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A Lax Functorial Definition of Open Dynamics

Stéphane DUGOWSON *

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Abstract. This paper provides a rewording in the language of lax-functors of the definition of open dynamics given in our systemic theory of interactivity previously exposed in [5], itself coming from [4] and [3].

To formulate this definition, we need first to give the definition of a *closed dynamic* on a small category \mathbf{C} : such a closed dynamic is a lax-functor from \mathbf{C} into the large 2-category of *states sets*, *transitions* and *constraint order* that we denote¹ by \mathbf{PY} . Using the 2-category that we denote by \mathbf{PY}_L or \mathbf{PY}^L , we then define in the same way L -multi-dynamics, where L refers to any non-empty set which we call the parametric set of the considered multi-dynamic. After having defined morphisms between multi-dynamics based on different parametric sets, we define open dynamics and morphisms between them. Finally, we give the definition of a realization of an open dynamic.

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1 The bicategories \mathbf{PY} and \mathbf{PY}_L

1.1 Transitions

For any sets U and V , we call any map $U \rightarrow \mathcal{P}(V)$ — or, equivalently, any binary relation $U \rightarrow V$ — a *transition from U to V* . We often write $\varphi : U \rightsquigarrow V$ to indicate that φ is such a transition.

Denoted $\psi \circ \varphi$, the composition of transitions $\varphi : U \rightsquigarrow V$ and $\psi : V \rightsquigarrow W$ is defined in the obvious way: for all $u \in U$

$$\psi \circ \varphi(u) = \bigcup_{v \in \varphi(u)} \psi(v) \subseteq W.$$

A transition $f : U \rightsquigarrow V$ is called

- *deterministic* if $\text{card}(f(u)) = 1$ for all $u \in U$,
- *quasi-deterministic*, if $\text{card}(f(u)) \leq 1$ for all $u \in U$.

A (quasi) deterministic transition $U \rightsquigarrow V$ is often denoted as a (partial) function $U \rightarrow V$.

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¹In [5], [3], [4], [2] and [1], the same category was denoted by \mathbf{P} .

1.2 The bicategory \mathbf{PY}

\mathbf{PY} is defined as the 2-category (and hence the bicategory) with objects the sets, and for each couple of sets (U, V) the category $\mathbf{PY}(U, V)$ being the *constraint* order on the set of transitions $U \rightsquigarrow V$ defined for all $\varphi, \psi \in \mathbf{PY}(U, V)$ by

$$\varphi \leq \psi \Leftrightarrow \varphi \supseteq \psi$$

where $\varphi \supseteq \psi$ means that for all $u \in U$, $\varphi(u) \supseteq \psi(u)$. If $\varphi \leq \psi$, we say that ψ is more *constraining* than φ , or that φ is *laxer* than ψ . Thus, there exists a 2-cell $\varphi \rightarrow \psi$ if and only if φ is laxer than ψ .

1.3 The bicategory \mathbf{PY}_L

In the same way, for any non-empty set L we define the 2-category (and hence the bicategory) denoted by \mathbf{PY}_L or \mathbf{PY}^L as the one with sets as 0-cells, and for each couple of sets (U, V) the category $\mathbf{PY}^L(U, V)$ being defined by

$$\mathbf{PY}^L(U, V) = (\mathbf{PY}(U, V))^L.$$

In other words, for a given domain U and a given codomain V , a 1-cells in \mathbf{PY}^L is an L -family $\varphi = (\varphi_\lambda)$ of transitions $\varphi_\lambda : U \rightsquigarrow V$. We'll sometimes write $\varphi : U \overset{\varphi}{\rightsquigarrow}_L V$ or $U \overset{\varphi}{\rightsquigarrow}_L V$ to say that φ is such a family.

