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Exact analytic solution of the incompressible Navier-Stokes equations

Fabien Salmon¹

Abstract—The Navier-Stokes equations, which govern fluid motions, are not resolved yet. This investigation relates to the application of the power series method to the incompressible Navier-Stokes equations. Power series with unknown coefficients substitute the variables into the equations. Then, the resulting recurrence relation allows the computation of the series coefficients from the initial and boundary conditions. The article begins with the resolution of an accessible multidimensional differential equation that illustrates the approach. Then, the exact analytic solution stemming from the application of the method to the Navier-Stokes equations is detailed. The existence and the convergence domain of the solution are discussed without concluding.

Keywords—Navier-Stokes, incompressible, analytic, exact solution, power series

I. INTRODUCTION

The Navier-Stokes equations describe the fluid motion. They were named in honour of Claude-Louis Navier and George Gabriel Stokes. At present, the general solution of this set of equations remains unknown. Except in specific cases with analytical solution, fluid motion problems request high performance computing.

Many studies handle the existence and uniqueness of solutions to the Navier-Stokes equations. The existence of weak solutions to the incompressible case were proved in [1]. Moreover, the weak solutions of the compressible equations were analysed with the aim of characterizing them [2], [3]. Regarding strong solutions, numerous studies investigated their existence as well as their properties [4], [5]. In the same way, scientists studied solutions of particular problems with specified boundary conditions [6].

This article does not concern such mathematical properties but features the formulation of an analytic solution to the incompressible Navier-Stokes equations. The resolution is based on the power series method [7] that consists in substituting variables by power series in the equations. The power series coefficients are then determined by solving the equations. This approach leads to an exact analytic solution as long as it exists.

First, an effortless 2D differential equation provides an overview of the method and difficulties. To make clear the restriction of this technique, the differences between this approach and a general one applicable to the simple example are indicated. Second, general issues about the incompressible Navier-Stokes equations are discussed as well as the assumptions required by the resolution method. Then, the analytic solution is given without detailing all hand calculations. This publication includes a discussion about the convergence domain.

II. METHOD DESCRIPTION

The method consists in seeking an analytic solution in several dimensions. Basically, it is the same approach as finding a series solution to a 1D differential equation, except it is not in one dimension anymore.

For reasons of brevity, the next section only contains several points of the mathematical resolution of the Navier-Stokes equations. Then, this part is dedicated to the presentation of the entire method through an application to an easy differential equation depending on two variables. This case displays the difficulties encountered with multidimensional series. The differential equation is

$$\frac{\partial f}{\partial x} - \frac{\partial f}{\partial y} = 0 \quad (1)$$

The solution is sought in $\mathbb{R}^+ \times \mathbb{R}^+$. Before solving the equation thanks to power series, the resolution is performed thanks to a substitution. By setting $\psi(x, y) = (u, v) = (x, x + y)$ and $f(x, y) = g(u, v)$,

$$\begin{cases} \frac{\partial f}{\partial x} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial x} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial x} = \frac{\partial g}{\partial u} + \frac{\partial g}{\partial v} \\ \frac{\partial f}{\partial y} = \frac{\partial g}{\partial u} \frac{\partial u}{\partial y} + \frac{\partial g}{\partial v} \frac{\partial v}{\partial y} = \frac{\partial g}{\partial v} \end{cases} \quad (2)$$

¹UMR CNRS 5295 I2M, Bordeaux University, France
Fabien.Salmon@u-bordeaux.fr

Then, the differential equation becomes

$$\frac{\partial g}{\partial u} = 0 \quad (3)$$

Thus, the solutions to the equation (1) are the functions depending on $x + y$: $f(x, y) = g(u, v) = \phi(v) = \phi(x + y)$.

Now, the power series method is used to provide an analytic solution to the equation (1) by introducing

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} a_{n,m} x^n y^m \quad (4)$$

By replacing f by its power series expansion in (1), the following relation between the coefficients is deduced:

$$a_{n+1,m} = \frac{m+1}{n+1} a_{n,m+1} \quad (5)$$

(5) corresponds to a recurrence relation with two indices. The coefficients must be expressed from the boundary conditions coefficients ($n = 0$ or $m = 0$). The resolution of this relation yields:

$$a_{n,m} = \frac{(m+n)!}{n!m!} a_{0,m+n} \quad (6)$$

So, the analytic solution of (1) is:

$$f(x, y) = \sum_{n=0}^{\infty} \sum_{m=0}^{\infty} \binom{m+n}{m} a_{0,m+n} x^n y^m \quad (7)$$

It does not look like to the solution formerly obtained and some modifications need be achieved. By setting $k = m + n$,

$$f(x, y) = \sum_{k=0}^{\infty} \sum_{m=0}^k \binom{k}{m} a_{0,k} x^{k-m} y^m \quad (8)$$

The binomial theorem simplifies the previous expression:

$$f(x, y) = \sum_{k=0}^{\infty} a_{0,k} (x + y)^k \quad (9)$$

Again, the solutions are the functions such that $f(x, y) = \phi(x + y)$ where ϕ derived from the boundary conditions. The coefficients $a_{0,k}$ correspond to the power series coefficients of $y \mapsto f(0, y)$:

$$f(0, y) = \sum_{m=0}^{\infty} a_{0,m} y^m \quad (10)$$

Therefore, the solution must admit a power series expansion in the neighbourhood of 0 at the boundaries. Thus, the specified boundary conditions must be analytic so that the solution can be analytic too. This obligation constitutes a prerequisite for multidimensional problems.

Moreover, the convergence domain of the series solution depends on the boundary conditions. First, an exponential boundary condition is assumed: $f(0, y) = e^y$. So, necessarily $a_{0,m} = \frac{1}{m!}$. The analytic solution is then

$$f_1(x, y) = \sum_{k=0}^{\infty} \frac{1}{k!} (x + y)^k = e^{x+y} \quad (11)$$

The convergence domain of the power series is \mathbb{R}^2 . Therefore, (11) is the global solution.

Now, $f(0, y) = \frac{1}{y+1}$ is imposed. In such a case, $a_{0,m} = (-1)^m$. The analytic solution is

$$f_2(x, y) = \sum_{k=0}^{\infty} (-1)^k (x + y)^k \quad (12)$$

The convergence domain is $D = \{(x, y) : |x + y| < 1\}$ in spite of the solution being clearly $(x, y) \mapsto \frac{1}{x+y+1}$ on $\mathbb{R}^+ \times \mathbb{R}^+$. Even for such a simple matter, various cases exist, depending on boundary conditions.

To summarize, the described method finds analytic solutions to differential equations. But, some solutions can be non-analytic and this investigation is not focused on them. The convergence domain of the series solution can also be more limited than the real solution domain. Moreover, for multidimensional problems, the analytic solution assumption requires that the boundary conditions be analytic as well. This restriction is not very constraining regarding the Navier-Stokes equations because the fluid boundary conditions are usually analytic.

III. APPLICATION TO THE INCOMPRESSIBLE NAVIER-STOKES EQUATIONS

This section concerns the application of the power series method to the incompressible Navier-Stokes equations:

$$\begin{cases} \nabla \cdot \mathbf{U} = 0 \\ \rho \left(\frac{\partial \mathbf{U}}{\partial t} + \mathbf{U} \cdot \nabla \mathbf{U} \right) = -\nabla P + \mu \nabla^2 \mathbf{U} + \rho \mathbf{g} \end{cases} \quad (13)$$

\mathbf{U} is the flow velocity, ρ is the constant flow density, P is the pressure, μ is the constant dynamic viscosity and \mathbf{g} is the sum of body forces. In what follows, the notations U^1 , U^0 and U^2 designate the x , y and z components of the velocity respectively. The notation U^0 for the second component stems from an arbitrary choice in the following development.

A. Mathematical approach

Each variable in the incompressible Navier-Stokes equations can be expressed as 4-variable power series. For instance, the x -velocity can be written:

$$U^1 = \sum_{\omega, p, q, r=0}^{\infty} u_{\omega, p, q, r}^1 t^\omega x^p y^q z^r \quad (14)$$

with the convention $\sum_{\omega, p, q, r=0}^{\infty} = \sum_{\omega=0}^{\infty} \sum_{p=0}^{\infty} \sum_{q=0}^{\infty} \sum_{r=0}^{\infty}$. It applies to U^0 , U^2 and P alike. The convergence domain of the x velocity component corresponds to $D_{U^1} = \{(t, x, y, z) \in \mathbb{R}^4 : \sum_{\omega, p, q, r=0}^{\infty} u_{\omega, p, q, r}^1 t^\omega x^p y^q z^r < \infty\}$. Similarly, D_{U^0} , D_{U^2} and D_P are defined. The convergence domains of the power series are assumed to be non-empty.

The resolution requires the body accelerations to be analytic functions

$$g^i = \sum_{\omega, p, q, r=0}^{\infty} g_{\omega, p, q, r}^i t^\omega x^p y^q z^r \quad (15)$$

with $i = 0, 1$ or 2 corresponding to the direction (such as the velocity). This assumption is not constraining because it is generally true. For instance, gravity often remains constant.

The initial conditions allow the computation of the coefficients when $\omega = 0$. Moreover, the velocity coefficients when at least one indice among p , q and r is worth 0 or 1 are calculated by the six velocity boundary conditions. As regards pressure, only the terms corresponding to $p = 0$ or $q = 0$ or $r = 0$ are derived from the boundary conditions.

Only the main steps of the calculation are described below. The first step consists in replacing variables by power series in the Navier-Stokes equations. First, the continuity equation becomes:

$$\sum_{\omega, p, q, r=0}^{\infty} [(p+1)u_{\omega, p+1, q, r}^1 + (q+1)u_{\omega, p, q+1, r}^0 + (r+1)u_{\omega, p, q, r+1}^2] t^\omega x^p y^q z^r = 0 \quad (16)$$

The coefficients of the power series in (16) are necessarily nil. It leads to:

$$u_{\omega, p, q, r}^0 = -\frac{p+1}{q} u_{\omega, p+1, q-1, r}^1 - \frac{r+1}{q} u_{\omega, p, q-1, r+1}^2 \quad \forall (\omega, p, r) \in \mathbb{N}^3, \forall q > 0 \quad (17)$$

The boundary conditions give the terms corresponding to $q = 0$.

Second, the three momentum equations must be studied. The non linear terms lead to products of series calculated thanks to the Cauchy product:

$$\left(\sum_{i=0}^{\infty} a_i x^i \right) \left(\sum_{j=0}^{\infty} b_j x^j \right) = \sum_{k=0}^{\infty} \left(\sum_{l=0}^k a_l b_{k-l} \right) x^k \quad (18)$$

Only one momentum equation is detailed because the two others are similar. For instance, the x -momentum equation yields

$$\begin{aligned} \frac{1}{p+1} \left\{ (\omega+1)u_{\omega+1, p, q+1, r+1}^1 + \sum_{k=0}^{\omega} \sum_{i=0}^p \sum_{j=0}^{q+1} \sum_{l=0}^{r+1} [(i+1)u_{\omega-k, p-i, q+1-j, r+1-l}^1 u_{k, i+1, j, l}^1 + \right. \\ \left. (j+1)u_{\omega-k, p-i, q+1-j, r+1-l}^1 u_{k, i, j+1, l}^1 + (l+1)u_{\omega-k, p-i, q+1-j, r+1-l}^1 u_{k, i, j, l+1}^1] - \nu [(p+1)(p+2)u_{\omega, p+2, q+1, r+1}^1 + \right. \\ \left. (q+2)(q+3)u_{\omega, p, q+3, r+1}^1 + (r+2)(r+3)u_{\omega, p, q+1, r+3}^1] - g_{\omega, p, q+1, r+1}^1 \right\} = -\frac{1}{\rho} P_{\omega, p+1, q+1, r+1} \quad (19) \end{aligned}$$

with $\omega \geq 0$, $p \geq 0$, $q \geq -1$ and $r \geq -1$.

