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Almost-crystallographic groups as quotients of Artin braid groups

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Abstract

Let $n, k \geq 3$. In this paper, we analyse the quotient group $B_n/\Gamma_k(P_n)$ of the Artin braid group B_n by the subgroup $\Gamma_k(P_n)$ belonging to the lower central series of the Artin pure braid group P_n . We prove that it is an almost-crystallographic group. We then focus more specifically on the case $k = 3$. If $n \geq 5$, and if $\tau \in \mathbb{N}$ is such that $\gcd(\tau, 6) = 1$, we show that $B_n/\Gamma_3(P_n)$ possesses torsion τ if and only if S_n does, and we prove that there is a one-to-one correspondence between the conjugacy classes of elements of order τ in $B_n/\Gamma_3(P_n)$ with those of elements of order τ in the symmetric group S_n . We also exhibit a presentation for the almost-crystallographic group $B_n/\Gamma_3(P_n)$. Finally, we obtain some 4-dimensional almost-Bieberbach subgroups of $B_3/\Gamma_3(P_3)$, we explain how to obtain almost-Bieberbach subgroups of $B_4/\Gamma_3(P_4)$ and $B_3/\Gamma_4(P_3)$, and we exhibit explicit elements of order 5 in $B_5/\Gamma_3(P_5)$.

1 Introduction

In this paper, we continue our study of quotients of the Artin braid group B_n by elements of the lower central series $(\Gamma_k(P_n))_{k \in \mathbb{N}}$ of the Artin pure braid group P_n . In the paper [GGO1], we analysed the group $B_n/\Gamma_2(P_n)$ in some detail, and we proved notably that it is a crystallographic group. Using different techniques, I. Marin generalised the results of [GGO1] to generalised braid groups associated to arbitrary complex reflection groups [Ma]. In the present paper, we show for all $n, k \geq 3$, the quotient $B_n/\Gamma_k(P_n)$ of B_n by $\Gamma_k(P_n)$ is an almost-crystallographic group, and we investigate more thoroughly the group $B_n/\Gamma_3(P_n)$. As in [GGO1], some natural questions that arise are the existence or not of torsion, the realisation of elements of finite order and that of finite subgroups, their conjugacy classes and the relation with other types of group, such as (almost-) crystallographic groups.

This paper is organised as follows. In Section 2, we recall some definitions and facts about the Artin braid groups, their quotients by the elements of the lower central series $(\Gamma_k(P_n))_{k \in \mathbb{N}}$ of P_n , and almost-crystallographic groups. In Section 3, we discuss the quotient $B_n/\Gamma_k(P_n)$, where $n, k \geq 3$. If G is a group, for all $q \in \mathbb{N}$, let $L_q(G)$ denote the lower central series quotient $\Gamma_q(G)/\Gamma_{q+1}(G)$. These quotients have been widely studied, see [Hal, MKS] for example. In the case of P_n , it is known that $L_q(P_n)$ is a free Abelian group of finite rank, and by [LVW, Theorem 4.6], its rank is given by:

$$\text{rank}(L_q(P_n)) = \frac{1}{q} \sum_{j=1}^{n-1} \sum_{d|q} \mu(d) j^{q/d} \quad (1)$$

where μ is the Möbius function. From this, it follows that the nilpotent group $P_n/\Gamma_k(P_n)$ of nilpotency class $k - 1$ is also torsion free (see Lemma 11(a)). Using (1), in Proposition 10, we calculate the Hirsch length of $P_n/\Gamma_k(P_n)$, which is equal to $\sum_{i=1}^{k-1} \text{rank}(L_i(P_n))$. In particular, the Hirsch length of $P_n/\Gamma_3(P_n)$ and $P_n/\Gamma_4(P_n)$ is equal to $\binom{n}{2} + \binom{n}{3}$ and $\binom{n}{2} + \binom{n}{3} + 2\binom{n+1}{4}$ respectively (see [CS, Theorem 1.1]). Using a criterion given in [De], we are then able to show that $B_n/\Gamma_k(P_n)$ is an almost-crystallographic group.

THEOREM 1. *Let $n, k \geq 3$. The group $B_n/\Gamma_k(P_n)$ is an almost-crystallographic group whose holonomy group is S_n and whose dimension is equal to $\sum_{q=1}^{k-1} \left(\frac{1}{q} \sum_{j=1}^{n-1} \sum_{d|q} \mu(d) j^{q/d} \right)$. In particular, the dimension of $B_n/\Gamma_3(P_n)$ (resp. of $B_n/\Gamma_4(P_n)$) is equal to $\binom{n}{2} + \binom{n}{3}$ (resp. to $\binom{n}{2} + \binom{n}{3} + 2\binom{n+1}{4}$).*

Torsion-free almost-crystallographic groups, or *almost-Bieberbach* groups, are of particular interest because they arise as fundamental groups of infra-nilmanifolds. Infra-nilmanifolds are manifolds that are finitely covered by a nilmanifold and represent a natural generalisation of flat manifolds. They play an important rôle in dynamical systems, notably in the study of expanding maps and Anosov diffeomorphisms [DD]. The reader may consult [De, Gr, Ru] for more information about these topics.

Another interesting problem is that of the nature of the finite-order elements of $B_n/\Gamma_k(P_n)$. Knowledge of the torsion of this group may be used for example to construct almost-Bieberbach subgroups. In this direction, we prove Theorem 2, which generalises [GGO1, Theorem 2].

THEOREM 2. *Let $n, k \geq 3$. Then the quotient group $B_n/\Gamma_k(P_n)$ has no elements of order 2 nor of order 3.*

From this, we are able to deduce Corollary 15, which proves the existence of almost-Bieberbach groups in $B_n/\Gamma_k(P_n)$ for all $n, k \geq 3$. In much of the rest of the paper, we focus our attention on the case $k = 3$. To study the torsion of $B_n/\Gamma_3(P_n)$, an explicit basis of $\Gamma_2(P_n)/\Gamma_3(P_n)$ is introduced in Section 3.2, and in Section 4.1, we partition this basis into the orbits of the action by conjugation of a certain element δ_n of $B_n/\Gamma_3(P_n)$. In Section 4.2, we exhibit presentations of $P_n/\Gamma_3(P_n)$ and $B_n/\Gamma_3(P_n)$. This enables us to study the finite-order elements of $B_n/\Gamma_3(P_n)$ and their conjugacy classes in Section 4.3. The following result shows that the torsion of $B_n/\Gamma_3(P_n)$ coincides with that of S_n if we remove the elements whose order is divisible by 2 or 3.

THEOREM 3. *Let $n \geq 5$, and let $\tau \in \mathbb{N}$ be such that $\gcd(\tau, 6) = 1$. Then the group $B_n/\Gamma_3(P_n)$ admit finite-order elements of torsion τ if and only if S_n does. Further, if $x \in S_n$ is of order τ , there exists $\alpha \in S_n$ of order τ such that $\bar{\sigma}(\alpha) = x$, in particular $\bar{\sigma}(\alpha)$ and x have the same cycle type.*

We end Section 4 with an analysis of the conjugacy classes of the finite-order elements of $B_n/\Gamma_3(P_n)$, the main result in this direction being the following.

THEOREM 4. *Let $n \geq 5$, and let α and β be two finite-order elements of $B_n/\Gamma_3(P_n)$ whose associated permutations have the same cycle type. Then α and β are conjugate in $B_n/\Gamma_3(P_n)$.*

In Section 5, we discuss some aspects of the quotients $B_n/\Gamma_3(P_n)$, where n is small. In Section 5.1, we obtain some almost-Bieberbach subgroups of $B_3/\Gamma_3(P_3)$ of dimension 4 that are the fundamental groups of orientable 4-dimensional infra-nilmanifolds, and in Section 5.2, we compute δ_5^5 in terms of the chosen basis of $\Gamma_2(P_5)/\Gamma_3(P_5)$. Using the constructions of Section 4.3, this allows us to exhibit explicit elements of order 5 in $B_5/\Gamma_3(P_5)$.

Another important question in our study is the existence and embedding of finite groups in $B_n/\Gamma_3(P_n)$. For cyclic groups, the answer is given by Theorem 3, and this may be generalised to Abelian groups in Corollary 24 using Lemma 19(b). We have proved recently in [GGO2] that Cayley-type theorems hold for $B_n/\Gamma_2(P_n)$ and $B_n/\Gamma_3(P_n)$, namely that if G is any finite group of odd order n (resp. of order n relatively prime with 6) then G embeds in $B_n/\Gamma_2(P_n)$ (resp. in $B_n/\Gamma_3(P_n)$). In the case of $B_n/\Gamma_2(P_n)$, the same result has been proved independently by V. Beck and I. Marin within the more general setting of complex reflection groups [BM]. In the same paper, we also show that with appropriate conditions on n and m , two families of groups of the form $G = \mathbb{Z}_n \rtimes_{\theta} \mathbb{Z}_m$ embed in $B_n/\Gamma_2(P_n)$ and $B_n/\Gamma_3(P_n)$.

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2 Preliminaries

In this section, we recall some definitions and results about Artin braid groups, (almost-) crystallographic groups, and the relations between them that will be used in this paper.

2.1 Artin braid groups

We start by recalling some facts about the Artin braid group B_n on n strings (see [Han] for more details). It is well known that B_n possesses a presentation with generators $\sigma_1, \dots, \sigma_{n-1}$ that are subject to the following relations:

$$\begin{cases} \sigma_i \sigma_j = \sigma_j \sigma_i \text{ for all } 1 \leq i < j \leq n-1 \text{ such that } |i-j| \geq 2 \\ \sigma_{i+1} \sigma_i \sigma_{i+1} = \sigma_i \sigma_{i+1} \sigma_i \text{ for all } 1 \leq i \leq n-2. \end{cases} \quad (2)$$

Let $\sigma: B_n \rightarrow S_n$ be the homomorphism defined on the given generators of B_n by $\sigma(\sigma_i) = (i, i+1)$ for all $1 \leq i \leq n-1$. Just as for braids, we read permutations from left to right so that if $\alpha, \beta \in S_n$, their product is defined by $\alpha \cdot \beta(i) = \beta(\alpha(i))$ for $i = 1, 2, \dots, n$. The pure braid group P_n on n strings is defined to be the kernel of σ , from which we obtain the following short exact sequence:

$$1 \rightarrow P_n \rightarrow B_n \xrightarrow{\sigma} S_n \rightarrow 1. \quad (3)$$

Let G be a group. If $g, h \in G$ then $[g, h] = ghg^{-1}h^{-1}$ will denote their commutator, and if H, K are subgroups of H then we set $[H, K] = \langle [h, k] \mid k \in H, k \in K \rangle$. The *lower central series* $\{\Gamma_i(G)\}_{i \in \mathbb{N}}$ of G is defined inductively by $\Gamma_1(G) = G$, and $\Gamma_{i+1}(G) = [G, \Gamma_i(G)]$ for all $i \in \mathbb{N}$. If $i = 2$, $\Gamma_2(G)$ is the *commutator subgroup* of G . For all $i, j \in \mathbb{N}$ with $j > i$, $\Gamma_j(G)$ is a normal subgroup of $\Gamma_i(G)$. Following P. Hall, for any group-theoretic property \mathcal{P} , G is said to be *residually* \mathcal{P} if for any (non-trivial) element $x \in G$, there exist a group H that possesses property \mathcal{P} and a surjective homomorphism $\varphi: G \rightarrow H$ such that $\varphi(x) \neq 1$. It is well known that a group G is *residually nilpotent* if and only if $\bigcap_{i \geq 1} \Gamma_i(G) = \{1\}$. The lower central series of groups and their successive quotients $\Gamma_i(G)/\Gamma_{i+1}(G)$ are isomorphism invariants, and have been widely studied using commutator calculus, in particular for free groups of finite rank [Hal, MKS]. Falk and Randell, and independently Kohno, investigated the lower central series of the pure braid group P_n , and proved that P_n is residually nilpotent [FR2, Ko].

