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To cite this version:

Yves Achdou, Manh-Khang Dao, Olivier Ley, Nicoletta Tchou. A Class of Infinite Horizon Mean Field Games on Networks. Networks and Heterogeneous Media, 2019, 14 (3), pp.537-566. 10.3934/nhm.2019021 . hal-01802120

HAL Id: hal-01802120

https://hal.science/hal-01802120v1

Submitted on 29 May 2018

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A Class of Infinite Horizon Mean Field Games on Networks

Yves Achdou [*], Manh-Khang Dao [†], Olivier Ley [‡], Nicoletta Tchou [§]

May 29, 2018

Abstract

We consider stochastic mean field games for which the state space is a network. In the ergodic case, they are described by a system coupling a Hamilton-Jacobi-Bellman equation and a Fokker-Planck equation, whose unknowns are the invariant measure m, a value function u, and the ergodic constant ρ. The function u is continuous and satisfies general Kirchhoff conditions at the vertices. The invariant measure m satisfies dual transmission conditions: in particular, m is discontinuous across the vertices in general, and the values of m on each side of the vertices satisfy special compatibility conditions. Existence and uniqueness are proven, under suitable assumptions.

1 Introduction and main results

Recently, an important research activity on mean field games (MFGs for short) has been initiated since the pioneering works [29, 30, 31] of Lasry and Lions (related ideas have been developed independently in the engineering literature by Huang-Caines-Malhamé, see for example [26, 25, 24]): it aims at studying the asymptotic behavior of stochastic differential games (Nash equilibria) as the number N of agents tends to infinity. In these models, it is assumed that the agents are all identical and that an individual agent can hardly influence the outcome of the game. Moreover, each individual strategy is influenced by some averages of functions of the states of the other agents. In the limit when $N \to +\infty$, a given agent feels the presence of the others through the statistical distribution of the states. Since perturbations of the strategy of a single agent do not influence the statistical states distribution, the latter acts as a parameter in the control problem to be solved by each agent. The delicate question of the passage to the limit is one of the main topics of the book of Carmona and Delarue, [11]. When the dynamics of the agents are independent stochastic processes, MFGs naturally lead to a coupled system of two partial differential equations (PDEs for short), a forward in time Kolmogorov or Fokker-Planck (FP) equation and a backward Hamilton-Jacobi-Bellman (HJB) equation. The unknown of this system is a pair of functions: the value function of the stochastic optimal control problem solved by a representative agent and the density of the distribution of states. In the infinite horizon limit, one obtains a system of two stationary PDEs.

A very nice introduction to the theory of MFGs is supplied in the notes of Cardaliaguet [10]. Theoretical results on the existence of classical solutions to the previously mentioned system of PDEs can be found in [29, 30, 31, 20, 22, 21]. Weak solutions have been studied in [31, 34, 35, 5]. The numerical approximation of these systems of PDEs has been discussed in [3, 1, 5].

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A network (or a graph) is a set of items, referred to as vertices (or nodes/crosspoints), with connections between them referred to as edges. In the recent years, there has been an increasing interest in the investigation of dynamical systems and differential equations on networks, in particular in connection with problems of data transmission and traffic management (see for example [18, 13, 15]). The literature on optimal control in which the state variable takes its values on a network is recent: deterministic control problems and related Hamilton-Jacobi equations were studied in [2, 27, 4, 28, 32, 33]. Stochastic processes on networks and related Kirchhoff conditions at the vertices were studied in [17, 16].

The present work is devoted to infinite horizon stochastic mean field games taking place on networks. The most important difficulty will be to deal with the transition conditions at the vertices. The latter are obtained from the theory of stochastic control in [17, 16], see Section 1.3 below. In [8], the first article on MFGs on networks, Camilli and Marchi consider a particular type of Kirchhoff condition at the vertices for the value function: this condition comes from an assumption which can be informally stated as follows: consider a vertex ν of the network and assume that it is the intersection of p edges $\Gamma_1, \ldots, \Gamma_p$, ; if, at time τ, the controlled stochastic process X_t associated to a given agent hits ν, then the probability that X_{τ^+} belongs to Γ_i is proportional to the diffusion coefficient in Γ_i. Under this assumption, it can be seen that the density of the distribution of states is continuous at the vertices of the network. In the present work, the above mentioned assumption is not made any longer. Therefore, it will be seen below that the value function satisfies more general Kirchhoff conditions, and accordingly, that the density of the distribution of states is no longer continuous at the vertices; the continuity condition is then replaced by suitable compatibility conditions on the jumps across the vertex. Moreover, as it will be explained in Remark 1.14 below, more general assumptions on the coupling costs will be made. Mean field games on networks with finite horizon will be considered in a forthcoming paper.

After obtaining the transmission conditions at the vertices for both the value function and the density, we shall prove existence and uniqueness of weak solutions of the uncoupled HJB and FP equations (in suitable Sobolev spaces). We have chosen to work with weak solutions because it is a convenient way to deal with existence and uniqueness in the stationary regime, but also because it is difficult to avoid it in the nonstationary case, see the forthcoming work on finite horizon MFGs. Classical arguments will then lead to the regularity of the solutions. Next, we shall establish the existence result for the MFG system by a fixed point argument and a truncation technique. Uniqueness will also be proved under suitable assumptions.

The present work is organized as follows: the remainder of Section 1 is devoted to setting the problem and obtaining the system of differential equations and the transmission conditions at the vertices. Section 2 contains useful results, first about some linear boundary value problems with elliptic equations, then on a pair of linear Kolmogorov and Fokker-Planck equations in duality. By and large, the existence of weak solutions is obtained by applying Banach-Necas-Babuška theorem to a special pair of Sobolev spaces referred to as V and W below and Fredholm's alternative, and uniqueness comes from a maximum principle. Section 3 is devoted to the HJB equation associated with an ergodic problem. Finally, the proofs of the main results of existence and uniqueness for the MFG system of differential equations are completed in Section 1.

1.1 Networks and function spaces

1.1.1 The geometry

A bounded network Γ (or a bounded connected graph) is a connected subset of $\mathbb{R}^n$ made of a finite number of bounded non-intersecting straight segments, referred to as edges, which connect nodes referred to as vertices. The finite collection of vertices and the finite set of closed edges are respectively denoted by $\mathcal{V} := \{\nu_i, i \in I\}$ and $\mathcal{E} := \{\Gamma_\alpha, \alpha \in \mathcal{A}\}$, where I and A are finite sets of indices contained in $\mathbb{N}$. We assume that for $\alpha, \beta \in \mathcal{A}$, if $\alpha \neq \beta$, then $\Gamma_\alpha \cap \Gamma_\beta$ is either empty or made of a single vertex. The length of Γ_α is denoted by

ℓ_α. Given $\nu_i \in \mathcal{V}$, the set of indices of edges that are adjacent to the vertex ν_i is denoted by $\mathcal{A}_i = \{\alpha \in \mathcal{A} : \nu_i \in \Gamma_\alpha\}$. A vertex ν_i is named a *boundary vertex* if $\sharp(\mathcal{A}_i) = 1$, otherwise it is named a *transition vertex*. The set containing all the boundary vertices is named the *boundary* of the network and is denoted by $\partial\Gamma$ hereafter.

The edges $\Gamma_\alpha \in \mathcal{E}$ are oriented in an arbitrary manner. In most of what follows, we shall make the following arbitrary choice that an edge $\Gamma_\alpha \in \mathcal{E}$ connecting two vertices ν_i and ν_j, with $i < j$ is oriented from ν_i toward ν_j: this induces a natural parametrization $\pi_\alpha : [0, \ell_\alpha] \to \Gamma_\alpha = [\nu_i, \nu_j]$:

$$\pi_\alpha(y) = (\ell_\alpha - y)\nu_i + y\nu_j \quad \text{for } y \in [0, \ell_\alpha]. \tag{1.1}$$

For a function $v : \Gamma \to \mathbb{R}$ and $\alpha \in \mathcal{A}$, we define $v_\alpha : (0, \ell_\alpha) \to \mathbb{R}$ by

$$v_\alpha(x) := v|_{\Gamma_\alpha} \circ \pi_\alpha(x), \qquad \text{for all } x \in (0, \ell_\alpha).$$

Remark 1.1. In what precedes, the edges have been arbitrarily oriented from the vertex with the smaller index toward the vertex with the larger one. Other choices are of course possible. In particular, by possibly dividing a single edge into two, adding thereby new artificial vertices, it is always possible to assume that for all vertices $\nu_i \in \mathcal{V}$,

$$\text{either } \pi_\alpha(\nu_i) = 0, \text{ for all } \alpha \in \mathcal{A}_i \text{ or } \pi_\alpha(\nu_i) = \ell_\alpha, \text{ for all } \alpha \in \mathcal{A}_i. \tag{1.2}$$

This idea was used by Von Below in [36]: some edges of Γ are cut into two by adding artificial vertices so that the new oriented network $\overline{\Gamma}$ has the property (1.2), see Figure 1 for an example.

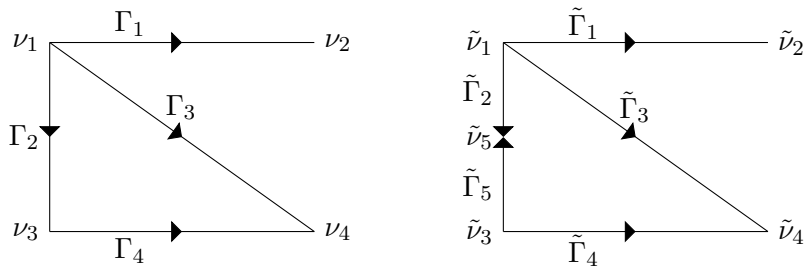

Figure 1: Left: the network Γ in which the edges are oriented toward the vertex with larger index (4 vertices and 4 edges). Right: a new network $\tilde{\Gamma}$ obtained by adding an artificial vertex (5 vertices and 5 edges): the oriented edges sharing a given vertex ν either have all their starting point equal ν, or have all their terminal point equal ν.

In Sections 1.2 and 1.3 below, especially when dealing with stochastic calculus, it will be convenient to assume that property (1.2) holds. In the remaining part of the paper, it will be convenient to work with the original network, i.e. without the additional artificial vertices and with the orientation of the edges that has been chosen initially.

1.1.2 Function spaces

The set of continuous functions on Γ is denoted by $C(\Gamma)$ and we set

$$PC(\Gamma) := \left\{ v : \Gamma \to \mathbb{R} \ : \ \text{for all } \alpha \in \mathcal{A}, \left| \begin{array}{l} v_\alpha \in C(0, \ell_\alpha) \\ v_\alpha \text{ can be extended by continuity to } [0, \ell_\alpha]. \end{array} \right. \right\}.$$

By the definition of piecewise continuous functions $v \in PC(\Gamma)$, for all $\alpha \in \mathcal{A}$, it is possible to extend $v|_{\Gamma_\alpha}$ by continuity at the endpoints of Γ_α: if $\Gamma_\alpha = [\nu_i, \nu_j]$, we set

$$v|_{\Gamma_\alpha}(x) = \begin{cases} v_\alpha\left(\pi_\alpha^{-1}(x)\right), & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \\ v_\alpha(0) := \lim\limits_{y \to 0^+} v_\alpha(y), & \text{if } x = \nu_i, \\ v_\alpha(\ell_\alpha) := \lim\limits_{y \to \ell_\alpha^-} v_\alpha(y), & \text{if } x = \nu_j. \end{cases} \tag{1.3}$$

For $m \in \mathbb{N}$, the space of m-times continuously differentiable functions on Γ is defined by

$$C^m(\Gamma) := \{v \in C(\Gamma) : v_\alpha \in C^m([0, \ell_\alpha]) \text{ for all } \alpha \in \mathcal{A}\},$$

and is endowed with the norm $\|v\|_{C^m(\Gamma)} := \sum_{\alpha \in \mathcal{A}} \sum_{k \le m} \left\|\partial^k v_\alpha\right\|_{L^\infty(0,\ell_\alpha)}$. For $\sigma \in (0,1)$, the space $C^{m,\sigma}(\Gamma)$, contains the functions $v \in C^m(\Gamma)$ such that $\partial^m v_\alpha \in C^{0,\sigma}([0, \ell_\alpha])$ for all $\alpha \in \mathcal{A}$; it is endowed with the norm $\|v\|_{C^{m,\sigma}(\Gamma)} := \|v\|_{C^m(\Gamma)} + \sup\limits_{\alpha \in \mathcal{A}} \sup\limits_{\substack{y \neq z \\ y,z \in [0,\ell_\alpha]}} \dfrac{|\partial^m v_\alpha(y) - \partial^m v_\alpha(z)|}{|y-z|^\sigma}$.

For a positive integer m and a function $v \in C^m(\Gamma)$, we set for $k \le m$,

$$\partial^k v(x) = \partial^k v_\alpha\left(\pi_\alpha^{-1}(x)\right) \text{ if } x \in \Gamma_\alpha \backslash \mathcal{V}. \tag{1.4}$$

Notice that $v \in C^k(\Gamma)$ is continuous on Γ but that the derivatives $\partial^l v$, $0 < l \le k$ are not defined at the vertices. For a vertex ν, we define $\partial_\alpha v(\nu)$ as the *outward* directional derivative of $v|_{\Gamma_\alpha}$ at ν as follows:

$$\partial_\alpha v(\nu) := \begin{cases} \lim\limits_{h \to 0^+} \dfrac{v_\alpha(0) - v_\alpha(h)}{h}, & \text{if } \nu = \pi_\alpha(0), \\ \lim\limits_{h \to 0^+} \dfrac{v_\alpha(\ell_\alpha) - v_\alpha(\ell_\alpha - h)}{h}, & \text{if } \nu = \pi_\alpha(\ell_\alpha). \end{cases} \tag{1.5}$$

For all $i \in I$ and $\alpha \in \mathcal{A}_i$, setting

$$n_{i\alpha} = \begin{cases} 1 & \text{if } \nu_i = \pi_\alpha(\ell_\alpha), \\ -1 & \text{if } \nu_i = \pi_\alpha(0), \end{cases} \tag{1.6}$$

we have

$$\partial_\alpha v(\nu_i) = n_{i\alpha}\, \partial v|_{\Gamma_\alpha}(\nu_i) = n_{i\alpha}\, \partial v_\alpha(\pi_\alpha^{-1}(\nu_i)). \tag{1.7}$$

Remark 1.2. Changing the orientation of the edge does not change the value of $\partial_\alpha v(\nu)$ in (1.5).