Because the second component of the velocity is fully determined by (17), the replacement of the pressure coefficients can be carried out thanks to the momentum equation along the y axis. It remains two large equations (20) and (21) combining u^1 , u^0 and u^2 coefficients.

$$\begin{aligned}
u_{\omega+1,p,q,r}^1 = & -\frac{p+1}{q(q-1)} \left[(p+2)u_{\omega+1,p+2,q-2,r}^1 + (r+1)u_{\omega+1,p+1,q-2,r+1}^2 \right] + \frac{\nu}{\omega+1} \left[\frac{(p+2)!}{p!} u_{\omega,p+2,q,r}^1 + \frac{(q+2)!}{q!} u_{\omega,p,q+2,r}^1 + \right. \\
& \left. \frac{(r+2)!}{r!} u_{\omega,p,q,r+2}^1 - \frac{(p+3)!}{q p!} u_{\omega,p+3,q-1,r}^0 - (p+1)(q+1)u_{\omega,p+1,q+1,r}^0 - \frac{(p+1)(r+2)!}{q r!} u_{\omega,p+1,q-1,r+2}^0 \right] - \frac{1}{\omega+1} \sum_{k=0}^{\omega} \\
& \left\{ \sum_{i=0}^p \sum_{j=0}^q \sum_{l=0}^r [(i+1)u_{\omega-k,p-i,q-j,r-l}^1 u_{k,i+1,j,l}^1 + (j+1)u_{\omega-k,p-i,q-j,r-l}^0 u_{k,i,j+1,l}^1 + (l+1)u_{\omega-k,p-i,q-j,r-l}^2 u_{k,i,j,l+1}^1] - \frac{p+1}{q} \right. \\
& \left. \sum_{i=0}^{p+1} \sum_{j=0}^{q-1} \sum_{l=0}^r [(i+1)u_{\omega-k,p+1-i,q-1-j,r-l}^1 u_{k,i+1,j,l}^0 + (j+1)u_{\omega-k,p+1-i,q-1-j,r-l}^0 u_{k,i,j+1,l}^0 + (l+1)u_{\omega-k,p+1-i,q-1-j,r-l}^2 u_{k,i,j,l+1}^0] \right\} \\
& + \frac{1}{\omega+1} \left(g_{\omega,p,q,r}^1 - \frac{p+1}{q} g_{\omega,p+1,q-1,r}^0 \right) \quad (20)
\end{aligned}$$

$$\begin{aligned}
u_{\omega+1,p,q,r}^2 = & -\frac{r+1}{q(q-1)} \left[(r+2)u_{\omega+1,p,q-2,r+2}^2 + (p+1)u_{\omega+1,p+1,q-2,r+1}^1 \right] + \frac{\nu}{\omega+1} \left[\frac{(r+2)!}{r!} u_{\omega,p,q,r+2}^2 + \frac{(q+2)!}{q!} u_{\omega,p,q+2,r}^2 + \right. \\
& \left. \frac{(p+2)!}{p!} u_{\omega,p+2,q,r}^2 - \frac{(r+3)!}{q r!} u_{\omega,p,q-1,r+3}^0 - (r+1)(q+1)u_{\omega,p,q+1,r+1}^0 - \frac{(r+1)(p+2)!}{q p!} u_{\omega,p+2,q-1,r+1}^0 \right] - \frac{1}{\omega+1} \sum_{k=0}^{\omega} \\
& \left\{ \sum_{i=0}^p \sum_{j=0}^q \sum_{l=0}^r [(i+1)u_{\omega-k,p-i,q-j,r-l}^1 u_{k,i+1,j,l}^2 + (j+1)u_{\omega-k,p-i,q-j,r-l}^0 u_{k,i,j+1,l}^2 + (l+1)u_{\omega-k,p-i,q-j,r-l}^2 u_{k,i,j,l+1}^2] - \frac{r+1}{q} \right. \\
& \left. \sum_{i=0}^p \sum_{j=0}^{q-1} \sum_{l=0}^{r+1} [(i+1)u_{\omega-k,p-i,q-1-j,r+1-l}^1 u_{k,i+1,j,l}^0 + (j+1)u_{\omega-k,p-i,q-1-j,r+1-l}^0 u_{k,i,j+1,l}^0 + (l+1)u_{\omega-k,p-i,q-1-j,r+1-l}^2 u_{k,i,j,l+1}^0] \right\} \\
& + \frac{1}{\omega+1} \left(g_{\omega,p,q,r}^2 - \frac{r+1}{q} g_{\omega,p,q-1,r+1}^0 \right) \quad (21)
\end{aligned}$$

(17) constitutes the third relation between the velocity coefficients. These three recurrence relations enable the calculation of the velocity. Then, the pressure coefficients are calculated from the velocities coefficients by a momentum equation such as (19).

B. Analytic solution

As per the section II, the next step consists in writing the coefficients $u_{\omega,p,q,r}^i$ as a function of the boundary coefficients $u_{\omega,p=0}^i$ or $1,q=0$ or $1,r=0$ or 1 and the initial ones $u_{\omega=0,p,q,r}^i$. The resolution of the equations (17), (20) and (21) leads to the following theorem:

Theorem 1: The velocity coefficients can be expressed as

$$\begin{aligned}
u_{\omega,p,q,r}^\alpha = & \sum_{i=0}^{\eta(p,q,r,\alpha,\omega,\omega-1)} \left\{ \left[\prod_{j=1}^{\omega} \prod_{k=1}^{2^{j-1}} \Gamma(\rho_1(i, 1, k, j, p, q, r, \alpha, \omega), \rho_0(i, 1, k, j, p, q, r, \alpha, \omega), \rho_2(i, 1, k, j, p, q, r, \alpha, \omega), \right. \right. \\
& \left. \left. \rho_\alpha(i, 1, k, j, p, q, r, \alpha, \omega), \rho_\omega(i, 1, k, j, p, q, r, \alpha, \omega), \Theta(k-2+2^{j-1}, i, p, q, r, \alpha, \omega, 0)) \right] \right. \\
& \left. \left[\prod_{k=1}^{2^\omega} \mathcal{J}(\aleph_1(i, 1, k, p, q, r, \alpha, \omega), \aleph_0(i, 1, k, p, q, r, \alpha, \omega), \aleph_2(i, 1, k, p, q, r, \alpha, \omega), \aleph_\alpha(i, 1, k, p, q, r, \alpha, \omega), \aleph_\omega(i, 1, k, p, q, r, \alpha, \omega)) \right] \right\} \quad (22)
\end{aligned}$$

with

$$\mathcal{J}(p, q, r, \alpha, \omega) = \begin{cases} 1 & \text{if } p < 0 \text{ or } q < 0 \text{ or } r < 0 \text{ or } \alpha < 0 \text{ or } \omega < 0 \\ u_{\omega,p,q,r}^\alpha & \text{otherwise} \end{cases} \quad (23)$$

The functions \aleph_i are such that the function \mathcal{J} in (22) always returns an initial or boundary coefficient. The pressure coefficients are calculated by (19). Then,

$$\begin{cases} U^\alpha = \sum_{\omega,p,q,r=0}^{\infty} u_{\omega,p,q,r}^\alpha t^\omega x^p y^q z^r \\ P = \sum_{\omega,p,q,r=0}^{\infty} P_{\omega,p,q,r} t^\omega x^p y^q z^r \end{cases} \quad (24)$$

satisfy the incompressible Navier-Stokes equations.

All the functions required to calculate the coefficients are defined in the appendix A. They progressively appear according to order of use in the construction of the functions in (22). The functions \aleph_i , ρ_i and Θ are recursive and no general expressions for these functions have been established because of their complexity. Consequently, they are very time consuming. A personal computer can only manage calculations for $\omega = 1$ and reasonable p, q, r until around 5. Parallel calculations on a powerful computer allow the computation of the coefficients for $\omega = 2$ and reasonable p, q, r as well. Regarding the coefficients with $\omega > 2$, the computational cost is prohibitive.

A check that no miscalculation occurs during the hand calculation has been performed until $\omega = 2$ with the Python code in appendix B. The absence of any error for low indices should substantiate that the relation (22) is correct for higher indices.

Because of the restrictive computational cost, a strategy need be set up to be able to apply this formulation to a physical case. A first option would be to find an expression for each function even if a personal attempt failed because of their apparent erratic behaviour. A second option would be the step by step storage of the functions values for all ω , α , p , q , r . Knowing all the functions values for $\omega < \omega_0$ would simplify the calculation of the functions for $\omega = \omega_0$ because it would only need one step instead of $\omega_0 - 1$ steps. This solution is the most common one to calculate recursive functions. Afterwards, each coefficient would be only a sum of the boundary coefficients. Numerically, it is easily parallelisable and only the sum size could constitute an obstacle.

C. Convergence domain

Despite investigations, no proof regarding the convergence domain has been established. The general expression of $u_{\omega,p,q,r}^{\alpha}$ seems prohibitive to estimate the convergence domain. A simpler formulation appears necessary to manage the theoretical calculation of the convergence domain. Therefore, the existence of an analytic solution cannot be maintained.

However, numerical tests can be undertaken for simple boundary conditions to get an idea of the convergence. Numerical tests on a personal computer show that polynomial boundary conditions yield many zero terms. Moreover, the other terms of the series tend to zero. Unfortunately, the calculation time prevents any conclusion.

IV. CONCLUSION

The investigation concerns the solution of the incompressible Navier-Stokes equations resulting from the application of the power series method. By assuming that the velocity and the pressure are analytic, the problem consists in calculating the power series coefficients. This method only provides an analytic solution if it exists.

The approach is commonly used in one dimension but rarely for multidimensional problems. Thus, it is first applied to an accessible differential equation with the aim of displaying some challenges. In particular, solving multidimensional equations need analytic boundary conditions.

Regarding the incompressible Navier-Stokes equations, an exact power series solution is given. The check can be performed thanks to a Python algorithm provided within the appendix B. Nonetheless, the convergence domain seems not easily computable and it is not investigated in the publication. Therefore, the existence of an analytic solution is not settled.

To complete this study, two major issues should be examined. First, the formulation of the solution is complex and makes the resolution very time-consuming. The recursive functions taking part in the solution must be enhanced to shorten computations. The quick fix could be the storage of each step one and for all. This achievement should contribute to a better understanding of the analytic solution. Second, the convergence domain must be studied theoretically. It should depend on the boundary conditions as well as the initial ones.

If the solution to the general incompressible Navier-Stokes equations is unique, then the solution provided in this publication is the exact solution within its convergence domain.

APPENDIX A

The appendix itemizes the functions necessary in the calculation of the analytic solution (22). Hereinafter, E is the floor function, \mathcal{H} is the Heaviside step function, δ is the Kronecker delta and $\mathcal{M}od(x, y)$ means x modulo y .

$$Ip(x) = \begin{cases} 2 & \text{if } E\left(\frac{x}{2}\right) = \frac{x}{2} \\ 1 & \text{if } E\left(\frac{x}{2}\right) \neq \frac{x}{2} \end{cases} \quad (25)$$

$$Ip_b(j, k) = \begin{cases} 0 & \text{if } j = 0 \\ Ip\left(E\left(\frac{k+1-2^{j-1}}{2^{j-1}}\right)\right) & \text{if } j \neq 0 \end{cases} \quad (26)$$

$$\mathcal{R}(x, y) = x - yE\left(\frac{x}{y}\right) \quad (27)$$

$$\xi(z, p, q, \alpha, n) = \begin{cases} -1 & \text{if } z = 0 \\ p + 2(\alpha - 1) & \text{if } z = 1 \\ p & \text{if } z = 2 \\ p + 2 - \alpha & \text{if } z = 3 \\ p + 1 & \text{if } z = 4 \\ p + (2 - \alpha)\left(2E\left(\frac{q}{2}\right) + 1\right) & \text{if } z = 5 \\ p + (-1)^{(\alpha+1)}2n + 3\alpha - 4 + 2(\alpha - 1)E\left(\frac{q}{2}\right) & \text{if } z = 6 \\ p + (-1)^{(\alpha+1)}2n + 2(\alpha - 1)E\left(\frac{q}{2}\right) & \text{if } z = 7 \\ p + (-1)^{(\alpha+1)}(2n - 1) + 2(\alpha - 1)E\left(\frac{q}{2}\right) & \text{if } z = 8 \end{cases} \quad (28)$$

$$\Delta(z, q) = \begin{cases} -1 & \text{if } z = 0 \\ q & \text{if } z = 1 \\ q + 2 & \text{if } z = 2 \\ q + 1 & \text{if } z = 3 \\ q & \text{if } z = 4 \\ q - 2E\left(\frac{q}{2}\right) + 1 & \text{if } z = 5 \\ q - 2E\left(\frac{q}{2}\right) + 1 & \text{if } z = 6 \\ q - 2E\left(\frac{q}{2}\right) + 2 & \text{if } z = 7 \\ q - 2E\left(\frac{q}{2}\right) + 2 & \text{if } z = 8 \end{cases} \quad (29)$$

$$\epsilon(z, q, r, \alpha, n) = \begin{cases} -1 & \text{if } z = 0 \\ r + 2(2 - \alpha) & \text{if } z = 1 \\ r & \text{if } z = 2 \\ r + \alpha - 1 & \text{if } z = 3 \\ r + 1 & \text{if } z = 4 \\ r + (\alpha - 1)\left(2E\left(\frac{q}{2}\right) + 1\right) & \text{if } z = 5 \\ r + (-1)^\alpha 2n - 3\alpha + 5 + 2(2 - \alpha)E\left(\frac{q}{2}\right) & \text{if } z = 6 \\ r + (-1)^\alpha 2n + 2(2 - \alpha)E\left(\frac{q}{2}\right) & \text{if } z = 7 \\ r + (-1)^\alpha(2n - 1) + 2(2 - \alpha)E\left(\frac{q}{2}\right) & \text{if } z = 8 \end{cases} \quad (30)$$

$$\kappa(z, \alpha) = \begin{cases} -1 & \text{if } z = 0 \\ \alpha & \text{if } z = 1 \\ \alpha & \text{if } z = 2 \\ 0 & \text{if } z = 3 \\ 3 - \alpha & \text{if } z = 4 \\ 0 & \text{if } z = 5 \\ 0 & \text{if } z = 6 \\ \alpha & \text{if } z = 7 \\ 3 - \alpha & \text{if } z = 8 \end{cases} \quad (31)$$

$$s_1(p, q, \alpha, a, n, m, b) = \begin{cases} p + (2 - \alpha) \left(2E\left(\frac{q}{2}\right) + 1 - b - 2n \right) & \text{if } a = 0 \\ p + (-1)^\alpha (2m - b) + (2 - \alpha)(2n + 1 - b) & \text{if } a = 1 \\ p + 2(-1)^\alpha m + (2 - \alpha)(2n + 3 - b) & \text{if } a = 2 \end{cases} \quad (32)$$

$$s_2(q, a, n, b) = \begin{cases} q - \left(2E\left(\frac{q}{2}\right) + 1 - b - 2n \right) & \text{if } a = 0 \\ q - 1 - 2n + b & \text{if } a = 1 \\ q - 2n - 3 + b & \text{if } a = 2 \end{cases} \quad (33)$$

