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► **To cite this version:**

Guillaume Carlier, Teresa Radice. Approximation of variational problems with a convexity constraint by PDEs of Abreu type. *Calculus of Variations and Partial Differential Equations*, 2019. hal-01802925

HAL Id: hal-01802925

<https://hal.science/hal-01802925>

Submitted on 29 May 2018

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Approximation of variational problems with a convexity constraint by PDEs of Abreu type

Guillaume Carlier* Teresa Radice †

April 30, 2018

Abstract

Motivated by some variational problems subject to a convexity constraint, we consider an approximation using the logarithm of the Hessian determinant as a barrier for the constraint. We show that the minimizer of this penalization can be approached by solving a second boundary value problem for Abreu's equation which is a well-posed nonlinear fourth-order elliptic problem. More interestingly, a similar approximation result holds for the initial constrained variational problem.

Keywords: Abreu equation, Monge-Ampère operator, calculus of variations with a convexity constraint.

MS Classification: 35G30, 49K30.

1 Introduction

Given Ω , a bounded, open, convex subset of \mathbb{R}^d with $d \geq 2$, $F : \Omega \times \mathbb{R} \rightarrow \mathbb{R}$ strictly convex in its second argument, and φ a uniformly convex and smooth function defined in a neighbourhood of Ω , we are interested in the variational problem with a convexity constraint:

$$\inf_{u \in \mathcal{S}[\varphi, \Omega]} \mathcal{J}_0(u) := \int_{\Omega} F(x, u(x)) dx \quad (1.1)$$

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where $\overline{S}[\varphi, \Omega]$ consists of all convex functions on Ω which admit a convex extension by φ in a neighbourhood of Ω . This is a way to express in some weak sense the boundary conditions

$$u = \varphi \text{ and } \partial_\nu u \leq \partial_\nu \varphi \text{ on } \partial\Omega, \quad (1.2)$$

where ν denotes the outward normal to $\partial\Omega$ and ∂_ν denotes the normal derivative.

Due to the convexity constraint, it is really difficult to write a tractable Euler-Lagrange equation for (1.1) (see [7], [2]). One may therefore wish to construct suitable penalizations for the convexity constraint which force the minimizers to somehow remain in the interior of the constraint and thus to be a critical point of the penalized functional. Since the seminal work of Trudinger and Wang [9, 10] on the prescribed affine mean curvature equation, the regularity of convex solutions of fourth-order nonlinear PDEs which are Euler-Lagrange equations of convex functionals involving the Hessian determinant have received a lot of attention. In particular, the Abreu equation which corresponds to the logarithm of the Hessian determinant has been studied by Zhou [11] in dimension 2 and more recently by Chau and Weinkove [3] and Le [5, 6] in higher dimensions. What the well-posedness and regularity results of these references in particular suggest is that a penalization involving the logarithm of the Hessian determinant should act as a good barrier for the convexity constraint in problems like (1.1). This was indeed confirmed numerically at a discretized level, see [1].

Our goal is precisely to show that one can indeed approximate (1.1) by a suitable boundary value problem for the Abreu equation. To do so, we first introduce a penalized version of (1.1) with a small parameter $\varepsilon > 0$:

$$\inf_{v \in \overline{S}[\varphi, \Omega]} \mathcal{J}_\varepsilon(v) := \mathcal{J}_0(v) - \varepsilon \mathfrak{F}_\Omega(v) \quad (1.3)$$

where, when $v \in \overline{S}[\varphi, \Omega]$ is smooth and strongly convex, (see section 2 for the definition for an arbitrary $v \in \overline{S}[\varphi, \Omega]$), $\mathfrak{F}_\Omega(v)$ is defined by

$$\mathfrak{F}_\Omega(v) := \int_\Omega \log(\det D^2 v).$$

Using the convexity of \mathcal{J}_ε setting $f(x, u) := \partial_u F(x, u)$, one can easily see that if u is smooth and uniformly convex up to $\partial\Omega$, and solves the first-boundary problem for Abreu equation

$$\varepsilon U^{ij} w_{ij} = f(x, u) \text{ in } \Omega, \quad u = \varphi \text{ and } \partial_\nu u = \partial_\nu \varphi \text{ on } \partial\Omega \quad (1.4)$$

where $w := \det(D^2u)^{-1}$ and U denotes the cofactor matrix of D^2u then it is indeed the solution of (1.3). It turns out however that the second-boundary value problem (where instead of prescribing both values of u and ∇u one rather prescribes u and $\det(D^2u)$ on $\partial\Omega$) is much more well-behaved, see [3, 5, 6] and it was indeed used as an approximation for the affine Plateau problem in [9]. We shall also consider an extra approximation parameter and a second-boundary value problem on a larger domain and show that it approximates correctly not only (1.3) but also the initial problem (1.1) as the parameter converges to zero.

The paper is organized as follows. Section 2 gives some preliminaries. In section 3, we show a Γ -convergence result for \mathcal{J}_ε . In section 4, we consider an approximation by a second boundary value problem on a ball B containing $\overline{\Omega}$, with a further penalization $\frac{1}{\delta}(u-\varphi)$ on $B \setminus \Omega$, for which we prove existence and uniqueness of a smooth solution. In section 5, we show that when $\delta \rightarrow 0$, we recover the minimizer of the problem from section 3. Finally, we also show full convergence of the second boundary value problem to the initial constrained variational problem (1.1) when $\delta = \delta_\varepsilon \rightarrow 0$ as $\varepsilon \rightarrow 0$, provided F satisfies a suitable uniform convexity assumption.