If for all $\alpha \in \mathcal{A}$, v_α is Lebesgue-integrable on $(0, \ell_\alpha)$, then the integral of v on Γ is defined by $\int_\Gamma v(x)\, dx = \sum_{\alpha \in \mathcal{A}} \int_0^{\ell_\alpha} v_\alpha(y)\, dy$. The space $L^p(\Gamma) = \{v : v|_{\Gamma_\alpha} \in L^p(\Gamma_\alpha) \text{ for all } \alpha \in \mathcal{A}\}$, $p \in [1, \infty]$, is endowed with the norm $\|v\|_{L^p(\Gamma)} := \left(\sum_{\alpha \in \mathcal{A}} \|v_\alpha\|^p_{L^p(0,\ell_\alpha)}\right)^{\frac{1}{p}}$ if $1 \le p < \infty$, and $\max_{\alpha \in \mathcal{A}} \|v_\alpha\|_{L^\infty(0,\ell_\alpha)}$ if $p = +\infty$. We shall also need to deal with functions on Γ whose restrictions to the edges are weakly-differentiable: we shall use the same notations for the weak derivatives. Let us introduce Sobolev spaces on Γ:

Definition 1.3. For any integer $s \ge 1$ and any real number $p \ge 1$, the Sobolev space $W^{s,p}(\Gamma)$ is defined as follows: $W^{s,p}(\Gamma) := \{v \in C(\Gamma) : v_\alpha \in W^{s,p}(0, \ell_\alpha) \text{ for all } \alpha \in \mathcal{A}\}$, and endowed with the norm $\|v\|_{W^{s,p}(\Gamma)} = \left(\sum_{k=1}^s \sum_{\alpha \in \mathcal{A}} \left\|\partial^k v_\alpha\right\|^p_{L^p(0,\ell_\alpha)} + \|v\|^p_{L^p(\Gamma)}\right)^{\frac{1}{p}}$. We also set $H^s(\Gamma) = W^{s,2}(\Gamma)$.

1.2 A class of stochastic processes on Γ

After rescaling the edges, it may be assumed that $\ell_\alpha = 1$ for all $\alpha \in \mathcal{A}$. Let $\mu_\alpha, \alpha \in \mathcal{A}$ and $p_{i\alpha}, i \in I, \alpha \in \mathcal{A}_i$ be positive constants such that $\sum_{\alpha \in \mathcal{A}_i} p_{i\alpha} = 1$. Consider also a real valued function $a \in PC(\Gamma)$.

As in Remark 1.1, we make the assumption (1.2) by possibly adding artificial nodes: if ν_i is such an artificial node, then $\sharp(\mathcal{A}_i) = 2$, and we assume that $p_{i\alpha} = 1/2$ for $\alpha \in \mathcal{A}_i$. The diffusion parameter μ has the same value on the two sides of an artificial vertex. Similarly, the function a does not have jumps across an artificial vertex.

Let us consider the linear differential operator:

$$\mathcal{L}u(x) = \mathcal{L}_\alpha u(x) := \mu_\alpha \partial^2 u(x) + a|_{\Gamma_\alpha}(x)\,\partial u(x), \quad \text{if } x \in \Gamma_\alpha, \tag{1.8}$$

with domain

$$D(\mathcal{L}) := \left\{ u \in C^2(\Gamma) : \sum_{\alpha \in \mathcal{A}_i} p_{i\alpha}\partial_\alpha u(\nu_i) = 0, \ \text{ for all } i \in I \right\}. \tag{1.9}$$

Remark 1.4. Note that in the definition of $D(\mathcal{L})$, the condition at boundary vertices boils down to a Neumann condition.

Freidlin and Sheu proved in [16] that

1. The operator $\mathcal{L}$ is the infinitesimal generator of a Feller-Markov process on Γ with continuous sample paths. The operators $\mathcal{L}_\alpha$ and the transmission conditions at the vertices
$$\sum_{\alpha \in \mathcal{A}_i} p_{i\alpha}\partial_\alpha u(\nu_i) = 0 \tag{1.10}$$
define such a process in a unique way, see also [17, Theorem 3.1]. The process can be written (X_t, α_t) where $X_t \in \Gamma_{\alpha_t}$. If $X_t = \nu_i$, $i \in I$, α_t is arbitrarily chosen as the smallest index in $\mathcal{A}_i$. Setting $x_t = \pi_{\alpha_t}(X_t)$ defines the process x_t with values in $[0, 1]$.
2. There exist
   (a) a one dimensional Wiener process W_t,
   (b) continuous non-decreasing processes $\ell_{i,t}$, $i \in I$, which are measurable with respect to the σ-field generated by (X_t, α_t),
   (c) continuous non-increasing processes $h_{i,t}$, $i \in I$, which are measurable with respect to the σ-field generated by (X_t, α_t),

   such that
$$\begin{aligned} dx_t &= \mu_{\alpha_t} dW_t + a_{\alpha_t}(x_t)dt + d\ell_{i,t} + dh_{i,t}, \\ \ell_{i,t} &\text{ increases only when } X_t = \nu_i \text{ and } x_t = 0, \\ h_{i,t} &\text{ decreases only when } X_t = \nu_i \text{ and } x_t = 1. \end{aligned} \tag{1.11}$$
3. The following Ito formula holds: for any real valued function $u \in C^2(\Gamma)$:
$$\begin{aligned} u(X_t) =& u(X_0) \\ &+ \sum_{\alpha \in \mathcal{A}} \int_0^t \mathbb{1}_{\{X_s \in \Gamma_\alpha \setminus \mathcal{V}\}} \left(\mu_\alpha \partial^2 u(X_s) + a(X_s)\partial u(X_s)ds + \sqrt{2\mu_\alpha}\partial u(X_s)dW_s \right) \\ &+ \sum_{i \in I} \sum_{\alpha \in \mathcal{A}_i} p_{i\alpha}\partial_\alpha u(\nu_i)(\ell_{i,t} + h_{i,t}). \end{aligned} \tag{1.12}$$

Remark 1.5. The assumption that all the edges have unit length is not restrictive, because we can always rescale the constants μ_α and the piecewise continuous function a. The Ito formula in (1.12) holds when this assumption is not satisfied.

Consider the invariant measure associated with the process X_t. We may assume that it is absolutely continuous with respect to the Lebesgue measure on Γ. Let m be its density:

$$\mathbb{E}[u(X_t)] := \int_\Gamma u(x)\,m(x)\,dx, \qquad \text{for all } u \in PC(\Gamma). \tag{1.13}$$

We focus on functions $u \in D(\mathcal{L})$. Taking the time-derivative of each member of (1.13), Ito's formula (1.12) and (1.10) lead to $\mathbb{E}\left[\mathbb{1}_{\{X_t \notin \mathcal{V}\}}\left(a\partial u(X_t) + \mu\partial^2 u(X_t)\right)\right] = 0$. This implies that

$$\int_\Gamma \left(a(x)\partial u(x) + \mu\partial^2 u(x)\right) m(x)dx = 0. \tag{1.14}$$

Since for $\alpha \in \mathcal{A}$, any smooth function on Γ compactly supported in $\Gamma_\alpha \backslash \mathcal{V}$ clearly belongs to $D(\mathcal{L})$, (1.14) implies that m satisfies

$$-\mu_\alpha \partial^2 m + \partial(ma) = 0 \tag{1.15}$$

in the sense of distributions in the edges $\Gamma_\alpha \backslash \mathcal{V}$, $\alpha \in \mathcal{A}$. This implies that there exists a real number c_α such that

$$-\mu_\alpha \partial m|_{\Gamma_\alpha} = -m|_{\Gamma_\alpha} a|_{\Gamma_\alpha} + c_\alpha. \tag{1.16}$$

So $m|_{\Gamma_\alpha}$ is C^1 regular, and (1.16) is true pointwise. Using this information and recalling (1.14), we find that, for all $u \in D(\mathcal{L})$,

$$\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha m|_{\Gamma_\alpha}(\nu_i)\,\partial_\alpha u(\nu_i) + \sum_{\beta\in\mathcal{A}} \int_{\Gamma_\beta} \partial u|_{\Gamma_\beta}(x)\Big(-\mu_\beta \partial m|_{\Gamma_\beta}(x) + a|_{\Gamma_\beta}(x) m|_{\Gamma_\beta}(x)\Big)dx = 0.$$

This and (1.16) imply that

$$\sum_{i\in I}\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha m|_{\Gamma_\alpha}(\nu_i)\,\partial_\alpha u(\nu_i) + \sum_{\beta\in\mathcal{A}} c_\beta \int_{\Gamma_\beta} \partial u|_{\Gamma_\beta}(x)dx = 0. \tag{1.17}$$

For all $i \in I$, it is possible to choose a function $u \in D(\mathcal{L})$ such that

1. $u(\nu_j) = \delta_{i,j}$ for all $j \in I$;
2. $\partial_\alpha u(\nu_j) = 0$ for all $j \in I$ and $\alpha \in \mathcal{A}_j$.

Using such a test-function in (1.17) implies that for all $i \in I$,

$$0 = \sum_{\beta\in\mathcal{A}} c_\beta \int_{\Gamma_\beta} \partial u|_{\Gamma_\beta}(x)dx = \sum_{j\in I}\sum_{\alpha\in\mathcal{A}_j} c_\alpha n_{j\alpha} u|_{\Gamma_\alpha}(\nu_j) = \sum_{\alpha\in\mathcal{A}_i} n_{i\alpha} c_\alpha, \tag{1.18}$$

where $n_{i\alpha}$ is defined in (1.6).

For all $i \in I$ and $\alpha, \beta \in \mathcal{A}_i$, it is possible to choose a function $u \in D(\mathcal{L})$ such that

1. u takes the same value at each vertex of Γ, thus $\int_{\Gamma_\delta} \partial u|_{\Gamma_\delta}(x)dx = 0$ for all $\delta \in \mathcal{A}$;
2. $\partial_\alpha u(\nu_i) = 1/p_{i\alpha}$, $\partial_\beta u(\nu_i) = -1/p_{i\beta}$ and all the other first order directional derivatives of u at the vertices are 0.

Using such a test-function in (1.17) yields

$$\frac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \frac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, \quad \text{for all } \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V},$$

in which

$$\gamma_{i\alpha} = \frac{p_{i\alpha}}{\mu_\alpha}, \quad \text{for all } i \in I, \alpha \in \mathcal{A}_i. \tag{1.19}$$

Next, for $i \in I$, multiplying (1.16) at $x = \nu_i$ by $n_{i\alpha}$ for all $\alpha \in \mathcal{A}_i$, then summing over all $\alpha \in \mathcal{A}_i$, we get $\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha \partial_\alpha m(\nu_i) - n_{i\alpha}\Big(m|_{\Gamma_\alpha}(\nu_i)\, a|_{\Gamma_\alpha}(\nu_i) - c_\alpha\Big) = 0$, and using (1.18), we obtain that

$$\sum_{\alpha\in\mathcal{A}_i} \mu_\alpha \partial_\alpha m(\nu_i) - n_{i\alpha} a|_{\Gamma_\alpha}(\nu_i)\, m|_{\Gamma_\alpha}(\nu_i) = 0, \quad \text{for all } i \in I. \tag{1.20}$$

Summarizing, we get the following boundary value problem for m (recalling that the coefficients $n_{i\alpha}$ are defined in (1.6)):

$$\begin{cases} -\mu_\alpha \partial^2 m + \partial (ma) = 0, & x \in (\Gamma_\alpha \backslash \mathcal{V}), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \mu_\alpha \partial_\alpha m (\nu_i) - n_{i\alpha} a|_{\Gamma_\alpha}(\nu_i) m|_{\Gamma_\alpha}(\nu_i) = 0, & \nu_i \in \mathcal{V}, \\ \dfrac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}. \end{cases} \tag{1.21}$$

1.3 Formal derivation of the MFG system on Γ

Consider a continuum of indistinguishable agents moving on the network Γ. The set of Borel probability measures on Γ is denoted by $\mathcal{P}(\Gamma)$. Under suitable assumptions, the theory of MFGs asserts that the distribution of states is absolutely continuous with respect to Lebesgue measure on Γ. Hereafter, m stands for the density of the distribution of states: $m \geq 0$ and $\int_\Gamma m(x)dx = 1$.

The state of a representative agent at time t is a time-continuous controlled stochastic process X_t in Γ, as defined in Section 1.2, where the control is the drift a_t, supposed to be of the form $a_t = a(X_t)$. The function $X \mapsto a(X)$ is the feedback.

For a representative agent, the optimal control problem is of the form:

$$\rho := \inf_{a_s} \liminf_{T \to +\infty} \frac{1}{T} \mathbb{E}_x \left[\int_0^T L(X_s, a_s) + V[m(\cdot, s)(X_s)] \, ds \right], \tag{1.22}$$

where $\mathbb{E}_x$ stands for the expectation conditioned by the event $X_0 = x$. The functions and operators involved in (1.22) will be described below.

Let us assume that there is an optimal feedback law, i.e. a function $a^\star$ defined on Γ which is sufficiently regular in the edges of the network, such that the optimal control at time t is given by $a_t^\star = a^\star(X_t)$. Then, almost surely if $X_t \in \Gamma_\alpha \backslash \mathcal{V}$, $d\pi_\alpha^{-1}(X_t) = a_\alpha^\star(\pi_\alpha^{-1}(X_t))dt + \sqrt{2\mu_\alpha} dW_t$. An informal way to describe the behavior of the process at the vertices is as follows: if X_t hits $\nu_i \in \mathcal{V}$, then it enters Γ_α, $\alpha \in \mathcal{A}_i$ with probability $p_{i\alpha} > 0$.

Let us discuss the ingredients in (1.22): the running cost depends separately on the control and on the distribution of states. The contribution of the distribution of states involves the coupling cost operator, which can either be nonlocal, i.e. $V : \mathcal{P}(\Gamma) \to \mathcal{C}^2(\Gamma)$, or local, i.e. $V[m](x) = F(m(x))$ assuming that m is absolutely continuous with respect to the Lebesgue measure, where $F : \mathbb{R}^+ \to \mathbb{R}$ is a continuous function.

The contribution of the control involves the Lagrangian L, i.e. a real valued function defined on $(\cup_{\alpha \in \mathcal{A}} \Gamma_\alpha \backslash \mathcal{V}) \times \mathbb{R}$. If $x \in \Gamma_\alpha \backslash \mathcal{V}$ and $a \in \mathbb{R}$, $L(x, a) = L_\alpha(\pi_\alpha^{-1}(x), a)$, where L_α is a continuous real valued function defined on $[0, \ell_\alpha] \times \mathbb{R}$. We assume that $\lim_{|a| \to \infty} \inf_{y \in \Gamma_\alpha} L_\alpha(y, a)/|a| = +\infty$. Further assumptions on L and V will be made below.

