



HAL
open science

A transformed stochastic Euler scheme for multidimensional transmission PDE

Pierre Etoré, Miguel Martinez

► **To cite this version:**

Pierre Etoré, Miguel Martinez. A transformed stochastic Euler scheme for multidimensional transmission PDE. 2019. hal-01800305v2

HAL Id: hal-01800305

<https://hal.science/hal-01800305v2>

Preprint submitted on 7 Jun 2019 (v2), last revised 8 Mar 2021 (v4)

HAL is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

A TRANSFORMED STOCHASTIC EULER SCHEME FOR MULTIDIMENSIONAL TRANSMISSION PDE

BY PIERRE ÉTORÉ[†], MIGUEL MARTINEZ[‡]

Laboratoire Jean-Kuntzmann[†] and Université Paris-Est, Marne-la-Vallée[‡]

In this paper we consider multi-dimensional Partial Differential Equations (PDE) of parabolic type in divergence form. The coefficient matrix of the divergence operator is assumed to be discontinuous along some smooth interface. At this interface, the solution of the PDE presents a compatibility transmission condition of its conormal derivatives (multi-dimensional diffraction problem). We prove an existence and uniqueness result for the solution and study its properties. In particular, we provide new estimates for the partial derivatives of the solution in the classical sense. We then construct a low complexity numerical Monte Carlo stochastic Euler scheme to approximate the solution of the PDE of interest. Using the afore mentioned estimates, we prove a convergence rate for our stochastic numerical method when the initial condition belongs to some iterated domain of the divergence form operator. Finally, we compare our results to classical deterministic numerical approximations and illustrate the accuracy of our method.

Introduction.

Statement of the problem. Given a finite time horizon T , a real valued function $x \mapsto u_0(x)$, and an elliptic symmetric matrix $x \mapsto a(x) \in \mathbb{R}^{d \times d}$, which is smooth except at the interface surfaces Γ between subdomains of \mathbb{R}^d , we consider the parabolic transmission problem (or diffraction) problem : find u from $[0, T] \times \mathbb{R}^d$ to \mathbb{R} satisfying

$$(0.1) \quad \left\{ \begin{array}{l} \partial_t u(t, x) - \nabla \cdot (a(x) \nabla_x u(t, x)) = 0, \quad \forall (t, x) \in (0, T] \times (\mathbb{R}^d \setminus \Gamma) \\ u(0, x) = u_0(x), \quad \forall x \in \mathbb{R}^d \\ \text{Compatibility transmission conditions along the interfaces surfaces.} \end{array} \right.$$

The objective of this paper is to provide an efficient stochastic numerical resolution method for the solution of (0.1).

Parabolic PDEs in divergence form and their probabilistic representations. Parabolic equations involving $\mathcal{L} = \nabla \cdot (a \nabla_x)$ have been a major preoccupation for mathematicians in the fifties and the sixties. We may cite the pioneering works of J.Nash [38, 39], E. De Giorgi [7], and J. Moser [36, 35, 34] that prove the continuity of the solution of the Cauchy problem attached to \mathcal{L} and also the celebrated paper by D.G. Aronson [1], which gives upper and lower Gaussian estimate bounds

*Research supported by the LABEX Bézout

MSC 2010 subject classifications: Primary 60H10, 65U05; secondary 65C05, 60J30, 60E07, 65R20

Keywords and phrases: Stochastic Differential Equations; Divergence Form Operators; Euler discretization scheme; Monte Carlo methods.

for the fundamental solution of the operator \mathcal{L} (for a more modern perspective on evolution PDEs involving divergence form operators of type \mathcal{L} see also [28]).

In the particular case where the matrix a is assumed to be discontinuous along the regular boundaries of some nice disjoint connected open sets in \mathbb{R}^d , a refined analysis of the parabolic equation may be found in the monograph [17], where the parabolic equation is interpreted as a diffraction problem with transmission conditions along the discontinuity boundaries.

Up to our knowledge, the first construction of a Markovian semi-group associated to $\mathcal{L} = \nabla \cdot (a \nabla)$ (in the general case where a is only supposed to be measurable) may be found in the seminal paper of D. Stroock [45] (see also [2], Chapter 7). To understand our difficulty, the conclusion of [45] is :

It should be obvious that the results obtained in this section can be used to construct a diffusion process on Ω corresponding to anyone of the semigroups discussed herein. In addition, the convergence results for the semigroups give rise to weak convergence of the corresponding measures on Ω .

It remains an open and challenging problem to provide a better probabilistic interpretation of these essentially analytic facts.