2 Preliminaries

In the sequel, Ω will be an open, bounded and convex subset of \mathbb{R}^d , $d \geq 2$. We are also given an open ball B containing $\overline{\Omega}$ and assume that the boundary datum φ satisfies for some $\lambda > 0$:

$$\varphi \in C^{3,1}(\overline{B}), \quad \varphi = 0 \text{ on } \partial B, \quad D^2\varphi \geq \lambda \text{id on } B. \quad (2.1)$$

We then define $\overline{\mathcal{S}}[\varphi, \Omega]$ as the set of convex functions on Ω , which, once extended by φ on $B \setminus \Omega$, are convex on B . Note that elements of $\overline{\mathcal{S}}[\varphi, \Omega]$ coincide with φ on $\partial\Omega$ and are Lipschitz continuous with Lipschitz constant at most $\|\nabla\varphi\|_{L^\infty(B)}$ so that $\overline{\mathcal{S}}[\varphi, \Omega]$ is compact for the topology of uniform convergence.

Finally, we assume that the integrand $F: (x, u) \in \Omega \times \mathbb{R} \mapsto F(x, u)$ in the definition of \mathcal{J}_0 in (1.1) is measurable with respect to x , strictly convex and differentiable with respect to u and such that that $F(\cdot, 0) \in L^1(\Omega)$ and $f(x, u) := \partial_u F(x, u)$ satisfies $f(\cdot, u) \in L^\infty(\Omega)$ for every $u \in \mathbb{R}$. These assumptions in particular guarantee that the convex functional \mathcal{J}_0 is everywhere continuous and Gâteaux differentiable on $\overline{\mathcal{S}}[\varphi, \Omega]$.

Following [9, 10, 11], let us recall how to define $\mathfrak{F}_\Omega(v)$ for an arbitrary convex function v on Ω , first recall that the subdifferential of v at $x \in \Omega$ is

given by

$$\partial v(x) := \{p \in \mathbb{R}^d : v(y) - v(x) \geq p \cdot (y - x), \forall y \in \Omega\}.$$

The Monge-Ampère measure of v , denoted $\mu[v]$ is then defined by

$$\mu[v](A) := |\partial v(A)|$$

for every Borel subset A of Ω . From the seminal results of Alexandrov (see [4]), $\mu[v]$ is indeed a Radon measure and $v \mapsto \mu[v]$ is weakly continuous in the sense that whenever v_n are convex functions which locally uniformly converge to v then

$$\limsup_n \mu[v_n](F) \leq \mu[v](F), \forall F \subset \Omega, \text{ closed}.$$

Decomposing the Monge-Ampère measure into its absolutely continuous part and its singular part (with respect to the Lebesgue measure \mathcal{L}^d) as

$$\mu[v] = \mu_r[v] + \mu_s[v], \mu_r[v] \ll \mathcal{L}^d, \mu_s[v] \perp \mathcal{L}^d.$$

Thanks to Alexandrov's theorem, v is differentiable twice a.e., at such points of twice differentiability, we denote by $\partial^2 v$ its Hessian matrix, Trudinger and Wang proved in [9] that $\det(\partial^2 v)$ is the density of $\mu_r[v]$ with respect to \mathcal{L}^d , and following their approach, one can define the functional \mathfrak{F}_Ω by

$$\mathfrak{F}_\Omega(v) := \int_\Omega \log(\det \partial^2 v(x)) dx, \forall v \in \overline{\mathcal{S}}[\varphi, \Omega]. \quad (2.2)$$

It is well-known that \mathfrak{F}_Ω is a concave functional and we refer to [8, 9, 11] for a proof of the useful properties of \mathfrak{F}_Ω recalled below in Lemmas 2.1 and 2.2

Lemma 2.1. *The functional $v \in \overline{\mathcal{S}}[\varphi, \Omega] \mapsto \mathfrak{F}_\Omega(v)$ defined in (2.2) is concave, upper semi-continuous for the topology of local uniform convergence and bounded from above on $\overline{\mathcal{S}}[\varphi, \Omega]$ with the explicit bound (where c_d denotes the measure of the unit ball of \mathbb{R}^d)*

$$\mathfrak{F}_\Omega(v) \leq C_{\Omega, \varphi} := |\Omega| \log \left(\frac{c_d \|\nabla \varphi\|_{L^\infty}^d}{|\Omega|} \right), \forall v \in \overline{\mathcal{S}}[\varphi, \Omega]. \quad (2.3)$$

As we shall also work on the larger domain B , it will be also convenient to consider for every open subset ω of B and every convex function u on B the concave functional

$$\mathfrak{F}_\omega(v) := \int_\omega \log(\det \partial^2 v(x)) dx. \quad (2.4)$$

Following the same lines as Lemma 6.4 in Trudinger-Wang [8], we also have:

Lemma 2.2. *If ω is an open subset of B with $\omega \subset\subset B$, then for every sequence of convex functions u_n converging locally uniformly on B to u , one has*

$$\limsup_n \mathfrak{F}_\omega(u_n) \leq \mathfrak{F}_\omega(u).$$

3 Logarithmic penalization

Given $\varepsilon > 0$, we consider

$$\inf_{v \in \overline{\mathcal{S}}[\varphi, \Omega]} \mathcal{J}_\varepsilon(v) := \mathcal{J}_0(v) - \varepsilon \mathfrak{F}_\Omega(v). \quad (3.1)$$

Since \mathcal{J}_ε is strictly convex and lsc on the convex compact set $\overline{\mathcal{S}}[\varphi, \Omega]$, we immediately have:

Proposition 3.1. *Problem (3.1) admits a unique solution v_ε .*

Arguing exactly as in [9, 11] by using Alexandrov's maximum principle, one can show:

Lemma 3.2. *Let $\varepsilon > 0$ and v_ε be the solution of (3.1) then $\mu_s[v_\varepsilon] = 0$ i.e. $\mu[v_\varepsilon]$ has no singular part.*

Remark 3.3. Let us remark that Lemma 3.2 enables one to express $-\mathfrak{F}_\Omega(v_\varepsilon)$ in an alternative way as the entropy of the push-forward of the Lebesgue measure on Ω by ∇v_ε . Also, thanks to Lemma 3.2, one can prove uniqueness of the solution of (3.1) when \mathcal{J}_0 is convex but not necessarily strictly convex.

In dimension 2, we actually even have a uniform local bound on $\det(\partial^2 v_\varepsilon)$:

Proposition 3.4. *Let $d = 2$, $\varepsilon > 0$ and v_ε be the solution of (3.1), then $\mu[v_\varepsilon] = \det(\partial^2 v_\varepsilon) \in L_{\text{loc}}^\infty(\Omega)$.*

Proof. It follows from Theorem 5.1 and Proposition 4.3 that v_ε is the uniform limit as $\delta \rightarrow 0$ of a sequence of smooth functions (v_ε^δ) in $\overline{\mathcal{S}}[\varphi, \Omega]$ such that, for every open subset ω with $\omega \subset\subset \Omega$, $\|\det(D^2 v_\varepsilon^\delta)\|_{L^\infty(\omega)} \leq C$ where C is a constant that depends on ε and ω but not on δ . By weak convergence of Monge-Ampère measures we deduce that $\det(\partial^2 v_\varepsilon) \in L_{\text{loc}}^\infty(\Omega)$. □

Let us now state a Γ -convergence result for \mathcal{J}_ε :

Proposition 3.5. *The family of functionals \mathcal{J}_ε defined on $\overline{\mathcal{S}}[\varphi, \Omega]$ equipped with the topology of uniform convergence Γ -converges to \mathcal{J}_0 in particular v_ε converges uniformly to the solution of (1.1).*

Proof. Assume u_ε is a family in $\overline{\mathcal{S}}[\varphi, \Omega]$ that converges uniformly as $\varepsilon \rightarrow 0$ to u , thanks to (2.3) and Fatou's Lemma, we have

$$\liminf_{\varepsilon} \mathcal{J}_\varepsilon(u_\varepsilon) \geq \liminf_{\varepsilon} (\mathcal{J}_0(u_\varepsilon) - \varepsilon C_{\Omega, \varphi}) \geq \mathcal{J}_0(u).$$

Given $u \in \overline{\mathcal{S}}[\varphi, \Omega]$, we now look for a recovery sequence $u_\varepsilon \in \overline{\mathcal{S}}[\varphi, \Omega]$ converging to u and such that $\limsup_{\varepsilon} \mathcal{J}_\varepsilon(u_\varepsilon) \leq \mathcal{J}_0(u)$, we simply take

$$u_\varepsilon := (1 - \varepsilon)u + \varepsilon\varphi$$

since $\partial^2 u_\varepsilon \geq \varepsilon D^2 \varphi$ we have

$$\mathfrak{F}_\Omega(u_\varepsilon) \geq d|\Omega| \log(\varepsilon) + \int_{\Omega} \log(\det(D^2 \varphi))$$

with the convexity of \mathcal{J}_0 , we then have

$$\limsup_{\varepsilon} \mathcal{J}_\varepsilon(u_\varepsilon) \leq \limsup_{\varepsilon} ((1 - \varepsilon)\mathcal{J}_0(u) + \varepsilon\mathcal{J}_0(\varphi) + O(\varepsilon \log(\varepsilon))) \leq \mathcal{J}_0(u).$$

