

# A remark on compact $H$ -surfaces into $\mathbb{R}^3$

Yuxin Ge and Frédéric Hélein

## 1 Introduction

Let  $\Omega$  be a smooth and bounded domain in  $\mathbb{R}^2$ . We consider the following system

$$\Delta u = u_x \wedge u_y, \quad \text{in } \Omega, \quad (1.1)$$

where  $u \in C^2(\Omega; \mathbb{R}^3)$  and subscripts denote partial differentiation with respect to coordinates. This equation characterizes surfaces of constant mean curvature  $H = \frac{1}{2}$  in  $\mathbb{R}^3$  in conformal representation. More precisely, any non constant smooth map  $u$  which is a solution of (1.1) and of the conformality condition

$$\omega := (|u_x|^2 - |u_y|^2 - 2i\langle u_x, u_y \rangle)dz \otimes dz = 0, \quad \text{in } \Omega, \quad (1.2)$$

(here  $dz = dx + idy$ ) parametrizes a branched immersed constant mean curvature surface in  $\mathbb{R}^3$ . For that reason (1.1) is called the  $H$ -system. The complex tensor  $\omega$  which appears in (1.2) is called the Hopf differential (see [8]). The first existence result for solutions to (1.1) and (1.2) was proved by H. Wente in [11].

In [7], the second author proposed a new variational approach for finding a solution to (1.1). For any pair of functions  $a, b \in H^1(\Omega)$ , we denote by  $\varphi := \tilde{ab}$  the unique solution in  $H^1(\Omega)$  of the Dirichlet problem

$$\begin{cases} -\Delta \varphi = \{a, b\}, & \text{in } \Omega \\ \varphi = 0, & \text{on } \partial\Omega, \end{cases} \quad (1.3)$$

where  $\{a, b\} = a_x b_y - a_y b_x$ . By a result of Brezis and Coron [3] based on an idea due to H. Wente [11] [12], we know that  $\varphi$  is continuous on  $\bar{\Omega}$  and

$$\|\varphi\|_{L^\infty(\Omega)} + \|\nabla \varphi\|_{L^2(\Omega)} \leq C_0(\Omega) \|\nabla a\|_{L^2(\Omega)} \|\nabla b\|_{L^2(\Omega)}. \quad (1.4)$$

Thus the following energy functional makes sense

$$E(a, b, \Omega) = \frac{\|\nabla a\|_{L^2(\Omega)}^2 + \|\nabla b\|_{L^2(\Omega)}^2}{2\|\nabla \varphi\|_{L^2(\Omega)}}, \quad \text{defined for } a, b \in H^1(\Omega) \setminus \{0\}.$$

The Euler-Lagrange equation satisfied by the critical points of this functional was derived in [7]. Through the substitution  $u := (\lambda a, \lambda b, \lambda^2 \varphi)$  for  $\lambda = -\sqrt{\frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2}}$ , this equation coincides with (1.1). The boundary conditions are

$$\varphi = \frac{\partial a}{\partial n} = \frac{\partial b}{\partial n} = 0 \quad \text{on } \partial\Omega, \quad (1.5)$$

where  $n = (n_1, n_2)$  is the normal vector on  $\partial\Omega$ . Moreover, in [6] the first author showed that the Hopf differential  $\omega$  is holomorphic and satisfies the boundary condition  $Im(\omega\nu^2) = 0$ , where  $\nu = n_1 + in_2$ . This implies in particular that  $\omega$  vanishes - i.e. (1.2) is true - if  $\Omega$  is simply connected.