Since then, there have been many works that try to provide this 'better probabilistic interpretation' in a multidimensional framework.

The construction of the associated Dirichlet process X . In the general case where the symmetric matrix a is only supposed to be elliptic and measurable, the theory of Dirichlet forms as exposed in the monograph by Fukushima [13, 12] gives surely the best possible answer to this question under very general hypothesis. The symmetric operator \mathcal{L} is naturally attached to its corresponding symmetric Dirichlet form, giving rise to a stochastic Dirichlet process X that is described as the addition of a continuous martingale and a continuous additive functional of zero energy. The theory ensures the validity of a Feynman-Kac type formula linking the solution of the Cauchy problem and the Dirichlet process X .

Going further in the analysis, A. Roskosz [43, 44] proves that X satisfies a Lyons-Zheng decomposition, namely X may be written as the solution of a complex stochastic equation that is the addition of three processes : a martingale, an increment of a reversed time martingale (whose quadratic variations depend on the unknown process X), and an additive functional involving the logarithm derivative of the fundamental solution of the parabolic operator evaluated at X .

This description permits to retrieve some kind of Itô formula for $\phi(X_T)$ where ϕ belongs to some 'good' Sobolev space.

From a numerical perspective, it seems clear that the Lyons-Zheng decomposition provides a description of X that is so strongly nonlinear (time reversion and logarithm derivative of the density of the unknown process), that is seems quite impossible to propose a stochastic numerical scheme for the process X at this stage.

However one may hope to describe in more detail the behavior of the trajectories of X when the coefficient matrix a – instead of being assumed to be only elliptic measurable – is now assumed to be very smooth outside discontinuities that take place along nice and smooth surfaces Γ .

In this perspective, many papers go deeper and manage to apply the stochastic calculus tools developed by the theory of Dirichlet forms to give a more precise answer to the description of processes related to X (see for example [48], [22], [49] and also the results included in the exercises of the monograph [12]). Applying the results of the theory of Dirichlet forms, we prove here that X is the solution of some stochastic differential equation that is the addition of the expected martingale term driven by some Brownian motion and the expected additive drift term (both terms involving the unknown process X), and an additive functional A^Γ that captures the behavior of the process at the discontinuity boundaries Γ . The additive functional is rigorously constructed through its

Revuz correspondence with some transformation of the natural surface measure of Γ involving the discontinuity jumps of a along Γ and the conormal derivatives.

Numerical Monte-Carlo methods in the one-dimensional case. When the underlying space is one dimensional and the discontinuity is at zero (Γ then reduces to the single point 0), the theory allows to identify A^Γ with $\frac{a(0+) - a(0-)}{a(0+) + a(0-)} L_t^0(X)$ and gives rise to the following stochastic differential equation for the process X

$$(0.2) \quad dX_t = \sigma(X_t)dB_t + \sigma\sigma'_\pm(X_t)dt + \frac{a(0+) - a(0-)}{a(0+) + a(0-)} dL_t^0(X).$$

Here $(B_t)_{t \geq 0}$ stands for some standard one dimensional Brownian motion constructed on some probability space $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ and we have $\sigma^2 = 2a$. The process $(L_t^0(X))_{t \geq 0}$ denotes the symmetric local time of the unknown process.

Under somewhat weaker conditions than those stated above, one can show that (0.2) has a unique strong solution $(X_t)_{t \geq 0}$, satisfying $X_0 = x_0$ \mathbb{P}^{x_0} - *a.s.*, which is moreover a strong Markov process: see Le Gall [19].

In this one dimensional context, the link between solutions of (0.2) and the solutions of parabolic PDEs with transmission conditions involving the operator \mathcal{L} has been thoroughly studied. One may refer to the overview [21], and the series of works [29, 30, 32], [23], and [8, 9] where stochastic numerical schemes are presented. Note that in most of these works the line-space is discretized and the scheme is in fact some rescaled random walk evolving on a space-time grid. The method proposed in [30] is an exception : it is an Euler type scheme that does not require any discretization of the underlying one dimensional state space.

Though somewhat different, all these one dimensional numerical schemes are constructed using this explicit representation of A^Γ as a local time. In particular, all the tools related to the theory of one dimensional local times for semi martingales (Itô -Tanaka formula, occupation time formula) are used in force to construct these numerical schemes and prove that there is indeed convergence in some sense towards the solution of (0.2).