Under suitable assumptions, the Ito calculus recalled in Section 1.2 and the dynamic programming principle lead to the following ergodic Hamilton-Jacobi equation on Γ, more precisely the following boundary value problem:

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \rho = V[m](x), & x \in (\Gamma_\alpha \backslash \mathcal{V}), \alpha \in \mathcal{A}, \\ \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & \nu_i \in \mathcal{V}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \nu_i \in \mathcal{V}, \\ \int_\Gamma v(x)dx = 0. \end{cases} \tag{1.23}$$

We refer to [29, 31] for the interpretation of the value function v and the ergodic cost ρ.

Let us comment the different equations in (1.23):

1. The Hamiltonian H is a real valued function defined on $(\cup_{\alpha\in\mathcal{A}}\Gamma_\alpha\backslash\mathcal{V})\times\mathbb{R}$. For $x\in\Gamma_\alpha\backslash\mathcal{V}$ and $p\in\mathbb{R}$,
$$H(x,p)=\sup_a\left\{-ap-L_\alpha\left(\pi_\alpha^{-1}(x),a\right)\right\},$$
The Hamiltonian is supposed to be C^1 and coercive with respect to p uniformly in x.
2. The second equation in (1.23) is a Kirchhoff transmission condition (or Neumann boundary condition if $\nu_i\in\partial\Gamma$); it is the consequence of the assumption on the behavior of X_s at vertices. It involves the positive constants $\gamma_{i\alpha}$ defined in (1.19).
3. The third condition means in particular that v is continuous at the vertices.
4. The fourth equation is a normalization condition.

If (1.22) has a smooth solution, then it provides a feedback law for the optimal control problem, i.e.
$$a^\star(x)=-\partial_pH(x,\partial v(x)).$$
At the MFG equilibrium, m is the density of the invariant measure associated with the optimal feedback law, so, according to Section 1.2, it satisfies (1.21), where a is replaced by $a^\star=-\partial_pH(x,\partial v(x))$. We end up with the following system:
$$\begin{cases}-\mu_\alpha\partial^2v+H(x,\partial v)+\rho=V([m]), & x\in\Gamma_\alpha\backslash\mathcal{V},\alpha\in\mathcal{A},\\ \mu_\alpha\partial^2m+\partial(m\partial_pH(x,\partial v))=0, & x\in\Gamma_\alpha\backslash\mathcal{V},\alpha\in\mathcal{A},\\ \sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha\partial_\alpha(\nu_i)=0, & \nu_i\in\mathcal{V},\\ \sum_{\alpha\in\mathcal{A}_i}\left[\mu_\alpha\partial_\alpha m(\nu_i)+n_{i\alpha}\partial_pH_\alpha\Big(\nu_i,\partial v|_{\Gamma_\alpha}(\nu_i)\Big)m|_{\Gamma_\alpha}(\nu_i)\right]=0, & \nu_i\in\mathcal{V},\\ v|_{\Gamma_\alpha}(\nu_i)=v|_{\Gamma_\beta}(\nu_i), & \alpha,\beta\in\mathcal{A}_i,\nu_i\in\mathcal{V},\\ \dfrac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}}=\dfrac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, & \alpha,\beta\in\mathcal{A}_i,\nu_i\in\mathcal{V},\\ \int_\Gamma v(x)\,dx=0,\quad\int_\Gamma m(x)\,dx=1,\quad m\geq 0. \end{cases}\tag{1.24}$$
At a vertex ν_i, $i\in I$, the transmission conditions for both v and m consist of $d_{\nu_i}=\sharp(\mathcal{A}_i)$ linear relations, which is the appropriate number of relations to have a well posed problem. If $\nu_i\in\partial\Gamma$, there is of course only one Neumann like condition for v and for m.

Remark 1.6. In [8], the authors assume that $\gamma_{i\alpha}=\gamma_{i\beta}$ for all $i\in I$, $\alpha,\beta\in\mathcal{A}_i$. Therefore, the density m does not have jumps across the transition vertices.

1.4 Assumptions and main results

1.4.1 Assumptions

Let $(\mu_\alpha)_{\alpha\in\mathcal{A}}$ be a family of positive numbers, and for each $i\in I$ let $(\gamma_{i\alpha})_{\alpha\in\mathcal{A}_i}$ be a family of positive numbers such that $\sum_{\alpha\in\mathcal{A}_i}\gamma_{i\alpha}\mu_\alpha=1$.
Consider the Hamiltonian $H:\Gamma\times\mathbb{R}\to\mathbb{R}$, with $H|_{\Gamma_\alpha}:\Gamma_\alpha\times\mathbb{R}\to\mathbb{R}$. We assume that, for some positive constants C_0,C_1,C_2 and $q\in(1,2]$,
$$H_\alpha\in C^1([0,\ell_\alpha]\times\mathbb{R});\tag{1.25}$$
$$H_\alpha(x,\cdot)\text{ is convex in }p\text{ for each }x\in[0,\ell_\alpha];\tag{1.26}$$
$$H_\alpha(x,p)\geq C_0|p|^q-C_1\text{ for }(x,p)\in[0,\ell_\alpha]\times\mathbb{R};\tag{1.27}$$
$$|\partial_pH_\alpha(x,p)|\leq C_2\left(|p|^{q-1}+1\right)\text{ for }(x,p)\in[0,\ell_\alpha]\times\mathbb{R}.\tag{1.28}$$

Remark 1.7. From (1.28), there exists a positive constant C_q such that
$$|H_\alpha(x,p)|\leq C_q(|p|^q+1),\quad\text{for all }(x,p)\in[0,\ell_\alpha]\times\mathbb{R}.\tag{1.29}$$

Below, we shall focus on local coupling operators V, namely

$$V[\tilde{m}](x) = F(m(x)) \text{ with } F \in C([0,+\infty);\mathbb{R}), \tag{1.30}$$

for all $\tilde{m}$ which are absolutely continuous with respect to the Lebesgue measure and such that $d\tilde{m}(x) = m(x)\,dx$. We shall also suppose that F is bounded from below, i.e., there exists a positive constant M such that

$$F(r) \geq -M, \quad \text{for all } r \in [0,+\infty). \tag{1.31}$$

1.4.2 Function spaces related to the Kirchhoff conditions

Let us introduce two function spaces on Γ, which will be the key ingredients in order to build weak solutions of (1.24).

Definition 1.8. We define two Sobolev spaces, $V := H^1(\Gamma)$, see Definition 1.3, and

$$W := \left\{ \begin{array}{ll} w : \Gamma \to \mathbb{R} : & w_\alpha \in H^1(0,\ell_\alpha) \text{ for all } \alpha \in \mathcal{A}, \\ & \dfrac{w|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{w|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I,\ \alpha,\beta \in \mathcal{A}_i \end{array} \right\} \tag{1.32}$$

which is also a Hilbert space, endowed with the norm $\|w\|_W = \left(\sum_{\alpha\in\mathcal{A}} \|w_\alpha\|^2_{H^1(0,\ell_\alpha)}\right)^{\frac{1}{2}}$.

Definition 1.9. Let the functions $\psi \in W$ and $\phi \in PC(\Gamma)$ be defined as follows:

$$\begin{cases} \psi_\alpha \text{ is affine on } (0,\ell_\alpha), \\ \psi|_{\Gamma_\alpha}(\nu_i) = \gamma_{i\alpha}, \text{ if } \alpha \in \mathcal{A}_i, \\ \psi \text{ is constant on the edges } \Gamma_\alpha \text{ which touch the boundary of } \Gamma. \end{cases} \tag{1.33}$$

$$\begin{cases} \phi_\alpha \text{ is affine on } (0,\ell_\alpha), \\ \phi|_{\Gamma_\alpha}(\nu_i) = \dfrac{1}{\gamma_{i\alpha}}, \text{ if } \alpha \in \mathcal{A}_i, \\ \phi \text{ is constant on the edges } \Gamma_\alpha \text{ which touch the boundary of } \Gamma. \end{cases} \tag{1.34}$$

Note that both functions ψ, ϕ are positive and bounded. We set $\overline{\psi} = \max_\Gamma \psi$, $\underline{\psi} = \min_\Gamma \psi$, $\overline{\phi} = \max_\Gamma \phi$, $\underline{\phi} = \min_\Gamma \phi$.

Remark 1.10. One can see that $v \in V \longmapsto v\psi$ is an isomorphism from V onto W and $w \in W \longmapsto w\phi$ is the inverse isomorphism.

Definition 1.11. Let the function space $\mathcal{W} \subset W$ be defined as follows:

$$\mathcal{W} := \left\{ \begin{array}{ll} m : \Gamma \to \mathbb{R} : & m_\alpha \in C^1([0,\ell_\alpha]) \text{ for all } \alpha \in \mathcal{A}, \\ & \dfrac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}} \text{ for all } i \in I, \alpha,\beta \in \mathcal{A}_i \end{array} \right\}. \tag{1.35}$$

1.4.3 Main result

Definition 1.12. A solution of the Mean Field Games system (1.24) is a triple $(v,\rho,m) \in C^2(\Gamma) \times \mathbb{R} \times \mathcal{W}$ such that (v,ρ) is a classical solution of

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x,\partial v) + \rho = F(m), & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \displaystyle\sum_{\alpha\in\mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & \text{if } \nu_i \in \mathcal{V}, \end{cases} \tag{1.36}$$

(note that v is continuous at the vertices from the definition of $C^2(\Gamma)$), and m satisfies

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} [\mu_\alpha \partial m \partial u + \partial(m\partial_p H(x,\partial v))\,u]\,dx = 0, \quad \text{for all } u \in V, \tag{1.37}$$

where V is given in Definition 1.8.

We are ready to state the main result:

Theorem 1.13. *If assumptions (1.25)-(1.28) and (1.30)-(1.31) are satisfied, then there exists a solution* $(v, m, \rho) \in C^2(\Gamma) \times \mathcal{W} \times \mathbb{R}$ *of (1.24). If* F *is locally Lipschitz continuous, then* $v \in C^{2,1}(\Gamma)$*. Moreover if* F *is strictly increasing, then the solution is unique.*

Remark 1.14. The proof of the existence result in [8] is valid only in the case when the coupling cost F is bounded.

Remark 1.15. The existence result in Theorem 1.13 holds if we assume that the coupling operator V is non local and regularizing, i.e., V is a continuous map from $\mathcal{P}$ to a bounded subset of $\mathcal{F}$, with $\mathcal{F} := \left\{ f : \Gamma \to \mathbb{R} : \ f|_{\Gamma_\alpha} \in C^{0,\sigma}(\Gamma_\alpha) \right\}$. The proof, omitted in what follows, is similar to that of Lemma 4.1 below.

2 Preliminary: A class of linear boundary value problems

This section contains elementary results on the solvability of some linear boundary value problems on Γ. To the best of our knowledge, these results are not available in the literature.

2.1 A first class of problems

We recall that the constants μ_α and $\gamma_{i\alpha}$ are defined in Section 1.2. Let λ be a positive number. We start with very simple linear boundary value problems, in which the only difficulty is the Kirchhoff condition:

$$\begin{cases} -\mu_\alpha \partial^2 v + \lambda v = f, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{2.1}$$

where $f \in W'$, W' is the topological dual of W.

Remark 2.1. We have already noticed that, if $\nu_i \in \partial\Gamma$, the last condition in (2.1) boils down to a standard Neumann boundary condition $\partial_\alpha v(\nu_i) = 0$, in which α is the unique element of $\mathcal{A}_i$. Otherwise, if $\nu_i \in \mathcal{V} \backslash \partial\Gamma$, the last condition in (2.1) is the Kirchhoff condition discussed above.

Definition 2.2. A weak solution of (2.1) is a function $v \in V$ such that

$$\mathscr{B}_\lambda(v, w) = \langle f, w \rangle_{W', W}, \quad \text{for all } w \in W, \tag{2.2}$$

where $\mathscr{B}_\lambda : V \times W \to \mathbb{R}$ is the bilinear form defined as follows:

$$\mathscr{B}_\lambda(v, w) = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial v \partial w + \lambda v w) \, dx.$$

Remark 2.3. Formally, (2.2) is obtained by testing the first line of (2.1) by $w \in W$, integrating by part the left hand side on each Γ_α and summing over $\alpha \in \mathcal{A}$. There is no contribution from the vertices, because of the Kirchhoff conditions on the one hand and on the other hand the jump conditions satisfied by the elements of W.

Remark 2.4. By using the fact that Γ_α are line segments, i.e. one dimensional sets and solving the differential equations, we see that if v is a weak solution of (2.1) with $f \in PC(\Gamma)$, then $v \in C^2(\Gamma)$.

Let us first study the homogeneous case, i.e. $f = 0$.

Lemma 2.5. *The function* $v = 0$ *is the unique solution of the following boundary value problem*

$$\begin{cases} -\partial^2 v + \lambda v = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, \ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{2.3}$$

Proof. Let $\mathcal{I}_i := \{k \in I : \ k \neq i; \ \nu_k \in \Gamma_\alpha \text{ for some } \alpha \in \mathcal{A}_i\}$ be the set of indices of the vertices which are connected to ν_i. By Remark 1.1, it is not restrictive to assume (in the remainder of the proof) that for all $k \in \mathcal{I}_i$, $\Gamma_\alpha = \Gamma_{\alpha_{ik}} = [\nu_i, \nu_k]$ is oriented from ν_i to ν_k.