The composition of 1-cells is naturally defined by $(\varphi_\lambda) \circ (\psi_\lambda) = ((\varphi_\lambda \circ \psi_\lambda)_\lambda)$, and 2-cells express the order

$$(\varphi \leq \psi) \Leftrightarrow (\forall \lambda \in L, \varphi_\lambda \leq \psi_\lambda).$$

2 Categories of closed dynamics and categories of multi-dynamics

2.1 The category $\mathcal{CD}_{\mathbf{C}}$ of closed dynamics on \mathbf{C}

Let \mathbf{C} be a small category viewed as a “discrete bicategory”², and α be a lax-functor from \mathbf{C} to \mathbf{PY} , that we denote³

$$\alpha : \mathbf{C} \rightarrow \mathbf{PY}.$$

We say that α is *disjunctive* if for all objects S and T in \mathbf{C} ,

$$S \neq T \Rightarrow S^\alpha \cap T^\alpha = \emptyset,$$

where we use the notation S^α to denote the set $\alpha(S)$. Likewise, for $d \in \vec{\mathbf{C}}$, we put $d^\alpha = \alpha(d)$.

Definition 1 (Closed dynamics on \mathbf{C}). We denote by $\mathcal{CD}_{\mathbf{C}}$ the category of *closed dynamics* on \mathbf{C} , that is the category

- whose objects are disjunctive lax-functors $\alpha : \mathbf{C} \rightarrow \mathbf{PY}$,

²For any objects S and T , $\mathbf{C}(S, T)$ is just a set, that is a discrete category.

³Keeping the notation we used in our previous papers about dynamics.

- whose arrows — called *dynamorphisms* — are lax-natural transformations $\delta : \alpha \rightsquigarrow \beta$.

The small category \mathbf{C} is called the *engine* of the dynamics $\alpha \in \text{Ob}(\mathcal{CD}_{\mathbf{C}})$.

The arrows $(S \xrightarrow{d} T) \in \vec{\mathbf{C}}$ are called *durations*.

Remark 1. In [3], section §2.1.2, the category $\mathcal{CD}_{\mathbf{C}}$ was denoted $\mathbf{DySC}_{(\mathbf{C})}$. Indeed, by definition of lax-functors and lax-natural transformations,

- a closed dynamic $\alpha : \mathbf{C} \rightarrow \mathbf{PY}$ has to satisfy those properties for objects S, T, \dots and composable arrows e, d of \mathbf{C} :

$$[\text{disjunctivity}] S \neq T \Rightarrow S^\alpha \cap T^\alpha = \emptyset,$$

$$[\text{lax composition}] (e \circ d)^\alpha \subseteq e^\alpha \odot d^\alpha,$$

$$[\text{lax identity}] (Id_S)^\alpha \subseteq Id_{S^\alpha},$$

- a dynamorphism $\delta : \alpha \rightsquigarrow \beta$ is such that

$$[\text{lax naturality}] \forall (S \xrightarrow{d} T) \in \vec{\mathbf{C}}, \delta_T \odot d^\alpha \subseteq d^\beta \odot \delta_S.$$

A closed dynamic $\alpha : \mathbf{C} \rightarrow \mathbf{PY}$ is said to be *deterministic* (resp. *quasi-deterministic*) if for every $(S \xrightarrow{d} T) \in \vec{\mathbf{C}}$ the transition d^α is deterministic (resp. quasi-deterministic). Likewise, a dynamorphism δ is said to be (*quasi-*)*deterministic* if all transitions δ_S are (*quasi-*)deterministic.

Definition 2. A deterministic closed dynamic is called a *clock*.

2.2 The category \mathcal{CD} of closed dynamics

Definition 3. We define the category \mathcal{CD} of all closed dynamics taking as *dynamorphisms* from $\alpha : \mathbf{C} \rightarrow \mathbf{PY}$ to $\beta : \mathbf{D} \rightarrow \mathbf{PY}$ couples (Δ, δ) with $\Delta : \mathbf{C} \rightarrow \mathbf{D}$ a functor, and $\delta : \alpha \rightsquigarrow (\beta \circ \Delta)$ a lax-natural transformation.

Remark 2. When $\mathbf{C} = \mathbf{D}$ we assume by default that $\Delta = Id_{\mathbf{C}}$. In other words, $\mathcal{CD}_{\mathbf{C}}$ is not a full subcategory of \mathcal{CD} .

2.3 The category $\mathcal{MD}_{(\mathbf{C}, L)}$ of L multi-dynamics on \mathbf{C}

Let L be a non-empty set.