A presentation of P_n is given by the set of generators $\{A_{i,j}\}_{1 \leq i < j \leq n}$, where:

$$A_{i,j} = \sigma_{j-1} \cdots \sigma_{i+1} \sigma_i^2 \sigma_{i+1}^{-1} \cdots \sigma_{j-1}^{-1}, \quad (4)$$

subject to the following relations that are expressed in terms of commutators (see [MK,

Remark 3.1, p. 56] or [Han, Chapter 1, Lemma 4.2]):

$$\begin{cases} [A_{r,s}, A_{i,j}] = 1 & \text{if } 1 \leq r < s < i < j \leq n \text{ or } 1 \leq r < i < j < s \leq n \\ [A_{r,s}, A_{r,j}] = [A_{s,j}^{-1}, A_{r,j}] & \text{if } 1 \leq r < s < j \leq n \\ [A_{r,s}, A_{s,j}] = [A_{s,j}^{-1}, A_{r,j}^{-1}] & \text{if } 1 \leq r < s < j \leq n \\ [A_{r,i}, A_{s,j}] = [[A_{i,j}^{-1}, A_{r,j}^{-1}], A_{s,j}] & \text{if } 1 \leq r < s < i < j \leq n. \end{cases} \quad (5)$$

For notational reasons, if $1 \leq i < j \leq n$, we set $A_{j,i} = A_{i,j}$, and if $A_{i,j}$ appears in a word of P_n with exponent $m_{i,j} \in \mathbb{Z}$, then we let $m_{j,i} = m_{i,j}$. It follows from the presentation (5) that $P_n/\Gamma_2(P_n)$ is isomorphic to $\mathbb{Z}^{n(n-1)/2}$, and that a basis of $P_n/\Gamma_2(P_n)$ is given by $\{A_{i,j}\}_{1 \leq i < j \leq n}$ where by abuse of notation, the $\Gamma_2(P_n)$ -coset of $A_{i,j}$ will also be denoted by $A_{i,j}$. For all $k \geq 2$, equation (3) gives rise to the following short exact sequence:

$$1 \longrightarrow P_n/\Gamma_k(P_n) \longrightarrow B_n/\Gamma_k(P_n) \xrightarrow{\bar{\sigma}} S_n \longrightarrow 1, \quad (6)$$

where $\bar{\sigma}: B_n/\Gamma_k(P_n) \longrightarrow S_n$ is the homomorphism induced by σ . In much of what follows, we shall be interested in the action by conjugation of $B_n/\Gamma_2(P_n)$ on $\Gamma_2(P_n)/\Gamma_3(P_n)$. For all $1 \leq k \leq n-1$ and $1 \leq i < j \leq n$, the action of B_n on P_n described in [GG01, equation (7)] may be rewritten as:

$$\sigma_k A_{i,j} \sigma_k^{-1} = \begin{cases} A_{i,j}^{-1} A_{i,j-1} A_{i,j} & \text{if } j = k+1 \text{ and } i < k \\ A_{i,j}^{-1} A_{i-1,j} A_{i,j} & \text{if } i = k+1 \\ A_{\sigma_k^{-1}(i), \sigma_k^{-1}(j)} & \text{otherwise,} \end{cases} \quad (7)$$

where by abuse of notation, we write $\sigma_k^{-1}(i) = \sigma(\sigma_k^{-1})(i)$, σ being as in equation (3). This action was used in [GG01] to prove the following proposition.

PROPOSITION 5 ([GG01, Proposition 12]). *Let $\alpha \in B_n/\Gamma_2(P_n)$, and let π be the permutation induced by α^{-1} . Then for all $1 \leq i < j \leq n$, $\alpha A_{i,j} \alpha^{-1} = A_{\pi(i), \pi(j)}$ in $P_n/\Gamma_2(P_n)$.*

For all $\alpha \in B_n/\Gamma_3(P_n)$, and all $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$, we claim that:

$$\alpha [A_{i,j}, A_{r,s}] \alpha^{-1} = [\alpha A_{i,j} \alpha^{-1}, \alpha A_{r,s} \alpha^{-1}] = [A_{\pi(i), \pi(j)}, A_{\pi(r), \pi(s)}] \text{ in } P_n/\Gamma_3(P_n), \quad (8)$$

where π is the permutation induced by α^{-1} , and by abuse of notation, $A_{i,j}$ and $A_{r,s}$ are considered as elements of $P_n/\Gamma_3(P_n)$. To see this, first note that the first equality of (8) clearly holds, and that for all $k \in \mathbb{N}$, we have the following short exact sequence:

$$1 \longrightarrow \Gamma_k(P_n)/\Gamma_{k+1}(P_n) \longrightarrow P_n/\Gamma_{k+1}(P_n) \longrightarrow P_n/\Gamma_k(P_n) \longrightarrow 1. \quad (9)$$

Taking $k = 2$ in (9) and using Proposition 5, there exist $\gamma_1, \gamma_2 \in \Gamma_2(P_n)/\Gamma_3(P_n)$ such that $\alpha A_{i,j} \alpha^{-1} = \gamma_1 A_{\pi(i), \pi(j)}$ and $\alpha A_{r,s} \alpha^{-1} = \gamma_2 A_{\pi(r), \pi(s)}$. The second equality of (8) then follows using standard commutator relations.

2.2 Almost-crystallographic groups

In this section, we recall briefly the definitions of almost-crystallographic and almost-Bieberbach groups, which are natural generalisations of crystallographic and Bieberbach groups, as well as a characterisation of almost-crystallographic groups. For more details about crystallographic groups, see [Ch, Section I.1.1], [De, Section 2.1] or [Wo, Chapter 3].

Given a connected and simply-connected nilpotent Lie group N , the group $\text{Aff}(N)$ of affine transformations of N is defined by $\text{Aff}(N) = N \rtimes \text{Aut}(N)$, and acts on N by:

$$(n, \varphi) \cdot m = n\varphi(m) \text{ for all } m, n \in N \text{ and } \varphi \in \text{Aut}(N).$$

DEFINITION 6 ([De, Sec. 2.2, p. 15]). Let N be a connected, simply-connected nilpotent Lie group, and consider a maximal compact subgroup C of $\text{Aut}(N)$. A uniform discrete subgroup E of $N \rtimes C$ is called an *almost-crystallographic group*, and its dimension is defined to be that of N . A torsion-free, almost-crystallographic group is called an *almost-Bieberbach group*, and the quotient space $E \backslash N$ is called an *infra-nilmanifold*. If further $E \subseteq N$, we say that the space $E \backslash N$ is a *nilmanifold*.

It is well known that infra-nilmanifolds are classified by their fundamental group that is almost-crystallographic [Au] Every almost-crystallographic subgroup E of the group $\text{Aff}(N)$ fits into an extension:

$$1 \longrightarrow \Lambda \longrightarrow E \longrightarrow F \longrightarrow 1, \quad (10)$$

where $\Lambda = E \cap N$ is a uniform lattice in N , and F is a finite subgroup of C known as the *holonomy group* of the corresponding infra-nilmanifold $E \backslash N$ [Au]. Let M be an infra-nilmanifold whose fundamental group E is almost-crystallographic. Following [GPS, Page 788], we recall the construction of a faithful linear representation associated with the extension (10). Suppose that the nilpotent lattice Λ is of class $c + 1$ i.e. $\Gamma_c(\Lambda) \neq 1$ and $\Gamma_{c+1}(\Lambda) = 1$. For $i = 1, \dots, c$, let $Z_i = \Gamma_i(\Lambda) / \Gamma_{i+1}(\Lambda)$ denote the factor groups of the lower central series $\{\Gamma_i(\Lambda)\}_{i=1}^{c+1}$ of Λ . We will assume further that these quotients are torsion free, since this will be the case for the groups that we will study in the following sections. Thus $Z_i \cong \mathbb{Z}^{k_i}$ for all $1 \leq i \leq c$ and for some $k_i > 0$. The *rank* or *Hirsch number* of Λ is equal to $\sum_{i=1}^c k_i$. The action by conjugation of E on Λ induces an action of E on Z_i which factors through an action of the group E/Λ (the holonomy group F), because Λ acts trivially on Z_i . This gives rise to a faithful representation $\theta_F: F \longrightarrow \text{GL}(n, \mathbb{Z})$ via the composition:

$$\theta_F: F \longrightarrow \text{GL}(k_1, \mathbb{Z}) \times \dots \times \text{GL}(k_c, \mathbb{Z}) \longrightarrow \text{GL}(n, \mathbb{Z}), \quad (11)$$

where n , which is the rank of Γ , is also equal to the dimension of N . Using [GPS, Remark 2.5], this representation will be used in Section 5.1 to decide whether M is orientable or not.

In order to prove Theorem 1, we shall use part of the algebraic characterisation of almost-crystallographic groups given in [De, Theorem 3.1.3] as follows.

THEOREM 7 ([De, Theorem 3.1.3]). *Let E be a polycyclic-by-finite group. Then E is almost-crystallographic if and only if it has a nilpotent subgroup, and possesses no non-trivial finite normal subgroups.*

3 The almost-crystallographic group $B_n/\Gamma_k(P_n)$

Let $n, k \geq 3$. In this section, we study the group $B_n/\Gamma_k(P_n)$. In Section 3.1, we start by recalling some results about the quotient groups $\Gamma_k(P_n)/\Gamma_{k+1}(P_n)$ that appear in (9). In Section 3.2, we prove Theorems 1 and 2 which state that the groups $B_n/\Gamma_k(P_n)$ are almost-crystallographic, and that $B_n/\Gamma_k(P_n)$ possesses no element of order 2 or 3 respectively. This allows us to prove in Corollary 15, which shows that if H is a subgroup of S_n whose order is not divisible by any prime other than 2 or 3 then the subgroup $\bar{\sigma}^{-1}(H)/\Gamma_k(P_n)$ of $B_n/\Gamma_k(P_n)$ is almost-Bieberbach.

3.1 The rank of the free Abelian group $\Gamma_k(P_n)/\Gamma_{k+1}(P_n)$

Let $n \geq 2$ and $k \geq 1$. The group $\Gamma_k(P_n)/\Gamma_{k+1}(P_n)$, which we shall denote by $L_k(P_n)$, is free Abelian of finite rank, and by [FR1, Theorem 4.2] and [Ko, Theorem 4.5], its rank is related to the Poincaré polynomial of certain hyperplane complements. Using Chen groups, Cohen and Suciu gave explicit formulæ for $\text{rank}(L_k(P_n))$ for $k \in \{1, 2, 3\}$ and all $n \geq 2$ [CS, Theorem 1.1 and page 46]. More generally, by [LVW, Theorem 4.6], the rank of $L_k(P_n)$ is given by equation (1). In practice, we may compute these ranks as follows. If $k \geq 2$, let k^* be the product of the distinct prime divisors of k . Then (1) may be rewritten as:

$$\text{rank}(L_k(P_n)) = \frac{1}{k} \sum_{j=1}^{n-1} \sum_{d|k^*} \mu(d) j^{k/d} = \frac{1}{k} \sum_{d|k^*} \mu(d) S_{k/d}(n), \quad (12)$$

where $S_r(n) = \sum_{j=1}^{n-1} j^r$. The number of summands in the expression $\frac{1}{k} \sum_{d|k^*} \mu(d) j^{k/d}$ is equal to 2^t .

PROPOSITION 8. *Let $n \geq 2$. Then $\text{rank}(L_k(P_n))$ is a polynomial of degree $k + 1$ in the variable n .*

Proof. By [N, Chapter XVI], the sum $S_{k/d}(n)$ is a polynomial of degree $\frac{k}{d} + 1$ in the variable n , so $\text{rank}(L_k(P_n))$ is also a polynomial in the variable n . The result follows by noting that $d = 1$ divides k^* . \square

REMARKS 9.

(a) For small values of r , a polynomial expression for $S_r(n)$ was computed for example in [AIK, Section 1.2] for $1 \leq r \leq 6$, and [N, Chapter XVI, page 296] or [Wi, Tables I and II] for $1 \leq r \leq 10$.

(b) Using the description given in equation (12), we have computed $\text{rank}(L_k(P_n))$ for all $1 \leq k \leq 10$. If $n = 2$ or 3 , we obtain the equalities $\text{rank}(L_2(P_n)) = \binom{n}{3}$ and $\text{rank}(L_3(P_n)) = 2 \binom{n+1}{4}$ given in [CS, Theorem 1.1]. For $3 \leq k \leq 10$, $\text{rank}(L_k(P_n))$ is the product of $\binom{n+1}{4}$ by a polynomial in the variable n of degree $k - 3$. The authors do not know whether this is true in general.

(c) The numbers $\text{rank}(L_k(P_n))$ may be expressed in terms of Bernoulli numbers since the latter are closely related to sums of powers of consecutive integers (see [AIK] for more information about Bernoulli numbers, and especially Formula (1.1) on page 1).

3.2 The quotient groups $B_n/\Gamma_k(P_n)$

The proofs of the first two results of this section are straightforward, but will be useful in the analysis of the group $B_n/\Gamma_k(P_n)$. First note that $P_n/\Gamma_k(P_n)$ is a nilpotent group of nilpotency class $k - 1$, and as we shall see in Lemma 11(a), it is torsion free.

PROPOSITION 10. *Let $n, k \geq 3$. The Hirsch length of the nilpotent group $P_n/\Gamma_k(P_n)$ is equal to $\sum_{q=1}^{k-1} \left(\frac{1}{q} \sum_{j=1}^{n-1} \sum_{d|q} \mu(d) j^{q/d} \right)$. In particular, the Hirsch length of $P_n/\Gamma_3(P_n)$ (resp. of $P_n/\Gamma_4(P_n)$) is equal to $\binom{n}{2} + \binom{n}{3}$ (resp. to $\binom{n}{2} + \binom{n}{3} + 2\binom{n+1}{4}$).*

Proof. Let $n, k \geq 3$. Since the Hirsch length of a nilpotent group is equal to the sum of the ranks of the consecutive lower central series quotients, the first part of the statement follows from equation (12). If $q \in \{3, 4\}$, the formulæ is then a consequence of those given in Remarks 9(b). \square

LEMMA 11.

(a) *Let $n, k \geq 2$. Then the group $P_n/\Gamma_k(P_n)$ is torsion free.*

(b) *Let $n \geq 3$, let $k \geq l \geq 1$, and let G be a finite group. If $B_n/\Gamma_k(P_n)$ possesses a (normal) subgroup isomorphic to G then $B_n/\Gamma_l(P_n)$ possesses a (normal) subgroup isomorphic to G . In particular, if p is prime, and if $B_n/\Gamma_l(P_n)$ has no p -torsion then $B_n/\Gamma_k(P_n)$ has no p -torsion.*

Proof.

(a) The proof is by induction on k . If $k = 2$ then $P_n/\Gamma_2(P_n) \cong \mathbb{Z}^{n(n-1)/2}$, which implies the result in this case. Suppose then that the result holds for some $k \geq 2$. Then the quotient of the short exact sequence (9) is torsion free by induction, and the kernel is torsion free by the results mentioned at the beginning of Section 3.1. It follows that $P_n/\Gamma_{k+1}(P_n)$ is also torsion free.