An important property of our problem is that the functional  $E$  and its critical points are preserved by conformal transformations of the domain  $\Omega$  (see [7]). So this variational problem depends only on the complex structure of  $\Omega$  and hence it also makes sense to consider the problem on a Riemann surface. The boundary conditions (1.5) allow us to construct a solution of (1.1) from a compact oriented Riemannian surface into  $\mathbb{R}^3$  by gluing together two copies of  $\Omega$ . More precisely, we construct  $N := \Omega \cup_{\partial\Omega} \tilde{\Omega}$ , where  $\tilde{\Omega}$  is a copy of  $\Omega$ , provided with opposite orientation and define a  $C^\infty$  map  $\tilde{u} : N \rightarrow \mathbb{R}^3$  by  $\tilde{u} = u$  on  $\Omega$  and  $\tilde{u} = (\lambda a, \lambda b, -\lambda^2\varphi)$  on  $\tilde{\Omega}$ . This map is a solution of the  $H$ -system (1.1) and its Hopf differential is holomorphic. Were this differential to vanish, i.e. if  $\omega = 0$ , then we would obtain a constant mean curvature branched immersion. Recall that Wente [13] (see also [1]) constructed an immersed constant mean curvature torus, which enjoys an invariance under an orthogonal symmetry with respect to a plane. Thus it has the form  $\Omega \cup_{\partial\Omega} \tilde{\Omega}$  as above, where  $\Omega$  is some annulus. This motivates the search for critical points of  $E$ . In [6], an existence result was derived for a perforated domain, provided the holes are small enough (in the same spirit as in [4]). Here we address the problem of a one-connected domain  $\Omega$ , i.e. a domain of the form  $U \setminus \bar{V}$  where  $U$  and  $V$  are smooth bounded simply connected open sets and  $\bar{V} \subset U$ , without smallness assumption on the hole  $V$ . Because any such domain is conformally equivalent to a radially symmetric annulus [2] and thanks to the invariance of the variational problem under conformal transformations, we shall restrict ourself to annuli without loss of generality. Our method relies on a minimization procedure on a subset of  $H^1 \times H^1$ , which is equivariant with respect to some finite group.

**Main Theorem.** *Consider the annulus  $\Omega := \{(x, y) \in \mathbb{R}^2, \quad r_0 < r = \sqrt{x^2 + y^2} < 1\}$  with  $0 < r_0 < 1$ . Then, there exists a critical point  $(a, b) \in H^1(\Omega, \mathbb{R}^2) \setminus \{(0, 0)\}$  of the energy functional  $E$ . Moreover  $(a, b)$  and  $\varphi := \tilde{a}\tilde{b}$  are smooth and satisfy the boundary conditions (1.5). Thus there exists a real number  $\lambda \neq 0$  such that the map  $u := (\lambda a, \lambda b, \lambda^2\varphi)$  is a solution of (1.1). Lastly the Hopf differential  $\omega$  of  $u$  has the form  $\omega = \frac{\tau}{z^2} dz \otimes dz$  for some real number  $\tau$ .  $\blacksquare$*

Unfortunately we are not able to prove that the map  $u$  is conformal. We expect that it should be so for some values of  $r_0$ . Indeed the parameter  $\tau$  characterizing  $\omega$  should vary with  $r_0$  (since the set of holomorphic quadratic differentials is, roughly speaking, the dual space of Teichmüller space). But we are still far from understanding how  $\tau$  could be related to  $r_0$ .

## 2 The Euler-Lagrange Equation

First, we note that  $\Omega$  is invariant under rotations. We define  $F_m = \{\Theta = (a, b) \in H^1(\Omega) \times H^1(\Omega), \quad \Theta \circ A = A \circ \Theta\}$  and we will prove existence of a minimum of  $E$  for  $m$  large enough, where  $A$  is the rotation of angle  $\frac{2\pi}{m}$  in  $\mathbb{R}^2$ .

**Lemma 1** Assume that  $(a, b) \in F_m$ . Then, the unique solution  $\varphi$  of (1.3) is invariant under  $A$ , that is,

$$\varphi \circ A = \varphi.$$

■

*Proof.* Clearly, we have

$$d\Theta = A^{-1} \cdot (d\Theta) \circ A \cdot A.$$

Thus,

$$\{a, b\} = \det(d\Theta) = \det(A^{-1}) \cdot [\det(d\Theta)] \circ A \cdot \det(A) = [\det(d\Theta)] \circ A = \{a, b\} \circ A,$$

since  $\det(A) = \det(A^{-1}) = 1$ . On the other hand, the unique solution  $\varphi$  of (1.3) is also the unique minimum of the following energy functional  $E_1$ :

$$E_1(\psi) = \frac{1}{2} \int_{\Omega} |\nabla \psi|^2 - \int_{\Omega} \{a, b\} \psi, \quad \text{defined for all } \psi \in H_0^1(\Omega).$$

Obviously,  $E_1(\varphi \circ A) = E_1(\varphi)$ . By the uniqueness of the minimizer, we deduce that

$$\varphi \circ A = \varphi.$$

■

In order to get the Euler-Lagrange equation of  $E$ , we first recall a technical lemma inspired by the work in [11] and proved in the Appendix in [3].