For $k \in \mathcal{I}_i$, $\Gamma_\alpha = [\nu_i, \nu_k]$, using the parametrization (1.1), the linear differential eqaution (2.3) in the edge Γ_α is

$$-v_\alpha''(y) + \lambda v_\alpha(y) = 0, \quad \text{in } (0, \ell_\alpha),$$

whose solution is

$$v_\alpha(y) = \zeta_\alpha \cosh\left(\sqrt{\lambda} y\right) + \xi_\alpha \sinh\left(\sqrt{\lambda} y\right), \tag{2.4}$$

with

$$\begin{cases} \zeta_\alpha = v_\alpha(0) = v(\nu_i), \\ \zeta_\alpha \cosh\left(\sqrt{\lambda}\ell_\alpha\right) + \xi_\alpha \sinh\left(\sqrt{\lambda}\ell_\alpha\right) = v_\alpha(\ell_\alpha) = v(\nu_k). \end{cases}$$

It follows that $\partial_\alpha v(\nu_i) = -\sqrt{\lambda}\xi_\alpha = -\dfrac{\sqrt{\lambda}}{\sinh\left(\sqrt{\lambda}\ell_\alpha\right)}\left[v(\nu_k) - v(\nu_i)\cosh\left(\sqrt{\lambda}\ell_\alpha\right)\right]$. Hence, the transmission condition in (2.3) becomes: for all $i \in I$,

$$0 = \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = \sum_{k \in \mathcal{I}_i} \frac{\sqrt{\lambda}\gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}}\cosh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)} v(\nu_i) - \sum_{k \in \mathcal{I}_i} \frac{\sqrt{\lambda}\gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}}}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)} v(\nu_k).$$

Therefore, we obtain a system of linear equations of the form $MU = 0$ with $M = (M_{ij})_{1 \le i,j \le N}$, $N = \sharp(I)$, and $U = (v(\nu_1), \ldots, v(\nu_N))^T$, where M is defined by

$$\begin{cases} M_{ii} &= \displaystyle\sum_{k \in \mathcal{I}_i} \gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}} \frac{\cosh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)} > 0, \\ M_{ik} &= \dfrac{-\gamma_{i\alpha_{ik}}\mu_{\alpha_{ik}}}{\sinh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right)} \le 0, \quad k \in \mathcal{I}_i, \\ M_{ik} &= 0, \quad k \notin \mathcal{I}_i. \end{cases}$$

For all $i \in I$, since $\cosh\left(\sqrt{\lambda}\ell_{\alpha_{ik}}\right) > 1$ for all $k \in \mathcal{I}_i$, the sum of the entries on each row is positive and M is diagonal dominant. Thus, M is invertible and $U = 0$ is the unique solution of the system. Finally, by solving the ODE in each edge Γ_β with $v_\beta(0) = v_\beta(\ell_\beta) = 0$, we get that $v = 0$ on Γ. □

Let us now study the non-homogeneous problems (2.1).

Lemma 2.6. *For any f in W', (2.1) has a unique weak solution v in V, see Definition 1.8. Moreover, there exists a constant C such that $\|v\|_V \le C\|f\|_{W'}$.*

Proof. First of all, we claim that for $\lambda_0 > 0$ large enough and any $f \in W'$, the problem

$$\mathscr{B}_\lambda(v, w) + \lambda_0(v, w) = \langle f, w\rangle_{W', W} \tag{2.5}$$

has a unique solution $v \in V$. Let us prove the claim. Let $v \in V$, then $\hat{w} := v\psi$ belongs to W, where ψ is given by Definition 1.9. Let us set $\overline{\partial\psi} := \max_\Gamma |\partial\psi|$ and $\underline{\psi} := \min_\Gamma \psi > 0$, ($\partial\psi$ is bounded, see Definition 1.9); we get

$$\begin{aligned} \mathscr{B}_\lambda(v, \hat{w}) + \lambda_0(v, \hat{w}) &= \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left[\mu_\alpha |\partial v|^2 \psi + \mu_\alpha (v\partial v)\partial\psi + (\lambda + \lambda_0) v^2 \psi\right] dx \\ &\ge \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left[\frac{\mu_\alpha \underline{\psi}}{2} |\partial v|^2 + \left(\lambda_0 \underline{\psi} - \frac{\mu_\alpha \overline{\partial\psi}^2}{2\underline{\psi}}\right) v^2\right] dx. \end{aligned} \tag{2.6}$$

When $\lambda_0 \geq \frac{\mu_\alpha}{2} + \frac{\mu_\alpha \overline{\partial \psi}^2}{2\underline{\psi}^2}$ for all $\alpha \in \mathcal{A}$, we obtain that $\mathscr{B}_\lambda(v, \hat{w}) + \lambda_0(v, \hat{w}) \geq \frac{\underline{\mu}\,\underline{\psi}}{2} \|v\|_V^2 \geq \frac{\underline{\mu}\,\underline{\psi}}{2C_\psi} \|v\|_V \|\hat{w}\|_W$, using the fact that, from Remark 1.10, there exists a positive constant C_ψ such that $\|v\psi\|_W \leq C_\psi \|v\|_V$ for all $v \in V$. This yields

$$\inf_{v \in V} \sup_{w \in W} \frac{\mathscr{B}_\lambda(v, w) + \lambda_0(v, w)}{\|v\|_V \|w\|_W} \geq \frac{\underline{\mu}\,\underline{\psi}}{2C_\psi}. \tag{2.7}$$

Using a similar argument for any $w \in W$ and $\hat{v} = w\phi$, where ϕ is given in Definition 1.9, we obtain that for λ_0 large enough, there exist a positive constant C_ϕ such that

$$\inf_{w \in W} \sup_{v \in V} \frac{\mathscr{B}_\lambda(v, w) + \lambda_0(v, w)}{\|w\|_W \|v\|_V} \geq \frac{\underline{\mu}\,\underline{\phi}}{2C_\phi}. \tag{2.8}$$

From (2.7) and (2.8), by the Banach-Necas-Babuška lemma (see [14] or [7]), for λ_0 large enough, for any $f \in W'$, there exists a unique solution $v \in V$ of (2.5) and $\|v\|_V \leq C \|f\|_{W'}$ for a positive constant C. Hence, our claim is proved.

Now, we fix λ_0 large enough and we define the continuous linear operator $\overline{R}_{\lambda_0} : W' \to V$ where $\overline{R}_{\lambda_0}(f) = v$ is the unique solution of (2.5). Since the injection $\mathcal{I}$ from V to W' is compact, then $\mathcal{I} \circ \overline{R}_{\lambda_0}$ is a compact operator from W' into W'. By the Fredholm alternative (see [19]), one of the following assertions holds:

$$\text{There exists } \overline{v} \in W' \backslash \{0\} \text{ such that } \left(Id - \lambda_0 \left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right) \overline{v} = 0. \tag{2.9}$$

$$\text{For any } g \in W', \text{ there exists a unique } \overline{v} \in W' \text{ such that } \left(Id - \lambda_0 \left(\mathcal{I} \circ \overline{R}_{\lambda_0}\right)\right) \overline{v} = g. \tag{2.10}$$

We claim that (2.10) holds. Indeed, assume by contradiction that (2.9) holds. Then there exists $\overline{v} \neq 0$ such that $\overline{v} \in V$ and $\mathcal{I} \circ \overline{R}_{\lambda_0} \overline{v} = \frac{\overline{v}}{\lambda_0}$. Therefore, $\overline{v} \in V$, and $\mathscr{B}_\lambda\left(\frac{\overline{v}}{\lambda_0}, w\right) + \lambda_0\left(\frac{\overline{v}}{\lambda_0}, w\right) = (\overline{v}, w)$, for all $w \in W$. This yields that $\mathscr{B}_\lambda(\overline{v}, w) = 0$ for all $w \in W$ and by Lemma 2.5, we get that $\overline{v} = 0$, which leads us to a contradiction. Hence, our claim is proved.

It is then classical to see that (2.10) implies that there exists a positive constant C such that for all $f \in W'$, (2.1) has a unique weak solution v and that $\|v\|_V \leq C \|f\|_{W'}$, see [12] for the details.

□

2.2 The Kolmogorov equation

Consider $b \in PC(\Gamma)$. This paragraph is devoted to the following boundary value problem including a Kolmogorov equation

$$\begin{cases} -\mu_\alpha \partial^2 v + b \partial v = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i, i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I. \end{cases} \tag{2.11}$$

Definition 2.7. A weak solution of (2.11) is a function $v \in V$ such that

$$\mathscr{A}^\star(v, w) = 0, \quad \text{for all } w \in W,$$

where $\mathscr{A}^\star : V \times W \to \mathbb{R}$ is the bilinear form defined by

$$\mathscr{A}^\star(v, w) := \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial v \partial w + b \partial v w)\, dx.$$

As in Remark 2.4, if v is a weak solution of (2.11), then $v \in C^2(\Gamma)$.

The uniqueness of solutions of (2.11) up to the addition of constants is obtained by using a maximum principle:

Lemma 2.8. *For $b \in PC(\Gamma)$, the solutions of (2.11) are the constant functions on Γ.*

Proof of Lemma 2.8. First of all, any constant function on Γ is a solution of (2.11). Now let v be a solution of (2.11) then $v \in C^2(\Gamma)$. Assume that the maximum of v over Γ is achieved in Γ_α; by the maximum principle, it is achieved at some endpoint ν_i of Γ_α. Without loss of generality, using Remark 1.1, we can assume that $\pi_\beta(\nu_i) = 0$ for all $\beta \in \mathcal{A}_i$. We have $\partial_\beta v(\nu_i) \geq 0$ for all $\beta \in \mathcal{A}_i$ because ν_i is the maximum point of v. Since all the coefficients $\gamma_{i\beta}, \mu_\beta$ are positive, by the Kirchhoff condition if ν_i is a transition vertex, or by the Neumann boundary condition if ν_i is a boundary vertex, we infer that $\partial_\beta v(\nu_i) = 0$ for all $\beta \in \mathcal{A}_i$. This implies that ∂v_β is a solution of the first order linear homogeneous differential equation $u' + b_\beta u = 0$, on $[0, \ell_\beta]$, with $u(0) = 0$. Therefore, $\partial v_\beta \equiv 0$ and v is constant on Γ_β for all $\beta \in \mathcal{A}_i$. We can propagate this argument, starting from the vertices connected to ν_i. Since the network Γ is connected and v is continuous, we obtain that v is constant on Γ. □

2.3 The dual Fokker-Planck equation

This paragraph is devoted to the dual boundary value problem of (2.11); it involves a Fokker-Planck equation:

$$\begin{cases} -\mu_\alpha \partial^2 m - \partial(bm) = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \dfrac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \left[n_{i\alpha} b|_{\Gamma_\alpha}(\nu_i)\, m|_{\Gamma_\alpha}(\nu_i) + \mu_\alpha \partial_\alpha m(\nu_i) \right] = 0, & i \in I, \end{cases} \tag{2.12}$$

where $b \in PC(\Gamma)$, with

$$m \geq 0, \quad \int_\Gamma m dx = 1. \tag{2.13}$$

First of all, let λ_0 be a nonnegative constant; for all $h \in V'$, we introduce the modified boundary value problem

$$\begin{cases} \lambda_0 m - \mu_\alpha \partial^2 m - \partial(bm) = h, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ \dfrac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \left[n_{i\alpha} b(\nu_i)\, m|_{\Gamma_\alpha}(\nu_i) + \mu_\alpha \partial_\alpha m(\nu_i) \right] = 0, & i \in I. \end{cases} \tag{2.14}$$

Definition 2.9. For $\lambda \in \mathbb{R}$, consider the bilinear form $\mathscr{A}_\lambda : W \times V \to \mathbb{R}$ defined by

$$\mathscr{A}_\lambda(m, v) = \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left[\lambda m v + (\mu_\alpha \partial m + bm) \partial v \right] dx.$$

A weak solution of (2.14) is a function $m \in W$ such that

$$\mathscr{A}_{\lambda_0}(m, v) = \langle h, v \rangle_{V', V}, \quad \text{for all } v \in V.$$

A weak solution of (2.12) is a function $m \in W$ such that

$$\mathscr{A}_0(m, v) := \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial m + bm) \partial v dx = 0, \quad \text{for all } v \in V. \tag{2.15}$$

Remark 2.10. Formally, to get (2.15), we multiply the first line of (2.12) by $v \in V$, integrate by part, sum over $\alpha \in \mathcal{A}$ and use the third line of (2.12) to see that there is no contribution from the vertices.

Theorem 2.11. *For any $b \in PC(\Gamma)$,*

- *(Existence) There exists a solution $\widehat{m} \in W$ of (2.12)-(2.13) satisfying*

$$\|\widehat{m}\|_W \leq C, \quad 0 \leq \widehat{m} \leq C, \tag{2.16}$$

  where the constant C depends only on $\|b\|_\infty$ and $\{\mu_\alpha\}_{\alpha\in A}$. Moreover, $\widehat{m}_\alpha \in C^1(0, \ell_\alpha)$ for all $\alpha \in \mathcal{A}$. Hence, $\widehat{m} \in \mathcal{W}$.
- *(Uniqueness) $\widehat{m}$ is the unique solution of (2.12)-(2.13).*
- *(Strictly positive solution) $\widehat{m}$ is strictly positive.*

Proof of existence in Theorem 2.11. We divide the proof of existence into three steps:

Step 1: Let λ_0 be a large positive constant that will be chosen later. We claim that for $\overline{m} \in L^2(\Gamma)$ and $h := \lambda_0\overline{m} \in L^2(\Gamma) \subset V'$, (2.14) has a unique solution $m \in W$. This allows us to define a linear operator as follows:

$$T : L^2(\Gamma) \longrightarrow W, \qquad T(\overline{m}) = m,$$

where m is the solution of (2.14) with $h = \lambda_0\overline{m}$. We are going to prove that T is well-defined and continuous, i.e, for all $\overline{m} \in L^2(\Gamma)$, (2.14) has a unique solution that depends continuously on $\overline{m}$. For $w \in W$, set $\widehat{v} := w\phi \in V$ where ϕ is given by Definition 1.9. We have

$$\begin{aligned}\mathscr{A}_{\lambda_0}(w, \widehat{v}) &= \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[\lambda_0\phi w^2 + (\mu_\alpha\partial w + bw)\,\partial(w\phi)\right] dx \\ &= \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[(\lambda_0\phi + b\partial\phi)\, w^2 + (\mu_\alpha\partial\phi + b\phi)\, w\partial w + \mu_\alpha\phi\,(\partial w)^2\right] dx.\end{aligned}$$

It follows that when λ_0 is large enough (larger than a constant that only depends on b, ϕ and μ_α), $\mathscr{A}_{\lambda_0}(w, \widehat{v}) \geq \widehat{C}_{\lambda_0}\|w\|_W^2$ for some positive constant $\widehat{C}_{\lambda_0}$. Moreover, by Remark 1.10, there exists a positive constant $\widehat{C}_\phi$ such that for all $w \in W$, we have $\|w\phi\|_V \leq C_\phi\|w\|_W$. This yields

$$\inf_{w\in W}\sup_{v\in V} \frac{\mathscr{A}_{\lambda_0}(w, v)}{\|v\|_V\|w\|_W} \geq \frac{\widehat{C}_{\lambda_0}}{C_\phi}.$$

Using similar arguments, for λ_0 large enough, there exist two positive constants C_{λ_0} and C_ψ such that

$$\inf_{v\in V}\sup_{w\in W} \frac{\mathscr{A}_{\lambda_0}(w, v)}{\|w\|_W\|v\|_V} \geq \frac{C_{\lambda_0}}{C_\psi}.$$

From Banach-Necas-Babuška lemma (see [14] or [7]), there exists a constant $\overline{C}$ such that for all $\overline{m} \in L^2(\Gamma)$, there exists a unique solution m of (2.14) with $h = \lambda_0\overline{m}$ and $\|m\|_W \leq \overline{C}\|\overline{m}\|_{L^2(\Gamma)}$. Hence, the map T is well-defined and continuous from $L^2(\Gamma)$ to W.