Definition 4 (L -multi-dynamics on \mathbf{C}). We denote by $\mathcal{MD}_{(\mathbf{C}, L)}$ the category

- with objects, called *L -multi-dynamics on \mathbf{C}* , the disjunctive⁴ lax functors

$$\alpha : \mathbf{C} \rightarrow \mathbf{PY}^L,$$

- with arrows $\delta : \alpha \rightsquigarrow \beta$ — called (\mathbf{C}, L) -*dynamorphisms* — given by⁵

$$\mathcal{MD}_{(\mathbf{C}, L)}(\alpha, \beta) = \bigcap_{\lambda \in L} \mathcal{CD}_{\mathbf{C}}(\alpha_\lambda, \beta_\lambda),$$

where for each $\lambda \in L$ we denote α_λ the closed dynamic associated in an obvious way with the multi-dynamic α for this λ .

⁴As in the closed dynamics case, a functor $\alpha : \mathbf{C} \rightarrow \mathbf{PY}^L$ is said to be disjunctive if for all objects S and T in \mathbf{C} , $S \neq T \Rightarrow S^\alpha \cap T^\alpha = \emptyset$.

⁵About the intersection used here, see below the remark 3.

Thus, by definition, and with the notation introduced in section §1.3, an L -multi-dynamic associates with each duration $(S \xrightarrow{d} T) \in \vec{\mathbf{C}}$ an L -family $S^\alpha \xrightarrow{\approx}^d_L T^\alpha$.

Remark 3. [About the expression $\bigcap_{\lambda \in L} \mathcal{CD}_{\mathbf{C}}(\alpha_\lambda, \beta_\lambda)$] If \mathbf{C} and \mathbf{E} are categories, and α, α', β and β' are functors $\mathbf{C} \rightarrow \mathbf{E}$ such that for all objects $C \in \mathbf{C}$ we have $\alpha(C) = \alpha'(C)$ and $\beta(C) = \beta'(C)$, then if $(\alpha, \beta) \neq (\alpha', \beta')$, the sets of natural transformations $\text{Nat}(\alpha, \beta)$ and $\text{Nat}(\alpha', \beta')$ are disjoint

$$\text{Nat}(\alpha, \beta) \cap \text{Nat}(\alpha', \beta') = \emptyset$$

because a natural transformation $\delta \in \text{Nat}(\alpha, \beta)$ has a unique domain α and a unique codomain β . Nevertheless, forgetting this information about domains and codomains and identifying δ just with a family of arrows $\left(\alpha(C) \xrightarrow{\delta_C} \beta(C) \right)_{C \in \mathbf{C}}$ belonging to $\vec{\mathbf{E}}$, we'll write $\delta \in \text{Nat}(\alpha, \beta)$ to say only that δ is natural in respect with α and β , that is

$$\forall (S \xrightarrow{d} T) \in \vec{\mathbf{C}}, \delta_T \circ \alpha(d) = \beta(d) \circ \delta_S.$$

Then, $\text{Nat}(\alpha, \beta) \cap \text{Nat}(\alpha', \beta')$ will designate the set of families of \mathbf{E} -arrows $\delta = ((\delta_C)_{C \in \mathbf{C}})$ which are natural in respect together with (α, β) and with (α', β') . In general, this kind of intersection will be non-empty.

The same remark can be made about lax transformations between lax functors between bicategories, and it is in this sense that we use expressions like $\bigcap_{\lambda \in L} \mathcal{CD}_{\mathbf{C}}(\alpha_\lambda, \beta_\lambda)$. To emphasize the forgetfulness of information about domains and codomains, we could use some notation like $\underline{\mathcal{CD}}_{\mathbf{C}}$ but it would be a little heavy, so we prefer not to do so.

Remark 4. Unsurprisingly, an L -multi-dynamic $\alpha : \mathbf{C} \rightarrow \mathbf{PY}^L$ is said to be (*quasi-*) *deterministic* if for every $\lambda \in L$ the closed dynamic α_λ is (*quasi-*)deterministic.