**Lemma 2** If  $\varphi \in H^1(\Omega) \cap L^\infty(\Omega)$ ,  $a \in H^1(\Omega) \cap L^\infty(\Omega)$ ,  $b \in H^1(\Omega)$  and  $\varphi a = 0$  on  $\partial\Omega$ , then we have

$$\int_{\Omega} \varphi \{a, b\} = \int_{\Omega} a \{b, \varphi\}.$$

■

The following result shows that critical points of  $E$  on  $F_m$  are also critical points of  $E$  on  $H^1 \times H^1$ .

**Lemma 3** Assume that  $H = (a, b) \in F_m$  is a minimizer of  $E$  on  $F_m$ . Then

1) there exists  $\lambda \in \mathbb{R}^*$  such that  $\Psi = (a_1, b_1, \varphi_1) = (\lambda a, \lambda b, \lambda^2 \varphi)$  satisfies equation (1.1).

2)  $\Psi$  verifies the boundary conditions (1.5).

3)  $\int_{\Omega} \nabla a \cdot \nabla b = 0$ .

4)  $\|\nabla a\|_{L^2} = \|\nabla b\|_{L^2}$ .

5) there exists  $c \in \mathbb{R}$  such that

$$\langle \partial_z \Psi, \partial_z \Psi \rangle = \frac{c}{z^2},$$

where  $\partial_z = \frac{1}{2}(\partial_x - i\partial_y)$ .

6)  $\Psi$  is regular on  $\bar{\Omega}$ .

■

*Proof.* The proof is very similar to the proof of Theorem 3.2 in [6]. We just need to adapt it to our equivariant setting.

Let  $\Lambda = (\alpha, \beta) \in F_m$ . Denote by  $\psi$  the unique solution of the following equation

$$\begin{cases} -\Delta\psi = \{\alpha, b\} + \{a, \beta\}, & \text{in } \Omega \\ \psi = 0, & \text{on } \partial\Omega. \end{cases} \quad (2.6)$$

We claim that  $\psi \circ A = \psi$ . Indeed, note that,

$$\{\alpha, b\} + \{a, \beta\} = \det(dH + d\Lambda) - \det(dH) - \det(d\Lambda)$$

and thus as in Lemma 1, we deduce that

$$(\{\alpha, b\} + \{a, \beta\}) \circ A = \{\alpha, b\} + \{a, \beta\}. \quad (2.7)$$

Hence, by the same argument as before, we establish the claim. Now set  $\Theta_t = \Theta + t\Lambda$ . Clearly,  $\Theta_t \in F_m$ . A direct calculation leads to

$$\begin{aligned} E(a_t, b_t, \Omega) &= \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2 + 2t \int_{\Omega} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) + O(t^2)}{2\sqrt{\|\nabla \varphi\|_2^2 + 2t \int_{\Omega} \nabla \varphi \cdot \nabla \psi + O(t^2)}} \\ &= \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2 + 2t \int_{\Omega} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) + O(t^2)}{2\left(\|\nabla \varphi\|_2 + \frac{t}{\|\nabla \varphi\|_2} \int_{\Omega} \varphi(\{\alpha, b\} + \{a, \beta\}) + O(t^2)\right)} \\ &= E(a, b, \Omega) \left(1 - \frac{t}{\|\nabla \varphi\|_2^2} \int_{\Omega} \varphi(\{\alpha, b\} + \{a, \beta\})\right. \\ &\quad \left. + \frac{2t}{\|\nabla a\|_2^2 + \|\nabla b\|_2^2} \int_{\Omega} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) + O(t^2)\right). \end{aligned} \quad (2.8)$$