Step 2: Let K be the set defined by

$$K := \left\{m \in L^2(\Gamma) : m \geq 0 \text{ and } \int_\Gamma m dx = 1\right\}.$$

We claim that $T(K) \subset K$ which means $\int_\Gamma m = 1$ and $m \geq 0$. Indeed, using $v = 1$ as a test function in (2.14), we have $\int_\Gamma m dx = \int_\Gamma \overline{m} dx = 1$. Next, consider the negative part m^- of m defined by $m^-(x) = -\mathbb{1}_{\{m(x)<0\}}m(x)$. Notice that $m^- \in W$ and $m^-\phi \in V$, where ϕ is given by Definition 1.9. Using $m^-\phi$ as a test function in (2.14) yields

$$\sum_{\alpha\in\mathcal{A}} -\int_{\Gamma_\alpha} \left[(\lambda_0\phi + b\partial\phi)\,(m^-)^2 + \mu_\alpha(\partial m^-)^2\phi + (\mu_\alpha\partial\phi + b\phi)\, m^-\partial m^-\right] dx = \int_\Gamma \lambda_0\overline{m}m^-\phi dx.$$

We can see that the right hand side is non-negative. Moreover, for λ_0 large enough (larger than the same constant as above, which only depends on b, ϕ and μ_α), the left hand side is non-positive. This implies that $m^- = 0$, and hence $m \geq 0$. Therefore, the claim is proved.

Step 3: We claim that T has a fixed point. Let us now focus on the case when $\overline{m} \in K$. Using $m\phi$ as a test function in (2.14) yields

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \left[(\lambda_0\phi + b\partial\phi)\, m^2 + \mu_\alpha (\partial m)^2 \phi + (\mu_\alpha \partial\phi + b\phi)\, m\, (\partial m) \right] dx = \int_\Gamma \lambda_0 \overline{m} m \phi dx. \quad (2.17)$$

Since $H^1(0, \ell_\alpha)$ is continuously embedded in $L^\infty(0, \ell_\alpha)$, there exists a positive constant C (independent of $\overline{m} \in K$) such that

$$\int_\Gamma \overline{m} m \phi dx \leq \int_\Gamma \overline{m} dx \, \|m\|_{L^\infty(\Gamma)} \overline{\phi} = \|m\|_{L^\infty(\Gamma)} \overline{\phi} \leq C \|m\|_W .$$

Hence, from (2.17), for λ_0 large enough, there exists a positive constant C_1 such that $C_1 \|m\|_W^2 \leq \lambda_0 C \|m\|_W$. Thus

$$\|m\|_W \leq \frac{\lambda_0 C}{C_1}. \quad (2.18)$$

Therefore, $T(K)$ is bounded in W. Since the bounded subsets of W are relatively compact in $L^2(\Gamma)$, $\overline{T(K)}$ is compact in $L^2(\Gamma)$. Moreover, we can see that K is closed and convex in $L^2(\Gamma)$. By Schauder fixed point theorem, see [19, Corollary 11.2], T has a fixed point $\widehat{m} \in K$ which is also a solution of (2.12) and $\|\widehat{m}\|_W \leq \lambda_0 C / C_1$.

Finally, from the differential equation in (2.12), for all $\alpha \in \mathcal{A}$, $(\widehat{m}'_\alpha + b_\alpha \widehat{m}_\alpha)' = 0$ on $(0, \ell_\alpha)$. Hence, there exists a constant C_α such that

$$\widehat{m}'_\alpha + b_\alpha \widehat{m}_\alpha = C_\alpha, \quad \text{for all } x \in (0, \ell_\alpha). \quad (2.19)$$

It follows that $\widehat{m}'_\alpha \in C([0, \ell_\alpha])$, for all $\alpha \in \mathcal{A}$. Hence $\widehat{m}_\alpha \in C^1([0, \ell_\alpha])$ for all $\alpha \in \mathcal{A}$. Thus, $\widehat{m} \in \mathcal{W}$. □

Remark 2.12. Let $m \in W$ be a solution of (2.12). If $b, \partial b \in PC(\Gamma)$, standard arguments yield that $m_\alpha \in C^2([0, \ell_\alpha])$ for all $\alpha \in \mathcal{A}$. Moreover, by Theorem 2.11, there exists a constant C which depends only on $\|b\|_\infty, \{\|\partial b_\alpha\|_\infty\}_{\alpha\in\mathcal{A}}$ and μ_α such that $\|m_\alpha\|_{C^2(0,\ell_\alpha)} \leq C$ for all $\alpha \in \mathcal{A}$.

Proof of the positivity in Theorem 2.11. From (2.13), $\widehat{m}$ is non-negative on Γ. Assume by contradiction that there exists $x_0 \in \Gamma_\alpha$ for some $\alpha \in \mathcal{A}$ such that $\widehat{m}|_{\Gamma_\alpha}(x_0) = 0$. Therefore, the minimum of $\widehat{m}$ over Γ is achieved at $x_0 \in \Gamma_\alpha$. If $x_0 \in \Gamma_\alpha \backslash \mathcal{V}$, then $\partial\widehat{m}(x_0) = 0$. In (2.19), we thus have $C_\alpha = 0$, and hence $\widehat{m}_\alpha$ satisfies

$$\widehat{m}'_\alpha + b_\alpha \widehat{m}_\alpha = 0, \quad \text{on } [0, \ell_\alpha],$$

with $\widehat{m}_\alpha\left(\pi_\alpha^{-1}(x_0)\right) = 0$. It follows that $\widehat{m}_\alpha \equiv 0$ and $\widehat{m}|_{\Gamma_\alpha}(\nu_i) = \widehat{m}|_{\Gamma_\alpha}(\nu_j) = 0$ if $\Gamma_\alpha = [\nu_i, \nu_j]$.

Therefore, it is enough to consider $x_0 \in \mathcal{V}$.

Now, from Remark 1.1, we may assume without loss of generality that $x_0 = \nu_i$ and $\pi_\beta(\nu_i) = 0$ for all $\beta \in \mathcal{A}_i$. We have the following two cases.

Case 1: if $x_0 = \nu_i$ is a transition vertex, then, since $\widehat{m}$ belongs to W, we get

$$\widehat{m}|_{\Gamma_\beta}(\nu_i) = \frac{\gamma_{i\beta}}{\gamma_{i\alpha}} \widehat{m}|_{\Gamma_\alpha}(\nu_i) = 0, \quad \text{for all } \beta \in \mathcal{A}_i. \quad (2.20)$$

This yields that ν_i is also a minimum point of $\widehat{m}|_{\Gamma_\beta}$ for all $\beta \in \mathcal{A}_i$. Thus $\partial_\beta \widehat{m}(\nu_i) \leq 0$ for all $\beta \in \mathcal{A}_i$. From the transmission condition in (2.12) which has a classical meaning from the regularity of $\widehat{m}$, $\partial_\beta \widehat{m}(\nu_i) = 0$, since all the coefficients μ_β are positive. From (2.19), for all $\beta \in \mathcal{A}_i$, we have

$$C_\beta = \widehat{m}'_\beta(0) + b_\beta(0)\widehat{m}_\beta(0) = 0.$$

Therefore, $\widehat{m}'_\beta(y) + b_\beta(y)\widehat{m}_\beta(y) = 0$, for all $y \in [0, \ell_\beta]$ with $\widehat{m}_\beta(0) = 0$. This implies that $\widehat{m}_\beta \equiv 0$ for all $\beta \in \mathcal{A}_i$. We can propagate the arguments from the vertices connected to ν_i. Since Γ is connected, we obtain that $\widehat{m} \equiv 0$ on Γ.

Case 2: if $x_0 = \nu_i$ is a boundary vertex, then the Robin condition in (2.12) implies that $\partial_\alpha \widehat{m}(\nu_i) = 0$ since μ_α is positive. From (2.19), we have $C_\alpha = 0$. Therefore, $\widehat{m}'_\alpha(y) + b_\alpha(y)\widehat{m}_\alpha(y) = 0$, for all $y \in [0, \ell_\alpha]$ with $\widehat{m}_\alpha(0) = 0$. This implies that $\widehat{m}(\nu_j) = 0$ where ν_j is the other endpoint of Γ_α. We are back to Case 1, so $\widehat{m} \equiv 0$ on Γ.

Finally, we have found that $\widehat{m} \equiv 0$ on Γ, in contradiction with $\int_\Gamma \widehat{m} dx = 1$. □

Now we prove uniqueness for (2.12)-(2.13).

Proof of uniqueness in Theorem 2.11. The proof of uniqueness is similar to the argument in [8, Proposition 13]. As in the proof of Lemma 2.6, we can prove that for λ_0 large enough, there exists a constant C such that for any $f \in V'$, there exists a unique $w \in W$ which satisfies

$$\mathscr{A}_{\lambda_0}(w, v) = \langle f, v \rangle_{V',V} \text{ for all } v \in V. \tag{2.21}$$

and $\|w\|_W \leq C \|f\|_{V'}$. This allows us to define the continuous linear operator

$$\begin{aligned} S_{\lambda_0} : L^2(\Gamma) &\longrightarrow W, \\ f &\longmapsto w, \end{aligned}$$

where w is a solution of (2.21). Then we define $R_{\lambda_0} = \mathcal{J} \circ S_{\lambda_0}$ where $\mathcal{J}$ is the injection from W in $L^2(\Gamma)$, which is compact. Obviously, R_{λ_0} is a compact operator from $L^2(\Gamma)$ into $L^2(\Gamma)$. Moreover, $m \in W$ is a solution of (2.12) if and only if $m \in \ker(Id - \lambda_0 R_{\lambda_0})$. By Fredholm alternative, see [19], $\dim\ker(Id - \lambda_0 R_{\lambda_0}) = \dim\ker\left(Id - \lambda_0 R^\star_{\lambda_0}\right)$.

In order to characterize $R^\star_{\lambda_0}$, we now consider the following boundary value problem for $g \in L^2(\Gamma) \subset W'$:

$$\begin{cases} \lambda_0 v - \mu_\alpha \partial^2 v + b\partial v = g, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i) & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I. \end{cases} \tag{2.22}$$

A weak solution of (2.22) is a function $v \in V$ such that

$$\mathscr{T}_{\lambda_0}(v, w) := \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\lambda_0 vw + \mu_\alpha \partial v \partial w + bw\partial v) dx = \int_\Gamma gw dx, \quad \text{for all } w \in W.$$

Using similar arguments as in the proof of existence in Theorem 2.11, we see that for λ_0 large enough and all $g \in L^2(\Gamma)$, there exists a unique solution $v \in V$ of (2.22). Moreover, there exists a constant C such that $\|v\|_V \leq C \|g\|_{L^2(\Gamma)}$ for all $g \in L^2(\Gamma)$. This allows us to define a continuous operator

$$\begin{aligned} T_{\lambda_0} : L^2(\Gamma) &\longrightarrow V, \\ g &\longmapsto v. \end{aligned}$$

Then we define $\tilde{R}_{\lambda_0} = \mathcal{I} \circ T_{\lambda_0}$ where $\mathcal{I}$ is the injection from V in $L^2(\Gamma)$. Since $\mathcal{I}$ compact, $\tilde{R}_{\lambda_0}$ is a compact operator from $L^2(\Gamma)$ into $L^2(\Gamma)$. For any $g \in L^2(\Gamma)$, set $v = T_{\lambda_0} g$. Noticing that $\mathscr{T}_{\lambda_0}(v, w) = \mathcal{A}_{\lambda_0}(w, v)$ for all $v \in V, w \in W$, we obtain that

$$(g, R_{\lambda_0} f)_{L^2(\Gamma)} = \mathscr{T}_{\lambda_0}(v, S_{\lambda_0} f) = \mathscr{A}_{\lambda_0}(S_{\lambda_0} f, v) = (f, v)_{L^2(\Gamma)} = (f, \tilde{R}_{\lambda_0} g)_{L^2(\Gamma)}.$$

Thus $R^\star_{\lambda_0} = \tilde{R}_{\lambda_0}$. But $\ker\left(Id - \lambda_0 \tilde{R}_{\lambda_0}\right)$ is the set of solutions of (2.11), which, from Lemma 2.8, consists of constant functions on Γ. This implies that $\dim\ker\left(Id - \lambda_0 R^\star_{\lambda_0}\right) = 1$ and then that $\dim\ker(Id - \lambda_0 R_{\lambda_0}) = \dim\ker\left(Id - \lambda_0 R^\star_{\lambda_0}\right) = 1$. Finally, since the solutions m of (2.12) are in $\ker(Id - \lambda_0 R_{\lambda_0})$ and satisfy the normalization condition $\int_\Gamma m dx = 1$, we obtain the desired uniqueness property in Theorem 2.11. □

3 Hamilton-Jacobi equation and the ergodic problem

3.1 The Hamilton-Jacobi equation

This section is devoted to the following boundary value problem including a Hamilton-Jacobi equation:

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \lambda v = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in A, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.1}$$

where λ is a positive constant and the Hamiltonian $H : \Gamma \times \mathbb{R} \to \mathbb{R}$ is defined in Section 1, except that, in (3.1) and the whole Section 3.1 below, the Hamiltonian contains the coupling term, i.e, $H(x, \partial v)$ in (3.1) plays the role of $H(x, \partial v) - F(m(x))$ in (1.24).

Definition 3.1.
- A classical solution of (3.1) is a function $v \in C^2(\Gamma)$ which satisfies (3.1) pointwise.
- A weak solution of (3.1) is a function $v \in V$ such that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (\mu_\alpha \partial v \partial w + H(x, \partial v) w + \lambda v w)\, dx = 0 \quad \text{for all } w \in W.$$

Proposition 3.2. *Assume that*

$$H_\alpha \in C([0, \ell_\alpha] \times \mathbb{R}), \tag{3.2}$$

$$|H(x, p)| \le C_2 \left(1 + |p|^2\right) \text{ for all } x \in \Gamma, p \in \mathbb{R}, \tag{3.3}$$

where C_2 is a positive constant. There exists a classical solution v of (3.1). Moreover, if H_α is locally Lipschitz with respect to both variables for all $\alpha \in \mathcal{A}$, then the solution v belongs to $C^{2,1}(\Gamma)$.