Contribution of this paper. However, when turning to the objective of constructing a stochastic numerical scheme for X in a multidimensional context (when $d \geq 2$), the description of A^Γ *via* its Revuz correspondence measure does not provide a direct natural way for the discretization of X (see however the Phd Thesis of L. Lenotre [24] and the walk on spheres algorithm in [5] in the special case of a diagonal coefficient matrix a constant outside the discontinuity boundary Γ). We also mention the work of [26], which attempts to interpret stochastically the deterministic Galerkin method using jump Markov Chains.

In this article, we propose to tackle the problem of the construction of a quite simple numerical scheme of Euler type for X from another perspective. Our starting point is the solution of the parabolic equation involving \mathcal{L} . Inspired by the proof of the convergence of the Euler scheme constructed in the one dimensional case in [32], we build a multidimensional Euler scheme that is purposely designed to capture the *multidimensional transmission conditions* of the parabolic PDE associated to \mathcal{L} .

The novelty of this approach is that we construct our scheme without being concerned at first sight by the description of the limiting process X . All our concern is to guarantee that the error between the expectation of our process – visualized through some very smooth arbitrary test function (belonging to the iterated domains of the operator \mathcal{L}) – and the corresponding solution of the parabolic equation (with the test function as initial condition) converges to zero.

The resulting numerical scheme may be viewed as an extension of the symmetrized Euler scheme of [4] for reflected diffusion with smooth coefficients to partially reflected diffusions with discontinuous coefficients, when no Skorokhod representation of the local time at the boundary Γ is available.

When turning to the proof of the convergence, we face the difficulty of getting global precise estimation bounds for the solution of the parabolic equation and its partial derivatives (up to order four in the space variable) outside the discontinuity boundary Γ for all strictly positive times (and not just for times t satisfying $t \geq \varepsilon$ for some $\varepsilon > 0$) and all the way up to the boundary (not only interior estimates). Unfortunately, the analysis performed in [17] (which is somewhat difficult to read and to understand in full detail) does not provide the refined estimations we need for our purpose (see also the recent work for the estimation of the gradient in [37] extending [27] to the parabolic case, but still for times after some $\varepsilon > 0$ and estimates depending on ε).

Instead of trying to adapt and extend the results in [17], we preferred to look at the results obtained for the solutions of *elliptic* divergence transmission problems involving \mathcal{L} (see [33] and also the rich monograph [3]). In this paper we extend the results of [33] to the parabolic case by performing the classical Hille-Yosida theorem and we perform the analysis in order to get global estimates. The price to pay is to strengthen strongly our assumptions on the regularity of a outside its discontinuity boundary Γ and to assume that the initial condition function (the test function) belongs to some order of the iterated domain of the operator \mathcal{L} . As expected, the orders of smoothness and iteration that we require increase strongly with the dimension d .

Then, using the classical Sobolev injections, we obtain all the estimates on the solution we need to prove the convergence of our stochastic numerical scheme.

Note that the analysis of the weak error, visualized through restricted test functions belonging to some large enough iteration of the domain of the underlying operator, seems quite natural. This is what is done for example in [4] for the symmetrized Euler scheme corresponding to reflected diffusions, where the spatial derivatives of the test functions are assumed to verify some compatibility conditions at the reflection boundary ; (whereas in [4] the compatibility conditions are fixed once and for all, the compatibility conditions we require depend crucially on the dimension d).

Finally, we prove that the weak error of our scheme is of order the square root of the time discretization step (improving slightly the results of [32] in the one dimensional case).

Organization of the paper. The paper is organized as follows.

In Section 1 we present the notations of the paper and our main assumptions. In Section 2 we study the parabolic transmission problem. Using classical tools of deterministic partial differential equation, we prove bounds on the solution and its partial derivatives when the initial condition is very smooth (i.e. belongs to iterated domains of the operator; see the result of Corollary 2.20). Up to our knowledge such results cannot be found in the existing literature on the subject. The proofs of many technical lemmas and propositions of this section are in fact transferred to an appendix section at the end of the paper. These proofs are technical and we believe they may be omitted at the first reading of the paper.

Using the theory of Dirichlet forms in Section 3 we proceed to construct the stochastic process X that is attached to the parabolic transmission problem. The main result of this section is the proof of a Skorokhod decomposition of the Hunt process associated to our transmission operator in divergence form.

In Section 4 we introduce our transformed Euler type stochastic numerical scheme, which captures the transmission condition at the boundary interface Γ (see the explanation figure 1). Using in force the results obtained in the former sections, we manage to prove a weak convergence result

towards the solution of the parabolic transmission problem in Section 5. We show that the convergence is of order the square root of the time discretization step. The precise result is stated in Theorem 5.1.