(b) Assume that $B_n/\Gamma_k(P_n)$ possesses a (normal) subgroup isomorphic to G . For all $j \geq 2$, we have a central extension of the form:

$$1 \longrightarrow \Gamma_{j-1}(P_n)/\Gamma_j(P_n) \longrightarrow B_n/\Gamma_j(P_n) \longrightarrow B_n/\Gamma_{j-1}(P_n) \longrightarrow 1. \quad (13)$$

As we mentioned above, the kernel of the short exact sequence (13) is torsion free, so the restriction of the homomorphism $B_n/\Gamma_j(P_n) \longrightarrow B_n/\Gamma_{j-1}(P_n)$ to this subgroup is injective, and $B_n/\Gamma_{j-1}(P_n)$ also has a (normal) subgroup isomorphic to G . The first part of the statement then follows by reverse induction on j . In particular, if $B_n/\Gamma_k(P_n)$ has p -torsion, the second part of the statement follows by taking $G = \mathbb{Z}_p$. \square

We now prove Theorem 1.

Proof of Theorem 1. Let $n, k \geq 3$. In order to prove that the group $B_n/\Gamma_k(P_n)$ is almost-crystallographic, by Theorem 7, it suffices to show that it does not have finite (non-trivial) normal subgroups. Suppose on the contrary that it possesses such a subgroup H . Taking $l = 2$ (resp. $l = 1$) in Lemma 11(b), it follows that $B_n/\Gamma_2(P_n)$ (resp. S_n) has a normal subgroup H' (resp. \widehat{H}) isomorphic to H . Hence the groups \widehat{H}, H' and H are either isomorphic to S_n , or to the alternating subgroup A_n , or if $n = 4$, to $\mathbb{Z}_2 \oplus \mathbb{Z}_2$. If $n \geq 4$, or if $n = 3$ and $\widehat{H} = S_3$, it follows that H' possesses 2-torsion, but this contradicts [GGO1, Theorem 2]. It remains to analyse the case $n = 3$ and $\widehat{H} = A_3 \cong \mathbb{Z}_3$. Then

H' has 3-torsion, and since the elements of $B_3/\Gamma_2(P_3)$ of order 3 are pairwise conjugate [GG01, Theorem 5], and H' is normal in $B_3/\Gamma_2(P_3)$, the only elements of $B_3/\Gamma_2(P_3)$ of order 3 are those belonging to H' , but this contradicts the fact that $B_3/\Gamma_2(P_3)$ possesses infinitely many elements of order 3 [GG01, Proposition 21]. This proves the first part of the statement of Theorem 1. For the second part, the dimension of the almost-crystallographic group $B_n/\Gamma_k(P_n)$ is equal to the Hirsch length of $P_n/\Gamma_k(P_n)$ given in Proposition 10, and the associated holonomy group is the quotient group $B_n/\Gamma_k(P_n)/P_n/\Gamma_k(P_n)$, which is isomorphic to S_n . \square

REMARKS 12.

(a) As in [GG01, Corollary 13], the first part of Theorem 1 may be generalised as follows: let $n, k \geq 3$, and let H be a subgroup of S_n . Then the group $\sigma^{-1}(H)/\Gamma_k(P_n)$ is an almost almost-crystallographic group whose holonomy group is H and whose dimension is equal to that of $B_n/\Gamma_k(P_n)$.

(b) Using Theorem 1, we may compute the dimension of the almost-crystallographic group $B_n/\Gamma_k(P_n)$. In Table 1, we exhibit these dimensions for small values of n and k . This gives us an idea about their growth in terms of k and n .

$k \backslash n$	3	4	5	6
2	3	6	10	15
3	4	10	20	35
4	6	20	50	105
5	9	41	131	336

Table 1: The dimension of the almost-crystallographic group $B_n/\Gamma_k(P_n)$ for $n = 3, 4, 5, 6$ and $k = 2, 3, 4, 5$.

In order to prove Theorem 2, we first recall the following lemma from [GG01].

LEMMA 13 ([GG01, Lemma 28]). *Let $k, n \geq 3$ and $r \geq 0$ such that k is odd and $r + k \leq n$, and define $\delta_{r,k}, \alpha_{r,k} \in B_n/\Gamma_2(P_n)$ by:*

$$\delta_{r,k} = \sigma_{r+k-1} \cdots \sigma_{r+\frac{k+1}{2}} \sigma_{r+\frac{k-1}{2}}^{-1} \cdots \sigma_{r+1}^{-1} \text{ and } \alpha_{r,k} = \sigma_{r+1} \cdots \sigma_{r+k-1}. \quad (14)$$

Then $\delta_{r,k}$ is of order k in $B_n/\Gamma_2(P_n)$, and satisfies:

$$\delta_{r,k} = (A_{r+\frac{k+1}{2}, r+k} A_{r+\frac{k+3}{2}, r+k} \cdots A_{r+k-1, r+k}) \alpha_{r,k}^{-1}. \quad (15)$$

Let $n \geq 3$. We now describe a basis of the free Abelian group $\Gamma_2(P_n)/\Gamma_3(P_n)$ of rank $\binom{n}{3}$ by Remark 9(b). This group is generated by the elements of the form $[A_{i,j}, A_{r,s}]$, where $1 \leq i < j \leq n$ and $1 \leq r < s \leq n$, but by equation (5), $[A_{i,j}, A_{r,s}] = 1$ in $\Gamma_2(P_n)/\Gamma_3(P_n)$ if $\text{Card}\{i, j, r, s\} \in \{2, 4\}$, so $\Gamma_2(P_n)/\Gamma_3(P_n)$ is generated by the elements of the form $[A_{i,j}, A_{r,s}]$, where $\text{Card}\{i, j, r, s\} = 3$. Let $\alpha_{i,j,k} = [A_{i,j}, A_{j,k}]$ for all $1 \leq i < j < k \leq n$. Using (5) once more, the following equalities hold in $\Gamma_2(P_n)/\Gamma_3(P_n)$:

$$[A_{i,k}, A_{j,k}] = \alpha_{i,j,k}^{-1}, [A_{i,j}, A_{i,k}] = \alpha_{i,j,k}^{-1} \text{ and } [A_{j,k}, A_{i,j}] = \alpha_{i,j,k}^{-1} \quad (16)$$

and if $\alpha_{i,j,k} = \alpha_{r,s,t}^{\pm 1}$ then $\{i, j, k\} = \{r, s, t\}$. So the set

$$\mathcal{B} = \{ \alpha_{i,j,k} \mid 1 \leq i < j < k \leq n \}, \quad (17)$$

generates $\Gamma_2(P_n)/\Gamma_3(P_n)$. Since $\text{Card}(\mathcal{B}) = \binom{n}{3} = \text{rank}(\Gamma_2(P_n)/\Gamma_3(P_n))$, it follows that \mathcal{B} is a basis for $\Gamma_2(P_n)/\Gamma_3(P_n)$, in particular, $\alpha_{i,j,k} \neq 1$. Applying equation (8) to the Artin generators of B_n , and using equation (16), the action of $B_n/\Gamma_3(P_n)$ on the elements of \mathcal{B} given in (17) is as follows:

$$\sigma_k \alpha_{r,s,t} \sigma_k^{-1} = \begin{cases} \alpha_{\sigma_k(r), \sigma_k(s), \sigma_k(t)} & \text{if } \sigma_k(r) < \sigma_k(s) < \sigma_k(t) \\ \alpha_{\sigma_k(s), \sigma_k(r), \sigma_k(t)}^{-1} & \text{if } \sigma_k(r) > \sigma_k(s) \\ \alpha_{\sigma_k(r), \sigma_k(t), \sigma_k(s)}^{-1} & \text{if } \sigma_k(s) > \sigma_k(t) \end{cases} \quad (18)$$

for all $1 \leq k \leq n-1$ and $1 \leq r < s < t \leq n$. Using the action of B_n on P_n given in equation (7) and equation (16), one may check that in $B_n/\Gamma_3(P_n)$:

$$\sigma_k A_{i,j} \sigma_k^{-1} = \begin{cases} A_{i,j-1} \alpha_{i,j-1,j}^{-1} & \text{if } j = k+1 \text{ and } i < k \\ A_{i-1,j} \alpha_{i-1,i,j}^{-1} & \text{if } i = k+1 \\ A_{\sigma_k(i), \sigma_k(j)} & \text{otherwise.} \end{cases} \quad (19)$$

REMARK 14. Consider the action by conjugation of $B_n/\Gamma_3(P_n)$ on $\Gamma_2(P_n)/\Gamma_3(P_n)$ described by (18). The restriction of this action to $P_n/\Gamma_3(P_n)$ on $\Gamma_2(P_n)/\Gamma_3(P_n)$ is trivial, and so the action of $B_n/\Gamma_3(P_n)$ factors through S_n . Further, if $n > 3$, the action of S_n is injective, *i.e.* if α is a non-trivial permutation, then the induced automorphism is different from the identity by (8), (16) and the fact that \mathcal{B} is a basis.

Proof of Theorem 2. Let $n, k \geq 3$. By [GGO1, Theorem 2], $B_n/\Gamma_2(P_n)$ has no 2-torsion, so applying Lemma 11(b), we conclude that this is also the case for $B_n/\Gamma_k(P_n)$. To complete the proof, using Lemma 11(b) once more, it suffices to prove that $B_n/\Gamma_3(P_n)$ has no 3-torsion. Suppose on the contrary that $B_n/\Gamma_3(P_n)$ possesses an element ρ of order 3. Let m denote the number of conjugacy classes of elements of $B_n/\Gamma_2(P_n)$ of order 3. By [GGO1, Theorem 5], m is equal to the number of conjugacy classes of elements of order 3 in S_n , so $m = \lfloor n/3 \rfloor \geq 1$, and using [GGO1, Proposition 29], representatives of the conjugacy classes of elements of order 3 in $B_n/\Gamma_2(P_n)$ are given by the elements of the form $\eta_t = \delta_{3t,3} \cdots \delta_{6,3} \delta_{3,3} \delta_{0,3}$, where $0 \leq t \leq m-1$. Conjugating ρ if necessary and using the short exact sequence (13), there exists $\theta \in \Gamma_2(P_n)/\Gamma_3(P_n)$ such that $\rho = \theta \eta_t$ for some $0 \leq t \leq m-1$ (here η_t is considered as an element of $B_n/\Gamma_3(P_n)$). Note that η_t acts by conjugation on $\bar{\sigma}^{-1}(S_3 \times 1 \times \cdots \times 1)$ and $\bar{\sigma}^{-1}(1 \times 1 \times 1 \times S_{n-3})$, where $\bar{\sigma}$ is as in (6). Further, by (8) and (16), we see that $\eta_t \alpha_{1,2,3} \eta_t^{-1} = \alpha_{1,2,3}$ in $B_n/\Gamma_3(P_n)$, and that if θ' is a word in the elements of $\mathcal{B} \setminus \{\alpha_{1,2,3}\}$ then $\eta_t \theta' \eta_t^{-1}$, written in reduced form as a word in $P_n/\Gamma_3(P_n)$, does not contain $\alpha_{1,2,3}$. Let $\theta = \alpha_{1,2,3}^l \theta_1$, where $l \in \mathbb{Z}$ and θ_1 is a word in the elements of $\mathcal{B} \setminus \{\alpha_{1,2,3}\}$. For all $i \in \{0, \dots, t\}$, we have:

$$\begin{aligned} \delta_{3i,3}^3 &= (\sigma_{3i+2} \sigma_{3i+1}^{-1})^3 = \sigma_{3i+2}^2 \sigma_{3i+2}^{-1} \sigma_{3i+1}^{-1} \sigma_{3i+2} \sigma_{3i+1}^{-1} \sigma_{3i+2} \sigma_{3i+1}^{-1} \\ &= \sigma_{3i+2}^2 \sigma_{3i+1} \sigma_{3i+2}^{-1} \sigma_{3i+1}^{-2} \sigma_{3i+2} \sigma_{3i+1}^{-1} = \sigma_{3i+2}^2 \sigma_{3i+1}^2 \sigma_{3i+2}^{-2} \sigma_{3i+1}^{-2} = [\sigma_{3i+1}^2, \sigma_{3i+2}^2]^{-1} \\ &= [A_{3i+1,3i+2}, A_{3i+2,3i+3}]^{-1} = \alpha_{3i+1,3i+2,3i+3}^{-1} \end{aligned}$$

and since the $\delta_{3i,3}$ commute pairwise, it follows that $\eta_t^3 = \alpha_{1,2,3}^{-1} \cdots \alpha_{3t+1,3t+2,3t+3}^{-1}$. Hence:

$$\begin{aligned} \rho^3 &= (\theta\eta_t)^3 = \alpha_{1,2,3}^l \theta_1 \eta_t \alpha_{1,2,3}^l \theta_1 \eta_t^{-1} \eta_t^2 \alpha_{1,2,3}^l \theta_1 \eta_t^{-2} \eta_t^3 \\ &= \alpha_{1,2,3}^{3l} \theta_1 \cdot \eta_t \theta_1 \eta_t^{-1} \cdot \eta_t^2 \theta_1 \eta_t^{-2} \cdot \alpha_{1,2,3}^{-1} \cdots \alpha_{3t+1,3t+2,3t+3}^{-1} \end{aligned}$$

in $\Gamma_2(P_n)/\Gamma_3(P_n)$. As we saw above, the words $\eta_t \theta_1 \eta_t^{-1}$ and $\eta_t^2 \theta_1 \eta_t^{-2}$ (written in reduced form in $P_n/\Gamma_3(P_n)$) do not contain $\alpha_{1,2,3}$, and since $\rho^3 = 1$, it follows by comparing the coefficients of $\alpha_{1,2,3}$ that $3l = 1$, which yields a contradiction. So $B_n/\Gamma_3(P_n)$ has no 3-torsion as required. \square

The following result is a consequence of Theorems 1 and 2 and the definition of almost-Bieberbach groups.