Remark 3.3. Assume (3.2) and that $v \in H^2(\Gamma) \subset V$ is a weak solution of (3.1). From the compact embedding of $H^2(0, \ell_\alpha)$ into $C^{1,\sigma}([0, \ell_\alpha])$ for all $\sigma \in (0, 1/2)$, we get $v \in C^{1,\sigma}(\Gamma)$. Therefore, from the differential equation in (3.1) $\mu_\alpha \partial^2 v_\alpha(\cdot) = H_\alpha(\cdot, \partial v_\alpha(\cdot)) + \lambda v_\alpha(\cdot) \in C([0, \ell_\alpha])$. It follows that v is a classical solution of (3.1).

Remark 3.4. Assume now that H is locally Lipschitz continuous and that $v \in H^2(\Gamma) \subset V$ is a weak solution of (3.1). From Remark 3.3, $v \in C^{1,\sigma}(\Gamma)$ for $\sigma \in (0, 1/2)$ and the function $-\lambda v_\alpha - H_\alpha(\cdot, \partial v_\alpha)$ belongs to $C^{0,\sigma}([0, \ell_\alpha])$. Then, from the first line of (3.1), $v \in C^{2,\sigma}(\Gamma)$. This implies that $\partial v_\alpha \in \mathrm{Lip}[0, \ell_\alpha]$ and using the differential equation again, we see that $v \in C^{2,1}(\Gamma)$.

Let us start with the case when H is a bounded Hamiltonian.

Lemma 3.5. *Assume (3.2) and for some $C_H > 0$,*

$$|H(x, p)| \le C_H, \quad \text{for all } (x, p) \in \Gamma \times \mathbb{R}. \tag{3.4}$$

There exists a classical solution v of (3.1). Moreover, if H_α is locally Lipschitz in $[0, \ell_\alpha] \times \mathbb{R}$ for all $\alpha \in \mathcal{A}$ then the solution v belongs to $C^{2,1}(\Gamma)$.

Proof of Lemma 3.5. For any $u \in V$, from Lemma 2.6, the following boundary value problem:

$$\begin{cases} -\mu_\alpha \partial^2 v + \lambda v = -H(x, \partial u), & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \sum\limits_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.5}$$

has a unique weak solution $v \in V$. This allows us to define the map $T : V \longrightarrow V$ by $T(u) := v$. Moreover, from Lemma 2.6, there exists a constant C such that

$$\|v\|_V \leq C \|H(x, \partial u)\|_{L^2(\Gamma)} \leq C C_H |\Gamma|^{1/2}, \tag{3.6}$$

where $|\Gamma| = \Sigma_{\alpha \in \mathcal{A}} \ell_\alpha$. Therefore, from the differential equation in (3.5),

$$\underline{\mu} \left\|\partial^2 v\right\|_{L^2(\Gamma)} \leq \lambda \|v\|_{L^2(\Gamma)} + \|H(x, \partial u)\|_{L^2(\Gamma)} \leq \lambda \|v\|_V + C_H |\Gamma|^{1/2} \leq (\lambda C + 1) C_H |\Gamma|^{1/2}, \tag{3.7}$$

where $\underline{\mu} := \min_{\alpha \in \mathcal{A}} \mu_\alpha$. From (3.6) and (3.7), $T(V)$ is a bounded subset of $H^2(\Gamma)$, see Definition 1.3. From the compact embedding of $H^2(\Gamma)$ into V, we deduce that $\overline{T(V)}$ is a compact subset of V.

Next, we claim that T is continuous from V to V. Assuming that

$$\begin{cases} u_n \to u, & \text{in } V, \\ v_n = T(u_n), & \text{for all } n, \\ v = T(u), \end{cases} \tag{3.8}$$

we need to prove that $v_n \to v$ in V. Since $\{v_n\}$ is uniformly bounded in $H^2(\Gamma)$, then, up to the extraction of a subsequence, $v_n \to \widehat{v}$ in $C^{1,\sigma}(\Gamma)$ for some $\sigma \in (0, 1/2)$. From (3.8), we have that $\partial u_n \to \partial u$ in $L^2(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$. This yields that, up to another extraction of a subsequence, $\partial u_n \to \partial u$ almost everywhere in Γ_α. Thus $H(x, \partial u_n) \to H(x, \partial u)$ in $L^2(\Gamma_\alpha)$ by Lebesgue dominated convergence theorem. Hence, $\widehat{v}$ is a weak solution of (3.5). Since the latter is unique, $\widehat{v} = v$ and we can conclude that the whole sequence v_n converges to v. The claim is proved.

From Schauder fixed point theorem, see [19, Corollary 11.2], T admits a fixed point which is a weak solution of (3.1). Moreover, recalling that $v \in H^2(\Gamma)$, we obtain that v is a classical solution of (3.1) from Remark 3.3.

Assume now that H is locally Lipschitz. Since $v_\alpha \in H^2(0, \ell_\alpha)$ for all $\alpha \in \mathcal{A}$, we may use Remark 3.4 and obtain that $v \in C^{2,1}(\Gamma)$. □

Lemma 3.6. *If $v, u \in C^2(\Gamma)$ satisfy*

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x, \partial v) + \lambda v \geq -\mu_\alpha \partial^2 u + H(x, \partial u) + \lambda u, & \text{if } x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in A, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) \geq \sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha u(\nu_i), & \text{if } \nu_i \in \mathcal{V}, \end{cases} \tag{3.9}$$

then $v \geq u$.

Proof of Lemma 3.6. The proof is reminiscent of an argument in [9]. Suppose by contradiction that $\delta := \max_\Gamma \{u - v\} > 0$. Let $x_0 \in \Gamma_\alpha$ be a maximum point of $u - v$. It suffices to consider the case when $x_0 \in \mathcal{V}$, since if $x_0 \in \Gamma \backslash \mathcal{V}$, then $u(x_0) > v(x_0)$, $\partial u(x_0) = \partial v(x_0)$, $\partial^2 u(x_0) \leq \partial^2 v(x_0)$, and we obtain a contradiction with the first line of (3.9).
Now consider the case when $x_0 = \nu_i \in \mathcal{V}$; from Remark 1.1, we can assume without restriction that $\pi_\alpha(0) = \nu_i$. Since $u - v$ achieves its maximum over Γ at ν_i, we obtain that $\partial_\beta u(\nu_i) \geq \partial_\beta v(\nu_i)$, for all $\beta \in \mathcal{A}_i$. From Kirchhoff conditions in (3.9), this implies that $\partial_\beta u(\nu_i) = \partial_\beta v(\nu_i)$, for all $\beta \in \mathcal{A}_i$. It follows that $\partial v_\alpha(0) = \partial u_\alpha(0)$. Using the first line of (3.9), we get that

$$-\mu_\alpha \left[\partial^2 v_\alpha(0) - \partial^2 u_\alpha(0)\right] \geq \underbrace{H_\alpha(0, \partial u_\alpha(0)) - H_\alpha(0, \partial v_\alpha(0))}_{=0} + \lambda (u_\alpha(0) - v_\alpha(0)) > 0.$$

Therefore, $u_\alpha - v_\alpha$ is locally strictly convex in $[0, \ell_\alpha]$ near 0 and its first order derivative vanishes at 0. This contradicts the fact that ν_i is the maximum point of $u - v$. □

We now turn to Proposition 3.2.

Proof of Proposition 3.2. We adapt the classical proof of Boccardo, Murat and Puel in [6]. First of all, we truncate the Hamiltonian as follows:

$$H_n(x,p) = \begin{cases} H(x,p), & \text{if } |p| \le n, \\ H\left(x, \dfrac{p}{|p|}n\right), & \text{if } |p| > n. \end{cases}$$

By Lemma 3.5, for all $n \in \mathbb{N}$, since $H_n(x,p)$ is continuous and bounded by $C_2\left(1+n^2\right)$, there exists a classical solution $v_n \in C^2(\Gamma)$ for the following boundary value problem

$$\begin{cases} -\mu_\alpha \partial^2 v + H_n(x, \partial v) + \lambda v = 0, & x \in \Gamma_\alpha \backslash \mathcal{V}, \alpha \in A, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \text{for all } \alpha, \beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I. \end{cases} \tag{3.10}$$

We wish to pass to the limit as n tend to $+\infty$; we first need to estimate v_n uniformly in n, successively in $L^\infty(\Gamma)$, $H^1(\Gamma)$ and $H^2(\Gamma)$.

Estimate in $L^\infty(\Gamma)$. Since $|H_n(x,p)| \le c\left(1+|p|^2\right)$ for all x, p, then $\varphi = -c/\lambda$ and $\overline{\varphi} = c/\lambda$ are respectively a sub- and super-solution of (3.10). Therefore, from Lemma 3.6, we obtain $|\lambda v_n| \le c$.

Estimate in V. For a positive constant K to be chosen later, we introduce $w_n := e^{Kv_n^2} v_n \psi \in W$, where ψ is given in Definition 1.9. Using w_n as a test function in (3.10) leads to

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \left(\mu_\alpha \partial v_n \partial w_n + \lambda v_n w_n\right) dx = -\int_\Gamma H_n(x, \partial v_n) w_n dx.$$

Since $|H_n(x,p)| \le c\left(1+p^2\right)$, we have

$$\begin{aligned} &\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} \left[(\mu_\alpha \psi)(\partial v_n)^2 + (\mu_\alpha 2K\psi) v_n^2 (\partial v_n)^2 + (\mu_\alpha \partial \psi) v_n \partial v_n + \lambda \psi v_n^2\right] dx \\ \le &\int_\Gamma e^{Kv_n^2} |H_n(x, \partial v_n)| |v_n \psi| dx \\ \le &\int_\Gamma c e^{Kv_n^2} \psi |v_n| dx + \int_\Gamma c\psi e^{Kv_n^2} |v_n| \psi (\partial v_n)^2 dx \\ \le &\int_\Gamma e^{Kv_n^2} \left(\lambda \psi v_n^2 + \psi \frac{c^2}{4\lambda}\right) dx + \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} \left[\frac{\mu_\alpha}{2} \psi (\partial v_n)^2 + \frac{c^2}{2\mu_\alpha} \psi (\partial v_n)^2 v_n^2\right] dx, \end{aligned}$$

where we have used Young inequalities. Since $\lambda > 0$ and $\psi > 0$, we deduce that

$$\begin{aligned} &\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} \left[\left(\frac{\mu_\alpha}{2}\psi\right)(\partial v_n)^2 + 2\psi\left(\mu_\alpha K - \frac{c^2}{4\mu_\alpha}\right) v_n^2 (\partial v_n)^2 + (\mu_\alpha \partial \psi) v_n \partial v_n\right] dx \\ \le &\frac{c^2}{4\lambda} \int_\Gamma e^{Kv_n^2} \psi dx. \end{aligned} \tag{3.11}$$

Next, choosing $K > (1 + c^2/4\underline{\mu})/\underline{\mu}$ yields that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} \left[\frac{\mu_\alpha}{2} \psi (\partial v_n)^2 + 2\psi v_n^2 (\partial v_n)^2 + (\mu_\alpha \partial \psi) v_n \partial v_n\right] dx \le C$$

for a positive constant C independent of n, because v_n is bounded by c/λ. Since ψ is bounded from below by a positive number and $\partial\psi$ is piecewise constant on Γ, we infer that $\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} e^{Kv_n^2} v_n^2 (\partial v_n)^2 \le \widetilde{C}$, where $\widetilde{C}$ is a positive constant independent on n. Using this information and (3.11) again, we obtain that $\int_\Gamma (\partial v_n)^2$ is bounded uniformly in n. There exists a constant $\overline{C}$ such that $\|v_n\|_V \le \overline{C}$ for all n.

Estimate in $H^2(\Gamma)$. From the differential equation in (3.10) and (3.3), we have

$$\underline{\mu}\left|\partial^2 v_n\right| \leq c + c\left|\partial v_n\right|^2 + \lambda\left|v_n\right|, \quad \text{for all } \alpha \in \mathcal{A}.$$

Thus $\partial^2 v_n$ is uniformly bounded in $L^1(\Gamma)$. This and the previous estimate on $\|\partial v_n\|_{L^2(\Gamma)}$ yield that ∂v_n is uniformly bounded in $L^\infty(\Gamma)$, from the continuous embedding of $W^{1,1}(0,\ell_\alpha)$ into $C([0,\ell_\alpha])$. Therefore, from (3.10), we get that $\partial^2 v_n$ is uniformly bounded in $L^\infty(\Gamma)$. This implies in particular that v_n is uniformly bounded in $W^{2,\infty}(\Gamma)$.

Hence, for any $\sigma \in (0,1)$, up to the extraction of a subsequence, there exists $v \in V$ such that $v_n \to v$ in $C^{1,\sigma}(\Gamma)$. This yields that $H_n(x,\partial v_n) \to H(x,\partial v)$ for all $x \in \Gamma$. By Lebesgue's Dominated Convergence Theorem, we obtain that v is a weak solution of (3.1), and since $v \in C^{1,\sigma}(\Gamma)$, by Remark 3.3, v is a classical solution of (3.1).