Finally, we conclude by presenting some numerical studies in Section 6. We compare results given by our procedure to stochastic numerical schemes that use a regularization of the coefficient a in the vicinity of the boundary, and to classical numerical deterministic scheme (using finite element methods for the discretization in space).

1. General notations and assumptions. For two points $x, y \in \mathbb{R}^d$ we denote by $\langle x, y \rangle$ their scalar product $\langle x, y \rangle = x^*y = \sum_{i=1}^d x_i y_i$.

For a point $x \in \mathbb{R}^d$ we denote by $|x|$ its euclidean norm i.e. $|x|^2 = \sum_{i=1}^d x_i^2 = \langle x, x \rangle$.

We denote by (e_1, \dots, e_d) the usual orthonormal basis of \mathbb{R}^d .

For two metric spaces E, F we will denote by $C(E; F)$ the set of continuous functions from E to F and, for $1 \leq p \leq \infty$, by $C^p(E; F)$ the set of functions in $C(E; F)$ that are p times differentiable with continuous derivatives.

We will denote by $C_c^p(E; F)$ the set of functions in $C^p(E; F)$ that have a compact support.

We will denote by $C_b^p(E; F)$ the set of functions in $C^p(E; F)$ that are continuous with bounded p first derivatives ($C_b(E; F)$ denotes the set of functions in $C(E; F)$ that are bounded).

If $F = \mathbb{R}$, we will sometimes simply write for instance $C(E)$ for $C(E; \mathbb{R})$, for the sake of conciseness.

For any multi-index $\alpha = (i_1, \dots, i_d) \in \mathbb{N}^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we note x^α the product $x_1^{i_1} \dots x_d^{i_d}$ and $|\alpha| = i_1 + \dots + i_d$. So that for $u \in C^{|\alpha|}(\mathbb{R}^d)$ we will denote $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$, or in short $\partial^\alpha u$, the partial derivative $\partial_{x_1}^{i_1} \dots \partial_{x_d}^{i_d} u$.

Let $U \subset \mathbb{R}^d$ an open subset. We will denote by $L^2(U)$ the set of square integrable functions from U to \mathbb{R} equipped with the usual norm and scalar product $\|\cdot\|_{L^2(U)}$ and $\langle \cdot, \cdot \rangle_{L^2(U)}$.

We denote $H^1(U)$ the usual Sobolev space $W^{1,2}(U)$, equipped with the norm

$$(1.1) \quad v \mapsto \left(\|v\|_{L^2(U)}^2 + \sum_{i=1}^d \|D_i v\|_{L^2(U)}^2 \right)^{1/2} =: \|v\|_{H^1(U)}$$

where $D_i v$ denotes the derivative in the distribution sense with respect to x_i of $v \in L^2(U)$. Note that for the sake of conciseness we will sometimes note $\nabla v = (D_1 v, \dots, D_d v)^*$ and thus

$$\|\nabla v\|_{L^2(U)}^2 = \sum_{i=1}^d \|D_i v\|_{L^2(U)}^2$$

for a function $v \in H^1(U)$.

We recall that the space $H_0^1(U) \subset H^1(U)$ can be defined as $H_0^1(U) = \overline{C_c^\infty(U; \mathbb{R})} = \overline{C_c^1(U; \mathbb{R})}$.

We denote $H^{-1}(U)$ the usual dual topological space of $H_0^1(U)$.

For $m \geq 2$, we denote $H^m(U)$ the usual Sobolev space $W^{m,2}(U) \subset L^2(U)$ of functions having m successive weak derivatives in $L^2(U)$ (and for a multi-index α with $|\alpha| \leq m$ and $u \in H^m(U)$ we denote $D^\alpha u$ such a weak derivative).

We will have to consider fractional Sobolev spaces - in fact $H^s(\Gamma)$, $s \in \{-\frac{1}{2}, \frac{1}{2}\}$, where Γ is some boundary of dimension $d - 1$ between domains included in \mathbb{R}^d . We recall here the definition of $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, based on Fourier transform. We denote $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly

decreasing $C^\infty(\mathbb{R}^n)$ functions, and $\mathcal{S}'(\mathbb{R}^n)$ the space of *temperate distributions* (see [33] p72 for details). For $u \in \mathcal{S}'(\mathbb{R}^n)$ we define $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi\xi^*x} u(x) dx$, $\xi \in \mathbb{R}^n$. Then for $u \in \mathcal{S}'(\mathbb{R}^n)$, \hat{u} is defined by extension, using

$$(1.2) \quad \langle \hat{u}, \varphi \rangle_{\mathcal{S}', \mathcal{S}} = \langle u, \hat{\varphi} \rangle_{\mathcal{S}', \mathcal{S}}, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n).$$

We thus have

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \right\}.$$

This space is equipped with the norm

$$v \mapsto \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi =: \|v\|_{H^s(\mathbb{R}^n)}.$$

In the case $s = 1$ the thus defined Sobolev space corresponds in fact exactly to our previous definition of $H^1(\mathbb{R}^n)$.