For fixed  $\theta_0 \in [0, 2\pi]$ , we consider the domain  $\Omega_{\theta_0} = \{(x, y), \quad r_0 < r < 1 \text{ and } \theta_0 < \theta < \theta_0 + \frac{2\pi}{m}\}$ . It follows from (2.7) and Lemma 1 that

$$[\varphi(\{\alpha, b\} + \{a, \beta\})] \circ A = \varphi(\{\alpha, b\} + \{a, \beta\}),$$

which implies

$$\int_{\Omega} \varphi(\{\alpha, b\} + \{a, \beta\}) = m \int_{\Omega_{\theta_0}} \varphi(\{\alpha, b\} + \{a, \beta\}). \quad (2.9)$$

On the other hand, we have

$$\begin{aligned} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) \circ A &= \text{tr}([d\Theta] \circ A \cdot [d\Lambda]^t \circ A) \\ &= \text{tr}(A \cdot d\Theta \cdot A^t \cdot A \cdot d\Lambda^t \cdot A^t) \\ &= \text{tr}(A \cdot d\Theta \cdot d\Lambda^t \cdot A^t) \\ &= \text{tr}(d\Theta \cdot d\Lambda^t \cdot A^t \cdot A) \\ &= \nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta, \end{aligned} \quad (2.10)$$

that is,

$$\int_{\Omega} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) = m \int_{\Omega_{\theta_0}} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta). \quad (2.11)$$

Combining (2.8) to (2.11), we obtain

$$\int_{\Omega_{\theta_0}} (\nabla a \cdot \nabla \alpha + \nabla b \cdot \nabla \beta) = \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2} \int_{\Omega_{\theta_0}} \varphi(\{\alpha, b\} + \{a, \beta\}).$$

In particular, if we set  $\alpha, \beta \in C_0^\infty(\Omega_{\theta_0})$ , we deduce from Lemma 2 that

$$\begin{cases} -\Delta a = \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2} \{b, \varphi\}, & \text{in } \Omega_{\theta_0} \\ -\Delta b = \frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2} \{\varphi, a\} & \text{in } \Omega_{\theta_0}. \end{cases} \quad (2.12)$$

Setting  $\lambda = -\sqrt{\frac{\|\nabla a\|_2^2 + \|\nabla b\|_2^2}{2\|\nabla \varphi\|_2^2}}$  and by arbitrariness of  $\theta_0$ , property 1) is demonstrated. Now, choosing  $\alpha, \beta \in C^\infty(\Omega_{\theta_0})$  with  $\alpha = \beta = 0$  on  $\Gamma_1 = \{(r, \theta), \theta = \theta_0 \text{ or } \theta = \theta_0 + \frac{2\pi}{m}\}$ , it follows from Lemma 2

$$\int_{\partial\Omega_{\theta_0}} \frac{\partial a}{\partial n} \cdot \alpha + \frac{\partial b}{\partial n} \cdot \beta = 0,$$

that is,  $\frac{\partial a}{\partial n} = \frac{\partial b}{\partial n} = 0$  on  $\partial\Omega_{\theta_0} \setminus \Gamma_1$ . By arbitrariness of  $\theta_0$ , we establish the property 2).

The properties 3) and 4) are just results of 1), 2) and Lemma 2.

Now, we choose any vector field  $X \in C^\infty(\Omega, \mathbb{R}^2)$  such that  $X \circ A = A \circ X$  and  $X \cdot n = 0$  on  $\partial\Omega$ . Let  $\sigma_t$  be the flow associated to  $X$ . Clearly,

$$\sigma_t \circ A = A \circ \sigma_t.$$

Therefore,  $\Theta \circ \sigma_t \in F_m$ . The proofs of 5) and 6) are the same as the proofs of Theorem 3.2 (vii) and (iv) in [6], respectively. ■

### 3 Study of a minimizing sequence

Through this section we analyze the behaviour of a minimizing sequence in the spirit of the theory developed in [6] (see also [5], [9] or [10]). First, we prove a useful fact.