COROLLARY 15. *Let $n \geq 3$ and let $k \geq 3$. Let H be a subgroup of S_n whose order is equal to $2^s 3^t$ for some $s, t \in \mathbb{N}$. Then $\sigma^{-1}(H)/\Gamma_k(P_n)$ is an almost-Bieberbach group. In particular, the groups $B_3/\Gamma_3(P_3)$ and $B_4/\Gamma_3(P_4)$ are almost-Bieberbach groups of dimension 4 and 10 respectively.*

Proof. The first part is a consequence of Theorems 1 and 2 and Definition 6. The second part follows by taking $H = S_n$ for $n \in \{3, 4\}$, and by applying the short exact sequence (6) and the results of Table 1. \square

4 Torsion elements, conjugacy classes and a presentation of $B_n/\Gamma_3(P_n)$

Let $n \in \mathbb{N}$. The main aim of this section is to prove Theorems 3 and 4 that describe the torsion elements of $B_n/\Gamma_3(P_n)$ and their conjugacy classes. We also exhibit a presentation of the quotient group $B_n/\Gamma_3(P_n)$ in Proposition 18.

4.1 A partition of the basis of $\Gamma_2(P_3)/\Gamma_3(P_3)$

Let δ_n be equal to the element $\delta_{0,n}$ of equation (14), considered as an element of the group $B_n/\Gamma_3(P_n)$. By Lemma 13, $\delta_n^n \in \Gamma_2(P_n)/\Gamma_3(P_n)$. We start by partitioning the basis \mathcal{B} of $\Gamma_2(P_n)/\Gamma_3(P_n)$ given in (17) into orbits for the action by conjugation by δ_n .

PROPOSITION 16. *Let $n \geq 5$. Then the basis \mathcal{B} of $\Gamma_2(P_n)/\Gamma_3(P_n)$ given in equation (17) is invariant with respect to the action of \mathbb{Z}_n given by conjugation by δ_n . Further, under this action:*

- (a) if $\gcd(n, 3) = 1$, \mathcal{B} is the disjoint union of $\frac{(n-1)(n-2)}{6}$ orbits each of length n .
- (b) if $\gcd(n, 3) \neq 1$, \mathcal{B} is the disjoint union of $\frac{n(n-3)}{6} + 1$ orbits, one of which is of length $\frac{n}{3}$, and the remaining $\frac{n(n-3)}{6}$ orbits are each of length n .

Proof. Let $n \geq 5$, and let $1 \leq i < j < k \leq n$. Using (8), (14) and (16), we see that:

$$\delta_n \alpha_{i,j,k} \delta_n^{-1} = \begin{cases} \alpha_{i-1,j-1,k-1} & \text{if } i \geq 2 \\ [A_{n,j-1}, A_{j-1,k-1}] = [A_{j-1,n}, A_{j-1,k-1}] = \alpha_{j-1,k-1,n} & \text{if } i = 1. \end{cases} \quad (20)$$

So the action of conjugation by δ_n permutes the elements of \mathcal{B} , which proves the first part of the statement. By (20), $\delta_n^n \alpha_{i,j,k} \delta_n^{-n} = \alpha_{i,j,k}$ for all $1 \leq i < j < k \leq n$. Suppose that $1 \leq l < n$ is such that $\delta_n^l \alpha_{i,j,k} \delta_n^{-l} = \alpha_{i,j,k}$. Then by (20), we have:

- (i) $i - l + n = k$, $j - l = i$ and $k - l = j$, or
- (ii) $i - l + n = j$, $j - l + n = k$ and $k - l = i$.

Summing the three equations in both cases, we see that $3 \mid n$. In particular:

- if $3 \nmid n$ then all of the orbits are of length n , and there are $\binom{n}{3}/n = \frac{(n-1)(n-2)}{6}$ orbits.
- if $3 \mid n$ then from (i) and (ii), we have either $l = \frac{n}{3}$ or $l = \frac{2n}{3}$. It follows that $j = i + \frac{n}{3}$ and $k = i + \frac{2n}{3}$, and for these values of i, j and k , we obtain a single orbit of length $\frac{n}{3}$. The remaining orbits are of length n , and so the number of such orbits is given by $(\binom{n}{3} - \frac{n}{3})/n$, which is equal to $\frac{n(n-3)}{6}$. \square

REMARK 17. Let $n = 3q + r$, where $q \in \mathbb{N}$ and $r \in \{0, 1, 2\}$. Let \mathcal{S} be the subset of \mathcal{B} defined by:

$$\mathcal{S} = \begin{cases} \{\alpha_{1,j,k} \mid 2 \leq j \leq q+1 \text{ and } 2j-1 \leq k \leq n-(j-1)\} & \text{if } r \neq 0 \\ \{\alpha_{1,j,k} \mid 2 \leq j \leq q \text{ and } 2j-1 \leq k \leq n-(j-1)\} \cup \{\alpha_{1,n/3+1, 2n/3+1}\} & \text{if } r = 0. \end{cases} \quad (21)$$

Then \mathcal{S} is a transversal for the action of δ_n on \mathcal{B} described in Proposition 16, and \mathcal{B} may be rewritten as:

$$\mathcal{B} = \begin{cases} \{\delta_n^t \alpha_{1,j,k} \delta_n^{-t} \mid 2 \leq j \leq q+1, 2j-1 \leq k \leq n-(j-1) \text{ and } 0 \leq t \leq n-1\} & \text{if } r \neq 0 \\ \{\delta_n^t \alpha_{1,j,k} \delta_n^{-t} \mid 2 \leq j \leq q, 2j-1 \leq k \leq n-(j-1) \text{ and } 0 \leq t \leq n-1\} \cup \\ \{\delta_n^t \alpha_{1,n/3+1, 2n/3+1} \delta_n^{-t} \mid 0 \leq t \leq n/3-1\} & \text{if } r = 0. \end{cases}$$

With the notation of Proposition 16, let $\mathcal{O}(n)$ be equal to the number of orbits of the action on \mathcal{B} by conjugation by δ_n , and let $\mathcal{T} = \{b_{i,1}\}_{1 \leq i \leq \mathcal{O}(n)}$ be a transversal for this action (for example, we may take \mathcal{T} to be the transversal \mathcal{S} defined in (21)). We choose \mathcal{T} so that $b_{\mathcal{O}(n),1}$ is a representative of the orbit of length $n/3$ if $3 \mid n$. For $1 \leq i \leq \mathcal{O}(n)$, let $q(i)$ be the length of the orbit of $b_{i,1}$, so that $q(i) = n/3$ if $3 \mid n$ and $i = \mathcal{O}(n)$, and $q(i) = n$ otherwise, and let:

$$b_{i,j} = \delta_n^{j-1} b_{i,1} \delta_n^{-(j-1)} \text{ for all } 1 \leq j \leq q(i). \quad (22)$$

Then $\mathcal{B} = \{b_{i,j} \mid 1 \leq i \leq \mathcal{O}(n) \text{ and } 1 \leq j \leq q(i)\}$.

4.2 A presentation of $B_n/\Gamma_3(P_n)$

In this section, we exhibit a presentation of $B_n/\Gamma_3(P_n)$ by applying the techniques of [Jo, Proposition 1, p. 139] to obtain a presentation of a group extension to the short exact sequence (9).

PROPOSITION 18. Let $n \geq 3$.

- (a) The group $P_n/\Gamma_3(P_n)$ has a presentation given by the generating set $\mathcal{X}_n = \{A_{i,j}\}_{1 \leq i < j \leq n} \cup \{\alpha_{r,s,t}\}_{1 \leq r < s < t \leq n}$, subject to the following relations:

(i) the $\alpha_{r,s,t}$ commute pairwise and with the $A_{i,j}$.

$$(ii) [A_{i,j}, A_{l,m}] = \begin{cases} \alpha_{i,j,m} & \text{if Card } \{i, j, l, m\} = 3 \text{ and } j = l \\ \alpha_{i,k,m}^{-1} & \text{if Card } \{i, j, l, m\} = 3, \text{ where } k = j \text{ if } i = l, \text{ and } k = l \text{ if } j = m \\ 1 & \text{if Card } \{i, j, l, m\} \in \{2, 4\}. \end{cases}$$

(b) The group $B_n/\Gamma_3(P_n)$ has a presentation given by the generating set $\{\sigma_k\}_{1 \leq k \leq n-1} \cup \mathcal{X}_n$, subject to the following relations:

- (i) relations (a)(i) and (a)(ii) emanating from those of $P_n/\Gamma_3(P_n)$.
- (ii) the Artin braid relations (2), viewed in $B_n/\Gamma_3(P_n)$.
- (iii) the conjugacy relations described in (18) and (19).

Proof. To prove parts (a) and (b), it suffices to apply [Jo, Proposition 1, p. 139] first to (9) taking $k = 2$, and then to (6) taking $k = 3$ using also part (a). \square

4.3 Torsion elements and conjugacy classes in $B_n/\Gamma_3(P_n)$

In this section, we prove Theorems 3 and 4. We start by proving the following lemma whose result generalises [GGO1, Theorem 3(a)].

LEMMA 19. Let $m, k \in \mathbb{N}$.

(a) Let $n \in \mathbb{N}$ be such that $2 \leq n \leq m$, and let $\iota: B_n \rightarrow B_m$ denote the injective homomorphism defined by $\iota(\sigma_i) = \sigma_i$ for all $1 \leq i \leq n-1$. Then the induced homomorphism $\bar{\iota}: B_n/\Gamma_k(P_n) \rightarrow B_m/\Gamma_k(P_m)$ is injective.

(b) Let $t \in \mathbb{N}$, let n_1, n_2, \dots, n_t be integers greater than or equal to 2 for which $\sum_{i=1}^t n_i \leq m$, and let $\zeta: B_{n_1} \times \dots \times B_{n_t} \rightarrow B_m$ denote the natural inclusion. Then the induced homomorphism $\bar{\zeta}: B_{n_1}/\Gamma_k(P_{n_1}) \times \dots \times B_{n_t}/\Gamma_k(P_{n_t}) \rightarrow B_m/\Gamma_k(P_m)$ is injective.

Proof. Let t, m and n be positive integers such that $2 \leq n \leq m$.

(a) If $k = 1$, the result is straightforward. So assume that $k \geq 2$. For $2 \leq n \leq m$, let $\iota_{n,m}: B_n \rightarrow B_m$ (resp. $\bar{\iota}_{n,m,k}: B_n/\Gamma_k(P_n) \rightarrow B_m/\Gamma_k(P_m)$) be the homomorphism ι (resp. $\bar{\iota}$) given in the statement. By equation (4), $\iota_{n,m}$ restricts to an injective homomorphism $\iota_{n,m}|_{P_n}: P_n \rightarrow P_m$ given by $\iota_{n,m}|_{P_n}(A_{i,j}) = A_{i,j}$ for all $1 \leq i < j \leq n$. For $n \leq q \leq m-1$, let $\iota_q: P_q \rightarrow P_{q+1}$ denote the homomorphism $\iota_{q,q+1}|_{P_q}$. We claim that the homomorphism $\bar{\iota}_{n,m}|_{\Gamma_k(P_n)/\Gamma_{k+1}(P_n)}: \Gamma_k(P_n)/\Gamma_{k+1}(P_n) \rightarrow \Gamma_k(P_m)/\Gamma_{k+1}(P_m)$ that is induced by $\iota_{n,m}|_{P_n}$ is injective for all $k \geq 2$. To see this, first note that since $\iota_{n,m}|_{P_n}$ is equal to the composition $\iota_{m-1} \circ \dots \circ \iota_n$, the homomorphism $\bar{\iota}_{n,m}|_{\Gamma_k(P_n)/\Gamma_{k+1}(P_n)}$ is equal to the composition $\bar{\iota}_{m-1}|_{\Gamma_k(P_{m-1})/\Gamma_{k+1}(P_{m-1})} \circ \dots \circ \bar{\iota}_n|_{\Gamma_k(P_n)/\Gamma_{k+1}(P_n)}$, where the homomorphism

$$\bar{\iota}_q|_{\Gamma_k(P_q)/\Gamma_{k+1}(P_q)}: \Gamma_k(P_q)/\Gamma_{k+1}(P_q) \rightarrow \Gamma_k(P_{q+1})/\Gamma_{k+1}(P_{q+1})$$

is induced by ι_q for all $n \leq q \leq m-1$, it suffices to prove that $\bar{\iota}_q|_{\Gamma_k(P_q)/\Gamma_{k+1}(P_q)}$ is injective. To do so, consider the Fadell-Neuwirth short exact sequence $1 \rightarrow \text{Ker}(p) \rightarrow P_{q+1} \xrightarrow{p} P_q \rightarrow 1$, where p is the homomorphism given geometrically by forgetting