The notion of a Lipschitz domain $U \subset \mathbb{R}^d$ (resp. of class C^k) with bounded boundary $\Gamma = \partial U$ is defined with the help of a system of local Lipschitz change of coordinates (resp. of class C^k ; see [33] Chap.3 pp89-90).

This allows to define the space $H^s(\Gamma)$ from $H^s(\mathbb{R}^{d-1})$, $0 \leq s \leq 1$, in the case U is Lipschitz with bounded boundary Γ (see again [33] Chap. 3, pp98-99).

In the case where U is Lipschitz with bounded boundary Γ , we denote $H^{-1/2}(\Gamma)$ and $\tilde{H}^{-1}(U)$ the dual spaces of respectively $H^{1/2}(\Gamma)$ and $H^1(U)$ (in coherence with respectively p98-99 and Theorem 3.30 p92 in [33]).

The surface measure $\zeta(dy)$ on the boundary Γ of U can be defined as follows. Consider a coordinate neighborhood

$$U \cap G = \{(x', x_d) \mid x_d < F(x')\} \cap G$$

with a Lipschitz function F . Denote by $\ell(dx')$ the Lebesgue measure on \mathbb{R}^{d-1} . Since F is differentiable a.e. with bounded ∇F , we may let for any Borel $B \subset \Gamma \cap G$

$$\zeta(B) = \int_{B_*} (1 + |\nabla F(x')|^2)^{1/2} \ell(dx') \quad , \quad B_* = \{x' \mid (x', F(x')) \in B\}.$$

Then, it is known that $\zeta(dy)$ a.e. point $x \in \Gamma$ admits a unit inward normal vector $\nu(x)$.

Consider $\mathbb{R}^d = \bar{D}_+ \cup D_-$ with D_+ and D_- two open connected subdomains separated by a transmission boundary Γ that is to say

$$\Gamma = \bar{D}_+ \cap \bar{D}_-.$$

By an assumption of type " Γ is bounded and Lipschitz (or of class C^k) " we will mean that both D_+ and D_- are Lipschitz (or C^k) domains, and that Γ is bounded. Note that in that case we shall consider D_+ (resp. D_-) as the interior (resp. exterior) domain. Note that D_- is then unbounded (although its boundary is bounded).

We shall encounter however the situation where D_\pm and Γ are unbounded but we will restrict ourselves to the case where $D_+ = \mathbb{R}_+^d = \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_d > 0\}$, $D_- = \mathbb{R}_-^d = \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_d < 0\}$ and $\Gamma = \mathbb{R}^{d-1} \times \{0\}$. This will be convenient to lead some of our proofs. In fact when we assert a result mentioning the curve Γ without further precision this will mean that either $D_\pm = \mathbb{R}_\pm^d$, either Γ is bounded and Lipschitz (for example in the Green identities of Subsubsection 2.1.2; see also Remark 2.11).

We will denote $\gamma : H^1(D_\pm) \rightarrow H^{1/2}(\Gamma)$ the usual trace operator on Γ (Γ is supposed to be bounded and Lipschitz).

On the surface Γ we introduce the following assumption.

ASSUMPTION 1.1. **(D)** : *The transmission boundary Γ is bounded and of class C^5 .*

We denote

$$D = D_+ \cup D_- = \mathbb{R}^d \setminus \Gamma \subset \mathbb{R}^d.$$

For $u \in C(\mathbb{R}^d; \mathbb{R})$ we denote for a point $y \in \Gamma$

$$u(y_\pm) = \lim_{z \rightarrow y, z \in D_\pm} u(z).$$

For $u \in C(\mathbb{R}^d; \mathbb{R}) \cap C^1(D; \mathbb{R})$ we denote $\nabla_x u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d})^*$ and, for a point $y \in \Gamma$

$$(1.3) \quad \nabla_x u_\pm(y) = \lim_{z \rightarrow y, z \in D_\pm} \nabla_x u(z),$$

if this limit exists.