**Lemma 4** *Assume that  $\Theta = (a, b) \in F_m$ . Then we have*

$$\int_{\Omega} a = \int_{\Omega} b = 0.$$

■

*Proof.* By definition of  $F_m$ , we have

$$\int_{\Omega} \Theta = \int_{\Omega} A^{-1} \circ \Theta \circ A = A^{-1} \int_{\Omega} \Theta \circ A = A^{-1} \int_{\Omega} \Theta.$$

We note that  $A^{-1}$  is a rotation. This implies

$$\int_{\Omega} \Theta = (0, 0).$$

■

Now we consider the minimum of energy functional  $E$ . Set  $G(\Omega) = \inf_{a, b \in H^1 \times H^1} E(a, b, \Omega)$  and  $G_m(\Omega) = \inf_{a, b \in F_m} E(a, b, \Omega)$ . Let  $(a_n, b_n, \varphi_n)$  be a minimizing sequence of  $E$  on  $F_m$ , that is,  $(a_n, b_n) \in F_m$ ,  $(a_n, b_n, \varphi_n)$  satisfying equation (1.3) and

$$E(a_n, b_n, \Omega) = G_m(\Omega) + o(1). \quad (3.13)$$

Without loss of generality, we can assume that  $\|\nabla \varphi_n\|_2 = 1$ . After extracting a subsequence, we may assume that

$$\begin{aligned} a_n &\longrightarrow \alpha \text{ weakly in } H^1 \text{ and strongly in } L^2, \\ b_n &\longrightarrow \beta \text{ weakly in } H^1 \text{ and strongly in } L^2, \\ \varphi_n &\longrightarrow \psi \text{ weakly in } H^1 \text{ and strongly in } L^2. \end{aligned}$$

Obviously,  $(\alpha, \beta) \in F_m$ . First, we recall a technical lemma.

**Lemma 5** (see [11], [6] and also [3]) *We assume that  $\varphi_n$  is a bounded sequence in  $H_0^1 \cap L^\infty$ . Let  $a_n \longrightarrow 0$  weakly in  $H^1$  and strongly in  $L^2$ . Then for every  $b \in H^1$ , we have*

$$\lim_{n \rightarrow \infty} \int \varphi_n \{a_n, b\} = 0. \quad (3.14)$$

■

We state the following result, analogous to Theorem 7.1 in [6].

**Lemma 6** *Under the above assumptions, we have that:*

(1) *if  $\psi = 0$ , then  $\alpha = \beta = 0$ ;*

*or*

(2) *if  $\psi \neq 0$ , then  $(\alpha, \beta, \psi)$  is a minimum of the energy  $E$  on  $F_m$ . Moreover, the following holds:*

$$\begin{aligned} a_n &\longrightarrow \alpha \text{ strongly in } H^1, \\ b_n &\longrightarrow \beta \text{ strongly in } H^1, \\ \varphi_n &\longrightarrow \psi \text{ strongly in } H^1. \end{aligned}$$

■

*Proof.* The proof is the same as the proof of Theorem 7.1 in [6], but here we work with equivariant maps.  $\blacksquare$

In the following, we will suppose that  $\psi = \alpha = \beta = 0$ . Denote by  $M(\mathbb{R}^2)$  the space of non-negative measures on  $\mathbb{R}^2$  with finite mass. Set  $\mu_n = \frac{1}{2}(|\nabla a_n|^2 + |\nabla b_n|^2)dx$  and  $\nu_n = |\nabla \varphi_n|^2 dx$ . We consider the extensions of  $\mu_n$  and  $\nu_n$  to all of  $\mathbb{R}^2$  by valuing 0 in  $\mathbb{R}^2 \setminus \Omega$ . Then  $\{\mu_n\}$  and  $\{\nu_n\}$  are bounded in  $M(\mathbb{R}^2)$ . Modulo a subsequence, we may assume that  $\mu_n \rightarrow \mu$ ,  $\nu_n \rightarrow \nu$  weakly in the sense of measures where  $\mu$  and  $\nu$  are bounded non-negative measure on  $\mathbb{R}^2$ .