the last string. Using the presentation given in equation (5), we obtain two well-known facts, first that ι_q is a section for p , and secondly that the resulting semi-direct product is almost direct, *i.e.* the action induced by P_q on the Abelianisation $\text{Ker}(p) / \Gamma_2(\text{Ker}(p))$ of $\text{Ker}(p)$ is trivial. It follows from [FR1, Theorem 3.1] that the induced sequence of homomorphisms $1 \longrightarrow \Gamma_k(\text{Ker}(p)) / \Gamma_{k+1}(\text{Ker}(p)) \longrightarrow \Gamma_k(P_{q+1}) / \Gamma_{k+1}(P_{q+1}) \xrightarrow{\bar{p}} \Gamma_k(P_q) / \Gamma_{k+1}(P_q) \longrightarrow 1$ is split short exact, and that the homomorphism $\bar{\iota}_q \Big|_{\Gamma_k(P_q) / \Gamma_{k+1}(P_q)}$ is a section for \bar{p} . In particular, $\bar{\iota}_q \Big|_{\Gamma_k(P_q) / \Gamma_{k+1}(P_q)}$ is injective, and the claim follows. Now consider the following commutative diagram of short exact sequences:

$$\begin{array}{ccccccccc} 1 & \longrightarrow & \Gamma_k(P_n) / \Gamma_{k+1}(P_n) & \longrightarrow & B_n / \Gamma_{k+1}(P_n) & \longrightarrow & B_n / \Gamma_k(P_n) & \longrightarrow & 1 \\ & & \downarrow \bar{\iota}_{n,m} \Big|_{\Gamma_k(P_n) / \Gamma_{k+1}(P_n)} & & \downarrow \bar{\iota}_{n,m,k+1} & & \downarrow \bar{\iota}_{n,m,k} & & \\ 1 & \longrightarrow & \Gamma_k(P_m) / \Gamma_{k+1}(P_m) & \longrightarrow & B_m / \Gamma_{k+1}(P_m) & \longrightarrow & B_m / \Gamma_k(P_m) & \longrightarrow & 1, \end{array}$$

where the rows are the short exact sequences given by the central extension (13). The statement of part (a) is then a consequence of applying the 5-Lemma to this diagram, the above claim, induction on k , and the fact that $\bar{\iota}: B_n / \Gamma_2(P_n) \longrightarrow B_m / \Gamma_2(P_m)$ is an injective homomorphism by [GGO1, Theorem 3(a)].

(b) Let n_1, \dots, n_t be integers greater than or equal to 2 such that $\sum_{i=1}^t n_i \leq m$, let $\sigma: B_{\sum_{i=1}^t n_i} \longrightarrow S_{\sum_{i=1}^t n_i}$ denote the usual homomorphism that to a braid associates its permutation, and let B_{n_1, \dots, n_t} denote the corresponding mixed braid group, namely the preimage under σ of the subgroup $S_{n_1} \times \dots \times S_{n_t}$ of $S_{\sum_{i=1}^t n_i}$. For $1 \leq i \leq t$, let $\varphi_i: B_{n_i} \longrightarrow B_{n_1, \dots, n_t}$ denote the embedding of B_{n_i} into the i^{th} factor of B_{n_1, \dots, n_t} . Note that ζ is equal to the following composition:

$$B_{n_1} \times \dots \times B_{n_t} \xrightarrow{\varphi_1 \times \dots \times \varphi_t} B_{n_1, n_2, \dots, n_t} \hookrightarrow B_{\sum_{i=1}^t n_i} \hookrightarrow B_m.$$

Note that φ_i induces a homomorphism $\bar{\varphi}_i: B_{n_i} / \Gamma_k(P_{n_i}) \longrightarrow B_{n_1, n_2, \dots, n_t} / \Gamma_k(P_{\sum_{i=1}^t n_i})$ because $\varphi_i(\Gamma_k(P_{n_i})) \subset \Gamma_k(P_{\sum_{i=1}^t n_i})$. Now let $\psi_i: B_{n_1, \dots, n_t} \longrightarrow B_{n_i} / \Gamma_k(P_{n_i})$ be the composition of the projection onto the i^{th} factor of B_{n_1, \dots, n_t} , followed by the canonical projection $B_{n_i} \longrightarrow B_{n_i} / \Gamma_k(P_{n_i})$. Under this composition, the normal subgroup $P_{\sum_{i=1}^t n_i}$ of B_{n_1, \dots, n_t} is sent to $P_{n_i} / \Gamma_k(P_{n_i})$, hence the normal subgroup $\Gamma_k(P_{\sum_{i=1}^t n_i})$ of B_{n_1, \dots, n_t} is sent to the trivial element of $B_{n_i} / \Gamma_k(P_{n_i})$, from which it follows that ψ_i induces a homomorphism $\bar{\psi}_i: B_{n_1, \dots, n_t} / \Gamma_k(P_{\sum_{i=1}^t n_i}) \longrightarrow B_{n_i} / \Gamma_k(P_{n_i})$. From the constructions of φ_i and ψ_i , we see that $\bar{\psi}_i \circ \bar{\varphi}_i = \text{Id}_{B_{n_i} / \Gamma_k(P_{n_i})}$ for all $1 \leq i \leq t$, and so the composition

$$\frac{B_{n_1}}{\Gamma_k(P_{n_1})} \times \dots \times \frac{B_{n_t}}{\Gamma_k(P_{n_t})} \xrightarrow{\bar{\varphi}_1 \times \dots \times \bar{\varphi}_t} \frac{B_{n_1, n_2, \dots, n_t}}{\Gamma_k(P_{\sum_{i=1}^t n_i})} \xrightarrow{\bar{\psi}_1 \times \dots \times \bar{\psi}_t} \frac{B_{n_1}}{\Gamma_k(P_{n_1})} \times \dots \times \frac{B_{n_t}}{\Gamma_k(P_{n_t})}$$

is the identity. Thus $\bar{\varphi}_1 \times \dots \times \bar{\varphi}_t$ is injective, and the composition

$$\frac{B_{n_1}}{\Gamma_k(P_{n_1})} \times \dots \times \frac{B_{n_t}}{\Gamma_k(P_{n_t})} \xrightarrow{\bar{\varphi}_1 \times \dots \times \bar{\varphi}_t} \frac{B_{n_1, n_2, \dots, n_t}}{\Gamma_k(P_{\sum_{i=1}^t n_i})} \longrightarrow \frac{B_{\sum_{i=1}^t n_i}}{\Gamma_k(P_{\sum_{i=1}^t n_i})} \longrightarrow \frac{B_m}{\Gamma_k(P_m)},$$

which is the homomorphism $\bar{\zeta}$ of the statement, may be seen to be injective using part (a) and the injectivity of the homomorphism $B_{n_1, \dots, n_t} \hookrightarrow B_{\sum_{i=1}^t n_i}$. \square

The following results allow us to decide whether certain elements of a group are of finite order or not. They will be used to obtain finite-order elements in $B_n/\Gamma_3(P_n)$, as well as in the proof of Theorem 3.

LEMMA 20. *Let $d \geq 2$, let G be a group that has no d -torsion, and let $\alpha \in G$. Suppose that $\sigma: G \rightarrow S_n$ is a surjective homomorphism whose kernel K is torsion free, and such that d divides the order of $\sigma(\alpha)$. Then for all $\theta \in K$, $\theta\alpha$ is of infinite order in G .*

Proof. Suppose on the contrary that there exists $\theta \in K$ for which $\theta\alpha$ is of finite order, m say, in G . Let q denote the order of $\sigma(\alpha)$. Since $\theta \in K$, $\text{Id} = \sigma((\theta\alpha)^m) = (\sigma(\alpha))^m$, and thus $q \mid m$. In a similar manner, we see that $\sigma((\theta\alpha)^q) = \text{Id}$, so $(\theta\alpha)^q \in K$. Now $1 = (\theta\alpha)^m = ((\theta\alpha)^q)^{m/q}$, and the fact that K is torsion free implies that $(\theta\alpha)^q = 1$, hence $q = m$. By hypothesis, $d \mid q$, and so $((\theta\alpha)^{q/d})^d = 1$. This implies that the order of $(\theta\alpha)^{q/d}$ divides d . On the other hand, the fact that $\sigma(\theta\alpha)$ is of order q implies that $\sigma((\theta\alpha)^{q/d})$ is of order d , and so the order of $(\theta\alpha)^{q/d}$ cannot be strictly less than d . Hence $(\theta\alpha)^{q/d}$ is of order d , but this contradicts the fact that G has no d -torsion. \square

PROPOSITION 21. *Let $n \geq 3$.*

- (a) *If n is even then for all $k \in \mathbb{N}$ and $\theta \in P_n/\Gamma_3(P_n)$, $(\theta\delta_n)^k \notin \Gamma_2(P_n)/\Gamma_3(P_n)$.*
(b) *Suppose that n is odd. Then the element δ_n^n belongs to $\Gamma_2(P_n)/\Gamma_3(P_n)$, and with the notation of (22), we have:*

$$\delta_n^n = \prod_{i=1}^{\mathcal{O}(n)} \prod_{j=1}^{q(i)} b_{i,j}^{m_{i,j}}, \quad (23)$$

where $m_{i,1} = m_{i,2} = \dots = m_{i,q(i)}$ for all $1 \leq i \leq \mathcal{O}(n)$. Further, if $3 \mid n$ then for all $k \in \mathbb{N}$ and $\theta \in P_n/\Gamma_3(P_n)$, $(\theta\delta_n)^k$ is non trivial in $B_n/\Gamma_3(P_n)$.

Proof.

(a) Let n be even, and assume on the contrary that $(\theta\delta_n)^k \in \Gamma_2(P_n)/\Gamma_3(P_n)$ for some $k \in \mathbb{N}$ and $\theta \in P_n/\Gamma_3(P_n)$. Let $f: B_n/\Gamma_3(P_n) \rightarrow B_n/\Gamma_2(P_n)$ be the homomorphism given in (13) with $j = 3$. Then $(\overline{\theta\delta_n})^k = 1$ in $B_n/\Gamma_2(P_n)$, where $\overline{\theta} = f(\theta)$ and $\overline{\delta_n} = f(\delta_n)$. We now apply Lemma 20 to the short exact sequence (6), taking $k = 2$, $G = B_n/\Gamma_2(P_n)$, $\alpha = \overline{\delta_n}$, and the homomorphism σ of that lemma to be $\overline{\sigma}$. Note that $\overline{\sigma}(\overline{\delta_n})$ is of order n , so $2 \mid n$, and $B_n/\Gamma_2(P_n)$ has no 2-torsion by [GGO1, Theorem 2]. Since $P_n/\Gamma_2(P_n)$ is torsion free, it follows from Lemma 20 that $\overline{\theta\delta_n}$ is of infinite order in $B_n/\Gamma_2(P_n)$, and we obtain a contradiction. We conclude that $(\theta\delta_n)^k \notin \Gamma_2(P_n)/\Gamma_3(P_n)$.

(b) Since $\overline{\delta_n}^n = 1$ in $B_n/\Gamma_2(P_n)$ by Lemma 13, it follows from the short exact sequence (13) with $j = 3$ that $\delta_n^n \in \Gamma_2(P_n)/\Gamma_3(P_n)$. Hence in terms of the basis \mathcal{B} of $\Gamma_2(P_n)/\Gamma_3(P_n)$ given by (17), and with the notation of (22), for all $1 \leq i \leq \mathcal{O}(n)$ and $1 \leq j \leq q(i)$, there exist $m_{i,j} \in \mathbb{Z}$ that are unique and for which (23) holds. The equality $m_{i,1} = m_{i,2} = \dots = m_{i,q(i)}$ follows by conjugating (23) by δ_n and using (22). The last part of the statement follows by applying Lemma 20 to the short exact sequence (6), where we take $G = B_n/\Gamma_3(P_n)$, $d = 3$ and $\alpha = \delta_n$, and using Theorem 2. \square

THEOREM 22. *Let $n \in \mathbb{N}$, let $m \leq n$, and let s be the largest divisor of m for which $\gcd(s, 6) = 1$. If $s > 1$, the group $B_n/\Gamma_3(P_n)$ possesses infinitely-many elements of order s .*

Proof. Let $1 \leq m \leq n$, and let s be the largest divisor of m for which $\gcd(s, 6) = 1$. Since we assume that $s > 1$, we must have $n \geq 5$. Further, the fact that $s \leq n$ implies using Lemma 19(a) that the homomorphism $\bar{l}: B_s/\Gamma_3(P_s) \rightarrow B_n/\Gamma_3(P_n)$ is injective. So it suffices to prove that if $n \in \mathbb{N}$ is relatively prime with 6 then $B_n/\Gamma_3(P_n)$ possesses elements of order n . In this case, δ_n^n is as given in (23), where $\mathcal{O}(n) = (n-1)(n-2)/6$ by Proposition 16(a), and $q(i) = n$ for all $1 \leq i \leq \mathcal{O}(n)$. Let:

$$\widehat{\delta}_n = \theta \delta_n, \quad (24)$$

where we take θ to be an element of $\Gamma_2(P_n)/\Gamma_3(P_n)$, so $\theta = \prod_{i=1}^{\mathcal{O}(n)} \prod_{j=1}^n b_{i,j}^{r_{i,j}}$, where $r_{i,j} \in \mathbb{Z}$ for all $1 \leq i \leq \mathcal{O}(n)$ and $1 \leq j \leq n$. Then by (22) and Proposition 21(b), we have:

$$\begin{aligned} \widehat{\delta}_n^n &= (\theta \delta_n)^n = \theta \cdot \delta_n \theta \delta_n^{-1} \cdot \delta_n^2 \theta \delta_n^{-2} \cdots \delta_n^{n-1} \theta \delta_n^{-(n-1)} \cdot \delta_n^n \\ &= \prod_{i=1}^{\mathcal{O}(n)} \left(\prod_{j=1}^n b_{i,j} \right)^{\sum_{j=1}^n r_{i,j}} \cdot \prod_{i=1}^{\mathcal{O}(n)} \prod_{j=1}^n b_{i,j}^{m_{i,1}} = \prod_{i=1}^{\mathcal{O}(n)} \left(\prod_{j=1}^n b_{i,j} \right)^{\sum_{j=1}^n r_{i,j} + m_{i,1}}. \end{aligned} \quad (25)$$

Taking $k = 3$ in (6), $\bar{\sigma}(\widehat{\delta}_n)$ is of order n in S_n , and it follows from this and (25) that $\widehat{\delta}_n$ is of order n if and only if:

$$\sum_{j=1}^n r_{i,j} = -m_{i,1} \text{ for all } i = 1, \dots, \mathcal{O}(n). \quad (26)$$

This system of equations has infinitely-many solutions in the $r_{i,j}$, which yields infinitely-many elements in $B_n/\Gamma_3(P_n)$ of order n . \square

REMARK 23. If $n \in \mathbb{N}$ and $\gcd(n, 6) = 1$, the proof of Theorem 22 shows how to obtain explicit finite-order elements of $B_n/\Gamma_3(P_n)$. However, as part of the process, we need to determine δ_n^n in terms of the elements of the basis \mathcal{B} . This seems to be an arduous computation in general, as the calculation given in Section 5.2 in the case $n = 5$ indicates.