$$s_3(q, r, \alpha, a, n, m, b) = \begin{cases} r + (\alpha - 1) \left(2E\left(\frac{q}{2}\right) + 1 - b - 2n \right) & \text{if } a = 0 \\ r + (-1)^{\alpha+1} (2m - b) + (\alpha - 1)(2n + 1 - b) & \text{if } a = 1 \\ r + 2(-1)^{\alpha+1} m + (\alpha - 1)(2n + 3 - b) & \text{if } a = 2 \end{cases} \quad (34)$$

$$B(d, q) = E \left(\frac{1}{4} \left(1 + E \left(\frac{d}{\frac{3}{2}E\left(\frac{q}{2}\right)^2 - \frac{3}{2}E\left(\frac{q}{2}\right) + 1} \right) \right) \right) \quad (35)$$

$$A(d, q) = \begin{cases} 0 & \text{if } \text{Mod} \left(d, \frac{3}{2}E\left(\frac{q}{2}\right)^2 - \frac{3}{2}E\left(\frac{q}{2}\right) + 1 \right) < \frac{1}{2}E\left(\frac{q}{2}\right) \left(E\left(\frac{q}{2}\right) + 1 \right) \\ 1 & \text{if } \frac{1}{2}E\left(\frac{q}{2}\right) \left(E\left(\frac{q}{2}\right) + 1 \right) - 1 < \text{Mod} \left(d, \frac{3}{2}E\left(\frac{q}{2}\right)^2 - \frac{3}{2}E\left(\frac{q}{2}\right) + 1 \right) < E\left(\frac{q}{2}\right)^2 \\ 2 & \text{if } \text{Mod} \left(d, \frac{3}{2}E\left(\frac{q}{2}\right)^2 - \frac{3}{2}E\left(\frac{q}{2}\right) + 1 \right) > E\left(\frac{q}{2}\right)^2 - 1 \end{cases} \quad (36)$$

$$\varphi(d, q) = E \left(\frac{d}{\frac{3}{2}E\left(\frac{q}{2}\right)^2 - \frac{3}{2}E\left(\frac{q}{2}\right) + 1} \right) - 3E \left(\frac{1}{4} \left(1 + E \left(\frac{d}{\frac{3}{2}E\left(\frac{q}{2}\right)^2 - \frac{3}{2}E\left(\frac{q}{2}\right) + 1} \right) \right) \right) \quad (37)$$

$$\gamma(d, q) = d + 1 - \left(\frac{3}{2}E\left(\frac{q}{2}\right)^2 - \frac{3}{2}E\left(\frac{q}{2}\right) + 1 \right) E \left(\frac{d}{\frac{3}{2}E\left(\frac{q}{2}\right)^2 - \frac{3}{2}E\left(\frac{q}{2}\right) + 1} \right) - \frac{1}{2} \left(A(d, q) E\left(\frac{q}{2}\right)^2 + A(d, q)(2 - A(d, q)) E\left(\frac{q}{2}\right) \right) \quad (38)$$

$$N(d, q) = E \left(\frac{1}{2} \left(1 + \sqrt{8\gamma(d, q) - 7} \right) \right) \quad (39)$$

$$M(d, q) = \gamma(d, q) - \frac{1}{2} \left((N(d, q) - 1)^2 + N(d, q) - 1 \right) \quad (40)$$

$$S(p, q, r, \alpha, d) = (s_1(p, q, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) + 1) (s_2(q, A(d, q), N(d, q), B(d, q)) + 1) (s_3(q, r, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) + 1) \quad (41)$$

$$Y_0(q, a, n, b) = \begin{cases} b - 1 + q - 2E\left(\frac{q}{2}\right) + 2n & \text{if } a = 0 \\ b - 1 + q - 2n & \text{if } a = 1 \\ b + q - 2n - 3 & \text{if } a = 2 \end{cases} \quad (42)$$

$$Y_1(p, q, \alpha, a, n, m, b) = \begin{cases} p + (2 - \alpha)(1 - b) + (2 - \alpha) \left(2E\left(\frac{q}{2}\right) - 2n \right) & \text{if } a = 0 \\ p + (\alpha - 1)(1 - b) + (-1)^\alpha (2m - 1) + 2n(2 - \alpha) & \text{if } a = 1 \\ p + (2 - \alpha)(1 - b) + (-1)^\alpha 2m + (2n + 2)(2 - \alpha) & \text{if } a = 2 \end{cases} \quad (43)$$

$$Y_2(q, r, \alpha, a, n, m, b) = \begin{cases} r + (\alpha - 1)(1 - b) + (\alpha - 1) \left(2E\left(\frac{q}{2}\right) - 2n\right) & \text{if } a = 0 \\ r + (2 - \alpha)(1 - b) + (-1)^{\alpha+1}(2m - 1) + 2n(\alpha - 1) & \text{if } a = 1 \\ r + (\alpha - 1)(1 - b) + (-1)^{\alpha+1}2m + (2n + 2)(\alpha - 1) & \text{if } a = 2 \end{cases} \quad (44)$$

$$H(q, a, n, m, b) = \begin{cases} \delta_{n,m}(-1)^{n+b+E\left(\frac{q}{2}\right)} & \text{if } a = 0 \\ (-1)^{n+b+\frac{1}{2}a(a-1)+\frac{1}{2}(a-1)(a-2)E\left(\frac{q}{2}\right)} \frac{(n-2+a)!}{(m-2+a)!(n-m)!} & \text{if } a > 0 \end{cases} \quad (45)$$

$$L_0(q, r, \alpha, d, i) = E\left(\frac{i-1}{s_3(q, r, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) + 1}\right) \quad (46)$$

$$L_1(q, r, \alpha, d, i) = E\left(\frac{i-1}{(s_3(q, r, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) + 1)(s_2(q, A(d, q), N(d, q), B(d, q)) + 1)}\right) \quad (47)$$

$$G(p, q, r, \alpha, d, i) = \frac{Y_0(q, A(d, q), N(d, q), B(d, q))!Y_1(p, q, \alpha, A(d, q), N(d, q), M(d, q), B(d, q))!Y_2(q, r, \alpha, A(d, q), N(d, q), M(d, q), B(d, q))!}{p!q!r!} \\ \left\{ \frac{1}{2}(\varphi(d, q) - 1)(\varphi(d, q) - 2) [L_0(q, r, \alpha, d, i) - \right. \\ \left. (s_2(q, A(d, q), N(d, q), B(d, q)) + 1)L_1(q, r, \alpha, d, i)] + \varphi(d, q)(2 - \varphi(d, q))L_1(q, r, \alpha, d, i) + \right. \\ \left. \frac{1}{2}\varphi(d, q)(\varphi(d, q) - 1)\text{Mod}(i - 1, s_3(q, r, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) + 1) + 1 \right\} \quad (48)$$

$$W_0(p, q, r, \alpha, d, i) = \frac{1}{2}(\varphi(d, q) - 1)(\varphi(d, q) - 2) + L_0(q, r, \alpha, d, i) - (s_2(q, A(d, q), N(d, q), B(d, q)) + 1)L_1(q, r, \alpha, d, i) \quad (49)$$

$$W_1(p, q, r, \alpha, d, i) = \varphi(d, q)(2 - \varphi(d, q)) + L_1(q, r, \alpha, d, i) \quad (50)$$

$$W_2(p, q, r, \alpha, d, i) = \frac{1}{2}\varphi(d, q)(\varphi(d, q) - 1) + \text{Mod}(i - 1, s_3(q, r, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) + 1) \quad (51)$$

$$V_0(q, r, \alpha, d, i) = Y_0(q, A(d, q), N(d, q), B(d, q)) - L_0(q, r, \alpha, d, i) + (s_2(q, A(d, q), N(d, q), B(d, q)) + 1)L_1(q, r, \alpha, d, i) \quad (52)$$

$$V_1(p, q, r, \alpha, d, i) = Y_1(p, q, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) - L_1(q, r, \alpha, d, i) \quad (53)$$

$$V_2(q, r, \alpha, d, i) = Y_2(q, r, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) - \text{Mod}(i - 1, s_3(q, r, \alpha, A(d, q), N(d, q), M(d, q), B(d, q)) + 1) \quad (54)$$

For the sake of simplicity, the notation X_i with $X_0 = q$, $X_1 = p$ and $X_2 = r$ is adopted.

$$F_0(\omega, p, q, r, \alpha) = \sum_{n=1}^{E\left(\frac{q}{2}\right)} (-1)^{E\left(\frac{q}{2}\right)} \binom{E\left(\frac{q}{2}\right) - 1}{n-1} \left(\frac{(X_\alpha + 2n)!(X_{3-\alpha} + 2E\left(\frac{q}{2}\right) - 2n)!}{p!q!r!} \right. \\ \left. u_{\omega, p+(-1)^{\alpha+1}2n+(\alpha-1)2E\left(\frac{q}{2}\right), q-2E\left(\frac{q}{2}\right), r+(-1)^\alpha 2n+2(2-\alpha)E\left(\frac{q}{2}\right)}^\alpha + \frac{(X_\alpha + 2n - 1)!(X_{3-\alpha} + 2E\left(\frac{q}{2}\right) - 2n + 1)!}{p!q!r!} \right. \\ \left. u_{\omega, p+(-1)^{\alpha+1}(2n-1)+(\alpha-1)2E\left(\frac{q}{2}\right), q-2E\left(\frac{q}{2}\right), r+(-1)^\alpha(2n-1)+2(2-\alpha)E\left(\frac{q}{2}\right)}^{3-\alpha} \right) \quad (55)$$

$$\begin{aligned}
\widetilde{F}_0(\omega, p, q, r, \alpha) &= \frac{1}{\omega} \sum_{n=1}^{E(\frac{q}{2})} (-1)^{n+1} \left(\frac{(q-2n+2)!(X_\alpha+2n-2)!}{X_\alpha!q!} g_{\omega-1, p+(2-\alpha)(2n-2), q-2n+2, r+(\alpha-1)(2n-2)}^\alpha \right. \\
&\quad \left. \frac{(q-2n+1)!(X_\alpha+2n-1)!}{X_\alpha!q!} g_{\omega-1, p+(2-\alpha)(2n-1), q-2n+1, r+(\alpha-1)(2n-1)}^0 \right) + \frac{1}{\omega} \sum_{n=1}^{E(\frac{q}{2})-1} \sum_{m=1}^n (-1)^n \\
&\quad \left(\binom{n-1}{m-1} \frac{(X_{3-\alpha}+2m-1)!(q-2n)!(X_\alpha-2m+2n+1)!}{p!q!r!} g_{\omega-1, p+(-1)^\alpha(2m-1)+2n(2-\alpha), q-2n, r+(-1)^{\alpha+1}(2m-1)+2(\alpha-1)n}^{3-\alpha} \right. \\
&\quad \left. \binom{n}{m} \frac{(X_{3-\alpha}+2m)!(q-1-2n)!(X_\alpha-2m+2n+1)!}{p!q!r!} g_{\omega-1, p+2m(-1)^\alpha+(2-\alpha)(2n+1), q-1-2n, r+2m(-1)^{\alpha+1}+(\alpha-1)(2n+1)}^0 \right) + \\
&\quad \frac{1}{\omega} \sum_{n=1}^{E(\frac{q}{2})-2} \sum_{m=1}^n (-1)^{n+1} \binom{n}{m} \frac{(X_{3-\alpha}+2m)!(q-2-2n)!(X_\alpha-2m+2n+2)!}{p!q!r!} \\
&\quad g_{\omega-1, p+2m(-1)^\alpha+(2-\alpha)(2n+2), q-2-2n, r+2m(-1)^{\alpha+1}+(\alpha-1)(2n+2)}^\alpha \quad (56)
\end{aligned}$$

Notations are required to simplify the following formulae. Three propositions are introduced:

$$P_1 : \text{“}\omega > 0, p > 1, q > 1, r > 1\text{”}$$

$$P_2 : \text{“}\omega < 0 \text{ or } p < 0 \text{ or } q < 0 \text{ or } r < 0\text{”}$$

$$P_3 : \text{“}\omega = 0 \text{ or } 0 \leq p \leq 1 \text{ or } 0 \leq q \leq 1 \text{ or } 0 \leq r \leq 1\text{”}$$