Assume now that H is locally Lipschitz. We may use Remark 3.4 and obtain that $v \in C^{2,1}(\Gamma)$. The proof is complete. □

3.2 The ergodic problem

For $f \in PC(\Gamma)$, we wish to prove the existence of $(v,\rho) \in C^2(\Gamma) \times \mathbb{R}$ such that

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x,\partial v) + \rho = f(x), & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha,\beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.12}$$

with the normalization condition

$$\int_\Gamma v dx = 0. \tag{3.13}$$

Theorem 3.7. *Assume* (1.25)-(1.27). *There exists a unique couple* $(v,\rho) \in C^2(\Gamma) \times \mathbb{R}$ *satisfying* (3.12)-(3.13), *with* $|\rho| \leq \max_{x\in\Gamma}|H(x,0) - f(x)|$. *There exists a constant* $\overline{C}$ *which only depends upon* $\|f\|_{L^\infty(\Gamma)}, \mu_\alpha$ *and the constants in* (1.27) *such that*

$$\|v\|_{C^2(\Gamma)} \leq \overline{C}. \tag{3.14}$$

Moreover, for some $\sigma \in (0,1)$, *if* $f_\alpha \in C^{0,\sigma}([0,\ell_\alpha])$ *for all* $\alpha \in \mathcal{A}$, *then* $(v,\rho) \in C^{2,\sigma}(\Gamma) \times \mathbb{R}$; *there exists a constant* $\overline{C}$ *which only depends upon* $\|f_\alpha\|_{C^{0,\sigma}([0,\ell_\alpha])}, \mu_\alpha$ *and the constants in* (1.27) *such that*

$$\|v\|_{C^{2.\sigma}(\Gamma)} \leq \overline{C}. \tag{3.15}$$

Proof of existence in Theorem 3.7. By Proposition 3.2, for any $\lambda > 0$, the following boundary value problem

$$\begin{cases} -\mu_\alpha \partial^2 v + H(x,\partial v) + \lambda v = f, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ v|_{\Gamma_\alpha}(\nu_i) = v|_{\Gamma_\beta}(\nu_i), & \alpha,\beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha v(\nu_i) = 0, & i \in I, \end{cases} \tag{3.16}$$

has a unique solution $v_\lambda \in C^2(\Gamma)$. Set $C := \max_\Gamma |f(\cdot) - H(\cdot,0)|$. The constant functions $\varphi := -C/\lambda$ and $\overline{\varphi} = C/\lambda$ are respectively sub- and super-solution of (3.16). By Lemma 3.6,

$$-C \leq \lambda v_\lambda(x) \leq C, \quad \text{for all } x \in \Gamma. \tag{3.17}$$

Next, set $u_\lambda := v_\lambda - \min_\Gamma v_\lambda$. We see that u_λ is the unique classical solution of

$$\begin{cases} -\mu_\alpha \partial^2 u_\lambda + H(x,\partial u_\lambda) + \lambda u_\lambda + \lambda \min_\Gamma v_\lambda = f, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \alpha \in \mathcal{A}, \\ u|_{\Gamma_\alpha}(\nu_i) = u|_{\Gamma_\beta}(\nu_i), & \alpha,\beta \in \mathcal{A}_i,\ i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha}\mu_\alpha \partial_\alpha u_\lambda(\nu_i) = 0, & i \in I. \end{cases} \tag{3.18}$$

Before passing to the limit as λ tends 0, we need to estimate u_λ in $C^2(\Gamma)$ uniformly with respect to λ. We do this in two steps:

Step 1: Estimate of $\|\partial u_\lambda\|_{L^q(\Gamma)}$. Using ψ as a test-function in (3.18), see Definition 1.9, and recalling that $\lambda u_\lambda + \lambda \min_\Gamma v_\lambda = \lambda v_\lambda$, we see that

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial u_\lambda \partial\psi dx + \int_\Gamma \left(H\left(x, \partial u_\lambda\right) + \lambda v_\lambda\right)\psi dx = \int_\Gamma f\psi dx.$$

From (1.27) and (3.17),

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial u_\lambda \partial\psi dx + \sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} C_0 \left|\partial u_\lambda\right|^q \psi dx \le \int_\Gamma \left(f + C + C_1\right)\psi dx.$$

On the other hand, since $q > 1$, $\psi \ge \underline{\psi} > 0$ and $\partial\psi$ is bounded, there exists a large enough positive constant C' such that

$$\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial u_\lambda \partial\psi dx + \frac{1}{2}\sum_{\alpha\in\mathcal{A}} \int_{\Gamma_\alpha} C_0 \left|\partial u_\lambda\right|^q \psi dx + C' > 0, \quad \text{for all } \lambda > 0.$$

Subtracting, we get $\frac{C_0}{2}\underline{\psi}\int_\Gamma \left|\partial u_\lambda\right|^q dx \le \int_\Gamma \left(f + C + C_1\right)\psi dx + C'$. Hence, for all $\lambda > 0$,

$$\|\partial u_\lambda\|_{L^q(\Gamma)} \le \widetilde{C}, \tag{3.19}$$

where $\widetilde{C} := \left[\left(2\int_\Gamma \left(|f| + C + C_1\right)\psi dx + 2C'\right)/\left(C_0\underline{\psi}\right)\right]^{1/q}$.

Step 2: Estimate of $\|u_\lambda\|_{C^2(\Gamma)}$. Since $u_\lambda = v_\lambda - \min_\Gamma v_\lambda$, there exists $\alpha \in \mathcal{A}$ and $x_\lambda \in \Gamma_\alpha$ such that $u_\lambda(x_\lambda) = 0$. For all $\lambda > 0$ and $x \in \Gamma_\alpha$, we have

$$\left|u_\lambda(x)\right| = \left|u_\lambda(x) - u_\lambda(x_\lambda)\right| \le \int_\Gamma \left|\partial u_\lambda\right| dx \le \|\partial u_\lambda\|_{L^q(\Gamma)} |\Gamma|^{q/(q-1)}.$$

From (3.19) and the latter inequality, we deduce that $\|u_\lambda|_{\Gamma_\alpha}\|_{L^\infty(\Gamma_\alpha)} \le \widetilde{C}|\Gamma|^{q/(q-1)}$. Let ν_i be a transition vertex which belongs to $\partial\Gamma_\alpha$. For all $\beta \in \mathcal{A}_i$, $y \in \Gamma_\beta$,

$$\left|u_\lambda(y)\right| \le \left|u_\lambda(y) - u_\lambda(\nu_i)\right| + \left|u_\lambda(\nu_i)\right| \le 2\widetilde{C}|\Gamma|^{q/(q-1)}.$$

Since the network is connected and the number of edges is finite, repeating the argument as many times as necessary, we obtain that there exists $M \in \mathbb{N}$ such that

$$\|u_\lambda\|_{L^\infty(\Gamma)} \le M\widetilde{C}|\Gamma|^{q/(q-1)}.$$

This bound is uniform with respect to $\lambda \in (0,1]$. Next, from (3.18) and (1.29), we get

$$\underline{\mu}\left|\partial^2 u_\lambda\right| \le \left|H\left(x, \partial u_\lambda\right)\right| + \left|\lambda v_\lambda\right| + |f| \le C_q\left(1 + \left|\partial u_\lambda\right|^q\right) + C + \|f\|_{L^\infty(\Gamma)}.$$

Hence, from (3.19), $\partial^2 u_\lambda$ is bounded in $L^1(\Gamma)$ uniformly with respect to $\lambda \in (0,1]$. From the continuous embedding of $W^{1,1}(0,\ell_\alpha)$ in $C([0,\ell_\alpha])$, we infer that $\partial u_\lambda|_{\Gamma_\alpha}$ is bounded in $C(\Gamma_\alpha)$ uniformly with respect to $\lambda \in (0,1]$. From the equation (3.18) and (3.17), this implies that u_λ is bounded in $C^2(\Gamma)$ uniformly with respect to $\lambda \in (0,1]$.

After the extraction of a subsequence, we may assume that when $\lambda \to 0^+$, the sequence u_λ converges to some function $v \in C^{1,1}(\Gamma)$ and that $\lambda \min v_\lambda$ converges to some constant ρ. Notice that v still satisfies the Kirchhoff conditions since $\partial u_\lambda|_{\Gamma_\alpha}(\nu_i) \to \partial v|_{\Gamma_\alpha}(\nu_i)$ as $\lambda \to 0^+$. Passing to the limit in (3.18), we get that the couple (v,ρ) satisfies (3.12) in the weak sense, then in the classical sense by using an argument similar to Remark 3.3. Adding a constant to v, we also get (3.13).

Furthermore, if for some $\sigma \in (0,1)$, $f|_{\Gamma_\alpha} \in C^{0,\sigma}(\Gamma_\alpha)$ for all $\alpha \in \mathcal{A}$, a bootstrap argument using the Lipschitz continuity of H on the bounded subsets of $\Gamma \times \mathbb{R}$ shows that u_λ is bounded in $C^{2,\sigma}(\Gamma)$ uniformly with respect to $\lambda \in (0,1]$. After a further extraction of a subsequence if necessary, we obtain (3.15). □

Proof of uniqueness in Theorem 3.7. Assume that there exist two solutions (v,ρ) and $(\tilde{v},\tilde{\rho})$ of (3.12)-(3.13). First of all, we claim that $\rho=\tilde{\rho}$. By symmetry, it suffices to prove that $\rho\geq\tilde{\rho}$. Let x_0 be a maximum point of $e:=\tilde{v}-v$. Using similar arguments as in the proof of Lemma 3.6, with λv and λu respectively replaced by ρ and $\tilde{\rho}$, we get $\rho\geq\tilde{\rho}$ and the claim is proved.

We now prove the uniqueness of v. Since H_α belongs to $C^1(\Gamma_\alpha\times\mathbb{R})$ for all $\alpha\in\mathcal{A}$, then e is a solution of $\mu_\alpha\partial^2 e_\alpha-\left[\int_0^1\partial_p H_\alpha\left(y,\theta\partial v_\alpha+(1-\theta)\,\partial\tilde{v}_\alpha\right)d\theta\right]\partial e_\alpha=0,\quad$ in $(0,\ell_\alpha)$, with the same transmission and boundary condition as in (3.12). By Lemma 2.8, e is a constant function on Γ. Moreover, from (3.13) , we know that $\int_\Gamma e dx=0$. This yields that $e=0$ on Γ. Hence, (3.12)-(3.13) has a unique solution. □

Remark 3.8. Since there exists a unique solution of (3.12)-(3.13), we conclude that the whole sequence $(u_\lambda,\lambda v_\lambda)$ in the proof of Theorem 3.7 converges to (v,ρ) as $\lambda\to 0$.

4 Proof of the main result

We first prove Theorem 1.13 when F is bounded.

Theorem 4.1. *Assume (1.25)-(1.28), (1.30) and that F is bounded. There exists a solution $(v,m,\rho)\in C^2(\Gamma)\times\mathcal{W}\times\mathbb{R}$ to the mean field games system (1.24). If F is locally Lipschitz continuous, then $v\in C^{2,1}(\Gamma)$. If furthermore F is strictly increasing, then the solution is unique.*

Proof of existence in Theorem 4.1. We adapt the proof of Camilli and Marchi in [8, Theorem 1]. For $\sigma\in(0,1/2)$ let us introduce the space

$$\mathcal{M}_\sigma=\left\{m:m_\alpha\in C^{0,\sigma}([0,\ell_\alpha])\text{ and }\frac{m|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}}=\frac{m|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}\text{ for all }i\in I\text{ and }\alpha,\beta\in\mathcal{A}_i\right\}$$

which, endowed with the norm $\|m\|_{\mathcal{M}_\sigma}=\|m\|_{L^\infty(\Gamma)}+\max\limits_{\alpha\in\mathcal{A}}\sup\limits_{y,z\in[0,\ell_\alpha],y\neq z}\dfrac{|m_\alpha(y)-m_\alpha(z)|}{|y-z|^\sigma}$, is a Banach space. Now consider the set

$$\mathcal{K}=\left\{m\in\mathcal{M}_\sigma:m\geq 0\text{ and }\int_\Gamma m dx=1\right\}$$

and observe that $\mathcal{K}$ is a closed and convex subset of $\mathcal{M}_\sigma$. We define a map $T:\mathcal{K}\to\mathcal{K}$ as follows: given $m\in\mathcal{K}$, set $f=F(m)$. By Theorem 3.7, (3.12)-(3.13) has a unique solution $(v,\rho)\in C^2(\Gamma)\times\mathbb{R}$. Next, for v given, we solve (2.12)-(2.13) with $b(\cdot)=\partial_p H(\cdot,\partial v(\cdot))\in PC(\Gamma)$. By Theorem 2.11, there exists a unique solution $\overline{m}\in\mathcal{K}\cap W$ of (2.12)-(2.13). We set $T(m)=\overline{m}$; we claim that T is continuous and has a precompact image. We proceed in several steps:

T is continuous. Let $m_n,m\in\mathcal{K}$ be such that $\|m_n-m\|_{\mathcal{M}_\sigma}\to 0$ as $n\to+\infty$; set $\overline{m}_n=T(m_n),\overline{m}=T(m)$. We need to prove that $\overline{m}_n\to\overline{m}$ in $\mathcal{M}_\sigma$. Let $(v_n,\rho_n),(v,\rho)$ be the solutions of (3.12)-(3.13) corresponding respectively to $f=F(m_n)$ and $f=F(m)$. Using estimate (3.14), we see that up to the extraction of a subsequence, we may assume that $(v_n,\rho_n)\to(\overline{v},\overline{\rho})$ in $C^1(\Gamma)\times\mathbb{R}$. Since $F(m_n)|_{\Gamma_\alpha}\to F(m)|_{\Gamma_\alpha}$ in $C(\Gamma_\alpha)$, $H_\alpha(y,(\partial v_n)_\alpha)\to H_\alpha(y,\partial\overline{v}_\alpha)$ in $C([0,\ell_\alpha])$, and since it is possible to pass to the limit in the transmission and boundary conditions thanks to the C^1-convergence, we obtain that $(\overline{v},\overline{\rho})$ is a weak (and strong by Remark 3.3) solution of (3.12)-(3.13). By uniqueness, $(\overline{v},\overline{\rho})=(v,\rho)$ and the whole sequence (v_n,ρ_n) converges.

Next, $\overline{m}_n=T(m_n),\overline{m}=T(m)$ are respectively the solutions of (2.12)-(2.13) corresponding to $b=\partial_p H(x,\partial v_n)$ and $b=\partial_p H(x,\partial v)$. From the estimate (2.16), since $\partial_p H(x,\partial v_n)$ is uniformly bounded in $L^\infty(\Gamma)$, we see that $\overline{m}_n$ is uniformly bounded in W. Therefore, up to the extraction of subsequence, $\overline{m}_n\rightharpoonup\widehat{m}$ in W and $\overline{m}_n\to\widehat{m}$ in $\mathcal{M}_\sigma$, because W is

compactly embedded in $\mathcal{M}_\sigma$ for $\sigma \in (0,1/2)$. It is easy to pass to the limit and find that $\widehat{m}$ is a solution of (2.12)-(2.13) with $b = \partial_p H(x, \partial v)$. From Theorem 2.11, we obtain that $\overline{m} = \widehat{m}$, and hence the whole sequence $\overline{m}_n$ converges to $\overline{m}$.

The image of T is precompact. Since $F \in C^0(\mathbb{R}^+;\mathbb{R})$ is a uniformly bounded function, we see that $F(m)$ is bounded in $L^\infty(\Gamma)$ uniformly with respect to $m \in \mathcal{K}$. From Theorem 3.7, there exists a constant $\overline{C}$ such that for all $m \in \mathcal{K}$, the unique solution v of (3.12)-(3.13) with $f = F(m)$ satisfies $\|v\|_{C^2(\Gamma)} \leq \overline{C}$. From Theorem 2.11, we obtain that $\overline{m} = T(m)$ is bounded in W by a constant independent of m. Since W is compactly embedded in $\mathcal{M}_\sigma$, for $\sigma \in (0,1/2)$ we deduce that T has a precompact image.