We are now able to prove Theorem 3.

Proof of Theorem 3. Let $n \geq 5$, and let $\tau \in \mathbb{N}$ be such that $\gcd(\tau, 6) = 1$. If β is an element of $B_n/\Gamma_3(P_n)$ of order τ then using (6) and the fact that $P_n/\Gamma_3(P_n)$ is torsion free by Lemma 11(a), it follows that $\bar{\sigma}(\beta)$ is an element of S_n of order τ . Conversely, suppose that $x \in S_n$ is an element of order τ , and let $\eta_1 \cdots \eta_t$ be the cycle decomposition of x . Then $\tau = \text{lcm}(n_1, \dots, n_t)$, where for $i = 1, \dots, t$, η_i is of length n_i . In particular, $\gcd(n_i, 6) = 1$ for all $i = 1, \dots, t$, and $\sum_{i=1}^t n_i \leq n$ by [Ho]. Let $y \in S_n$ be such that

$$yxy^{-1} = (1, \dots, n_1)(n_1 + 1, \dots, n_1 + n_2) \cdots (n_1 + \cdots + n_{t-1} + 1, \dots, n_1 + \cdots + n_t). \quad (27)$$

Since $\bar{\sigma}$ is surjective, there exists $\rho \in B_n/\Gamma_3(P_n)$ such that $\bar{\sigma}(\rho) = y$. For $i = 1, \dots, t$, let $\widehat{\delta}_{n_i} \in B_{n_i}/\Gamma_3(P_{n_i})$ be as defined in (24), where the coefficients of θ satisfy (26). From the proof of Theorem 22, $\widehat{\delta}_{n_i}$ is of order n_i . Let $\beta = \bar{\zeta}(\widehat{\delta}_{n_1}, \dots, \widehat{\delta}_{n_t})$ in $B_n/\Gamma_3(P_n)$. By taking $k = 3$ in Lemma 19(b), we see that β is of order τ , and since the permutation associated to $\widehat{\delta}_{n_i}$ is the n_i -cycle $(1, \dots, n_i)$, it follows that $\bar{\sigma}(\beta) = yxy^{-1}$, and hence $\bar{\sigma}(\alpha) = x$, where $\alpha = \rho^{-1}\beta\rho$. In particular, $B_n/\Gamma_3(P_n)$ has elements of order τ if and only if S_n does. \square

One consequence of Theorem 3 is the classification of the isomorphism classes of the finite cyclic subgroups of $B_n/\Gamma_3(P_n)$. Using Lemma 19(b), we may also characterise the isomorphism classes of the finite Abelian subgroups of $B_n/\Gamma_3(P_n)$ in a manner similar to that of $B_n/\Gamma_2(P_n)$ given in [GGO1, Theorem 6].

COROLLARY 24. *Let $n \geq 3$. Then there is a one-to-one correspondence between the isomorphism classes of the finite Abelian subgroups of $B_n/\Gamma_3(P_n)$ and those of the finite Abelian subgroups of S_n whose order is relatively prime with 6.*

Proof. The proof is similar to that of [GGO1, Theorem 6], using Lemma 19(b) and the finite-order elements of the form (24) constructed in the proof of Theorem 22. \square

To end this section, we prove Theorem 4, which says that if $n \geq 5$, two finite-order elements of $B_n/\Gamma_3(P_n)$ are conjugate if and only if their associated permutations are conjugate.

Proof of Theorem 4. Let $n \geq 5$, and let α and β be two finite-order elements of $B_n/\Gamma_3(P_n)$ whose associated permutations have the same cycle type. Conjugating β in $B_n/\Gamma_3(P_n)$ if necessary, we may suppose that β is as in the proof of Theorem 3, in particular, it is of order $\tau = \text{lcm}(n_1, \dots, n_t)$, where $\text{gcd}(n_i, 6) = 1$ for all $i = 1, \dots, t$, and $\bar{\sigma}(\beta)$ is equal to the permutation given in (27). To complete the proof of the theorem, it suffices to show that if $\alpha \in B_n/\Gamma_3(P_n)$ is any finite-order element such that $\bar{\sigma}(\alpha)$ has the same cycle type as $\bar{\sigma}(\beta)$, then α and β are conjugate in $B_n/\Gamma_3(P_n)$. Conjugating α if necessary, we may suppose further that $\bar{\sigma}(\alpha) = \bar{\sigma}(\beta)$. Since $\text{Ker}(\bar{\sigma}) = P_n/\Gamma_3(P_n)$ is torsion free by Lemma 11(a), it follows that α is also of order τ .

Let \mathcal{B}' be the union of the elements of the basis \mathcal{B} defined in (17) and their inverses. From (8) and (16), β acts on \mathcal{B}' by conjugation, and if $\beta\alpha_{i,j,k}\beta^{-1} = \alpha_{r,s,t}^{-1}$ for some $1 \leq i < j < k \leq n$ and $1 \leq r < j < s \leq t$, then either $i = n_1 + \dots + n_m + 1 < j \leq n_1 + \dots + n_{m+1} < k$ for some $0 \leq m < t$, or $i < j = n_1 + \dots + n_m + 1 < k \leq n_1 + \dots + n_{m+1}$ for some $1 \leq m < t$. From this and the fact that n_l is odd for all $l = 1, \dots, t$, we see that under this action, the orbits of $\alpha_{i,j,k}$ and $\alpha_{i,j,k}^{-1}$ are disjoint for all $1 \leq i < j < k \leq n$, and that the orbit of $\alpha_{i,j,k}^{-1}$ is obtained by taking inverses of the elements of the orbit of $\alpha_{i,j,k}$. We then choose a transversal $\{b_{i,1}, b_{i,1}^{-1} \mid 1 \leq i \leq q\}$ for this action on \mathcal{B}' , where for $1 \leq i \leq q$, $b_{i,1} \in \mathcal{B}$, and we let s_i denote the length of the orbit of $b_{i,1}$. Observe that s_i divides τ . Then $\mathcal{B}'' = \{b_{i,j} \mid 1 \leq i \leq q \text{ and } 1 \leq j \leq s_i\}$ is a basis of $\Gamma_2(P_n)/\Gamma_3(P_n)$, where:

$$b_{i,j} = \beta^{j-1}b_{i,1}\beta^{-(j-1)} \text{ for all } 1 \leq i \leq q \text{ and } 1 \leq j \leq s_i. \quad (28)$$

As in the proof of Proposition 21(a), let $f: B_n/\Gamma_3(P_n) \rightarrow B_n/\Gamma_2(P_n)$ be the projection given in (13) with $j = 3$, and let $\bar{\sigma}': B_n/\Gamma_2(P_n) \rightarrow S_n$ be the homomorphism given in (6) with $k = 2$. Since $\text{Ker}(f) = \Gamma_2(P_n)/\Gamma_3(P_n)$ is torsion free, $f(\alpha)$ and $f(\beta)$ are elements of $B_n/\Gamma_2(P_n)$ of order τ . Further, $\bar{\sigma} = \bar{\sigma}' \circ f$, so $\bar{\sigma}'(f(\alpha)) = \bar{\sigma}'(f(\beta))$, and applying [GGO1, Theorem 5], there exists $\xi \in B_n/\Gamma_2(P_n)$ such that $\xi f(\alpha)\xi^{-1} = f(\beta)$. Since f is surjective, there exists $\zeta' \in B_n/\Gamma_3(P_n)$ such that $f(\zeta'\alpha\zeta'^{-1}) = f(\beta)$. So conjugating α if necessary, there exists $\theta \in \Gamma_2(P_n)/\Gamma_3(P_n)$ such that $\theta\beta = \alpha$.

It suffices to show that there exists $\Omega \in \Gamma_2(P_n)/\Gamma_3(P_n)$ such that $\Omega\alpha\Omega^{-1} = \beta$, or equivalently that:

$$\theta = (\beta\Omega\beta^{-1})\Omega^{-1}, \quad (29)$$

using the fact that θ commutes with Ω . Let $\theta = \prod_{\substack{1 \leq i \leq q \\ 1 \leq j \leq s_i}} b_{i,j}^{r_{i,j}}$, and $\Omega = \prod_{\substack{1 \leq i \leq q \\ 1 \leq j \leq s_i}} b_{i,j}^{x_{i,j}}$, where $r_{i,j}, x_{i,j} \in \mathbb{Z}$ for all $1 \leq i \leq q$ and $1 \leq j \leq s_i$. Since $\theta\beta = \alpha$, the elements β and $\theta\beta$ are of order τ , and so:

$$1 = (\theta\beta)^\tau = \theta(\beta\theta\beta^{-1}) \cdots (\beta^{\tau-1}\theta\beta^{-(\tau-1)})\beta^\tau = \theta(\beta\theta\beta^{-1}) \cdots (\beta^{\tau-1}\theta\beta^{-(\tau-1)}). \quad (30)$$

From the construction of the basis \mathcal{B}'' , it follows from (30) that:

$$1 = \theta(\beta\theta\beta^{-1}) \cdots (\beta^{\tau-1}\theta\beta^{-(\tau-1)}) = \prod_{1 \leq i \leq q} \left(\prod_{1 \leq j \leq s_i} b_{i,j} \right)^{\tau(\sum_{1 \leq j \leq s_i} r_{i,j})/s_i},$$

from which we conclude that $\sum_{1 \leq j \leq s_i} r_{i,j} = 0$ for all $1 \leq i \leq q$. In a similar manner, (29) may be written as:

$$\prod_{\substack{1 \leq i \leq q \\ 1 \leq j \leq s_i}} b_{i,j}^{r_{i,j}} = \prod_{\substack{1 \leq i \leq q \\ 1 \leq j \leq s_i}} b_{i,j}^{x_{i,j-1} - x_{i,j}},$$

where the index $j-1$ of $x_{i,j-1}$ is taken modulo s_i . So for all $i = 1, \dots, q$, we obtain a system of equations $x_{i,j-1} - x_{i,j} = r_{i,j}$ for all $1 \leq j \leq s_i$ that is subject to the compatibility condition $\sum_{1 \leq j \leq s_i} r_{i,j} = 0$, and it may be seen easily that each such system admits a solution. Using (29), we conclude that α and β are conjugate as required. \square

5 Some examples with a small numbers of strings

In this section, we study a couple of examples where the number of strings is small. In Section 5.1, we determine, up to conjugacy, the almost-Bieberbach subgroups of $B_3/\Gamma_3(P_3)$ that contain $P_3/\Gamma_3(P_3)$, and we identify them using the classification of [De]. In Section 5.2, we calculate explicitly δ_5^5 in $\Gamma_2(P_5)/\Gamma_3(P_5)$ in terms of the basis \mathcal{B} . This example illustrates the computational difficulties that we encounter with respect to the constructions of Proposition 21 and Theorem 22. We finish the paper with a remark concerning the study of the quotients $B_n/\Gamma_k(P_n)$ for $k > 3$.

5.1 Some almost-Bieberbach subgroups of $B_3/\Gamma_3(P_3)$

In this section, we describe the almost-Bieberbach groups $\sigma^{-1}(H)/\Gamma_3(P_3)$, where H is a subgroup of S_3 . Recall that representatives of the conjugacy classes of subgroups of S_3 are given by $\{\text{Id}\}$, $\langle(1,2)\rangle$, $\langle(1,2,3)\rangle$ and S_3 . As we shall see now in Theorem 25, these groups are of dimension 4 and their holonomy group is H , and that any subgroup of $B_3/\Gamma_3(P_3)$ containing $P_3/\Gamma_3(P_3)$ is in fact of the form $\sigma^{-1}(H)/\Gamma_3(P_3)$, where H is a subgroup of S_3 .

THEOREM 25.

(a) Let K be a subgroup of $B_3/\Gamma_3(P_3)$ that contains $P_3/\Gamma_3(P_3)$. Then K is conjugate to $\sigma^{-1}(H)/\Gamma_3(P_3)$, where H is one of the subgroups $\{\text{Id}\}$, $\langle(1,2)\rangle$, $\langle(1,2,3)\rangle$ or S_3 of S_3 .