□

4 Second boundary value approximation

Having Proposition 3.5 in mind, we now fix the value of ε . Throughout this section, to simplify notations, we therefore take $\varepsilon = 1$ and we are interested in approximating the solution of

$$\inf_{v \in \overline{\mathcal{S}}[\varphi, \Omega]} \mathcal{J}_1(v) := \int_{\Omega} F(x, v(x)) dx - \mathfrak{F}_\Omega(v), \quad (4.1)$$

by a second-boundary value problem for Abreu equation. More precisely given $\delta > 0$, we consider

$$U^{ij} w_{ij} = f_\delta(x, u), \quad \text{in } B, \quad u = \varphi, \quad w = \psi \text{ on } \partial B \quad (4.2)$$

where $\psi := \det((D^2 \varphi)^{-1})$ and

$$f_\delta(x, u) := \begin{cases} f(x, u) & \text{if } x \in \Omega \\ \frac{1}{\delta}(u - \varphi(x)) & \text{if } x \in B \setminus \Omega \end{cases}$$

and as before $w = \det(D^2u)^{-1}$ and U is the cofactor matrix of D^2u . In view of (4.2) and the definition of f_δ , it is natural to introduce the functional defined over convex functions on B by

$$\mathcal{J}_1^\delta(v) := \int_{\Omega} F(x, v(x)) dx + \frac{1}{2\delta} \int_{B \setminus \Omega} (v - \varphi)^2 - \mathfrak{F}_B(v)$$

where

$$\mathfrak{F}_B(v) := \int_B \log(\det(\partial^2 v))$$

so that

$$\mathcal{J}_1^\delta(v) = \mathcal{J}_1(v) + \frac{1}{2\delta} \int_{B \setminus \Omega} (v - \varphi)^2 - \int_{B \setminus \Omega} \log(\det(\partial^2 v)). \quad (4.3)$$

4.1 A priori estimates for the second boundary value problem

Following a similar convexity argument as in Lemma 2.2 in Chau and Weinkove [3], we first have

Proposition 4.1. *Let u be a smooth and uniformly convex solution of (4.2), then*

$$\max_B |u| + \int_{\partial B} |\partial_\nu u|^d + \frac{1}{\delta} \int_{B \setminus \Omega} |u - \varphi|^2 \leq C \quad (4.4)$$

for some constant C only depending on B , $\|\varphi\|_{C^{3,1}(\overline{B})}$ and the constant λ in (2.1).

Proof. First observe that by convexity and (2.1), $u < 0$ in B and $\partial_\nu u > 0$ on ∂B . Define $\tilde{u} := \varphi$, \tilde{U} as the cofactor matrix of $D^2\varphi$, $\tilde{w} := \det(D^2\varphi)^{-1}$ and $\tilde{f} := \tilde{U}^{ij} \tilde{w}_{ij}$ (whose L^∞ norm only depends on $\|\varphi\|_{C^{3,1}(\overline{B})}$ and the constant λ in (2.1)) we have by the concavity of \mathfrak{F}_B , (4.2) and the monotonicity of

$f(x, \cdot)$:

$$\begin{aligned}
0 &\geq (\mathfrak{F}'_B(u) - \mathfrak{F}'_B(\tilde{u}))(u - \tilde{u}) \\
&= \int_B (U^{ij}w_{ij} - \tilde{U}^{ij}\tilde{w}_{ij})(u - \varphi) + \int_{\partial B} \psi(U^{ij} - \tilde{U}^{ij})\partial_i(u - \varphi)\nu_j \\
&= \int_{\Omega} f(x, u)(u - \varphi) - \int_B \tilde{f}(u - \varphi) + \frac{1}{\delta} \int_{B \setminus \Omega} (u - \varphi)^2 \\
&\quad + \int_{\partial B} \psi(U^{\nu\nu} - \tilde{U}^{\nu\nu})\partial_\nu(u - \varphi) \\
&\geq \int_{\Omega} f(x, \varphi)(u - \varphi) - \int_B \tilde{f}(u - \varphi) + \frac{1}{\delta} \int_{B \setminus \Omega} (u - \varphi)^2 \\
&\quad + \int_{\partial B} \psi(U^{\nu\nu} - \tilde{U}^{\nu\nu})\partial_\nu(u - \varphi)
\end{aligned}$$

where, in the last line, we have used the fact that $\nabla u - \nabla \varphi = \partial_\nu(u - \varphi)\nu$ on ∂B and set $U^{\nu\nu} = U\nu \cdot \nu$, $\tilde{U}^{\nu\nu} = \tilde{U}\nu \cdot \nu$. Using the fact that $f(x, \varphi)$, \tilde{f} , φ , $\nabla \varphi$ and \tilde{U} are bounded, we thus get

$$\frac{1}{\delta} \int_{B \setminus \Omega} (u - \varphi)^2 + \int_{\partial B} \psi U^{\nu\nu} \partial_\nu u \leq C \left(1 + \int_B |u| + \int_{\partial B} \partial_\nu u + \int_{\partial B} U^{\nu\nu} \right). \quad (4.5)$$

Denoting by R the radius of B and by the same argument as in Lemma 2.2 in [3], one has

$$U^{\nu\nu} = \frac{1}{R^{d-1}} \partial_\nu u^{d-1} + E \text{ with } |E| \leq C(1 + \partial_\nu u^{d-2}) \text{ on } \partial B. \quad (4.6)$$

Moreover since u is convex and $u = \varphi = 0$ on ∂B , one has

$$\max_B |u| = -\min_B u \leq 2R \partial_\nu u(x) \text{ for all } x \in \partial B. \quad (4.7)$$