For convenience, the notation $F = F(z, \omega, p, q, r, \alpha, n)$ is used for the next equation,

$$F = \begin{cases} F_0(\omega, p, q, r, \alpha) + \widetilde{F}_0(\omega, p, q, r, \alpha) & \text{if } z = 0 \text{ and } P_1 \text{ is true} \\ u_{\omega, p, q, r}^\alpha & \text{if } z = 0 \text{ and } P_1 \text{ is false} \\ \frac{\nu}{\omega} (X_{3-\alpha} + 1)(X_{3-\alpha} + 2) & \text{if } z = 1 \text{ and } n = 1 \\ \frac{\nu}{\omega} (q + 1)(q + 2) & \text{if } z = 2 \text{ and } n = 1 \\ -\frac{\nu}{\omega} (q + 1)(X_\alpha + 1) & \text{if } z = 3 \text{ and } n = 1 \\ -\frac{\nu}{\omega} (p + 1)(r + 1)\mathcal{H}(q - 4) & \text{if } z = 4 \text{ and } n = 1 \\ \frac{\nu}{\omega} (-1)^{E(\frac{q}{2})} \frac{(X_\alpha + 2E(\frac{q}{2}) + 1)!(q - 2E(\frac{q}{2}) + 1)}{X_\alpha!q!} & \text{if } z = 5 \text{ and } n = 1 \\ \frac{\nu}{\omega} (-1)^{E(\frac{q}{2})} \binom{E(\frac{q}{2})}{n-1} \frac{(X_\alpha + 2n - 1)!(X_{3-\alpha} + 2E(\frac{q}{2}) - 2n + 2)!(1 + q - 2E(\frac{q}{2}))}{p!q!r!} & \text{if } z = 6 \\ 2\frac{\nu}{\omega} (-1)^{E(\frac{q}{2})+1} \binom{E(\frac{q}{2})-1}{n-1} \frac{(X_\alpha + 2n)!(X_{3-\alpha} + 2E(\frac{q}{2}) - 2n)!(1 + 2q - 4E(\frac{q}{2}))}{p!q!r!} & \text{if } z = 7 \\ 2\mathcal{H}(q - 4)\frac{\nu}{\omega} (-1)^{E(\frac{q}{2})+1} \binom{E(\frac{q}{2})-1}{n-1} \frac{(X_\alpha + 2n - 1)!(X_{3-\alpha} + 2E(\frac{q}{2}) - 2n + 1)!(1 + 2q - 4E(\frac{q}{2}))}{p!q!r!} & \text{if } z = 8 \\ 0 & \text{otherwise} \end{cases} \quad (57)$$

$$D(p, q, r, \alpha, \tau) = \sum_{j=1}^{9(\frac{q}{2})^2 - 9E(\frac{q}{2}) + 6} \mathcal{H} \left(\frac{\tau - 1}{\sum_{k=0}^{j-1} S(p, q, r, \alpha, k)} - 1 \right) \quad (58)$$

$$\mathcal{I}(p, q, r, \alpha, \tau) = \tau - \sum_{j=1}^{9E(\frac{q}{2})^2 - 9E(\frac{q}{2}) + 6} S(p, q, r, \alpha, j - 1) \mathcal{H} \left(\frac{\tau - 1}{\sum_{k=0}^{j-1} S(p, q, r, \alpha, k)} - 1 \right) \quad (59)$$

$$\begin{aligned}
\sigma(p, q, r, \alpha) &= \left[2E\left(\frac{q}{2}\right)^4 + (4X_\alpha + 6X_{3-\alpha} + 8)E\left(\frac{q}{2}\right)^3 + (9X_\alpha X_{3-\alpha} + 1.5X_\alpha + 3X_{3-\alpha} - 8)E\left(\frac{q}{2}\right)^2 - \right. \\
&\quad \left. (9X_{3-\alpha} X_\alpha + 5.5X_\alpha + 9X_{3-\alpha} + 2)E\left(\frac{q}{2}\right) + 6X_\alpha X_{3-\alpha} + 6X_\alpha + 9X_{3-\alpha} + 9 \right] q - 3.2E\left(\frac{q}{2}\right)^5 - (6X_\alpha + 8X_{3-\alpha} + 8)E\left(\frac{q}{2}\right)^4 + \\
&\quad (6X_{3-\alpha} + 6X_\alpha - 10X_{3-\alpha} X_\alpha + 24)E\left(\frac{q}{2}\right)^3 + (14X_{3-\alpha} + 9X_\alpha + 22.5X_\alpha X_{3-\alpha} - 7)E\left(\frac{q}{2}\right)^2 - \\
&\quad (9X_\alpha + 12X_{3-\alpha} + 12.5X_\alpha X_{3-\alpha} - 2.2)E\left(\frac{q}{2}\right) + 3(X_{3-\alpha} + 1)(X_\alpha + 1) - 8E\left(\frac{q}{2}\right) \quad (60)
\end{aligned}$$

$$\mathcal{T}(p, q, r, \alpha, j) = j + 1 - \sigma(p, q, r, \alpha)E\left(\frac{j}{\sigma(p, q, r, \alpha)}\right) \quad (61)$$

$$\chi(\omega, p, q, r, \alpha, i) = i - \sigma(p, q, r, \alpha)\omega \quad (62)$$

$$\mathcal{N}(\omega, p, q, r, \alpha, i) = \begin{cases} 1 & \text{if } \chi(\omega, p, q, r, \alpha, i) = 0 \\ E\left(\frac{\chi(\omega, p, q, r, \alpha, i) - 1}{8}\right) + 1 & \text{otherwise} \end{cases} \quad (63)$$

$$\mathcal{F}(p, q, r, \alpha, \omega) = \begin{cases} 0 & \text{if } p < 2 \text{ or } q < 2 \text{ or } r < 2 \text{ or } \omega \leq 0 \\ 1 & \text{if } a = 0 \text{ and } q = 2 \\ \sigma(p, q, r, \alpha)\omega + 8E\left(\frac{q}{2}\right) & \text{if } a \neq 0, \omega > 0 \text{ and } p, q, r > 1 \\ \sigma(p+1, q-1, r, 1)\omega + \sigma(p, q-1, r+1, 2)\omega + 16E\left(\frac{q-1}{2}\right) + 1 & \text{otherwise} \end{cases} \quad (64)$$

Two propositions need be introduced afresh:

$$R_1 : \text{"}i < \sigma(p, q, r, \alpha)\omega\text{"} \quad (65)$$

$$R_2 : \text{"}i \leq \mathcal{F}(p+1, q-1, r, 1, \omega)\text{"} \quad (66)$$

To minimize the size of the following formulae, "elif" will mean "else if" (same notation as the Python language).

$$T_0(\omega, p, q, r, \alpha, i) = \begin{cases} -1 & \text{if } P_2 \text{ is true} \\ q & \text{if } P_3 \text{ is true and } P_2 \text{ is false} \\ W_0(p, q, r, \alpha, D(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i)), \mathcal{I}(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i))) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is true} \\ \Delta(\chi(\omega, p, q, r, \alpha, i) - 8(\mathcal{N}(\omega, p, q, r, \alpha, i) - 1), q) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is false} \\ T_0(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ T_0(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (67)$$

$$T_1(\omega, p, q, r, \alpha, i) = \begin{cases} -1 & \text{if } P_2 \text{ is true} \\ p & \text{if } P_3 \text{ is true and } P_2 \text{ is false} \\ W_1(p, q, r, \alpha, D(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i)), \mathcal{I}(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i))) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is true} \\ \xi(\chi(\omega, p, q, r, \alpha, i) - 8(\mathcal{N}(\omega, p, q, r, \alpha, i) - 1), p, q, \alpha, \mathcal{N}(\omega, p, q, r, \alpha, i)) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is false} \\ T_1(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ T_1(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (68)$$

$$T_2(\omega, p, q, r, \alpha, i) = \begin{cases} -1 & \text{if } P_2 \text{ is true} \\ r & \text{if } P_3 \text{ is true and } P_2 \text{ is false} \\ W_2(p, q, r, \alpha, D(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i)), \mathcal{I}(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i))) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is true} \\ \epsilon(\chi(\omega, p, q, r, \alpha, i) - 8(\mathcal{N}(\omega, p, q, r, \alpha, i) - 1), q, r, \alpha, \mathcal{N}(\omega, p, q, r, \alpha, i)) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is false} \\ T_2(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ T_2(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (69)$$

$$\beta(\omega, p, q, r, \alpha, i) = \begin{cases} \alpha & \text{if } P_2 \vee P_3 \text{ is true} \\ B(d, q)(\alpha + A(d, q)(A(d, q) - 2)(2\alpha - 3)) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is true} \\ \kappa(\chi(\omega, p, q, r, \alpha, i) - 8(\mathcal{N}(\omega, p, q, r, \alpha, i) - 1), \alpha) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is false} \\ \beta(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ \beta(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (70)$$

$$\mathcal{E}(\omega, p, q, r, \alpha, i) = \begin{cases} -1 & \text{if } P_2 \text{ is true} \\ \omega & \text{if } P_3 \text{ is true and } P_2 \text{ is false} \\ E\left(\frac{i}{\sigma(p, q, r, \alpha)}\right) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } E\left(\frac{i}{\sigma(p, q, r, \alpha)}\right) < \omega \\ \omega - 1 & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } \omega \leq E\left(\frac{i}{\sigma(p, q, r, \alpha)}\right) \\ \mathcal{E}(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ \mathcal{E}(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (71)$$

$$Z_0(\omega, p, q, r, \alpha, i) = \begin{cases} -1 & \text{if } P_2 \vee P_3 \text{ is true} \\ V_0(q, r, \alpha, D(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i)), \mathcal{I}(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i))) & \text{elif } \alpha = 1 \text{ or } 2 \\ Z_0(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ Z_0(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (72)$$

$$Z_1(\omega, p, q, r, \alpha, i) = \begin{cases} -1 & \text{if } P_2 \vee P_3 \text{ is true} \\ V_1(p, q, r, \alpha, D(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i)), \mathcal{I}(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i))) & \text{elif } \alpha = 1 \text{ or } 2 \\ Z_1(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ Z_1(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (73)$$

$$Z_2(\omega, p, q, r, \alpha, i) = \begin{cases} -1 & \text{if } P_2 \vee P_3 \text{ is true} \\ V_2(p, q, r, \alpha, D(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i)), \mathcal{I}(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i))) & \text{elif } \alpha = 1 \text{ or } 2 \\ Z_2(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ Z_2(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (74)$$

$$\Phi(\omega, p, q, r, \alpha, i) = \begin{cases} \alpha & \text{if } P_2 \vee P_3 \text{ is true} \\ \varphi(D(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i)), q) & \text{elif } \alpha = 1 \text{ or } 2 \\ \Phi(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ \Phi(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (75)$$

$$\Upsilon(\omega, p, q, r, \alpha, i) = \begin{cases} -1 & \text{if } P_2 \vee P_3 \text{ is true} \\ \omega - 1 - E\left(\frac{i}{\sigma(p, q, r, \alpha)}\right) & \text{elif } \alpha = 1 \text{ or } 2 \\ \Upsilon(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ \Upsilon(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{otherwise} \end{cases} \quad (76)$$

$$\Gamma(\omega, p, q, r, \alpha, i) = \begin{cases} 1 & \text{if } P_3 \text{ is true} \\ \frac{1}{\omega} G(p, q, r, \alpha, D(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i)), \mathcal{I}(p, q, r, \alpha, \mathcal{T}(p, q, r, \alpha, i))) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is true} \\ \frac{1}{\omega} (\chi(\omega, p, q, r, \alpha, i) - 8(\mathcal{N}(\omega, p, q, r, \alpha, i) - 1), \omega, p, q, r, \alpha, \mathcal{N}(\omega, p, q, r, \alpha, i)) & \text{elif } (\alpha = 1 \text{ or } 2) \text{ and } R_1 \text{ is false} \\ -\frac{p+1}{q} \Gamma(\omega, p+1, q-1, r, 1, i) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is true} \\ -\frac{r+1}{q} \Gamma(\omega, p, q-1, r+1, 2, i - \mathcal{F}(p+1, q-1, r, 1, \omega) - 1) & \text{elif } \alpha = 0 \text{ and } R_2 \text{ is false} \end{cases} \quad (77)$$

$$\eta(p, q, r, \alpha, \omega, ind) = \begin{cases} 0 & \text{if } P_2 \vee P_3 \text{ is true} \\ 1 & \text{if } \alpha = 0 \text{ and } q = 2 \\ \mathcal{F}(p, q, r, \alpha, \omega) & \text{if } P_2 \vee P_3 \text{ is false and } ind = 0 \\ \mathcal{F}(p, q, r, \alpha, \omega) \sum_{k=0}^{\omega} \{[\eta(T_1(\omega, p, q, r, \alpha, k), T_0(\omega, p, q, r, \alpha, k), T_2(\omega, p, q, r, \alpha, k), \\ \beta(\omega, p, q, r, \alpha, k), \mathcal{E}(\omega, p, q, r, \alpha, k), ind - 1) + 1] \\ [\eta(Z_1(\omega, p, q, r, \alpha, k), Z_0(\omega, p, q, r, \alpha, k), Z_2(\omega, p, q, r, \alpha, k), \\ \Phi(\omega, p, q, r, \alpha, k), \Upsilon(\omega, p, q, r, \alpha, k), ind - 1) + 1]\} - 1 & \text{otherwise} \end{cases} \quad (78)$$

$$\tilde{\eta}_1(p, q, r, \alpha, \omega, k, ind) = \begin{cases} \mathcal{F}(T_1(\omega, p, q, r, \alpha, k), T_0(\omega, p, q, r, \alpha, k), T_2(\omega, p, q, r, \alpha, k), \\ \beta(\omega, p, q, r, \alpha, k), \mathcal{E}(\omega, p, q, r, \alpha, k)) & \text{if } ind = 0 \\ \eta(T_1(\omega, p, q, r, \alpha, k), T_0(\omega, p, q, r, \alpha, k), T_2(\omega, p, q, r, \alpha, k), \\ \beta(\omega, p, q, r, \alpha, k), \mathcal{E}(\omega, p, q, r, \alpha, k), ind) & \text{if } ind > 0 \end{cases} \quad (79)$$

$$\tilde{\eta}_2(p, q, r, \alpha, \omega, k, ind) = \begin{cases} \mathcal{F}(Z_1(\omega, p, q, r, \alpha, k), Z_0(\omega, p, q, r, \alpha, k), Z_2(\omega, p, q, r, \alpha, k), \\ \Phi(\omega, p, q, r, \alpha, k), \Upsilon(\omega, p, q, r, \alpha, k)) & \text{if } ind = 0 \\ \eta(Z_1(\omega, p, q, r, \alpha, k), Z_0(\omega, p, q, r, \alpha, k), Z_2(\omega, p, q, r, \alpha, k), \\ \Phi(\omega, p, q, r, \alpha, k), \Upsilon(\omega, p, q, r, \alpha, k), ind) & \text{if } ind > 0 \end{cases} \quad (80)$$

$$\mathcal{S}\tilde{\eta}(j, p, q, r, \alpha, \omega, ind) = \sum_{k=0}^j [\tilde{\eta}_1(p, q, r, \alpha, \omega, k, ind) + 1][\tilde{\eta}_2(p, q, r, \alpha, \omega, k, ind) + 1] \quad (81)$$