(b) Consider the four subgroups of S_3 given in part (a).

(i) If $H = \{\text{Id}\}$, the group $\sigma^{-1}(H)/\Gamma_3(P_3) = P_3/\Gamma_3(P_3)$ has a presentation whose generators are $a = A_{1,3}$, $b = A_{2,3}$, $c = A_{1,2}$ and $d = [A_{1,2}, A_{2,3}]$, and that are subject to the following relations:

$$\begin{array}{lll}
(1) [b, a] = d & (3) [c, b] = d & (5) [d, b] = 1 \\
(2) [c, a] = d^{-1} & (4) [d, a] = 1 & (6) [d, c] = 1.
\end{array}$$

For each of the remaining groups, a presentation is obtained by adding extra generators and relations to those of $P_3/\Gamma_3(P_3)$ given in (i). In each case, we will just indicate these extra generators and relations.

(ii) If $H = \langle(1, 2)\rangle$, the group $\sigma^{-1}(H)/\Gamma_3(P_3)$ has a presentation with one extra generator $\alpha = \sigma_1$ and five extra relations:

$$\begin{array}{lll}
(1) \alpha^2 = c & (3) \alpha a \alpha^{-1} = b & (5) \alpha c \alpha^{-1} = c. \\
(2) \alpha d \alpha^{-1} = d^{-1} & (4) \alpha b \alpha^{-1} = ad^{-1} &
\end{array}$$

(iii) If $H = \langle(1, 2, 3)\rangle$, the group $\sigma^{-1}(H)/\Gamma_3(P_3)$ has one extra generator $\alpha = \sigma_2 \sigma_1^{-1}$ and five extra relations:

$$\begin{array}{lll}
(1) \alpha^3 = d^{-1} & (3) \alpha a \alpha^{-1} = bd & (5) \alpha c \alpha^{-1} = a. \\
(2) \alpha d \alpha^{-1} = d & (4) \alpha b \alpha^{-1} = cd^{-1} &
\end{array}$$

(iv) If $H = S_3$, the group $\sigma^{-1}(H)/\Gamma_3(P_3) = B_3/\Gamma_3(P_3)$ has two extra generators $\alpha = \sigma_2 \sigma_1$ and $\beta = \sigma_1$ and eleven extra relations:

$$\begin{array}{lll}
(1) \alpha^3 = abc & (5) \alpha a \alpha^{-1} = b & (9) \beta b \beta^{-1} = ad^{-1} \\
(2) \beta^2 = c & (6) \alpha b \alpha^{-1} = c & (10) \beta c \beta^{-1} = c \\
(3) \alpha d \alpha^{-1} = d & (7) \alpha c \alpha^{-1} = a & (11) \beta \alpha \beta^{-1} = b^{-1} \alpha^2. \\
(4) \beta d \beta^{-1} = d^{-1} & (8) \beta a \beta^{-1} = b &
\end{array}$$

Proof.

(a) First suppose that K_1 and K_2 are subgroups of $B_3/\Gamma_3(P_3)$ that contain $P_3/\Gamma_3(P_3)$, and for which $\bar{\sigma}(K_1) = \bar{\sigma}(K_2)$. We claim that $K_1 = K_2$. To see this, let $x \in K_1$. Since $\bar{\sigma}(K_1) = \bar{\sigma}(K_2)$, there exists $y \in K_2$ such that $\bar{\sigma}(x) = \bar{\sigma}(y)$, and so there exists $z \in P_3/\Gamma_3(P_3)$ such that $y^{-1}x = z$. But $P_3/\Gamma_3(P_3) \subset K_1 \cap K_2$ by hypothesis, so $x = yz \in K_2$, which proves that $K_1 \subset K_2$. A similar argument shows that $K_2 \subset K_1$, which proves the claim. So if K is a subgroup $B_3/\Gamma_3(P_3)$ that contains $P_3/\Gamma_3(P_3)$, $K = \bar{\sigma}^{-1}(\bar{\sigma}(K))$, in particular $K = \bar{\sigma}^{-1}(H)$, where $H = \bar{\sigma}(K)$ is a subgroup of S_3 . Since all such subgroups are normal in S_3 , with the exception of those of order 2, to complete the proof of part (a), it suffices to show that if K_1 and K_2 are subgroups of $B_3/\Gamma_3(P_3)$ that contain $P_3/\Gamma_3(P_3)$, and for which $\bar{\sigma}(K_1) = \langle(1, 2)\rangle$ and $\bar{\sigma}(K_2) = \tau$, where $\tau \in \{(1, 3), (2, 3)\}$, then K_1 and K_2 are conjugate in $B_3/\Gamma_3(P_3)$. To see this, let $\tau' \in S_3$ be such that $\tau'(1, 2)\tau'^{-1} = \tau$, and let $\tilde{\tau}' \in B_3/\Gamma_3(P_3)$ be such that $\bar{\sigma}(\tilde{\tau}') = \tau'$. Then $\tilde{\tau}'K_1\tilde{\tau}'^{-1}$ contains $\text{Ker}(\bar{\sigma}) = P_3/\Gamma_3(P_3)$, and $\bar{\sigma}(\tilde{\tau}'K_1\tilde{\tau}'^{-1}) = \bar{\sigma}(K_2) = \langle\tau\rangle$. The result then follows from the first part of the proof.

(b) The case (i) follows from Proposition 18(a). For cases (ii)–(iv) we apply the techniques of [Jo, Proposition 1, p. 139] to the extension

$$1 \longrightarrow P_3/\Gamma_3(P_3) \longrightarrow \sigma^{-1}(H)/\Gamma_3(P_3) \longrightarrow H \longrightarrow 1.$$

The extra relations involving the action by conjugacy of H on the kernel follow from equations (18) and (19). \square

By Remarks 12(a), the groups of the form $\sigma^{-1}(H)/\Gamma_3(P_3)$ described in Theorem 25 are almost-crystallographic. Further, since $\text{Ker}(\bar{\sigma})$ is torsion free by Lemma 11(a) and the torsion of S_3 divides 6, it follows from Theorem 2 that these groups are also almost-Bieberbach. Using Theorem 25, we now identify these groups with those given in the classification of 4-dimensional almost-Bieberbach groups with 2-step nilpotent subgroup given in [De, Section 7.2]. Note that by [GPS, Remark 2.5], if M is an infra-nilmanifold whose fundamental group is E , then it is *orientable* if and only if the image of the representation $\theta_F: F \rightarrow \text{GL}(n, \mathbb{Z})$ given by equation (11) is contained in $\text{SL}(n, \mathbb{Z})$.

COROLLARY 26. *Let H be a subgroup of S_3 . Then $\sigma^{-1}(H)/\Gamma_3(P_3)$ is a 4-dimensional almost-Bieberbach group with 2-step nilpotent subgroup, is the fundamental group of an orientable 4-infra-nilmanifold X_H , and is isomorphic to:*

- (a) group number 1, $Q = P1$, given in [De, p. 169] with $k_1 = k_3 = 1$ and $k_2 = -1$ if $H = \{1\}$.
- (b) group number 9, $Q = Cc$, given in [De, pp. 173-174] with $k_1 = 1, k_2 = -1, k_3 = k_4 = 0$ and non-trivial action if $H = \langle(1, 2)\rangle$.
- (c) group number 146, $Q = R3$, given in [De, p. 207] with $k_1 = k_2 = 1$ and $k_3 = k_4 = -1$ if $H = \langle(1, 2, 3)\rangle$.
- (d) group number 161, $Q = R3c$, given in [De, p. 209] with $k_1 = 1$ and $k_2 = k_3 = k_4 = k_5 = 0$ if $H = S_3$.

REMARK 27. If G is a group and $a, b \in G$, then the notation used in [De] for the commutator $[a, b]$, namely $[a, b] = a^{-1}b^{-1}ab$, is different to that used in this paper. However $[a, b] \equiv [a^{-1}, b^{-1}]$ modulo $\Gamma_3(G)$, so the difference in notation does not cause any problems in the identification of our subgroups with those of [De].

Proof of Corollary 26. By Remarks 9(b) and Proposition 10, the groups $P_3/\Gamma_2(P_3)$ and $\Gamma_2(P_3)/\Gamma_3(P_3)$ are torsion free and their respective ranks are 3 and 1, and the Hirsch length of $P_3/\Gamma_3(P_3)$ is equal to 4. So by Corollary 15, $\sigma^{-1}(H)/\Gamma_3(P_3)$ is a 4-dimensional almost-Bieberbach group with 2-step nilpotent subgroup. In order to identify the group $\sigma^{-1}(H)/\Gamma_3(P_3)$ with the corresponding group of [De] for each subgroup H of S_3 , it suffices to use Theorem 25 and to apply the classification of [De, Section 7.2], where the k_i are as given in the statement of parts (a)–(d).

It remains to show that for each subgroup H of S_3 , the manifold X_H are orientable. To see this, by the paragraph preceding the statement of Corollary 26, it suffices to show that the image of each representation $\theta_H: H \rightarrow \text{GL}(4, \mathbb{Z})$ lies in $\text{SL}(4, \mathbb{Z})$. We exhibit the matrices $\theta_H(\alpha)$ and $\theta_H(\beta)$, where α and β are the generators given in Theorem 25 that act on the (ordered) elements a, b, c and d . If $H = \{\text{Id}\}$, the representation θ_H is clearly trivial, and the result follows. For the remaining cases, consider the elements $M_1 = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & -1 \end{pmatrix}$ and $M_2 = \begin{pmatrix} 0 & 0 & 1 & 0 \\ 1 & 0 & 0 & 0 \\ 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 \end{pmatrix}$ of $\text{SL}(4, \mathbb{Z})$. If $H = \langle(1, 2)\rangle$, $\theta_H(\alpha) = M_1$, if $H = \langle(1, 2, 3)\rangle$, $\theta_H(\alpha) = M_2$, and if $H = S_3$, $\theta_H(\alpha) = M_2$ and $\theta_H(\beta) = M_1$. \square

REMARK 28. Since S_4 has 11 non-conjugate subgroups (including the trivial group and the whole group), then in a similar manner, we may show that there are eleven non-isomorphic almost-Bieberbach subgroups of $B_4/\Gamma_3(P_4)$ of the form $\sigma^{-1}(H)/\Gamma_3(P_4)$ with holonomy group H , each of dimension 10 using Proposition 10, where H runs through the subgroups of S_4 .

5.2 Some explicit finite-order elements in $B_5/\Gamma_3(P_5)$

By [GGO1, Corollary 4], $B_5/\Gamma_2(P_5)$ possesses elements of order 3 and 5. From Theorem 2, $B_5/\Gamma_3(P_5)$ does not have elements of order 3, but Theorem 22 implies that there exist elements of order 5. Further, if $\gcd(n, 6) = 1$, elements of order n in $B_n/\Gamma_3(P_n)$ may be determined explicitly using the construction given in the proof of Theorem 22 provided we are able to compute δ_n^n in terms of the elements of the basis \mathcal{B} of the group $\Gamma_2(P_n)/\Gamma_3(P_n)$ described in (17). We now carry out this calculation in the case $n = 5$.

Using equation (7) and with the notation of Lemma 13, we start by describing the action by conjugation of $\alpha_5^{-1} = \alpha_{0,5}^{-1}$, where $\alpha_{0,5} = \sigma_1\sigma_2\sigma_3\sigma_4$, on the elements of the generating set $\{A_{i,j}\}_{1 \leq i < j \leq 5}$ of P_5 :

$$\alpha_5^{-1} = \sigma_4^{-1}\sigma_3^{-1}\sigma_2^{-1}\sigma_1^{-1} : \begin{cases} A_{1,2} \mapsto [A_{1,2}A_{1,3}A_{1,4}, A_{1,5}]A_{1,5} \\ A_{1,3} \mapsto [A_{1,2}A_{2,3}A_{2,4}, A_{2,5}]A_{2,5} \\ A_{1,4} \mapsto [A_{1,3}A_{2,3}A_{3,4}, A_{3,5}]A_{3,5} \\ A_{1,5} \mapsto [A_{1,4}A_{2,4}A_{3,4}, A_{4,5}]A_{4,5} \\ A_{i,j} \mapsto A_{i-1,j-1} \text{ if } 2 \leq i < j \leq 5. \end{cases} \quad (31)$$

From equation (17), the following 10 elements form a basis of $\Gamma_2(P_5)/\Gamma_3(P_5)$:

$$\begin{aligned} a_1 &= [A_{1,2}, A_{2,3}], a_2 = [A_{1,2}, A_{2,5}], a_3 = [A_{1,4}, A_{4,5}], a_4 = [A_{3,4}, A_{4,5}] \\ a_5 &= [A_{2,3}, A_{3,4}], b_1 = [A_{1,2}, A_{2,4}], b_2 = [A_{1,3}, A_{3,5}], b_3 = [A_{2,4}, A_{4,5}] \\ b_4 &= [A_{1,3}, A_{3,4}], b_5 = [A_{2,3}, A_{3,5}], \end{aligned} \quad (32)$$

and that using (8) and (16), under the action by conjugation by α_5^{-1} (considered as an element of $B_5/\Gamma_3(P_5)$), this basis splits into two orbits of length 5 of the form:

$$a_1 \mapsto a_2 \mapsto a_3 \mapsto a_4 \mapsto a_5 \text{ and } b_1 \mapsto b_2 \mapsto b_3 \mapsto b_4 \mapsto b_5.$$