Putting together (4.5), (4.6), (4.7) and the fact that $\inf_{\partial B} \psi > 0$, we obtain

$$\int_{\partial B} (\partial_\nu u)^d \leq C \left(1 + \int_{\partial B} (\partial_\nu u)^{d-1} \right)$$

which gives a bound on $\|\partial_\nu u\|_{L^d(\partial B)}$ hence also on $\max_B |u|$ by (4.7) so that finally the bound on $\delta^{-1} \int_{B \setminus \Omega} (u - \varphi)^2$ follows from the latter bounds and (4.5). \square

4.2 Existence and uniqueness of a smooth uniformly convex solution

Thanks to Theorem 1.1 in [3], a Leray-Schauder degree argument and the a priori estimate (4.4), one easily deduces the following:

Theorem 4.2. *For every $\delta > 0$, the second boundary value problem (4.2) admits a unique uniformly convex solution which is $W^{4,p}(B)$ for every $p \in [1, +\infty)$.*

Proof. Let $D := \{u \in C(\overline{B}), \|u\|_{C(\overline{B})} \leq C + 1\}$ where C is the constant from (4.4). For $t \in [0, 1]$ and $u \in D$, it follows from Theorem 1.1 in [3] that there exists a unique $W^{4,p}$ for every $p \in [1, \infty)$ and uniformly convex solution of

$$V^{ij}w_{ij} = tf_\delta(x, u), w = \det(D^2v)^{-1} \text{ in } B, \quad v = \varphi, w = \psi \text{ on } \partial B \quad (4.8)$$

where V denotes the cofactor matrix of D^2v . We denote by $v = T_t(u)$ the solution of (4.8). Moreover, by Theorem 2.1 of [3], for every $\alpha \in (0, 1)$ there are a priori bounds on $\|v\|_{C^{3,\alpha}}$ and on $\sup_B(\det(D^2v) + \det(D^2v)^{-1})$ that only depend on $C, \alpha, \delta, \|\varphi\|_{C^{3,1}}$ and the constant λ in (2.1). Therefore $(t, u) \in [0, 1] \times D \mapsto T_t(u)$ is continuous on $[0, 1] \times D$ and T_t is compact in $C(\overline{B})$ for every $t \in [0, 1]$. Since T_0 is constant and by (4.4) it has a unique fixed point in D , again by (4.4), T_t has no fixed point on ∂D , it thus follows from the Leray-Schauder Theorem that T_1 has a fixed point in D , this proves the existence claim for (4.2).

Finally, uniqueness follows from the same argument as in Lemma 7.1 from [10] where it is proven that two smooth solutions actually have the same gradient on ∂B and then are the minimizers of the same strictly convex minimization problem hence coincide. □

In dimension $d = 2$, following the argument of Remark 4.2 of Trudinger and Wang [9] and taking advantage of the fact that the right-hand side of the Abreu equation (4.2) does not depend on δ on Ω , we have the following local bound (which we have used in the proof of Proposition 3.4):

Proposition 4.3. *Let $d = 2$ and u be the solution of (4.2) then for every open set $\omega \subset\subset \Omega$, $\|\det(D^2u)\|_{L^\infty(\omega)}$ is bounded independently of δ .*

Proof. Let $B_r := B_r(0) \subset\subset \Omega$, and observe that thanks to (4.4) both $\|f_\delta(\cdot, u(\cdot))\|_{L^\infty(\Omega)} = \|f(\cdot, u(\cdot))\|_{L^\infty(\Omega)}$ and $\|\nabla u\|_{L^\infty(\Omega)}$ are bounded independently of δ . Define then $\eta(x) := \frac{1}{2}(r^2 - |x|^2)$ and consider $z := \log(w) -$

$2\log(\eta) - \frac{1}{2}|\nabla u|^2$, by construction z achieves its minimum at an interior point x_0 of B_r at such a point, we have

$$\frac{\nabla w}{w} = 2\frac{\nabla\eta}{\eta} + D^2u\nabla u. \quad (4.9)$$

We also have

$$z_{ij} = \frac{w_{ij}}{w} - \frac{w_i w_j}{w^2} + 2\frac{\delta_{ij}}{\eta} + 2\frac{\eta_i \eta_j}{\eta^2} - u_{ijk}u_k - u_{ik}u_{jk}, \quad (4.10)$$

multiplying by $wU = [D^2u]^{-1}$, using $wU^{ij}z_{ij} \geq 0$ at x_0 , $U^{ij}w_{ij} = f(x, u) \leq C$ and the identities

$$wU^{ij}u_{ik}u_{jk} = u_{ii} = \Delta u, \quad wU^{ij}u_{ijk}u_k = -\frac{w_k}{w}u_k = -\frac{\nabla w}{w} \cdot \nabla u, \quad (4.11)$$

(the second identity is classically obtained by first differentiating the relation $-\log(w) = \log(\det D^2u)$ and then taking the scalar product with ∇u) as well as the fact that $\text{Tr}(U) = \Delta u$ in dimension $d = 2$, we get

$$0 \leq C - wU \frac{\nabla w}{w} \cdot \frac{\nabla w}{w} + 2\frac{w}{\eta}\Delta u + 2wU \frac{\nabla\eta}{\eta} \cdot \frac{\nabla\eta}{\eta} + \frac{\nabla w}{w} \cdot \nabla u - \Delta u. \quad (4.12)$$