$$\theta(i, p, q, r, \alpha, \omega, n, ind) = \begin{cases} i & \text{if } ind = 0 \\ j \text{ such as } i \in [\mathcal{S}\tilde{\eta}(j-1, p, q, r, \alpha, \omega, ind-1); \mathcal{S}\tilde{\eta}(j, p, q, r, \alpha, \omega, ind-1)] & \text{if } ind > 0 \text{ and } n = 0 \\ E\left(\frac{i - \mathcal{S}\tilde{\eta}(\theta(i, p, q, r, \alpha, \omega, 0, ind), p, q, r, \alpha, \omega, ind-1)}{\tilde{\eta}_2(p, q, r, \alpha, \omega, \theta(i, p, q, r, \alpha, \omega, 0, ind), ind-1) + 1}\right) & \text{if } ind > 0 \text{ and } n = 1 \\ \mathcal{R}(i - \mathcal{S}\tilde{\eta}(\theta(i, p, q, r, \alpha, \omega, 0, ind), p, q, r, \alpha, \omega, ind-1), \\ \tilde{\eta}_2(p, q, r, \alpha, \omega, \theta(i, p, q, r, \alpha, \omega, 0, ind), ind-1) + 1) & \text{if } ind > 0 \text{ and } n = 2 \end{cases} \quad (82)$$

$$C_0(m, h, l, i, p, q, r, \alpha, \omega) = \begin{cases} q & \text{if } E\left(\frac{\ln(m+1)}{\ln(2)}\right) - h < 1 \text{ or } l > E\left(\frac{\ln(m+1)}{\ln(2)}\right) - 1 - h \\ T_0(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & = 1 \\ Z_0(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & \neq 1 \end{cases} \quad (83)$$

$$C_1(m, h, l, i, p, q, r, \alpha, \omega) = \begin{cases} p & \text{if } E\left(\frac{\ln(m+1)}{\ln(2)}\right) - h < 1 \text{ or } l > E\left(\frac{\ln(m+1)}{\ln(2)}\right) - 1 - h \\ T_1(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & = 1 \\ Z_1(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & \neq 1 \end{cases} \quad (84)$$

$$C_2(m, h, l, i, p, q, r, \alpha, \omega) = \begin{cases} r & \text{if } E\left(\frac{\ln(m+1)}{\ln(2)}\right) - h < 1 \text{ or } l > E\left(\frac{\ln(m+1)}{\ln(2)}\right) - 1 - h \\ T_2(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & = 1 \\ Z_2(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & \neq 1 \end{cases} \quad (85)$$

$$C_\alpha(m, h, l, i, p, q, r, \alpha, \omega) = \begin{cases} \alpha & \text{if } E\left(\frac{\ln(m+1)}{\ln(2)}\right) - h < 1 \text{ or } l > E\left(\frac{\ln(m+1)}{\ln(2)}\right) - 1 - h \\ \beta(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & = 1 \\ \Phi(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & \neq 1 \end{cases} \quad (86)$$

$$C_\omega(m, h, l, i, p, q, r, \alpha, \omega) = \begin{cases} \omega & \text{if } E\left(\frac{\ln(m+1)}{\ln(2)}\right) - h < 1 \text{ or } l > E\left(\frac{\ln(m+1)}{\ln(2)}\right) - 1 - h \\ \mathcal{E}(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & = 1 \\ \Upsilon(C_\omega(m, h, l+1, i, p, q, r, \alpha, \omega), C_1(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_0(m, h, l+1, i, p, q, r, \alpha, \omega), C_2(m, h, l+1, i, p, q, r, \alpha, \omega), \\ C_\alpha(m, h, l+1, i, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{m+1-2^{l+1+h}}{2^{l+1+h}}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{m+1-2^{l+h}}{2^{l+h}}\right)\right) \\ & \neq 1 \end{cases} \quad (87)$$

$$\Theta(m, i, p, q, r, \alpha, \omega, h) = \begin{cases} \theta(i, p, q, r, \alpha, \omega, Ip_b(h, m), \omega - 1) & \text{if } h = E\left(\frac{\ln(m+1)}{\ln(2)}\right) \\ \theta(\Theta(m, i, p, q, r, \alpha, \omega, h+1), C_1(m, h, 0, i, p, q, r, \alpha, \omega), \\ C_0(m, h, 0, i, p, q, r, \alpha, \omega), C_2(m, h, 0, i, p, q, r, \alpha, \omega), C_\alpha(m, h, 0, i, p, q, r, \alpha, \omega), \\ C_\omega(m, h, 0, i, p, q, r, \alpha, \omega), Ip_b(h, m), \omega - 1 + h - E\left(\frac{\ln(m+1)}{\ln(2)}\right)) & \text{otherwise} \end{cases} \quad (88)$$

$$\rho_0(i, l, k, j, p, q, r, \alpha, \omega) = \begin{cases} q & \text{if } j < 2 \\ T_0(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } l < j-1 \\ Z_0(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & \neq 1 \text{ and } l < j-1 \\ T_0(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j-1 \leq l \\ Z_0(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j-1 \leq l \end{cases} \quad (89)$$

$$\rho_1(i, l, k, j, p, q, r, \alpha, \omega) = \begin{cases} p & \text{if } j < 2 \\ T_1(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } l < j-1 \\ Z_1(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & \neq 1 \text{ and } l < j-1 \\ T_1(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j-1 \leq l \\ Z_1(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j-1 \leq l \end{cases} \quad (90)$$

$$\rho_2(i, l, k, j, p, q, r, \alpha, \omega) = \begin{cases} r & \text{if } j < 2 \\ T_2(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } l < j - 1 \\ Z_2(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & \neq 1 \text{ and } l < j - 1 \\ T_2(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j - 1 \leq l \\ Z_2(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j - 1 \leq l \end{cases} \quad (91)$$

$$\rho_\alpha(i, l, k, j, p, q, r, \alpha, \omega) = \begin{cases} \alpha & \text{if } j < 2 \\ \beta(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } l < j - 1 \\ \Phi(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & \neq 1 \text{ and } l < j - 1 \\ \beta(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j - 1 \leq l \\ \Phi(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j - 1 \leq l \end{cases} \quad (92)$$

$$\rho_\omega(i, l, k, j, p, q, r, \alpha, \omega) = \begin{cases} \omega & \text{if } j < 2 \\ \mathcal{E}(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } l < j-1 \\ \Upsilon(\rho_\omega(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_1(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_0(i, l+1, k, j, p, q, r, \alpha, \omega), \rho_2(i, l+1, k, j, p, q, r, \alpha, \omega), \\ \rho_\alpha(i, l+1, k, j, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & \neq 1 \text{ and } l < j-1 \\ \mathcal{E}(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j-1 \leq l \\ \Upsilon(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1+2^{j-1}-2^l}{2^l}\right), i, p, q, r, \alpha, \omega, 0\right)) & \text{elif } Ip\left(E\left(\frac{k-1+2^{j-1}}{2^{l-1}} - 1\right)\right) \\ & = 1 \text{ and } j-1 \leq l \end{cases} \quad (93)$$

$$\aleph_0(i, l, k, p, q, r, \alpha, \omega) = \begin{cases} T_0(\aleph_\omega(i, l+1, k, p, q, r, \alpha, \omega), \aleph_1(i, l+1, k, p, q, r, \alpha, \omega), \\ \aleph_0(i, l+1, k, p, q, r, \alpha, \omega), \aleph_2(i, l+1, k, p, q, r, \alpha, \omega), \\ \aleph_\alpha(i, l+1, k, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1}{2^l}\right) - 1 + 2^{\omega-1}, i, p, q, r, \alpha, \omega, 0\right)) & \text{if } Ip\left(E\left(\frac{k-1}{2^{l-1}} + 1\right)\right) \\ & = 1 \text{ and } l < \omega \\ Z_0(\aleph_\omega(i, l+1, k, p, q, r, \alpha, \omega), \aleph_1(i, l+1, k, p, q, r, \alpha, \omega), \\ \aleph_0(i, l+1, k, p, q, r, \alpha, \omega), \aleph_2(i, l+1, k, p, q, r, \alpha, \omega), \\ \aleph_\alpha(i, l+1, k, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1}{2^l}\right) - 1 + 2^{\omega-1}, i, p, q, r, \alpha, \omega, 0\right)) & \text{if } Ip\left(E\left(\frac{k-1}{2^{l-1}} + 1\right)\right) \\ & \neq 1 \text{ and } l < \omega \\ T_0(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1}{2^l}\right) - 1 + 2^{\omega-1}, i, p, q, r, \alpha, \omega, 0\right)) & \text{if } Ip\left(E\left(\frac{k-1}{2^{l-1}} + 1\right)\right) \\ & = 1 \text{ and } \omega \leq l \\ Z_0(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1}{2^l}\right) - 1 + 2^{\omega-1}, i, p, q, r, \alpha, \omega, 0\right)) & \text{if } Ip\left(E\left(\frac{k-1}{2^{l-1}} + 1\right)\right) \\ & \neq 1 \text{ and } \omega \leq l \end{cases} \quad (94)$$

$$\aleph_1(i, l, k, p, q, r, \alpha, \omega) = \begin{cases} T_1(\aleph_\omega(i, l+1, k, p, q, r, \alpha, \omega), \aleph_1(i, l+1, k, p, q, r, \alpha, \omega), \\ \aleph_0(i, l+1, k, p, q, r, \alpha, \omega), \aleph_2(i, l+1, k, p, q, r, \alpha, \omega), \\ \aleph_\alpha(i, l+1, k, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1}{2^l}\right) - 1 + 2^{\omega-1}, i, p, q, r, \alpha, \omega, 0\right)) & \text{if } Ip\left(E\left(\frac{k-1}{2^{l-1}} + 1\right)\right) \\ & = 1 \text{ and } l < \omega \\ Z_1(\aleph_\omega(i, l+1, k, p, q, r, \alpha, \omega), \aleph_1(i, l+1, k, p, q, r, \alpha, \omega), \\ \aleph_0(i, l+1, k, p, q, r, \alpha, \omega), \aleph_2(i, l+1, k, p, q, r, \alpha, \omega), \\ \aleph_\alpha(i, l+1, k, p, q, r, \alpha, \omega), \Theta\left(E\left(\frac{k-1}{2^l}\right) - 1 + 2^{\omega-1}, i, p, q, r, \alpha, \omega, 0\right)) & \text{if } Ip\left(E\left(\frac{k-1}{2^{l-1}} + 1\right)\right) \\ & \neq 1 \text{ and } l < \omega \\ T_1(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1}{2^l}\right) - 1 + 2^{\omega-1}, i, p, q, r, \alpha, \omega, 0\right)) & \text{if } Ip\left(E\left(\frac{k-1}{2^{l-1}} + 1\right)\right) \\ & = 1 \text{ and } \omega \leq l \\ Z_1(\omega, p, q, r, \alpha, \Theta\left(E\left(\frac{k-1}{2^l}\right) - 1 + 2^{\omega-1}, i, p, q, r, \alpha, \omega, 0\right)) & \text{if } Ip\left(E\left(\frac{k-1}{2^{l-1}} + 1\right)\right) \\ & \neq 1 \text{ and } \omega \leq l \end{cases} \quad (95)$$