**Lemma 7** *Under assumptions of Lemma 6, if  $\psi = \alpha = \beta = 0$ , then we have*

$$G_m(\Omega) \geq \sqrt{m}G(\Omega).$$

$\blacksquare$

*Proof.* Clearly,  $\mu(\mathbb{R}^2 \setminus \bar{\Omega}) = \nu(\mathbb{R}^2 \setminus \bar{\Omega}) = 0$ . Choose  $\xi \in C^\infty(\mathbb{R}^2)$ . Denote by  $\psi_n$  the unique solution of equation (1.3) for  $a = \xi a_n$  and  $b = \xi b_n$ , that is

$$\begin{cases} -\Delta \psi_n = \{\xi a_n, \xi b_n\}, & \text{in } \Omega \\ \psi = 0, & \text{on } \partial\Omega. \end{cases}$$

A computation using the same arguments as in the proof of Lemma 7.5 in [6] gives

$$\lim_{n \rightarrow \infty} \|\nabla(\psi_n - \xi^2 \varphi_n)\|_2 = 0.$$

Hence, we obtain

$$G(\Omega)\|\nabla(\xi^2 \varphi_n)\|_2 + o(1) = G(\Omega)\|\nabla \psi_n\|_2 \leq \frac{1}{2}(\|\nabla(\xi a_n)\|_2^2 + \|\nabla(\xi b_n)\|_2^2).$$

Passing to the limit as  $n \rightarrow \infty$ , there holds

$$G(\Omega)\sqrt{\int \xi^4 d\nu} \leq \int \xi^2 d\mu, \quad \forall \xi \in C_0^\infty(\mathbb{R}^2). \quad (3.15)$$

By approximation, therefore,

$$G(\Omega)\sqrt{\nu(E)} \leq \mu(E) \quad (E \subset \mathbb{R}^2, E \text{ Borel}). \quad (3.16)$$

Now since  $\mu$  is a finite measure, the set

$$D \equiv \{x \in \bar{\Omega}, \mu(\{x\}) > 0\}$$

is at most countable. We can therefore write  $D = \{x_j\}_{j \in J}$ ,  $\mu_{x_j} = \mu(\{x_j\})$  ( $j \in J$ ) so that

$$\mu \geq \sum_{j \in J} \mu_{x_j} \delta_{x_j}.$$

Since (3.16) implies  $\nu$  is absolutely continuous relative to  $\mu$ , we can write

$$\nu(E) = \int_E h d\mu \quad (E \text{ Borel}), \quad (3.17)$$

where

$$h(x) \equiv \lim_{r \rightarrow 0} \frac{\nu(B(x, r))}{\mu(B(x, r))}, \quad (3.18)$$

this limit existing for  $\mu$ -a.e.  $x \in \mathbb{R}^2$ . On the other hand, from (3.16), we have

$$(G(\Omega))^2 \frac{\nu(B(x, r))}{\mu(B(x, r))} \leq \mu(B(x, r)),$$

provided  $\mu(B(x, r)) \neq 0$ . Thus we infer

$$h = 0 \quad \mu\text{-a.e. } x \in \mathbb{R}^2 \setminus D. \quad (3.19)$$

Finally, define  $\nu_{x_j} \equiv h(x_j)\mu_{x_j}$ . Then we get

$$\nu = \sum_{j \in J} \nu_{x_j} \delta_{x_j},$$

and

$$G(\Omega) \sqrt{\nu_{x_j}} \leq \mu_{x_j}.$$

However, by symmetry of functions in  $F_m$ , we have  $A^i x_j \in D$  for  $i = 1, 2, \dots, m-1$  provided  $x_j \in D$ . Consequently, by suitably relabelling the  $x_j$ , we may assume that  $x_j \in \Omega_{\theta_0}$  (where  $\Omega_{\theta_0}$  is defined as in the proof of Lemma 3) for  $j \in \{1, \dots, k\} = J'$  and  $k = \text{card}(J)m^{-1}$  and

$$\nu = \sum_{i=0}^{m-1} \sum_{j \in J'} \nu_{x_j} \delta_{A^i x_j} \quad \text{and} \quad \mu \geq \sum_{i=0}^{m-1} \sum_{j \in J'} \mu_{x_j} \delta_{A^i x_j}.$$

