2}u_{\delta}))) + \int_{\partial B} U_{\delta}^{\nu\nu} \psi \partial_{\nu} (u_{\delta} - \varphi) \end{aligned}$$

It follows from Lemma 5.2 below that

$$\liminf_{\delta \to 0} \int_{B \setminus \Omega} (\log(\det(D^2 \varphi)) - \log(\det(D^2 u_{\delta})))) \ge 0.$$

We now have to pay attention to the boundary term, we know from (4.6) that $\theta_{\delta} := \psi U_{\delta}^{\nu\nu}$ satisfies $0 \leq \theta_{\delta} \leq C(1 + (\partial_{\nu}u_{\delta})^{d-1})$ so that thanks to (4.4), θ_{δ} is bounded in $L^{\frac{d}{d-1}}(\partial B)$, up to an extraction we may therefore assume that it weakly converges in $L^{\frac{d}{d-1}}(\partial B)$ to some nonnegative function θ . By convexity we also have that for $\tau > 0$

$$\partial_{\nu} u_{\delta}(x) \ge D_{\tau,\nu} u_{\delta}(x) := \frac{1}{\tau} \Big(u_{\delta}(x - \tau \nu(x)) - u_{\delta}(x - 2\tau \nu(x)) \Big), \ \forall x \in \partial B$$

For small fixed $\tau > 0$ note that $D_{\tau,\nu}u_{\delta}$ is bounded independently of δ thanks to (4.4) and that it converges as $\delta \to 0$ pointwise to $D_{\tau,\nu}\varphi$, we thus have

$$\liminf_{\delta \to 0} \int_{\partial B} \theta_{\delta} \partial_{\nu} (u_{\delta} - \varphi) \ge \liminf_{\delta \to 0} \int_{\partial B} \theta_{\delta} (D_{\tau,\nu} u_{\delta} - \partial_{\nu} \varphi)$$
$$= \int_{\partial B} \theta (D_{\tau,\nu} \varphi - \partial_{\nu} \varphi)$$

where in the last line we have passed to the limit using the fact that we have the product of a weakly convergent sequence with a strongly convergent sequence. Now letting $\tau \to 0$ and using the smoothness of φ , we deduce that

$$\liminf_{\delta \to 0} \int_{\partial B} U_{\delta}^{\nu\nu} \psi \partial_{\nu} (u_{\delta} - \varphi) \ge 0.$$

Since \mathcal{J}_1 is lower semi-continuous thanks to Lemma 2.2, we can conclude that

$$\mathcal{J}_1(v) \ge \liminf_{\delta \to 0} \mathcal{J}_1(u_\delta) \ge \mathcal{J}_1(u)$$

so that u solves (4.1) and by uniqueness of the minimizer there is in fact convergence of the whole sequence.

In the previous proof we have used:

Lemma 5.2. Let u_{δ} be the unique smooth strictly convex solution of (4.2) as before, then

$$\limsup_{\delta \to 0} \int_{B \setminus \Omega} \log(\det(D^2 u_{\delta})) \le \int_{B \setminus \Omega} \log(\det(D^2 \varphi)).$$

Proof. The key point here is the estimate $\int_B \Delta u_{\delta} = \int_{\partial B} \partial_{\nu} u_{\delta} \leq C$ which follows from (4.4). Let ω be an arbitrary Borel subset of B, we have (for some constant C varying from a line to another):

$$\int_{\omega} \log(\det(D^2 u_{\delta})) \leq C(|\omega| + \int_{\omega} \det(D^2 u_{\delta})^{1/2d})$$
$$\leq C\left(|\omega| + \int_{\omega} \sqrt{\Delta u_{\delta}}\right) \leq C\left(|\omega| + |\omega|^{1/2} \left(\int_{B} \Delta u_{\delta}\right)^{1/2}\right)$$
$$= C\left(|\omega| + |\omega|^{1/2} \left(\int_{\partial B} \partial_{\nu} u_{\delta}\right)^{1/2}\right)$$

so that

$$\int_{\omega} \log(\det(D^2 u_{\delta})) \le C(|\omega| + |\omega|^{1/2}).$$
(5.1)