APPENDIX B

A Python code comparing the exact analytic solution (22) with (17), (20) and (21) is supplied. The algorithm should yield 0. By replacing all dollar signs \$ by a tabulation, the code can be directly copied and run.

```
from __future__ import division
import numpy as np
from numpy import *
from multiprocessing import Pool
import random
$
##### USER PARAMETERS #####
sequential=True # False for parallelized python code - True for sequential calculation
formal=False # False to use random values - True to perform a formal calculation (sympy is required)
nb_proc=0 # Processors number minus 1 (nb_proc=0 for sequential calculation)
nu_value=1e-5 # m^2/s - Kinematic viscosity - Pointless for a formal calculation
w=1 # Same indices as in the article u^a_(w,p,q,r)
p=2
q=4
r=2
a=1
#####
$
if formal==True:
$import sympy
$from sympy import *
$listu=symbols("u_0:20_0:20_0:20_0:20")
$listv=symbols("v_0:20_0:20_0:20_0:20")
$listw=symbols("w_0:20_0:20_0:20_0:20")
$listgu=symbols("gu_0:20_0:20_0:20_0:20")
$listgv=symbols("gv_0:20_0:20_0:20_0:20")
$listgw=symbols("gw_0:20_0:20_0:20_0:20")
$nu=symbols("nu")
else:
$nu=nu_value
$
# Definition of the random/formal boundary conditions
N=20
u=np.zeros((N,3,N,N,N))
g=np.zeros((N,3,N,N,N))
$
for k in range(0,N):
$for i in range(0,N):
$$for j in range(0,N):
$$$for l in range(0,N):
$$$$if formal==True:
$$$$$g[k,0,i,j,1],g[k,1,i,j,1],g[k,2,i,j,1]=listgv[1+10*j+100*i+1000*k],listgu[1+10*j+100*i+1000*k],listgw[1+10*j+100*i+1000*k]
$$$$$u[k,0,i,j,1],u[k,1,i,j,1],u[k,2,i,j,1]=listv[1+10*j+100*i+1000*k],listu[1+10*j+100*i+1000*k],listw[1+10*j+100*i+1000*k]
$$$$else:
$$$$$u[k,0,i,j,1],u[k,1,i,j,1],u[k,2,i,j,1]=random.uniform(-1,1),random.uniform(-1,1),random.uniform(-1,1)
$$$$$g[k,0,i,j,1],g[k,1,i,j,1],g[k,2,i,j,1]=random.uniform(-1,1),random.uniform(-1,1),random.uniform(-1,1)
$
def fact(n): # Factorial
$res=1
$for i in range(2,int(n)+1):
$$res=res*i
$return res
$
def Heaviside(x): # Heaviside step function
$if x<0:
$$return 0
$else:
$$return 1
$
def kronecker(a,b): # Kronecker delta
$return 1 if abs(a-b)<1e-7 else 0
$
# Definition of the functions of the appendix A in the same order
def Ip(x):
$return 2 if int(x/2)==x/2 else 1
$
def Ip_b(j,k):
$return 0 if j==0 else Ip(int((k+1-2**(j-1))/(2**(j-1))))
$
def R(x,y):
$return x-y*int(x/y)
$
def ksi(z,p,q,a,n):
$if z==0:
$$return -1
$elif z==1:
$$result=p+2*(a-1)
$elif z==2:
$$result=p
$elif z==3:
$$result=p+2-a
$elif z==4:
$$result=p+1
$elif z==5:
$$result=p+(2-a)*(2*int(q/2)+1)
$elif z==6:
$$result=p+(-1)*(a+1)*2*n+3*a-4+2*(a-1)*int(q/2)
$elif z==7:
$$result=p+(-1)*(a+1)*2*n+(a-1)*2*int(q/2)
$elif z==8:
$$result=p+(-1)*(a+1)*(2*n-1)+(a-1)*2*int(q/2)
$return result
$
def Delta(z,q):
$if z==0:
$$return -1
$elif z==1 or z==4:
$$result=q
$elif z==2:
$$result=q+2
$elif z==3:
$$result=q+1
$elif z==5 or z==6:
$$result=q-2*int(q/2)+1
$elif z==7 or z==8:
$$result=q-2*int(q/2)+2
$return result
$
```

```

def epsilon(z,q,r,a,n):
$if z==0:
$$return -1
$elif z==1:
$$result=r+2*(2-a)
$elif z==2:
$$result=r
$elif z==3:
$$result=r+a-1
$elif z==4:
$$result=r+1
$elif z==5:
$$result=r+(a-1)*(2*int(q/2)+1)
$elif z==6:
$$result=r+(-1)**a*2*n+5-3*a+(2-a)*2*int(q/2)
$elif z==7:
$$result=r+(-1)**a*2*n+(2-a)*2*int(q/2)
$elif z==8:
$$result=r+(-1)**a*(2*n-1)+2*(2-a)*int(q/2)
$return result
$
def kappa(z,a):
$if z==0:
$$return -1
$elif z==1 or z==2 or z==7:
$$return a
$elif z==3 or z==5 or z==6:
$$return 0
$elif z==4 or z==8:
$$return 3-a
$
def s1(p,q,a,a_value,n,m,b):
$return p+(2-a)*(2*int(q/2)+1-b-2*n) if a_value==0 else (p+(-1)**a*(2*m-b)+(2-a)*(2*n+1-b) if a_value==1 else p+(-1)**a*2*m+(2-a)*(2*n+3-b))
$
def s2(q,a_value,n,b):
$return q-(2*int(q/2)+1-b-2*n) if a_value==0 else (q-1-2*n+b if a_value==1 else q-3-2*n+b)
$
def s3(q,r,a,a_value,n,m,b):
$return r+(a-1)*(2*int(q/2)+1-b-2*n) if a_value==0 else (r+(-1)**(a+1)*(2*m-b)+(a-1)*(2*n+1-b) if a_value==1 else r+(-1)**(a+1)*2*m+(a-1)*(2*n+3-b))
$
def B(d,q):
$return int(0.25*(1+int(d/(1+1.5*int(q/2))+2-1.5*int(q/2))))
$
def A(d,q):
$m=1.5*int(q/2)+2-1.5*int(q/2)+1
$delta=d % m
$return 0 if delta<(int(q/2)+2*int(q/2))/2 else (1 if (delta>(int(q/2)+2*int(q/2))/2-1 and delta<int(q/2)+2) else 2)
$
def phi(d,q):
$return int(d/(1+1.5*int(q/2)+2-1.5*int(q/2)))-3*int(0.25*(1+int(d/(1+1.5*int(q/2))+2-1.5*int(q/2))))
$
def gamma(d,q):
$a_value=A(d,q)
$return d+1-(1.5*int(q/2)+2-1.5*int(q/2)+1)*int(d/(1.5*int(q/2)+2-1.5*int(q/2)+1))-0.5*(a_value*int(q/2)+a_value*(2-a_value)*int(q/2))
$
def N(d,q):
$return int((1+(8*gamma(d,q)-7)**0.5)/2)
$
def M(d,q):
$n_value=N(d,q)
$return gamma(d,q)-0.5*((n_value-1)**2+n_value-1)
$
def S(p,q,r,a,d):
$a_value, n_value, m_value, b_value =A(d,q),N(d,q),M(d,q),B(d,q)
$return (s1(p,q,a,a_value,n_value,m_value,b_value)+1)*(s2(q,a_value,n_value,b_value)+1)*(s3(q,r,a,a_value,n_value,m_value,b_value)+1)
$
def Y0(q,a_value,n,b):
$return b-1+q-2*int(q/2)+2*n$if a_value==0 else (b-1+q-2*n if a_value==1 else b+q-2*n-3)
$
def Y1(p,q,a,a_value,n,m,b):
$return p+(2-a)*(1-b)+(2-a)*(2*int(q/2)-2*n) if a_value==0 else (p+(a-1)*(1-b)+(-1)**a*(2*m-1)+2*n*(2-a) if a_value==1 else p+(2-a)*(1-b)+(2-a)*(2+2*n)+(-1)**a*2*m)
$
def Y2(q,r,a,a_value,n,m,b):
$return r+(a-1)*(1-b)+(a-1)*(2*int(q/2)-2*n) if a_value==0 else (r+(2-a)*(1-b)+(-1)**(a+1)*(2*m-1)+2*n*(a-1) if a_value==1 else r+(a-1)*(1-b)+(-1)**(a+1)*2*m+(a-1)*(2*n+2))
$
def H(q,a_value,n,m,b):
$return kronecker(n,m)*(-1)**(n+b*int(q/2)) if a_value==0 else (-1)**(n+b+0.5*a_value*(a_value-1)+0.5*(a_value-1)*(a_value-2)*int(q/2))*fact(n-2+a_value)/(fact(m-2+a_value)
$*fact(n-m))
$
def L0(q,r,a,d,i):
$return int((i-1)/(s3(q,r,a,A(d,q),N(d,q),M(d,q),B(d,q))+1))
$
def L1(q,r,a,d,i):
$return int((i-1)/((s3(q,r,a,A(d,q),N(d,q),M(d,q),B(d,q))+1)*(s2(q,A(d,q),N(d,q),B(d,q))+1)))
$
def G(p,q,r,a,d,i):
$a_value, n_value, m_value, b_value =A(d,q),N(d,q),M(d,q),B(d,q)
$mod=(i-1) % (s3(q,r,a,a_value,n_value,m_value,b_value)+1)
$return H(q,a_value,n_value,m_value,b_value)*fact(Y0(q,a_value,n_value,b_value))*fact(Y1(p,q,a,a_value,n_value,m_value,b_value))*fact(Y2(q,r,a,a_value,n_value,m_value,
b_value))
$*(0.5*(phi(d,q)-1)*(phi(d,q)-2)*(L0(q,r,a,d,i)-L1(q,r,a,d,i)*(s2(q,a_value,n_value,b_value)+1))+phi(d,q)*(2-phi(d,q))*L1(q,r,a,d,i)+0.5*phi(d,q)*(phi(d,q)-1)*mod+1)/\
$fact(p)*fact(q)*fact(r)
$
def W0(p,q,r,a,d,i):
$return 0.5*(phi(d,q)-1)*(phi(d,q)-2)+L0(q,r,a,d,i)-L1(q,r,a,d,i)*(s2(q,A(d,q),N(d,q),B(d,q))+1)
$
def W1(p,q,r,a,d,i):
$return phi(d,q)*(2-phi(d,q))+L1(q,r,a,d,i)
$
def W2(p,q,r,a,d,i):
$mod=(i-1) % (s3(q,r,a,A(d,q),N(d,q),M(d,q),B(d,q))+1)
$return 0.5*phi(d,q)*(phi(d,q)-1)*mod
$
def V0(q,r,a,d,i):
$a_value, n_value, m_value, b_value =A(d,q),N(d,q),M(d,q),B(d,q)
$return Y0(q,a_value,n_value,b_value)-L0(q,r,a,d,i)+(s2(q,a_value,n_value,b_value)+1)*L1(q,r,a,d,i)
$
def V1(p,q,r,a,d,i):
$return Y1(p,q,a,A(d,q),N(d,q),M(d,q),B(d,q))-L1(q,r,a,d,i)
$
def V2(q,r,a,d,i):
$a_value, n_value, m_value, b_value =A(d,q),N(d,q),M(d,q),B(d,q)
$mod=(i-1) % (s3(q,r,a,a_value,n_value,m_value,b_value)+1)
$return Y2(q,r,a,a_value,n_value,m_value,b_value)-mod