End of the proof. We can apply Schauder fixed point theorem (see [19, Corollary 11.2]) to conclude that the map T admits a fixed point m. By Theorem 2.11, we get $m \in \mathcal{W}$. Hence, there exists a solution $(v, m, \rho) \in C^2(\Gamma) \times \mathcal{W} \times \mathbb{R}$ to the mean field games system (1.24). If F is locally Lipschitz continuous, then $v \in C^{2,1}(\Gamma)$ from the final part of Theorem 3.7. □

Proof of uniqueness in Theorem 4.1. We assume that F is strictly increasing and that there exist two solutions (v_1, m_1, ρ_1) and (v_2, m_2, ρ_2) of (1.24). We set $\overline{v} = v_1 - v_2, \overline{m} = m_1 - m_2$ and $\overline{\rho} = \rho_1 - \rho_2$ and write the equations for $\overline{v}, \overline{m}$ and $\overline{\rho}$

$$\begin{cases} -\mu_\alpha \partial^2 \overline{v} + H(x, \partial v_1) - H(x, \partial v_2) + \overline{\rho} - (F(m_1) - F(m_2)) = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \\ -\mu_\alpha \partial^2 \overline{m} - \partial(m_1 \partial_p H(x, \partial v_1)) + \partial(m_2 \partial_p H(x, \partial v_2)) = 0, & \text{in } \Gamma_\alpha \backslash \mathcal{V}, \\ \overline{v}|_{\Gamma_\alpha}(\nu_i) = \overline{v}|_{\Gamma_\beta}(\nu_i), \quad \dfrac{\overline{m}|_{\Gamma_\alpha}(\nu_i)}{\gamma_{i\alpha}} = \dfrac{\overline{m}|_{\Gamma_\beta}(\nu_i)}{\gamma_{i\beta}}, & \alpha, \beta \in \mathcal{A}_i, i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} \gamma_{i\alpha} \mu_\alpha \partial_\alpha \overline{v}(\nu_i) = 0, & i \in I, \\ \displaystyle\sum_{\alpha \in \mathcal{A}_i} n_{i\alpha} \left[m_1|_{\Gamma_\alpha}(\nu_i) \partial_p H(\nu_i, \partial v_1|_{\Gamma_\alpha}(\nu_i)) - m_2|_{\Gamma_\alpha}(\nu_i) \partial_p H(\nu_i, \partial v_2|_{\Gamma_\alpha}(\nu_i)) \right] & \\ \displaystyle + \sum_{\alpha \in \mathcal{A}_i} \mu_\alpha \partial_\alpha \overline{m}(\nu_i) = 0, & i \in I, \\ \displaystyle\int_\Gamma \overline{v} dx = 0, \quad \int_\Gamma \overline{m} dx = 0. \end{cases} \tag{4.1}$$

Multiplying the equation for $\overline{v}$ by $\overline{m}$ and integrating over Γ_α, we get

$$\int_{\Gamma_\alpha} \mu_\alpha \partial \overline{v} \partial \overline{m} + [H(x, \partial v_1) - H(x, \partial v_2) + \overline{\rho} - (F(m_1) - F(m_2))] \overline{m} dx - [\mu_\alpha \overline{m}_\alpha \partial \overline{v}_\alpha]_0^{\ell_\alpha} = 0. \tag{4.2}$$

Multiplying the equation for $\overline{m}$ by $\overline{v}$ and integrating over Γ_α, we get

$$\begin{aligned} &\int_{\Gamma_\alpha} \mu_\alpha \partial \overline{v} \partial \overline{m} + [m_1 \partial_p H(x, \partial v_1) - m_2 \partial_p H(x, \partial v_2)] \partial \overline{v} dx \\ &- \left[\overline{v}|_{\Gamma_\alpha} (\mu_\alpha \partial \overline{m}|_{\Gamma_\alpha} + m_1|_{\Gamma_\alpha} \partial_p H(x, \partial v_1|_{\Gamma_\alpha}) - m_2|_{\Gamma_\alpha} \partial_p H(x, \partial v_2|_{\Gamma_\alpha})) \right]_0^{\ell_\alpha} = 0. \end{aligned} \tag{4.3}$$

Subtracting (4.2) to (4.3), summing over $\alpha \in \mathcal{A}$, assembling the terms corresponding to a same vertex ν_i and taking into account the transmission and the normalization condition for $\overline{v}$ and $\overline{m}$, we obtain

$$\begin{aligned} 0 = &\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} (m_1 - m_2)[F(m_1) - F(m_2)] dx \\ &+ \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} m_1 [H(x, \partial v_2) - H(x, \partial v_1) + \partial_p H(x, \partial v_1) \partial \overline{v}] dx \\ &+ \sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} m_2 [H(x, \partial v_1) - H(x, \partial v_2) - \partial_p H(x, \partial v_2) \partial \overline{v}] dx. \end{aligned}$$

Since F is strictly monotone then the first sum is non-negative. Moreover, by the convexity of H and the positivity of m_1, m_2, the last two sums are non-negative. Therefore, we have that $m_1 = m_2$. From Theorem 3.7, we finally obtain $v_1 = v_2$ and $\rho_1 = \rho_2$. □

Proof of Theorem 1.13 for a general coupling F. We only need to modify the proof of existence.

We now truncate the coupling function as follows:

$$F_n(r) = \begin{cases} F(r), & \text{if } |r| \leq n, \\ F\left(\frac{r}{|r|}n\right), & \text{if } |r| \geq n. \end{cases}$$

Then F_n is continuous, bounded below by $-M$ as in (1.31) and bounded above by some constant C_n. By Theorem 4.1, for all $n \in \mathbb{N}$, there exists a unique solution $(v_n, m_n, \rho_n) \in C^2(\Gamma) \times \mathcal{W} \times \mathbb{R}$ of the mean field game system (1.24) where F is replaced by F_n. We wish to pass to the limit as $n \to +\infty$. We proceed in several steps:

Step 1: ρ_n is bounded from below. Multiplying the HJB equation in (1.24) by m_n and the Fokker-Planck equation in (1.24) by v_n, using integration by parts and the transmission conditions, we obtain that

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v_n \partial m_n dx + \int_\Gamma H(x, \partial v_n) m_n dx + \rho_n = \int_\Gamma F_n(m_n) m_n dx, \tag{4.4}$$

and

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v_n \partial m_n dx + \int_\Gamma \partial_p H(x, \partial v_n) m_n \partial v_n dx = 0. \tag{4.5}$$

Subtracting the two equations, we obtain

$$\rho_n = \int_\Gamma F_n(m_n) m_n dx + \int_\Gamma [\partial_p H(x, \partial v_n) \partial v_n - H(x, \partial v_n)] m_n dx. \tag{4.6}$$

In what follows, the constant C may vary from line to line but remains independent of n. From (1.26), we see that $\partial_p H(x, \partial v_n) \partial v_n - H(x, \partial v_n) \geq -H(x, 0) \geq -C$. Therefore

$$\rho_n \geq \int_\Gamma F_n(m_n) m_n dx - C \int_\Gamma m_n dx = \int_\Gamma F_n(m_n) m_n dx - C. \tag{4.7}$$

Hence, since $F_n + M \geq 0$ and $\int_\Gamma m_n dx = 1$, we get that ρ_n is bounded from below by $-M - C$ independently of n.

Step 2: ρ_n and $\int_\Gamma F_n(m_n) dx$ are uniformly bounded. By Theorem 2.11, there exists a positive solution $w \in W$ of (2.12)-(2.13) with $b = 0$. It yields

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial w \partial u dx = 0, \text{for all } u \in V, \qquad \text{and} \qquad \int_\Gamma w dx = 1.$$

Multiplying the HJB equation of (1.24) by w, using integration by parts and the Kirchhoff condition, we get

$$\underbrace{\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v_n \partial w dx}_{=0} + \int_\Gamma H(x, \partial v_n) w dx + \rho_n \underbrace{\int_\Gamma w dx}_{=1} = \int_\Gamma F_n(m_n) w dx.$$

This implies, using (1.27), (2.16) and $F_n + M \geq 0$,

$$\begin{aligned} \rho_n &= \int_\Gamma F_n(m_n) w dx - \int_\Gamma H(x, \partial v_n) w dx \\ &\leq \|w\|_{L^\infty(\Gamma)} \int_\Gamma (F_n(m_n) + M) dx - M - \int_\Gamma (C_0 |\partial v_n|^q - C_1) w dx \\ &\leq C \int_\Gamma F_n(m_n) dx + C - \int_\Gamma C_0 |\partial v_n|^q w dx. \end{aligned} \tag{4.8}$$

Thus, by (4.7), we have

$$-M-C \leq \int_{\Gamma} F_n(m_n)\, m_n dx - C \leq \rho_n \leq C \int_{\Gamma} F_n(m_n)\, dx + C. \tag{4.9}$$

Let $K > 0$ be a constant to be chosen later. We have

$$\begin{aligned}\int_{\Gamma} F_n(m_n)\, dx &\leq \frac{1}{K} \int_{m_n \geq K} [F_n(m_n) + M]\, m_n dx + \sup_{0 \leq r \leq K} F(r) \int_{m_n \leq K} dx \\ &\leq \frac{1}{K} \int_{\Gamma} F_n(m_n)\, m_n dx + \frac{M}{K} + C_K, \end{aligned} \tag{4.10}$$

where C_K is independent of n. Choosing $K = 2C$ where C is the constant in (4.9), we get by combining (4.10) with (4.9) that $\int_{\Gamma} F_n(m_n) m_n \leq C$. Using (4.10) again, we obtain $\int_{\Gamma} F_n(m_n) dx \leq C$. Hence, from (4.9), we conclude that $|\rho_n| + |\int_{\Gamma} F_n(m_n) dx| \leq C$.

Step 3: Prove that $F_n(m_n)$ is uniformly integrable and v_n and m_n are uniformly bounded respectively in $C^1(\Gamma)$ and W. Let E be a measurable with $|E| = \eta$. By (4.10) with Γ is replaced by E, we have

$$\begin{aligned}\int_E F_n(m_n) m_n dx &\leq \frac{1}{K} \int_{E \cap \{m_n \geq K\}} F_n(m_n)\, m_n dx + \frac{M}{K} + \sup_{0 \leq r \leq K} F(r) \int_{E \cap \{m_n \leq K\}} dx \\ &\leq \frac{C+M}{K} + C_K \eta, \end{aligned}$$

since $\int_E F_n(m_n) m_n dx \leq C$ and $\sup_{0 \leq r \leq K} F_n(r) \leq \sup_{0 \leq r \leq K} F(r) := C_K$. Therefore, for all $\varepsilon > 0$, we may choose K such that $(C+M)/K \leq \varepsilon/2$ and then η such that $C_K \eta \leq \varepsilon/2$ and get

$$\int_E F_n(m_n)\, dx \leq \varepsilon, \quad \text{for all } E \text{ which satisfies } |E| \leq \eta,$$

which proves the uniform integrability of $\{F_n(m_n)\}_n$.

Next, since ρ_n and $\int_{\Gamma} F_n(m_n)\, dx$ are uniformly bounded, we infer from (4.8) that ∂v_n is uniformly bounded in $L^q(\Gamma)$. Since by the condition $\int_{\Gamma} v_n dx = 0$, there exists $\overline{x}_n$ such that $v_n(\overline{x}_n) = 0$, we infer from the latter bound that v_n is uniformly bounded in $L^{\infty}(\Gamma)$. Using the HJB equation in (1.24) and Remark 1.7, we get

$$\mu_\alpha |\partial^2 v_n| \leq |H(x, \partial v_n)| + |F_n(m_n)| + |\rho_n| \leq C_q(|\partial v_n|^q + 1) + |F_n(m_n)| + |\rho_n|.$$

We obtain that $\partial^2 v_n$ is uniformly bounded in $L^1(\Gamma)$, which implies that v_n is uniformly bounded in $C^1(\Gamma)$. Therefore the sequence of functions $C_q(|\partial v_n|^q + 1) + |F_n(m_n)| + |\rho_n|$ is uniformly integrable, and so is $\partial^2 v_n$. This implies that ∂v_n is equicontinuous. Hence, $\{v_n\}$ is relatively compact in $C^1(\Gamma)$ by Arzelà-Ascoli's theorem. Finally, from the Fokker-Planck equation and Theorem 2.11, since $\partial_p H(x, \partial v_n)$ is uniformly bounded in $L^{\infty}(\Gamma)$, we obtain that m_n is uniformly bounded in W.

Step 4: Passage to the limit

From Step 1 and 2, since $\{\rho_n\}$ is uniformly bounded, there exists $\rho \in \mathbb{R}$ such that $\rho_n \to \rho$ up to the extraction of subsequence. From Step 3, there exists $m \in W$ such that $m_n \rightharpoonup m$ in W and $m_n \to m$ almost everywhere, up to the extraction of subsequence. Also from Step 3, since $F_n(m_n)$ is uniformly integrable, from Vitali theorem, $\lim_{n\to\infty} \int_{\Gamma} F_n(m_n)\, \tilde{w} dx = \int_{\Gamma} F(m)\, \tilde{w} dx$, for all $\tilde{w} \in W$. From Step 3, up to the extraction of subsequence, there exists $v \in C^1(\Gamma)$ such that $v_n \to v$ in $C^1(\Gamma)$. Hence, (v, ρ, m) satisfies the weak form of the MFG system:

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial v \partial \tilde{w} dx + \int_{\Gamma} (H(x, \partial v) + \rho)\, \tilde{w} dx = \int_{\Gamma} F(m)\, \tilde{w} dx, \quad \text{for all } \tilde{w} \in W,$$

and

$$\sum_{\alpha \in \mathcal{A}} \int_{\Gamma_\alpha} \mu_\alpha \partial m \partial \tilde{v} dx + \int_{\Gamma} \partial_p H(x, \partial v)\, m \partial \tilde{v} dx = 0, \quad \text{for all } \tilde{v} \in V.$$

Finally, we prove the regularity for the solution of (1.24) . Since $m \in W$, $m|_{\Gamma_\alpha} \in C^{0,\sigma}$ for some constant $\sigma \in (0, 1/2)$ and all $\alpha \in \mathcal{A}$. By Theorem 3.7, $v \in C^2(\Gamma)$ ($v \in C^{2,\sigma}(\Gamma)$ if F is locally Lipschitz continuous). Then, by Theorem 2.11, we see that $m \in \mathcal{W}$. If F is locally Lipschitz continuous, this implies that $v \in C^{2,1}(\Gamma)$. We also obtain that v and m satisfy the Kirchhoff and transmission conditions in (1.24). The proof is done. □

Acknowledgement. The authors were partially supported by ANR project ANR-16-CE40-0015-01.

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