In order to obtain an element α of finite order in $B_5/\Gamma_3(B_5)$, by the construction of the proof of Theorem 22, it suffices to compute δ_5^5 in terms of the basis of $\Gamma_2(P_5)/\Gamma_3(P_5)$ given in (32). Let:

$$c_1 = [A_{1,2}A_{2,3}A_{2,4}, A_{2,5}], c_2 = [A_{1,5}A_{1,2}A_{1,3}, A_{1,4}] \text{ and } c_3 = [A_{1,2}A_{1,3}A_{1,4}, A_{1,5}].$$

In $\Gamma_2(P_5)/\Gamma_3(P_5)$, we have $c_1c_2c_3 = b_1^{-1}b_2^{-1}b_3^{-1}b_4^{-1}b_5^{-1}$. To see this, recall that if a, b and c are elements of a group G , we have a Witt-Hall identity $[ab, c] = [a, [b, c]][b, c][a, c]$ [MKS, Theorem 5.1]. So using equation (16), in $\Gamma_2(P_5)/\Gamma_3(P_5)$ we obtain:

$$\begin{aligned} c_1c_2c_3 &= [A_{1,2}A_{2,3}A_{2,4}, A_{2,5}][A_{1,5}A_{1,2}A_{1,3}, A_{1,4}][A_{1,2}A_{1,3}A_{1,4}, A_{1,5}] \\ &= [A_{1,2}, A_{2,5}][A_{2,3}, A_{2,5}][A_{2,4}, A_{2,5}][A_{1,5}, A_{1,4}][A_{1,2}, A_{1,4}][A_{1,3}, A_{1,4}] \\ &\quad \cdot [A_{1,2}, A_{1,5}][A_{1,3}, A_{1,5}][A_{1,4}, A_{1,5}] \\ &= [A_{1,2}, A_{2,5}][A_{2,3}, A_{3,5}]^{-1}[A_{2,4}, A_{4,5}]^{-1}[A_{1,4}, A_{1,5}]^{-1}[A_{1,2}, A_{2,4}]^{-1}[A_{1,3}, A_{3,4}]^{-1} \\ &\quad \cdot [A_{1,2}, A_{2,5}]^{-1}[A_{1,3}, A_{3,5}]^{-1}[A_{1,4}, A_{1,5}] \\ &= [A_{2,3}, A_{3,5}]^{-1}[A_{2,4}, A_{4,5}]^{-1}[A_{1,2}, A_{2,4}]^{-1}[A_{1,3}, A_{3,4}]^{-1}[A_{1,3}, A_{3,5}]^{-1} \\ &= b_1^{-1}b_2^{-1}b_3^{-1}b_4^{-1}b_5^{-1}. \end{aligned}$$

By equation (15), we have $\delta_5 = A_{3,5}A_{4,5}\alpha_5^{-1}$, and using equation (31) and Proposition 18(a)(ii), as well as the fact that $\alpha_5^{-1}c_1\alpha_5 = c_2$, in $P_5/\Gamma_3(P_5)$, we have:

$$\begin{aligned}
\delta_5^5 &= (A_{3,5}A_{4,5}\alpha_5^{-1})^5 \\
&= A_{3,5}A_{4,5}(\alpha_5^{-1}A_{3,5}A_{4,5}\alpha_5)(\alpha_5^{-2}A_{3,5}A_{4,5}\alpha_5^2)(\alpha_5^{-3}A_{3,5}A_{4,5}\alpha_5^3)(\alpha_5^{-4}A_{3,5}A_{4,5}\alpha_5^4)\alpha_5^{-5} \\
&= A_{3,5}A_{4,5}A_{2,4}A_{3,4}A_{1,3}A_{2,3}c_1A_{2,5}A_{1,2}(\alpha_5^{-1}c_1A_{2,5}\alpha_5)c_3A_{1,5}\alpha_5^{-5} \\
&= c_1c_2c_3A_{3,5}A_{4,5}A_{2,4}A_{3,4}A_{1,3}A_{2,3}A_{2,5}\underline{A_{1,2}}A_{1,4}A_{1,5}\alpha_5^{-5} \\
&= c_1c_2c_3a_2^{-1}a_1^{-1}a_1b_1^{-1}A_{1,2}A_{3,5}A_{4,5}A_{2,4}A_{3,4}\underline{A_{1,3}}A_{2,3}A_{2,5}A_{1,4}A_{1,5}\alpha_5^{-5} \\
&= c_1c_2c_3a_2^{-1}b_1^{-1}b_4^{-1}b_2^{-1}A_{1,2}A_{1,3}A_{3,5}A_{4,5}A_{2,4}A_{3,4}A_{2,3}A_{2,5}\underline{A_{1,4}}A_{1,5}\alpha_5^{-5} \\
&= c_1c_2c_3a_2^{-1}b_1^{-1}b_4^{-1}b_2^{-1}b_4b_1a_3^{-1}A_{1,2}A_{1,3}A_{1,4}A_{3,5}A_{4,5}A_{2,4}A_{3,4}A_{2,3}A_{2,5}\underline{A_{1,5}}\alpha_5^{-5} \\
&= c_1c_2c_3a_2^{-1}b_2^{-1}a_3^{-1}a_2a_3b_2A_{1,2}A_{1,3}A_{1,4}A_{1,5}A_{3,5}A_{4,5}A_{2,4}A_{3,4}\underline{A_{2,3}}A_{2,5}\alpha_5^{-5} \\
&= c_1c_2c_3a_5^{-1}a_5b_5^{-1}A_{1,2}A_{1,3}A_{1,4}A_{1,5}A_{2,3}A_{3,5}A_{4,5}\underline{A_{2,4}}A_{3,4}A_{2,5}\alpha_5^{-5} \\
&= c_1c_2c_3b_5^{-1}b_3^{-1}A_{1,2}A_{1,3}A_{1,4}A_{1,5}A_{2,3}A_{2,4}A_{3,5}A_{4,5}A_{3,4}\underline{A_{2,5}}\alpha_5^{-5} \\
&= c_1c_2c_3b_5^{-1}b_3^{-1}b_3b_5A_{1,2}A_{1,3}A_{1,4}A_{1,5}A_{2,3}A_{2,4}A_{2,5}A_{3,5}A_{4,5}\underline{A_{3,4}}\alpha_5^{-5} \\
&= c_1c_2c_3a_4^{-1}a_4A_{1,2}A_{1,3}A_{1,4}A_{1,5}A_{2,3}A_{2,4}A_{2,5}A_{3,4}A_{3,5}A_{4,5}\alpha_5^{-5} \\
&= c_1c_2c_3\alpha_5^5\alpha_5^{-5} = b_1^{-1}b_2^{-1}b_3^{-1}b_4^{-1}b_5^{-1}. \tag{33}
\end{aligned}$$

The idea of the above computation is first to eliminate all of the terms involving α_5 using equation (31), and then to move each of the underlined terms to the left one-by-one in order to create the word $A_{1,2}A_{1,3}A_{1,4}A_{1,5}A_{2,3}A_{2,4}A_{2,5}A_{3,4}A_{3,5}A_{4,5}$, which we know to be the full twist braid α_5^5 in P_5 , and so in $P_5/\Gamma_3(P_5)$. In doing so, we introduce basis elements of $\Gamma_2(P_5)/\Gamma_3(P_5)$ given by (32), perhaps written in one of the forms of equation (16). Note that the result is coherent with that of Proposition 21, and with the notation of that proposition, we have m . Setting $\theta = \prod_{i=1}^2 \prod_{j=1}^5 b_{i,j}^{r_{i,j}}$, where $b_{1,j} = a_j$, $b_{2,j} = b_j$ and $r_{i,j} \in \mathbb{Z}$ for all $i = 1, 2$ and $j = 1, \dots, 5$, using the notation of Proposition 21(b), and applying the construction of Theorem 22, by (33) and (26), we have $m_{1,j} = 0$, $m_{2,j} = -1$, and $\theta\delta_5$ is of order 5 if and only if $\sum_{j=1}^5 r_{1,j} = 0$ and $\sum_{j=1}^5 r_{2,j} = 1$. So to obtain an explicit element θ , it suffices to pick any integers satisfying these two relations. For example, if $r_{2,1} = 1$ and $r_{2,2} = \dots = r_{2,5} = r_{1,1} = \dots = r_{1,5} = 0$ then the element $b_1\delta_5 = [A_{1,2}, A_{2,4}](\sigma_4\sigma_3\sigma_2^{-1}\sigma_1^{-1})$ is of order 5 in $B_5/\Gamma_3(P_5)$.

REMARK 29. The study of the quotients $B_n/\Gamma_k(P_n)$ for $k > 3$ does not appear to be an easy problem. Our approach requires a description of a basis of $\Gamma_k(P_n)/\Gamma_k(P_n)$. For example, if $n = 3$ and $k = 4$, a long and arduous calculation show that a basis of the group $\Gamma_3(P_3)/\Gamma_4(P_3)$, which is free Abelian of rank 2 by Remark 9(b), is given by $\{[[A_{1,2}, A_{2,3}], A_{1,3}], [[A_{2,3}, A_{1,3}], A_{1,2}]\}$. Since S_3 has 4 subgroups (up to isomorphism), arguing as in Remark 28, we may exhibit 4 non-isomorphic almost-Bieberbach subgroups of $B_3/\Gamma_4(P_3)$ of dimension 6 of the form $\sigma^{-1}(H)/\Gamma_4(P_3)$ with holonomy group H , where H is a subgroup of S_3 and σ is as in (3).

References

- [AIK] T. Arakawa, T. Ibukiyama and M. Kaneko, Bernoulli numbers and zeta functions, with an appendix by D. Zagier, *Springer Monographs in Mathematics*, Tokyo (2014).
- [Au] L. Auslander, Bieberbach's theorems on space groups and discrete uniform subgroups of Lie groups. *Ann. Math.* **71** (1960), 579–590.
- [BM] V. Beck and I. Marin, Torsion subgroups of quasi-abelianized braid groups, preprint, [arXiv:1709.01853](https://arxiv.org/abs/1709.01853).
- [Ch] L. Charlap, Bieberbach groups and flat manifolds, Springer-Verlag, New York (1986).
- [CS] D. C. Cohen and A. I. Suciú, The Chen groups of the pure braid group, The Cech centennial (Boston, MA, 1993), Amer. Math. Soc., Providence, RI, *Contemp. Math.* **181** (1995), 45–64.
- [De] K. Dekimpe, Almost-Bieberbach groups: affine and polynomial structures, Springer Lecture Notes in Mathematics **1639**, Berlin (1996).
- [DD] K. Dekimpe and J. Deré, Existence of Anosov diffeomorphisms on infra-nilmanifolds modeled on free nilpotent lie groups, *Top. Methods Nonlinear Analysis* **46** (2014), 165–189.
- [FR1] M. Falk and R. Randell, The lower central series of a fiber-type arrangement. *Invent. Math.* **82** (1985), 77–88.
- [FR2] M. Falk and R. Randell, Pure braid groups and products of free groups, *Contemp. Math.* **78** (1988), 217–228.
- [GPS] A. Gasior, N. Petrosyan and A. Szczepański, Spin structures on almost-flat manifolds, *Algebr. Geom. Topol.* **16** (2016), 783–796.
- [GGO1] D. L. Gonçalves, J. Guaschi and O. Ocampo, A quotient of the Artin braid groups related to crystallographic groups, *J. Algebra* **474** (2017), 393–423.
- [GGO2] D. L. Gonçalves, J. Guaschi and O. Ocampo, Embeddings of finite groups in $B_n/\Gamma_k(P_n)$ for $k = 2, 3$, preprint, 2018.
- [Gr] M. Gromov, Almost flat manifolds, *J. Differential Geom.* **13** (1978), 231–241.
- [Hal] M. Hall, The theory of groups, Macmillan, New York, 1959.
- [Han] V. L. Hansen, Braids and coverings: selected topics, London Math. Soc. Student Text **18**, Cambridge University Press, 1989.
- [Ho] M. Hoffman, An invariant of finite Abelian groups, *Amer. Math. Monthly* **94** (1987), 664–666.
- [Jo] D. L. Johnson, Presentation of groups, LMS Lecture Notes **22** (1976), Cambridge University Press.
- [Ko] T. Kohno, Série de Poincaré-Koszul associée aux groupes de tresses pures, *Invent. Math.* **82** (1985), 57–75.

- [LVW] J. Y. Li, V. Vershinin and J. Wu, Brunnian braids and Lie algebras, *J. Algebra* **439** (2015), 270–293.
- [MKS] W. Magnus, A. Karrass and D. Solitar, Combinatorial group theory – presentations of groups in terms of generators and relations, Dover Publications Inc., New York (1976).
- [Ma] I. Marin, Crystallographic groups and flat manifolds from complex reflection groups, *Geom. Dedicata* **182** (2016), 233–247.
- [MK] K. Murasugi and B. I. Kurpita, A study of braids, Mathematics and its Applications **484**, Kluwer Academic Publishers, Dordrecht, 1999.
- [N] N. Nielsen, *Traité élémentaire des nombres de Bernoulli*, Gauthier-Villars, Paris (1923).
- [Ru] E. Ruh, Almost flat manifolds, *J. Diff. Geom.* **17** (1982), 1–14.
- [Wi] E. Witmer, The sums of powers of integers, *Amer. Math. Monthly* **42** (1935), 540–548.
- [Wo] J. A. Wolf, Spaces of constant curvature, sixth edition, AMS Chelsea Publishing, vol. **372**, 2011.