2.4 The category $\mathcal{MD}_{\mathbf{C}}$ of multi-dynamics on \mathbf{C}

Let $\alpha : \mathbf{C} \rightarrow \mathbf{PY}^L$ and $\beta : \mathbf{C} \rightarrow \mathbf{PY}^M$ be multi-dynamics on \mathbf{C} . We define a *\mathbf{C} -dynamorphism* $\delta : \alpha \rightsquigarrow \beta$ as a couple $(\theta, \underline{\delta})$ with $\theta : L \rightarrow M$ a map, and $\underline{\delta} \in \bigcap_{\lambda \in L} \mathcal{CD}_{\mathbf{C}}(\alpha_\lambda, \beta_{\theta(\lambda)})$. The lax natural part $\underline{\delta}$ of such a dynamorphism δ is often itself simply denoted δ . Thus, to be a dynamorphism, (θ, δ) must satisfy the lax naturality condition

$$\forall \lambda \in L, \forall (S \xrightarrow{d} T) \in \vec{\mathbf{C}}, \delta_T \circ d_\lambda^\alpha \subseteq d_{\theta(\lambda)}^\beta \circ \delta_S.$$

Definition 5. We define the category $\mathcal{MD}_{\mathbf{C}}$ taking as objects all multi-dynamics on \mathbf{C} , and as arrows all \mathbf{C} -dynamorphisms between them.

Naturally, a dynamorphism δ is said to be (*quasi-*)*deterministic* if for every object $S \in \mathbf{C}$, δ_S is (*quasi-*)deterministic.

Remark 5. Closed dynamics on \mathbf{C} can be seen as a full subcategory of $\mathcal{MD}_{\mathbf{C}}$, and we can in particular consider dynamorphisms between closed dynamics on \mathbf{C} and multi-dynamics on \mathbf{C} .

2.5 The category \mathcal{MD} of multi-dynamics

Let $\alpha : \mathbf{C} \rightarrow \mathbf{PY}^L$ and $\beta : \mathbf{D} \rightarrow \mathbf{PY}^M$ be multi-dynamics with possibly different engines. We define a *dynamorphism* $\alpha \looparrowright \beta$ as a triple (θ, Δ, δ) with

- $\theta : L \rightarrow M$ a map,
- $\Delta : \mathbf{C} \rightarrow \mathbf{D}$ a functor,
- $\delta \in \bigcap_{\lambda \in L} \mathcal{CD}_{\mathbf{C}}(\alpha_\lambda, \beta_{\theta(\lambda)} \circ \Delta)$.

The last condition is equivalent to

$$(\Delta, \delta) \in \bigcap_{\lambda \in L} \mathcal{CD}(\alpha_\lambda, \beta_{\theta(\lambda)}),$$

i.e. the lax naturality condition

$$\forall \lambda \in L, \forall (S \xrightarrow{d} T) \in \vec{\mathbf{C}}, \delta_T \odot d_\lambda^\alpha \subseteq (\Delta d)_{\theta(\lambda)}^\beta \odot \delta_S$$

has to be satisfied.

Definition 6. The category \mathcal{MD} of multi-dynamics is defined taking as objects all multi-dynamics, and as arrows all dynamorphisms between them.

3 The category \mathcal{OD} of open dynamics

3.1 Definition

Definition 7. An *open dynamic* A with engine \mathbf{C} is the data

$$A = \left((\alpha : \mathbf{C} \rightarrow \mathbf{PY}^L) \overset{\rho}{\looparrowright} (\mathbf{h} : \mathbf{C} \rightarrow \mathbf{Sets}) \right)$$

of

- a non-empty set L ,
- a multi-dynamic $\alpha \in \mathcal{MD}_{(\mathbf{C}, L)} \subset \mathcal{MD}_{\mathbf{C}}$,
- a clock $\mathbf{h} \in \mathcal{CD}_{\mathbf{C}} \subset \mathcal{MD}_{\mathbf{C}}$,
- a deterministic dynamorphism $\rho \in \mathcal{MD}_{\mathbf{C}}(\alpha, \mathbf{h})$ called *datation*.