```

```

$
def F0(w,p,q,r,a):
$result=0
$for n in range(1,int(q/2)+1):
$if a==1:
$$result+(-1)**int(q/2)*fact(int(q/2)-1)/(fact(n-1)*fact(int(q/2)-n))*(fact(p+2*n)*fact(r+2*int(q/2)-2*n)*u[w,a,p+(-1)**(a+1)*2*n+(a-1)*2*int(q/2),q-2*int(q/2),\
$$r+(-1)**(a+1)*2*n+2*(2-a)*int(q/2)]/(fact(r)*fact(p)*fact(q))+fact(p+2*n-1)*fact(r+2*int(q/2)-2*n+1)*u[w,3-a,p+(-1)**(a+1)*(2*n-1)+(a-1)*2*int(q/2),q-2*int(q/2),\
$$r+(-1)**(a+1)*(2*n-1)+2*(2-a)*int(q/2)]/(fact(r)*fact(p)*fact(q))
$elif a==2:
$$result+(-1)**int(q/2)*fact(int(q/2)-1)/(fact(n-1)*fact(int(q/2)-n))*(fact(r+2*n)*fact(p+2*int(q/2)-2*n)*u[w,a,p+(-1)**(a+1)*2*n+(a-1)*2*int(q/2),q-2*int(q/2),\
$$r+(-1)**(a+1)*2*n+2*(2-a)*int(q/2)]/(fact(r)*fact(p)*fact(q))+fact(r+2*n-1)*fact(p+2*int(q/2)-2*n+1)*u[w,3-a,p+(-1)**(a+1)*(2*n-1)+(a-1)*2*int(q/2),q-2*int(q/2),\
$$r+(-1)**(a+1)*(2*n-1)+2*(2-a)*int(q/2)]/(fact(r)*fact(p)*fact(q))
$return result
$
def tildeF0(w,p,q,r,a):
$result=0
$if a==1:
$for n in range(1,int(q/2)+1):
$$result+(-1)**(n+1)*(fact(q-2*n+2)*fact(p+2*n-2)*g[w-1,a,p+(2-a)*(2*n-2),q-2*n+2,r+(a-1)*(2*n-2)]/(fact(q)*fact(p))-fact(q-2*n+1)*fact(p+2*n-1)*\
$$g[w-1,0,p+(2-a)*(2*n-1),q-2*n+1,r+(a-1)*(2*n-1)]/(fact(q)*fact(p)))
$for n in range(1,int(q/2)+1):
$for m in range(1,n+1):
$$$$result+(-1)**n*((fact(n-1)/(fact(m-1)*fact(n-m)))*fact(r+2*m-1)*fact(q-2*n)*fact(p-2*m+2*n+1)*g[w-1,3-a,p+(-1)**(2*m-1)+(2-a)*2*n,q-2*n,r+(-1)**(a+1)*(2*m-1)+\
$$$$n*(a-1)]/(fact(p)*fact(q)*fact(r))-fact(n)/(fact(m)*fact(n-m))*fact(r+2*m)*fact(q-1-2*n)*fact(p-2*m+2*n+1)*g[w-1,0,p+(-1)**(2*m)*(2-a)*(2*n+1),q-1-2*n,\
$$$$r+(-1)**(a+1)*2*m*(a-1)*(2*n+1)]/(fact(p)*fact(q)*fact(r)))
$for n in range(1,int(q/2)-1):
$for m in range(1,n+1):
$$$$result+(-1)**(n+1)*fact(n)/(fact(m)*fact(n-m))*fact(r+2*m)*fact(q-2-2*n)*fact(p-2*m+2*n+2)*g[w-1,a,p+(-1)**(2*m)*(2-a)*(2*n+2),q-2-2*n,r+(-1)**(a+1)*2*m*\
$$$$n*(a-1)*(2*n+2)]/(fact(p)*fact(q)*fact(r))
$elif a==2:
$for n in range(1,int(q/2)+1):
$$result+(-1)**(n+1)*(fact(q-2*n+2)*fact(r+2*n-2)*g[w-1,a,p+(2-a)*(2*n-2),q-2*n+2,r+(a-1)*(2*n-2)]/(fact(q)*fact(r))-fact(q-2*n+1)*fact(r+2*n-1)*g[w-1,0,p+(2-a)*(2*n-1),\
$$q-2*n+1,r+(a-1)*(2*n-1)]/(fact(q)*fact(r)))
$for n in range(1,int(q/2)+1):
$for m in range(1,n+1):
$$$$result+(-1)**n*((fact(n-1)/(fact(m-1)*fact(n-m)))*fact(p+2*m-1)*fact(q-2*n)*fact(r-2*m+2*n+1)*g[w-1,3-a,p+(-1)**(2*m-1)+(2-a)*2*n,q-2*n,r+(-1)**(a+1)*(2*m-1)+\
$$$$n*(a-1)]/(fact(p)*fact(q)*fact(r))-fact(n)/(fact(m)*fact(n-m))*fact(p+2*m)*fact(q-1-2*n)*fact(r-2*m+2*n+1)*g[w-1,0,p+(-1)**(2*m)*(2-a)*(2*n+1),q-1-2*n,\
$$$$r+(-1)**(a+1)*2*m*(a-1)*(2*n+1)]/(fact(p)*fact(q)*fact(r)))
$for n in range(1,int(q/2)-1):
$for m in range(1,n+1):
$$$$result+(-1)**(n+1)*fact(n)/(fact(m)*fact(n-m))*fact(p+2*m)*fact(q-2-2*n)*fact(r-2*m+2*n+2)*g[w-1,a,p+(-1)**(2*m)*(2-a)*(2*n+2),q-2-2*n,\
$$$$r+(-1)**(a+1)*2*m*(a-1)*(2*n+2)]/(fact(p)*fact(q)*fact(r))
$return result/w
$
def F(z,w,p,q,r,a,n):
$result=0
$P1=w>0 and p>1 and q>1 and r>1
$if z==0 and P1:
$return F0(w,p,q,r,a)+tildeF0(w,p,q,r,a)
$elif z==0 and (not P1):
$return u[w,a,p,q,r]
$elif z==1 and n==1:
$if a==1:
$$result=nu*(r+1)*(r+2)
$elif a==2:
$$result=nu*(p+1)*(p+2)
$elif z==2 and n==1:$
$$result=nu*(q+1)*(q+2)
$elif z==3 and n==1:
$if a==1:
$$result=nu*(q+1)*(p+1)
$elif a==2:
$$result=nu*(q+1)*(r+1)
$elif z==4 and n==1:
$$result=nu*Heaviside(q-4)*(p+1)*(r+1)
$elif z==5 and n==1:
$if a==1:
$$result=nu*(-1)**(int(q/2))*fact(p+2*int(q/2)+1)*(q-2*int(q/2)+1)/(fact(p)*fact(q))
$elif a==2:
$$result=nu*(-1)**(int(q/2))*fact(r+2*int(q/2)+1)*(q-2*int(q/2)+1)/(fact(r)*fact(q))
$elif z==6:
$if a==1:
$$result=nu*(-1)**int(q/2)*fact(int(q/2))/(fact(n-1)*fact(int(q/2)-n+1))*fact(p+2*n-1)*fact(r+2*int(q/2)-2*n+2)*(1+q-2*int(q/2))/(fact(p)*fact(r)*fact(q))
$elif a==2:
$$result=nu*(-1)**int(q/2)*fact(int(q/2))/(fact(n-1)*fact(int(q/2)-n+1))*fact(r+2*n-1)*fact(p+2*int(q/2)-2*n+2)*(1+q-2*int(q/2))/(fact(p)*fact(r)*fact(q))
$elif z==7:
$if a==1:
$$result=nu*(-1)**(int(q/2)+1)*2*fact(int(q/2)-1)/(fact(n-1)*fact(int(q/2)-n))*fact(p+2*n)*fact(r+2*int(q/2)-2*n)*(1+2*q-4*int(q/2))/(fact(p)*fact(r)*fact(q))
$elif a==2:
$$result=nu*(-1)**(int(q/2)+1)*2*fact(int(q/2)-1)/(fact(n-1)*fact(int(q/2)-n))*fact(r+2*n)*fact(p+2*int(q/2)-2*n)*(1+2*q-4*int(q/2))/(fact(p)*fact(r)*fact(q))
$elif z==8:
$if a==1:
$$result=nu*(-1)**(int(q/2)+1)*2*Heaviside(q-4)*fact(int(q/2)-1)/(fact(n-1)*fact(int(q/2)-n))*fact(p+2*n-1)*fact(r+2*int(q/2)-2*n+1)*(1+2*q-4*int(q/2))/\
$$$fact(p)*fact(r)*fact(q)
$elif a==2:
$$result=nu*(-1)**(int(q/2)+1)*2*Heaviside(q-4)*fact(int(q/2)-1)/(fact(n-1)*fact(int(q/2)-n))*fact(r+2*n-1)*fact(p+2*int(q/2)-2*n+1)*(1+2*q-4*int(q/2))/\
$$$fact(p)*fact(r)*fact(q)
$return result/w
$
def D(p,q,r,a,tau):
$d,ressl=0,0
$for j in range(1,9*int(q/2)**2-9*int(q/2)+7):
$$ressl+=S(p,q,r,a,j-1)
$$d+=Heaviside((tau-1)/ressl-1)
$if (tau-1)<ressl: # In order to make this function less costly
$$break
$return d
$
def I(p,q,r,a,tau):
$l,ressl=0,0
$for j in range(1,9*int(q/2)**2-9*int(q/2)+7):
$$ressl+=S(p,q,r,a,j-1)
$$l+=S(p,q,r,a,j-1)*Heaviside((tau-1)/ressl-1)
$if (tau-1)<ressl: # Same reason
$$break
$return tau-l
$
def sigma(p,q,r,a): #int(round()) in order to have an integer despite numerical inaccuracies
$if a==1:
$return int(round(q*(2*int(q/2)**4+(4*p+6*r+8)*int(q/2)**3+(9*r+p+1.5*p+3*r-8)*int(q/2)**2-(9*r+p+5.5*p+9*r+2)*int(q/2)+6*r+p+6*p+9*r+9)-3.2*int(q/2)**5-\
$$*(6*p+8*r+8)*int(q/2)**4+(6*r+2+10*r+p+6*p)*int(q/2)**3+(14*r+7+22.5*r+p+9*p)*int(q/2)**2-(12.5*r+p+9*p+12*r-2.2)*int(q/2)+3*(r+1)*(p+1)-8*int(q/2)))
$else:
$return sigma(r,q,p,1) #interchanging of p and r when a=2
$
def T(p,q,r,a,j):

```

```

$return j+1-sigma(p,q,r,a)*int(j/sigma(p,q,r,a))
$
def chi(w,p,q,r,a,i):
$return i-w*sigma(p,q,r,a)
$
def NN(w,p,q,r,a,i):
$return 1 if chi(w,p,q,r,a,i)==0 else int((chi(w,p,q,r,a,i)-1)/8)+1
$
def FF(p,q,r,a,w):
$if p<2 or q<2 or r<2 or w<=0:
$$return 0
$elif a==0 and q==2:
$$return 1
$elif a!=0 and w>0 and p>1 and q>1 and r>1:
$return sigma(p,q,r,a)*w+8*int(q/2)
$else:
$return sigma(p+1,q-1,r,1)*w+sigma(p,q-1,r+1,2)*w+16*int((q-1)/2)+1
$
def T0(w,p,q,r,a,i):
$if p<0 or q<0 or r<0 or w<0:
$$return -1
$elif p<2 or q<2 or r<2 or w==0:
$return q
$elif (a==1 or a==2) and i<sigma(p,q,r,a)*w:
$return W0(p,q,r,a,D(p,q,r,a,T(p,q,r,a,i)),I(p,q,r,a,T(p,q,r,a,i)))
$elif (a==1 or a==2) and i>=sigma(p,q,r,a)*w:
$return Delta(chi(w,p,q,r,a,i)-8*(NN(w,p,q,r,a,i)-1),q)
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$return T0(w,p+1,q-1,r,1,i)
$else:
$return T0(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$$
def T1(w,p,q,r,a,i):
$if p<0 or q<0 or r<0 or w<0:
$$return -1
$elif p<2 or q<2 or r<2 or w==0:
$return p
$elif (a==1 or a==2) and i<sigma(p,q,r,a)*w:
$return W1(p,q,r,a,D(p,q,r,a,T(p,q,r,a,i)),I(p,q,r,a,T(p,q,r,a,i)))
$elif (a==1 or a==2) and i>=sigma(p,q,r,a)*w:
$return ksi(chi(w,p,q,r,a,i)-8*(NN(w,p,q,r,a,i)-1),p,q,a,NN(w,p,q,r,a,i))
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$return T1(w,p+1,q-1,r,1,i)
$else:
$return T1(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$$
def T2(w,p,q,r,a,i):
$if p<0 or q<0 or r<0 or w<0:
$$return -1
$elif p<2 or q<2 or r<2 or w==0:
$return r
$elif (a==1 or a==2) and i<sigma(p,q,r,a)*w:
$return W2(p,q,r,a,D(p,q,r,a,T(p,q,r,a,i)),I(p,q,r,a,T(p,q,r,a,i)))
$elif (a==1 or a==2) and i>=sigma(p,q,r,a)*w:
$return epsilon(chi(w,p,q,r,a,i)-8*(NN(w,p,q,r,a,i)-1),q,r,a,NN(w,p,q,r,a,i))
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$return T2(w,p+1,q-1,r,1,i)
$else:
$return T2(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$
def beta(w,p,q,r,a,i):
$if p<2 or q<2 or r<2 or w<1:
$$return a
$elif (a==1 or a==2) and i<sigma(p,q,r,a)*w:
$$d=D(p,q,r,a,T(p,q,r,a,i))
$return B(d,q)*(a+A(d,q)*(A(d,q)-2)*(2*a-3))
$elif (a==1 or a==2) and i>=sigma(p,q,r,a)*w:
$return kappa(chi(w,p,q,r,a,i)-8*(NN(w,p,q,r,a,i)-1),a)
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$return beta(w,p+1,q-1,r,1,i)
$else:
$return beta(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$
def E(w,p,q,r,a,i):
$if p<0 or q<0 or r<0 or w<0:
$$return -1
$elif p<2 or q<2 or r<2 or w==0:
$return w
$elif (a==1 or a==2) and int(i/sigma(p,q,r,a))<w:
$return int(i/sigma(p,q,r,a))
$elif (a==1 or a==2) and w<=int(i/sigma(p,q,r,a)):
$return w-1
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$return E(w,p+1,q-1,r,1,i)
$else:
$return E(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$
def Z0(w,p,q,r,a,i):
$if p<2 or q<2 or r<2 or w<1:
$$return -1
$elif a==1 or a==2:
$return V0(q,r,a,D(p,q,r,a,T(p,q,r,a,i)),I(p,q,r,a,T(p,q,r,a,i)))
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$return Z0(w,p+1,q-1,r,1,i)
$else:
$return Z0(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$
def Z1(w,p,q,r,a,i):
$if p<2 or q<2 or r<2 or w<1:
$$return -1
$elif a==1 or a==2:
$return V1(p,q,r,a,D(p,q,r,a,T(p,q,r,a,i)),I(p,q,r,a,T(p,q,r,a,i)))
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$return Z1(w,p+1,q-1,r,1,i)
$else:
$return Z1(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$
def Z2(w,p,q,r,a,i):
$if p<2 or q<2 or r<2 or w<1:
$$return -1
$elif a==1 or a==2:
$return V2(q,r,a,D(p,q,r,a,T(p,q,r,a,i)),I(p,q,r,a,T(p,q,r,a,i)))
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$return Z2(w,p+1,q-1,r,1,i)
$else:

```

```

$$return Z2(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$
def Phi(w,p,q,r,a,i):
$if p<2 or q<2 or r<2 or w<1:
$$return a
$elif a==1 or a==2:
$$return phi(D(p,q,r,a,T(p,q,r,a,i)),q)
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$$return Phi(w,p+1,q-1,r,1,i)
$else:
$$return Phi(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$
def Upsilon(w,p,q,r,a,i):
$if p<2 or q<2 or r<2 or w<1:
$$return -1
$elif a==1 or a==2:
$$return w-1-int(i/sigma(p,q,r,a))
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$$return Upsilon(w,p+1,q-1,r,1,i)
$else:
$$return Upsilon(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)
$
def Gamma(w,p,q,r,a,i):
$if p<2 or q<2 or r<2 or w<1:
$$return 1
$elif (a==1 or a==2) and int(i/sigma(p,q,r,a))<w:
$$return 1/w*G(p,q,r,a,D(p,q,r,a,T(p,q,r,a,i)),I(p,q,r,a,T(p,q,r,a,i)))
$elif (a==1 or a==2) and w<int(i/sigma(p,q,r,a)):
$$return F(chi(w,p,q,r,a,i)-8*(NN(w,p,q,r,a,i)-1),w,p,q,r,a,NN(w,p,q,r,a,i))
$elif a==0 and i<=FF(p+1,q-1,r,1,w):
$$return -(p+1)*Gamma(w,p+1,q-1,r,1,i)/q
$else:
$$return -(r+1)*Gamma(w,p,q-1,r+1,2,i-FF(p+1,q-1,r,1,w)-1)/q
$
def eta(p,q,r,a,w,index):
$if w==0 or p<2 or q<2 or r<2:
$$return 0
$elif a==0 and q==2:
$$return 1
$elif index==0:
$$return FF(p,q,r,a,w)
$else:
$$res=0
$$for k in range(0,int(FF(p,q,r,a,w))+1):
$$res += (eta(T1(w,p,q,r,a,k),T0(w,p,q,r,a,k),T2(w,p,q,r,a,k),beta(w,p,q,r,a,k),E(w,p,q,r,a,k),index-1) + 1) * \
$$$(eta(Z1(w,p,q,r,a,k),Z0(w,p,q,r,a,k),Z2(w,p,q,r,a,k),Phi(w,p,q,r,a,k),Upsilon(w,p,q,r,a,k),index-1) + 1)
$$return res-1
$
def eta1(p,q,r,a,w,k,index):
$if index==0:
$$return FF(T1(w,p,q,r,a,k),T0(w,p,q,r,a,k),T2(w,p,q,r,a,k),beta(w,p,q,r,a,k),E(w,p,q,r,a,k))
$else:
$$return eta(T1(w,p,q,r,a,k),T0(w,p,q,r,a,k),T2(w,p,q,r,a,k),beta(w,p,q,r,a,k),E(w,p,q,r,a,k),index)
$
def eta2(p,q,r,a,w,k,index):
$if index==0:
$$return FF(Z1(w,p,q,r,a,k),Z0(w,p,q,r,a,k),Z2(w,p,q,r,a,k),phi(D(p,q,r,a,T(p,q,r,a,k)),q),Upsilon(w,p,q,r,a,k))
$else:
$$return eta(Z1(w,p,q,r,a,k),Z0(w,p,q,r,a,k),Z2(w,p,q,r,a,k),phi(D(p,q,r,a,T(p,q,r,a,k)),q),Upsilon(w,p,q,r,a,k),index)
$
def Seta(j,p,q,r,a,w,index):
$res=0
$for k in range(0,j+1):
$$res+=eta1(p,q,r,a,w,k,index)+1*(eta2(p,q,r,a,w,k,index)+1)
$return res
$
def theta(i,p,q,r,a,w,n,index):
$if index==0:
$$return i
$else:
$$if n==0: #Different from the article. Modifications with the aim of making this function less costly. Same result of course
$$if i<(eta1(p,q,r,a,w,0,index-1)+1)*(eta2(p,q,r,a,w,0,index-1)+1):
$$$$return 0
$$$$else:
$$$$res=0
$$$$for j in range(0,FF(p,q,r,a,w)+1):
$$$$res=res
$$$$res+=eta1(p,q,r,a,w,j,index-1)+1*(eta2(p,q,r,a,w,j,index-1)+1)
$$$$if i > res-1 and i < res:
$$$$p=j
$$$$break
$$$$return p
$$$$n=1:
$$$$if i<(eta1(p,q,r,a,w,0,index-1)+1)*(eta2(p,q,r,a,w,0,index-1)+1):
$$$$res=0
$$$$j=0
$$$$else:
$$$$res=0
$$$$for k in range(0,FF(p,q,r,a,w)+1):
$$$$res=res
$$$$res+=eta1(p,q,r,a,w,k,index-1)+1*(eta2(p,q,r,a,w,k,index-1)+1)
$$$$if i>res-1 and i<res:
$$$$j=k
$$$$break
$$$$return int((i-res)/(eta2(p,q,r,a,w,j,index-1)+1))
$else:
$$if i<(eta1(p,q,r,a,w,0,index-1)+1)*(eta2(p,q,r,a,w,0,index-1)+1):
$$$$res=0
$$$$j=0
$$$$else:
$$$$res=0
$$$$for k in range(0,FF(p,q,r,a,w)+1):
$$$$res=res
$$$$res+=eta1(p,q,r,a,w,k,index-1)+1*(eta2(p,q,r,a,w,k,index-1)+1)
$$$$if i>res-1 and i<res:
$$$$j=k
$$$$break
$$$$return R(i-res,eta2(p,q,r,a,w,j,index-1)+1)
$
def C0(m,h,l,i,p,q,r,a,w):
$if int(log(m+1)/log(2))-h<1 or l>int(log(m+1)/log(2))-l-h:
$$return q
$elif Ip(int((m+1-2**(l+h))/(2**(l+h))))=1:
$$return T0(Cw(m,h,l+1,i,p,q,r,a,w),C1(m,h,l+1,i,p,q,r,a,w),C0(m,h,l+1,i,p,q,r,a,w),C2(m,h,l+1,i,p,q,r,a,w),Ca(m,h,l+1,i,p,q,r,a,w),\
$$Thet(int((m+1-2**(l+h))/(2**(l+h))))),i,p,q,r,a,w,0)

```



```

$elif l<j-1 and aa==1:
$$return E(rhow(i,l+1,k,j,p,q,r,a,w), rho1(i,l+1,k,j,p,q,r,a,w), rho0(i,l+1,k,j,p,q,r,a,w), rho2(i,l+1,k,j,p,q,r,a,w), rhoa(i,l+1,k,j,p,q,r,a,w), \
$$TheT(int((k-1+2*(j-1)-2**l)/2**l), i,p,q,r,a,w,0))
$elif l<j-1 and aa!=1:
$$return Upsilon(rhow(i,l+1,k,j,p,q,r,a,w), rho1(i,l+1,k,j,p,q,r,a,w), rho0(i,l+1,k,j,p,q,r,a,w), rho2(i,l+1,k,j,p,q,r,a,w), rhoa(i,l+1,k,j,p,q,r,a,w), \
$$TheT(int((k-1+2*(j-1)-2**l)/2**l), i,p,q,r,a,w,0))
$elif j-1<l and aa==1:
$$return E(w,p,q,r,a,TheT(int((k-1+2*(j-1)-2**l)/2**l), i,p,q,r,a,w,0))
$else:
$$return Upsilon(w,p,q,r,a,TheT(int((k-1+2*(j-1)-2**l)/2**l), i,p,q,r,a,w,0))
$
def aleph0(l,k,i,p,q,r,a,w):
$aa=Ip(1+int((k-1)/2**(l-1)))
$if l<w and aa==1:
$$return T0(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif l<w and aa!=1:
$$return Z0(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif w<l and aa==1:
$$return T0(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$else:
$$return Z0(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$
def aleph1(l,k,i,p,q,r,a,w):
$aa=Ip(1+int((k-1)/2**(l-1)))
$if l<w and aa==1:
$$return T1(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif l<w and aa!=1:
$$return Z1(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif w<l and aa==1:
$$return T1(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$else:
$$return Z1(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$
def aleph2(l,k,i,p,q,r,a,w):
$aa=Ip(1+int((k-1)/2**(l-1)))
$if l<w and aa==1:
$$return T2(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif l<w and aa!=1:
$$return Z2(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif w<l and aa==1:
$$return T2(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$else:
$$return Z2(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$
def alepha(l,k,i,p,q,r,a,w):
$aa=Ip(1+int((k-1)/2**(l-1)))
$if l<w and aa==1:
$$return beta(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif l<w and aa!=1:
$$return Phi(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif w<l and aa==1:
$$return beta(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$else:
$$return Phi(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$
def alephw(l,k,i,p,q,r,a,w):
$aa=Ip(1+int((k-1)/2**(l-1)))
$if l<w and aa==1:
$$return E(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif l<w and aa!=1:
$$return Upsilon(alephw(l+1,k,i,p,q,r,a,w), aleph1(l+1,k,i,p,q,r,a,w), aleph0(l+1,k,i,p,q,r,a,w), aleph2(l+1,k,i,p,q,r,a,w), alepha(l+1,k,i,p,q,r,a,w), \
$$TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$elif w<l and aa==1:
$$return E(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$else:
$$return Upsilon(w,p,q,r,a,TheT(2*(w-1)-1+int((k-1)/2**l), i,p,q,r,a,w,0))
$
def J(w,a,p,q,r):
$return 1 if (p<0 or q<0 or r<0 or a<0 or w<0) else u[w,a,p,q,r]
$
# Function which calculates the coefficient u[w,a,p,q,r]
def Function(nb): # Parallelized function for the calculation of u[w,a,p,q,r] | nb is the processors number - 1
$result=0
$if nb==0:
$$a_list=[i for i in range(0,int(intervalle)+1)]
$elif nb==nb_proc-1:
$$a_list=[i for i in range(nb*int(intervalle)+1,N_max+1)]
$else:
$$a_list=[i for i in range(nb*int(intervalle)+1,(nb+1)*int(intervalle)+1)]
$for i in a_list:
$$res=1
$$for j in range(1,w+1):
$$for k in range(1,2*(j-1)+1):
$$sv1=rhow(1,l,k,j,p,q,r,a,w)
$$sv1f vl<l: # If this is the case, Gamma returns 1. There is no need to calculate the others
$$res=res*sv1
$$else:
$$res=res*Gamma(v1,rho1(i,l,k,j,p,q,r,a,w), rho0(i,l,k,j,p,q,r,a,w), rho2(i,l,k,j,p,q,r,a,w), rhoa(i,l,k,j,p,q,r,a,w), TheT(k-2+2*(j-1), i,p,q,r,a,w,0))
$for k in range(1,2*(w+1)):
$$sv2=alephw(1,k,i,p,q,r,a,w)
$$sv2f v2<0:
$$res=res*sv2
$$else:
$$res=res*J(v2,alepha(1,k,i,p,q,r,a,w), aleph1(1,k,i,p,q,r,a,w), aleph0(1,k,i,p,q,r,a,w), aleph2(1,k,i,p,q,r,a,w))
$result+=res
$return result
$
# Calculation
N_max=int(eta(p,q,r,a,w,w-1))
if nb_proc==0:
$intervalle=N_max
else:
$intervalle=N_max/nb_proc
if sequential==True:
$AnalyticResult=Function(nb_proc)
else:

```

```

$if __name__=="__main__":
$$chunks=[i for i in range(0,nb_proc)]
$$pool=Pool(nb_proc)
$$result=pool.map(Function,chunks)
$AnalyticResult=sum(result)
$
# Verification with the equations 18-21-22 which are less time consuming
def fu(w,p,q,r):
$if q>1 and p>1 and r>1 and w>0:
$$res=-(p+1)*((r+1)*fw(w,p+1,q-2,r+1)+(p+2)*fw(w,p+2,q-2,r))/(q*(q-1))+(nu/w)*(fact(p+2)*fw(w-1,p+2,q,r)/fact(p)+(q+2)*(q+1)*fw(w-1,p,q+2,r)+fact(r+2)*fw(w-1,p,q,r+2)/fact(r)-\
$$fact(p+3)*fw(w-1,p+3,q-1,r)/(q*fact(p))-(p+1)*(q+1)*fw(w-1,p+1,q+1,r)-fact(r+2)*(p+1)*fv(w-1,p+1,q-1,r+2)/(q*fact(r))
$$for k in range(0,w):
$$for i in range(0,p+2):
$$for j in range(0,q):
$$$$for l in range(0,r+1):
$$$$$res+= (p+1)/(w*q)*((i+1)*fw(w-1-k,p+1-i,q-1-j,r-1)*fv(k,i+1,j,l)+(j+1)*fv(w-1-k,p+1-i,q-1-j,r-1)*fv(k,i,j+1,l)+(l+1)*fw(w-1-k,p+1-i,q-1-j,r-1)*fv(k,i,j,l+1))
$$for i in range(0,p+1):
$$for j in range(0,q+1):
$$$$for l in range(0,r+1):
$$$$$res=res-(1/w)*((i+1)*fw(w-1-k,p-i,q-j,r-1)*fw(k,i+1,j,l)+(j+1)*fv(w-1-k,p-i,q-j,r-1)*fw(k,i,j+1,l)+(l+1)*fw(w-1-k,p-i,q-j,r-1)*fw(k,i,j,l+1))
$$res+=1/w*(g[w-1,l,p,q,r]-(p+1)*g[w-1,0,p,q-1,r]/q)
$else:
$$res=u[w,1,p,q,r]
$return res
$
def fw(w,p,q,r):
$if q>1 and p>1 and r>1 and w>0:
$$res=-(r+1)*((r+2)*fw(w,p,q-2,r+2)+(p+1)*fw(w,p+1,q-2,r+1))/(q*(q-1))+(nu/w)*(fact(p+2)*fw(w-1,p+2,q,r)/fact(p)+(q+2)*(q+1)*fw(w-1,p,q+2,r)+fact(r+2)*fw(w-1,p,q,r+2)/fact(r)-\
$$fact(r+3)*fw(w-1,p,q-1,r+3)/(q*fact(r))-(r+1)*(q+1)*fv(w-1,p,q+1,r+1)-(r+1)*fact(p+2)*fw(w-1,p+2,q-1,r+1)/(q*fact(p))
$$for k in range(0,w):
$$for i in range(0,p+1):
$$for j in range(0,q):
$$$$for l in range(0,r+2):
$$$$$res+= (r+1)/(w*q)*((i+1)*fw(w-1-k,p-i,q-1-j,r+1-1)*fv(k,i+1,j,l)+(j+1)*fv(w-1-k,p-i,q-1-j,r+1-1)*fv(k,i,j+1,l)+(l+1)*fw(w-1-k,p-i,q-1-j,r+1-1)*fv(k,i,j,l+1))
$$for i in range(0,p+1):
$$for j in range(0,q+1):
$$$$for l in range(0,r+1):
$$$$$res=res-(1/w)*((i+1)*fw(w-1-k,p-i,q-j,r-1)*fw(k,i+1,j,l)+(j+1)*fv(w-1-k,p-i,q-j,r-1)*fw(k,i,j+1,l)+(l+1)*fw(w-1-k,p-i,q-j,r-1)*fw(k,i,j,l+1))
$$res+=1/w*(g[w-1,2,p,q,r]-(r+1)*g[w-1,0,p,q-1,r+1]/q)
$else:
$$res=u[w,2,p,q,r]
$return res
$
def fv(w,p,q,r):
$if q>1 and p>1 and r>1 and w>0:
$$res=-(p+1)*fw(w,p+1,q-1,r)/q-(r+1)*fw(w,p,q-1,r+1)/q
$else:
$$res=u[w,0,p,q,r]
$return res
$
RightResult=(fv(w,p,q,r) if a==0 else (fu(w,p,q,r) if a==1 else fw(w,p,q,r)))
if formal==True:
$print(simplify(RightResult-AnalyticResult)) # Should be nil with numerical inaccuracies
else:
$print(RightResult-AnalyticResult) # Should be nil with numerical inaccuracies

```

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