Using (4.9) and using again $wU = [D^2u]^{-1}$, we then obtain

$$wU \frac{\nabla w}{w} \cdot \frac{\nabla w}{w} = 4wU \frac{\nabla\eta}{\eta} \cdot \frac{\nabla\eta}{\eta} + D^2u\nabla u \cdot \nabla u + 4\frac{\nabla\eta}{\eta} \cdot \nabla u \quad (4.13)$$

and

$$\frac{\nabla w}{w} \cdot \nabla u = 2\frac{\nabla\eta}{\eta} \cdot \nabla u + D^2u\nabla u \cdot \nabla u. \quad (4.14)$$

Replacing (4.13), (4.14) in (4.12), multiplying by η and rearranging gives

$$\Delta u(\eta - 2w) \leq C\eta - 2\frac{w}{\eta}U\nabla\eta \cdot \nabla\eta - 2\nabla\eta \cdot \nabla u \leq C\eta + \|\nabla\eta\|_{L^\infty} \|\nabla u\|_{L^\infty(\Omega)} \leq C' \quad (4.15)$$

If $\eta(x_0) \geq 4w(x_0)$, (4.15) gives $\eta(x_0)\Delta u(x_0) \leq 2C'$ and since $\Delta u(x_0)w(x_0)^{1/2} \geq 2$ we get the desired lower bound on the minimum of $\eta^{-2}w$. In the remaining case $w(x_0) \geq \frac{1}{4}\eta(x_0) \geq \frac{\eta^2(x_0)}{2r^2}$ and we reach the same conclusion. This gives a local lower bound on w i.e. the desired local upper bound on $\det(D^2u)$. \square

5 Convergence

5.1 Letting $\delta \rightarrow 0$ for fixed ε

In this paragraph, we fix ε (and thus normalize it to $\varepsilon = 1$ as we did in the whole of section 4).

Theorem 5.1. *Let u_δ be the unique smooth strictly convex solution of (4.2), then u_δ converges uniformly on Ω to the unique minimizer of (4.1) as $\delta \rightarrow 0^+$.*

Proof. We already know from (4.4) that (possibly up to an extraction) u_δ converges locally uniformly on B to some convex u and it also follows from (4.4) that $u \in \overline{S}[\varphi, \Omega]$. Let $v \in \overline{S}[\varphi, \Omega]$ (extended by φ on $B \setminus \Omega$), thanks to (4.2) and the convexity of \mathcal{J}_1^δ we first have

$$\mathcal{J}_1^\delta(v) - \mathcal{J}_1^\delta(u_\delta) \geq \int_{\partial B} U_\delta^{\nu\nu} \psi \partial_\nu(u_\delta - \varphi)$$

i.e.

$$\begin{aligned} \mathcal{J}_1(v) - \mathcal{J}_1(u_\delta) &\geq \frac{1}{2\delta} \int_{B \setminus \Omega} (u_\delta - \varphi)^2 + \int_{B \setminus \Omega} (\log(\det(D^2\varphi)) - \log(\det(D^2u_\delta))) \\ &\quad + \int_{\partial B} U_\delta^{\nu\nu} \psi \partial_\nu(u_\delta - \varphi) \\ &\geq \int_{B \setminus \Omega} (\log(\det(D^2\varphi)) - \log(\det(D^2u_\delta))) + \int_{\partial B} U_\delta^{\nu\nu} \psi \partial_\nu(u_\delta - \varphi). \end{aligned}$$

It follows from Lemma 5.2 below that

$$\liminf_{\delta \rightarrow 0} \int_{B \setminus \Omega} (\log(\det(D^2\varphi)) - \log(\det(D^2u_\delta))) \geq 0.$$

We now have to pay attention to the boundary term, we know from (4.6) that $\theta_\delta := \psi U_\delta^{\nu\nu}$ satisfies $0 \leq \theta_\delta \leq C(1 + (\partial_\nu u_\delta)^{d-1})$ so that thanks to (4.4), θ_δ is bounded in $L^{\frac{d}{d-1}}(\partial B)$, up to an extraction we may therefore assume that it weakly converges in $L^{\frac{d}{d-1}}(\partial B)$ to some nonnegative function θ . By convexity we also have that for $\tau > 0$

$$\partial_\nu u_\delta(x) \geq D_{\tau, \nu} u_\delta(x) := \frac{1}{\tau} \left(u_\delta(x - \tau\nu(x)) - u_\delta(x - 2\tau\nu(x)) \right), \quad \forall x \in \partial B$$

For small fixed $\tau > 0$ note that $D_{\tau, \nu} u_\delta$ is bounded independently of δ thanks to (4.4) and that it converges as $\delta \rightarrow 0$ pointwise to $D_{\tau, \nu} \varphi$, we thus have

$$\begin{aligned} \liminf_{\delta \rightarrow 0} \int_{\partial B} \theta_\delta \partial_\nu(u_\delta - \varphi) &\geq \liminf_{\delta \rightarrow 0} \int_{\partial B} \theta_\delta (D_{\tau, \nu} u_\delta - \partial_\nu \varphi) \\ &= \int_{\partial B} \theta (D_{\tau, \nu} \varphi - \partial_\nu \varphi) \end{aligned}$$

where in the last line we have passed to the limit using the fact that we have the product of a weakly convergent sequence with a strongly convergent sequence. Now letting $\tau \rightarrow 0$ and using the smoothness of φ , we deduce that

$$\liminf_{\delta \rightarrow 0} \int_{\partial B} U_\delta^{\nu\nu} \psi \partial_\nu (u_\delta - \varphi) \geq 0.$$