Take 0 < R' < R with Ω contained in $B_{R'}$ (recall R is the radius of B), we then have, thanks to Lemma 2.2, the fact that $\log(\det(D^2\varphi))$ is bounded

and (5.1):

$$\begin{split} \limsup_{\delta \to 0} \mathfrak{F}_{B \setminus \Omega}(u_{\delta}) &= \limsup_{\delta \to 0} \mathfrak{F}_{B \setminus \overline{\Omega}}(u_{\delta}) \\ &\leq \limsup_{\delta \to 0} \mathfrak{F}_{B_{R'} \setminus \overline{\Omega}}(u_{\delta}) + \limsup_{\delta \to 0} \mathfrak{F}_{B \setminus B_{R'}}(u_{\delta}) \\ &\leq \mathfrak{F}_{B_{R'} \setminus \overline{\Omega}}(\varphi) + C(|B \setminus B_{R'}| + |B \setminus B_{R'}|^{1/2}) \\ &\leq \mathfrak{F}_{B \setminus \Omega}(\varphi) + C'(|B \setminus B_{R'}| + |B \setminus B_{R'}|^{1/2}). \end{split}$$

The desired result follows by letting R' tend to R.

5.2 Full convergence

We now take $\delta = \delta_{\varepsilon} > 0$ with

$$\lim_{\varepsilon \to 0^+} \delta_{\varepsilon} = 0, \tag{5.2}$$

i.e. we only have a single small parameter ε and we consider the second-boundary value problem

$$\varepsilon U_{\varepsilon}^{ij} w_{ij}^{\varepsilon} = g_{\varepsilon}(x, u_{\varepsilon}), \quad \text{in } B, \ u_{\varepsilon} = \varphi, \ w^{\varepsilon} = \psi \text{ on } \partial B$$
 (5.3)

where $\psi := \det((D^2 \varphi)^{-1}),$

$$g_{\varepsilon}(x,u) := \begin{cases} f(x,u) \text{ if } x \in \Omega\\ \frac{1}{\delta_{\varepsilon}}(u - \varphi(x)) \text{ if } x \in B \setminus \Omega \end{cases},$$

 $w^{\varepsilon} = \det(D^2 u_{\varepsilon})^{-1}$ and U_{ε} is the cofactor matrix of $D^2 u_{\varepsilon}$. We further assume that there is an $\alpha > 0$ such that

$$(f(x,u) - f(x,v))(u-v) \ge \alpha(u-v)^2, \ \forall (u,v) \in \mathbb{R}^d, \text{ and a.e. } x \in \Omega$$
 (5.4)

which amounts to say that the integrand F is uniformly convex in its second argument. Under these assumptions, we have a *full* convergence result:

Theorem 5.3. Let u_{ε} be the unique smooth strictly convex solution of (5.3), then u_{ε} converges uniformly on Ω to the unique minimizer of (1.1) as $\varepsilon \to 0^+$.

Proof. Step 1: a priori estimates. The first step of the proof is similar to the proof of Proposition 4.1. Again define $\tilde{u} := \varphi$, \tilde{U} as the cofactor matrix

of $D^2\varphi$, $\widetilde{w} := \det(D^2\varphi)^{-1}$ and $\widetilde{f}_{\varepsilon} := \varepsilon \widetilde{U}^{ij}\widetilde{w}_{ij}$. We then have together with (5.4):

$$0 \geq \varepsilon(\mathfrak{F}'_{B}(u_{\varepsilon}) - \mathfrak{F}'_{B}(\widetilde{u}))(u_{\varepsilon} - \widetilde{u})$$

$$\geq \int_{\Omega} (f(x,\varphi) - \widetilde{f}_{\varepsilon})(u_{\varepsilon} - \varphi) + \alpha \int_{\Omega} (u_{\varepsilon} - \varphi)^{2} + \frac{1}{\delta_{\varepsilon}} \int_{B \setminus \Omega} (u_{\varepsilon} - \varphi)^{2} + \varepsilon \int_{\partial B} \psi(U_{\varepsilon}^{\nu\nu} - \widetilde{U}^{\nu\nu}) \partial_{\nu}(u_{\varepsilon} - \varphi)$$

thanks to the fact that $f(x, \varphi) - \tilde{f}_{\varepsilon}$ is bounded uniformly with respect to ε , using Young's inequality and invoking (4.6), we get

$$\int_{\Omega} (u_{\varepsilon} - \varphi)^2 + \frac{1}{\delta_{\varepsilon}} \int_{B \setminus \Omega} (u_{\varepsilon} - \varphi)^2 + \varepsilon \int_{\partial B} (\partial_{\nu} u_{\varepsilon})^d \le C.$$
(5.5)