According to the definitions given in § 2.4.2 of [3] and § 1.2.2 of [5], a *dynamorphism* from an open dynamic

$$A = \left((\alpha : \mathbf{C} \rightarrow \mathbf{PY}^L) \overset{\rho}{\looparrowright} (\mathbf{h} : \mathbf{C} \rightarrow \mathbf{Sets}) \right)$$

to an open dynamic

$$B = \left((\beta : \mathbf{D} \rightarrow \mathbf{PY}^M) \overset{\tau}{\looparrowright} (\mathbf{k} : \mathbf{D} \rightarrow \mathbf{Sets}) \right)$$

is a quadruplet $(\theta, \Delta, \delta, \varepsilon)$ with

- $(\theta, \Delta, \delta) \in \mathcal{MD}(\alpha, \beta)$,
- $(\Delta, \varepsilon) \in \mathcal{CD}(\mathbf{h}, \mathbf{k})$,
- this lax synchronization condition satisfied:

$$\forall S \in \dot{\mathbf{C}}, \tau_{\Delta_S} \odot \delta_S \subseteq \varepsilon_S \odot \rho_S.$$

We'll denote by \mathcal{OD} the category of all open dynamics, with dynamorphisms as arrows.

3.2 Realizations of an open dynamic

With the definitions given above, the other notions of our systemic theory of interactivity keep the same definitions as previously given in [5]: parametric quotients, interactions, connectivity structures of an interaction, dynamical families, open dynamics generated by a dynamical family, ...

In particular, given an open dynamic $A = \left((\alpha : \mathbf{C} \rightarrow \mathbf{PY}^L) \overset{\rho}{\dashv} (\mathbf{h} : \mathbf{C} \rightarrow \mathbf{Sets}) \right)$ we have

Definition 8. A *realization* (or a *solution*) of A is a quasi-deterministic dynamorphism $(\mathfrak{s} : \mathbf{h} \dashv \alpha) \in \mathcal{MD}_{\mathbf{C}}(\mathbf{h}, \alpha)$ satisfying the lax condition $\rho \odot \mathfrak{s} \subseteq Id_{\mathbf{h}}$.

In other words⁶, such a realization is a couple $\mathfrak{s} = (\lambda, \sigma)$ with $\lambda \in L$ and $\sigma \in \mathcal{CD}_{\mathbf{C}}(\mathbf{h}, \alpha_{\lambda})$ which is a partial function $\sigma : st(\mathbf{h}) \rightarrow st(\alpha)$ defined on a subset $D_{\sigma} \subset st(\mathbf{h})$ and satisfying these properties:

$$\begin{aligned} \forall t \in D_{\sigma}, \rho(\sigma(t)) &= t, \\ \forall S \in \dot{\mathbf{C}}, \forall t \in S^{\mathbf{h}} \cap D_{\sigma}, \sigma(t) &\in S^{\alpha}, \\ \forall (S \xrightarrow{d} T) \in \vec{\mathbf{C}}, \forall t \in S^{\mathbf{h}}, (d^{\mathbf{h}}(t) \in D_{\sigma} &\Rightarrow (t \in D_{\sigma} \text{ and } \sigma(d^{\mathbf{h}}(t)) \in d_{\lambda}^{\alpha}(\sigma(t))))). \end{aligned}$$

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References

- [1] Stéphane Dugowson. Introduction aux dynamiques catégoriques connectives, décembre 2011. <http://hal.archives-ouvertes.fr/hal-00654494/fr/>.
- [2] Stéphane Dugowson. *Dynamiques connectives (Une introduction aux notions connectives : espaces, représentations, feuilletages et dynamiques catégoriques)*. Éditions Universitaires Européennes, 2012.

⁶See [5], § 1.3.1.

- [3] Stéphane Dugowson. Dynamiques sous-catégoriques ouvertes en interaction (définitions et théorème de stabilité), 9 août 2015.
<https://hal.archives-ouvertes.fr/hal-01183519>.
- [4] Stéphane Dugowson. Interaction des dynamiques graphiques ouvertes (définitions), 7 août 2015.
<https://hal.archives-ouvertes.fr/hal-01177450>.
- [5] Stéphane Dugowson. Dynamiques en interaction : une introduction à la théorie des dynamiques sous-fonctorielles ouvertes (hal-01357009), 31 août 2016. <https://hal.archives-ouvertes.fr/hal-01357009>.

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