Since \mathcal{J}_1 is lower semi-continuous thanks to Lemma 2.2, we can conclude that

$$\mathcal{J}_1(v) \geq \liminf_{\delta \rightarrow 0} \mathcal{J}_1(u_\delta) \geq \mathcal{J}_1(u)$$

so that u solves (4.1) and by uniqueness of the minimizer there is in fact convergence of the whole sequence. \square

In the previous proof we have used:

Lemma 5.2. *Let u_δ be the unique smooth strictly convex solution of (4.2) as before, then*

$$\limsup_{\delta \rightarrow 0} \int_{B \setminus \Omega} \log(\det(D^2 u_\delta)) \leq \int_{B \setminus \Omega} \log(\det(D^2 \varphi)).$$

Proof. The key point here is the estimate $\int_B \Delta u_\delta = \int_{\partial B} \partial_\nu u_\delta \leq C$ which follows from (4.4). Let ω be an arbitrary Borel subset of B , we have (for some constant C varying from a line to another):

$$\begin{aligned} \int_\omega \log(\det(D^2 u_\delta)) &\leq C(|\omega| + \int_\omega \det(D^2 u_\delta)^{1/2d}) \\ &\leq C\left(|\omega| + \int_\omega \sqrt{\Delta u_\delta}\right) \leq C\left(|\omega| + |\omega|^{1/2} \left(\int_B \Delta u_\delta\right)^{1/2}\right) \\ &= C\left(|\omega| + |\omega|^{1/2} \left(\int_{\partial B} \partial_\nu u_\delta\right)^{1/2}\right) \end{aligned}$$

so that

$$\int_\omega \log(\det(D^2 u_\delta)) \leq C(|\omega| + |\omega|^{1/2}). \quad (5.1)$$

Take $0 < R' < R$ with Ω contained in $B_{R'}$ (recall R is the radius of B), we then have, thanks to Lemma 2.2, the fact that $\log(\det(D^2 \varphi))$ is bounded

and (5.1):

$$\begin{aligned}
\limsup_{\delta \rightarrow 0} \mathfrak{F}_{B \setminus \Omega}(u_\delta) &= \limsup_{\delta \rightarrow 0} \mathfrak{F}_{B \setminus \bar{\Omega}}(u_\delta) \\
&\leq \limsup_{\delta \rightarrow 0} \mathfrak{F}_{B_{R'} \setminus \bar{\Omega}}(u_\delta) + \limsup_{\delta \rightarrow 0} \mathfrak{F}_{B \setminus B_{R'}}(u_\delta) \\
&\leq \mathfrak{F}_{B_{R'} \setminus \bar{\Omega}}(\varphi) + C(|B \setminus B_{R'}| + |B \setminus B_{R'}|^{1/2}) \\
&\leq \mathfrak{F}_{B \setminus \Omega}(\varphi) + C'(|B \setminus B_{R'}| + |B \setminus B_{R'}|^{1/2}).
\end{aligned}$$

The desired result follows by letting R' tend to R . \square

5.2 Full convergence

We now take $\delta = \delta_\varepsilon > 0$ with

$$\lim_{\varepsilon \rightarrow 0^+} \delta_\varepsilon = 0, \quad (5.2)$$

i.e. we only have a single small parameter ε and we consider the second-boundary value problem

$$\varepsilon U_\varepsilon^{ij} w_{ij}^\varepsilon = g_\varepsilon(x, u_\varepsilon), \quad \text{in } B, \quad u_\varepsilon = \varphi, \quad w^\varepsilon = \psi \text{ on } \partial B \quad (5.3)$$

where $\psi := \det((D^2\varphi)^{-1})$,

$$g_\varepsilon(x, u) := \begin{cases} f(x, u) & \text{if } x \in \Omega \\ \frac{1}{\delta_\varepsilon}(u - \varphi(x)) & \text{if } x \in B \setminus \Omega \end{cases},$$

$w^\varepsilon = \det(D^2 u_\varepsilon)^{-1}$ and U_ε is the cofactor matrix of $D^2 u_\varepsilon$. We further assume that there is an $\alpha > 0$ such that

$$(f(x, u) - f(x, v))(u - v) \geq \alpha(u - v)^2, \quad \forall (u, v) \in \mathbb{R}^d, \quad \text{and a.e. } x \in \Omega \quad (5.4)$$

which amounts to say that the integrand F is uniformly convex in its second argument. Under these assumptions, we have a *full* convergence result:

Theorem 5.3. *Let u_ε be the unique smooth strictly convex solution of (5.3), then u_ε converges uniformly on Ω to the unique minimizer of (1.1) as $\varepsilon \rightarrow 0^+$.*