On the other hand, we have  $\nu(\bar{\Omega}) = 1$  and  $\mu(\bar{\Omega}) = G_m(\Omega)$ . This implies

$$\begin{aligned} G_m(\Omega) &= \mu(\bar{\Omega}) \geq m \sum_{j \in J'} \mu_{x_j} \geq m \sum_{j \in J'} G(\Omega) \sqrt{\nu_{x_j}} \\ &\geq m G(\Omega) \sqrt{\sum_{j \in J'} \nu_{x_j}} = \sqrt{m} G(\Omega). \end{aligned}$$

■

## 4 Proof of the main Theorem

In view of Lemma 6, the result follows if there is no concentration (i.e. case (1) in Lemma 6 for minimizing sequences does not occur). By Lemma 7, a sufficient condition for that is to assume  $G_m(\Omega) < \sqrt{m}G(\Omega)$ . For this purpose, we set  $a(x, y) = x$  and  $b(x, y) = y$ . It



is obvious to see that  $(a, b) \in F_m$  and  $E(a, b, \Omega) > 0$ . For any fixed  $r_0 > 0$ , we can choose some  $m \in \mathbb{N}$  such that

$$\sqrt{m}G(\Omega) > E(a, b, \Omega) \geq G_m(\Omega).$$

Thus, the main Theorem is proved.

*Acknowledgements.* The authors thank the referee for his valuable remarks on a first version of this paper.

## References

- [1] U. Abresh, *Constant mean curvature tori in term of elliptic functions*, J. Reine Angew. Math. **374**, 169-192 (1987).
- [2] L. V. Ahlfors, *Complex analysis*, Mcgraw-Hill, New York (1966).
- [3] H. Brezis and J. M. Coron, *Multiple solutions of H-systemes and Rellich's conjecture*, Comm. Pure. Appl. Math, **37** (1984) 149-187.
- [4] J. M. Coron, *Topologie et cas limite des injections de Sobolev*, C. R. Acad. Sc. Paris, **299**, Ser. I (1984) 209-212.
- [5] L. C. Evans, *Weak convergence methods for nonlinear partial differential equations*, Regional conference series in mathematics, **74** (1990).
- [6] Y. Ge, *Estimations of the best constant involving the  $L^2$  norm in Wente's inequality and compact H-surfaces in Euclidean space*, Control, Optimisation and Calculus of Variations, Vol. **3**, (1998) 263-300.
- [7] F. Hélein, *Applications harmoniques, lois de conservation et repère mobile*, Diderot éditeur, Paris-New York, 1996 (see Chapter III).
- [8] J. Jost, *Two-dimensional geometric variational problems*, Wiley (1991).
- [9] P. L. Lions, *The concentration-compactness principle in the calculus of variations: The limit case. Part I and Part II*, Rev. Mat. Ibero. **1**(1) (1985) 145-201 and **1**(2) (1985) 45-121.

- [10] M. Struwe, *Variational Methods*, Springer, Berlin-Heidelberg-New York-Tokyo (1990).
- [11] H. Wente, *An existence Theorem for surfaces of constant mean curvature*, J. Math. Anal. Appl, **26** (1969) 318-344.
- [12] H. Wente, *Large solutions to the volume constrained Plateau problem*, Arch. rat. Mech. Anal. **75**, 59-77 (1980).
- [13] H. Wente, *Counter-example to a conjecture of H. Hopf*, Pacific J. Math. **121**, 193-243 (1986).

Y. Ge ( [ge@cmla.ens-cachan.fr](mailto:ge@cmla.ens-cachan.fr) )  
C.M.L.A., E.N.S de Cachan  
61, avenue du Président Wilson  
94235 Cachan Cedex, France  
and  
Département de Mathématiques  
Faculté de Sciences et Technologie  
Université Paris XII-Val de Marne  
61, avenue du Général de Gaulle  
94010 Créteil Cedex, France

F. Hélein ( [helein@cmla.ens-cachan.fr](mailto:helein@cmla.ens-cachan.fr) )  
C.M.L.A., E.N.S de Cachan  
61, avenue du Président Wilson  
94235 Cachan Cedex, France