Step 2: convergence. Thanks to (5.5), up to taking a subsequence of vanishing ε_n , we may assume that u_{ε} converges locally uniformly in B to some u such that $u = \varphi$ in $B \setminus \Omega$ so that the restriction of u to Ω belongs to $\overline{S}[\varphi, \Omega]$. For every v convex on B such that $v = \varphi$ on ∂B , define

$$\widetilde{\mathcal{J}}_{\varepsilon}(v) := \int_{\Omega} F(x, v(x)) \mathrm{d}x + \frac{1}{2\delta_{\varepsilon}} \int_{B \setminus \Omega} (v - \varphi)^2 - \varepsilon \int_{B} \log(\det(\partial^2 v)).$$

Let then $v \in \overline{S}[\varphi, \Omega]$ (extended by φ on $B \setminus \Omega$), we then have

$$\widetilde{\mathcal{J}}_{\varepsilon}(v) - \widetilde{\mathcal{J}}_{\varepsilon}(u_{\varepsilon}) \ge \varepsilon \int_{\partial B} \psi U_{\varepsilon}^{\nu\nu} \partial_{\nu} (u_{\varepsilon} - \varphi)$$

hence

$$\mathcal{J}_0(v) \ge \liminf_{\varepsilon} \mathcal{J}_0(u_{\varepsilon}) + \liminf_{\varepsilon} \varepsilon(\mathfrak{F}_B(v) - \mathfrak{F}_B(u_{\varepsilon})) - \limsup_{\varepsilon} \varepsilon \int_{\partial B} \psi U_{\varepsilon}^{\nu\nu} \partial_{\nu} \varphi.$$

Arguing as in the proof of Proposition 3.5, we may actually assume that $\mathfrak{F}_{\Omega}(v) > -\infty$ so that $\liminf_{\varepsilon} \varepsilon \mathfrak{F}_{B}(v) \geq 0$. As for an upper bound for $\varepsilon \mathfrak{F}_{B}(u_{\varepsilon})$ we use the fact that thanks to (5.5), we have $\int_{\partial B} \partial_{\nu} u_{\varepsilon} \leq C \varepsilon^{-1/d}$ and argue in a similar way as in the proof of Lemma 5.2, to obtain

$$\varepsilon \mathfrak{F}_B(u_{\varepsilon}) \le C\varepsilon (1 + \int_B \det(D^2 u_{\varepsilon})^{1/d}) \le C\varepsilon (1 + \int_{\partial B} \partial_{\nu} u_{\varepsilon}) \le C(\varepsilon + \varepsilon^{1-1/d}),$$

which yields

$$\liminf_{\varepsilon} \varepsilon(\mathfrak{F}_B(v) - \mathfrak{F}_B(u_{\varepsilon})) \ge 0.$$

Thanks to (4.6), we have

$$\int_{\partial B} \psi U_{\varepsilon}^{\nu\nu} \partial_{\nu} \varphi \leq C \int_{\partial B} (1 + (\partial_{\nu} u_{\varepsilon})^{d-1})$$

but, thanks to (5.5) and Hölder's inequality, we deduce

$$\varepsilon \int_{\partial B} (\partial_{\nu} u_{\varepsilon})^{d-1} \le C \varepsilon^{\frac{1}{d}}$$

so that

$$\mathcal{J}_0(v) \ge \liminf_{\varepsilon} \mathcal{J}_0(u_{\varepsilon}) = \mathcal{J}_0(u)$$

hence u solves (1.1) (and the whole family u_{ε} converges uniformly on Ω to u by uniqueness of the minimizer of \mathcal{J}_0 on $\overline{S}[\varphi, \Omega]$).

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