Proof. Step 1: a priori estimates. The first step of the proof is similar to the proof of Proposition 4.1. Again define $\tilde{u} := \varphi$, \tilde{U} as the cofactor matrix

of $D^2\varphi$, $\tilde{w} := \det(D^2\varphi)^{-1}$ and $\tilde{f}_\varepsilon := \varepsilon\tilde{U}^{ij}\tilde{w}_{ij}$. We then have together with (5.4):

$$\begin{aligned} 0 &\geq \varepsilon(\mathfrak{F}'_B(u_\varepsilon) - \mathfrak{F}'_B(\tilde{u}))(u_\varepsilon - \tilde{u}) \\ &\geq \int_\Omega (f(x, \varphi) - \tilde{f}_\varepsilon)(u_\varepsilon - \varphi) + \alpha \int_\Omega (u_\varepsilon - \varphi)^2 + \frac{1}{\delta_\varepsilon} \int_{B \setminus \Omega} (u_\varepsilon - \varphi)^2 \\ &\quad + \varepsilon \int_{\partial B} \psi(U_\varepsilon^{\nu\nu} - \tilde{U}^{\nu\nu}) \partial_\nu (u_\varepsilon - \varphi) \end{aligned}$$

thanks to the fact that $f(x, \varphi) - \tilde{f}_\varepsilon$ is bounded uniformly with respect to ε , using Young's inequality and invoking (4.6), we get

$$\int_\Omega (u_\varepsilon - \varphi)^2 + \frac{1}{\delta_\varepsilon} \int_{B \setminus \Omega} (u_\varepsilon - \varphi)^2 + \varepsilon \int_{\partial B} (\partial_\nu u_\varepsilon)^d \leq C. \quad (5.5)$$

Step 2: convergence. Thanks to (5.5), up to taking a subsequence of vanishing ε_n , we may assume that u_ε converges locally uniformly in B to some u such that $u = \varphi$ in $B \setminus \Omega$ so that the restriction of u to Ω belongs to $\overline{\mathcal{S}}[\varphi, \Omega]$. For every v convex on B such that $v = \varphi$ on ∂B , define

$$\tilde{\mathcal{J}}_\varepsilon(v) := \int_\Omega F(x, v(x)) dx + \frac{1}{2\delta_\varepsilon} \int_{B \setminus \Omega} (v - \varphi)^2 - \varepsilon \int_B \log(\det(\partial^2 v)).$$

Let then $v \in \overline{\mathcal{S}}[\varphi, \Omega]$ (extended by φ on $B \setminus \Omega$), we then have

$$\tilde{\mathcal{J}}_\varepsilon(v) - \tilde{\mathcal{J}}_\varepsilon(u_\varepsilon) \geq \varepsilon \int_{\partial B} \psi U_\varepsilon^{\nu\nu} \partial_\nu (u_\varepsilon - \varphi)$$

hence

$$\mathcal{J}_0(v) \geq \liminf_\varepsilon \mathcal{J}_0(u_\varepsilon) + \liminf_\varepsilon \varepsilon(\mathfrak{F}_B(v) - \mathfrak{F}_B(u_\varepsilon)) - \limsup_\varepsilon \varepsilon \int_{\partial B} \psi U_\varepsilon^{\nu\nu} \partial_\nu \varphi.$$

Arguing as in the proof of Proposition 3.5, we may actually assume that $\mathfrak{F}_\Omega(v) > -\infty$ so that $\liminf_\varepsilon \varepsilon \mathfrak{F}_B(v) \geq 0$. As for an upper bound for $\varepsilon \mathfrak{F}_B(u_\varepsilon)$ we use the fact that thanks to (5.5), we have $\int_{\partial B} \partial_\nu u_\varepsilon \leq C\varepsilon^{-1/d}$ and argue in a similar way as in the proof of Lemma 5.2, to obtain

$$\varepsilon \mathfrak{F}_B(u_\varepsilon) \leq C\varepsilon(1 + \int_B \det(D^2 u_\varepsilon)^{1/d}) \leq C\varepsilon(1 + \int_{\partial B} \partial_\nu u_\varepsilon) \leq C(\varepsilon + \varepsilon^{1-1/d}),$$

which yields

$$\liminf_\varepsilon \varepsilon(\mathfrak{F}_B(v) - \mathfrak{F}_B(u_\varepsilon)) \geq 0.$$

Thanks to (4.6), we have

$$\int_{\partial B} \psi U_\varepsilon^{\nu\nu} \partial_\nu \varphi \leq C \int_{\partial B} (1 + (\partial_\nu u_\varepsilon)^{d-1})$$

but, thanks to (5.5) and Hölder's inequality, we deduce

$$\varepsilon \int_{\partial B} (\partial_\nu u_\varepsilon)^{d-1} \leq C \varepsilon^{\frac{1}{d}}$$

so that

$$\mathcal{J}_0(v) \geq \liminf_\varepsilon \mathcal{J}_0(u_\varepsilon) = \mathcal{J}_0(u)$$

hence u solves (1.1) (and the whole family u_ε converges uniformly on Ω to u by uniqueness of the minimizer of \mathcal{J}_0 on $\overline{S}[\varphi, \Omega]$).

□

Acknowledgments: G.C. is grateful to the Agence Nationale de la Recherche for its support through the project MAGA (ANR-16-CE40-0014).

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