For a vector field $G \in C(\mathbb{R}^d; \mathbb{R}^d) \cap C^1(D; \mathbb{R}^d)$ we denote by $\nabla \cdot G$ its divergence i.e. $\nabla \cdot G = \sum_{i=1}^d \frac{\partial G_i}{\partial x_i}$.

For $G \in C^2(D; \mathbb{R})$ and $x \in D$ we denote $\mathbf{H}[G](x)$ the Hessian matrix of G at point x .

Let $a(x) = (a_{ij}(x))_{i,j \in \{1, \dots, d\}}$ be a symmetric matrix valued and time homogeneous diffusion coefficient.

If $a_{ij} \in C^1(D; \mathbb{R})$ for all $1 \leq i, j \leq d$ and $u \in C(\mathbb{R}^d; \mathbb{R}) \cap C^2(D; \mathbb{R})$ we denote

$$(1.4) \quad \mathcal{L}u(x) = \nabla \cdot (a(x) \nabla_x u(x)), \quad \forall x \in D.$$

In the whole paper the coefficients of the function matrix a are always assumed to be measurable and bounded. We assume

ASSUMPTION 1.2. **(B)** : *There exists $\Lambda_* \in (0, \infty)$ such that*

$$(1.5) \quad \forall x \in \mathbb{R}^d, \forall i, j \in \{1, \dots, d\}, \quad |a_{ij}(x)| \leq \Lambda_*.$$

Let us denote $\Lambda := d \Lambda_*$ and $\Lambda^* = \Lambda^2 = d^2 \Lambda_*^2$.

Then, we have for any $x \in \mathbb{R}^d$ and $\xi \in \mathbb{R}^d$

$$(1.6) \quad \xi^* a(x) \xi = \sum_{1 \leq i, j \leq d} a_{ij}(x) \xi_i \xi_j \leq \Lambda_* \sum_{1 \leq i, j \leq d} \xi_i \xi_j \leq \frac{\Lambda_*}{2} \sum_{1 \leq i, j \leq d} (|\xi_i|^2 + |\xi_j|^2) = \Lambda_* d |\xi|^2 = \Lambda |\xi|^2,$$

and

$$(1.7) \quad |a(x) \xi|^2 = \sum_{1 \leq i \leq d} \left| \sum_{1 \leq j \leq d} a_{ij}(x) \xi_j \right|^2 \leq \Lambda_*^2 d^2 \sum_{1 \leq j \leq d} |\xi_j|^2 = \Lambda^* |\xi|^2.$$

We will also often make the following ellipticity assumption

ASSUMPTION 1.3. **(E)** : *There exists $\lambda \in (0, \infty)$ such that*

$$(1.8) \quad \forall x \in \mathbb{R}^d, \quad \forall \xi \in \mathbb{R}^d, \quad \lambda |\xi|^2 \leq \xi^* a(x) \xi.$$

REMARK 1.4. *Under the ellipticity assumption **(E)** each diagonal term a_{ii} of the matrix-valued coefficient $a : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ has a uniformly bounded inverse (see Lemma 4.17 in [33]).*

In the sequel we will frequently note a_{\pm} the restrictions of a to D_{\pm} . More precisely, by an assumption of type "the coefficients a_{ij} satisfy $(a_{\pm})_{ij} \in C_b^p(D_{\pm})$ " we will mean that the restriction of each a_{ij} to D_+ (resp. D_-) coincides on D_+ (resp. D_-) with a function \tilde{a}_{ij}^+ of class $C_b^p(\mathbb{R}^d)$ (resp. \tilde{a}_{ij}^-). So that for any $x \in \Gamma$ we can give a sense for example to $a_+(x)$: it is $\lim_{z \rightarrow x, z \in D_+} a(z) = \tilde{a}^+(x)$ (denoting $\tilde{a}^+(x) = (\tilde{a}_{ij}^+(x))_{1 \leq i, j \leq d}$).

Note that under **(E)** we can assert that for any $x \in D$ we have

$$(1.9) \quad a_{\pm}(x) = P_{\pm}^*(x) D_{\pm}(x) P_{\pm}(x)$$

with $P_{\pm}(x)$ some orthogonal matrices and $D_{\pm}(x)$ some diagonal matrices with strictly positive eigenvalues.

For a point $x \in \Gamma$ we denote by $\nu(x) \in \mathbb{R}^d$ the unit normal to Γ at point x , pointing to D_+ . Assume the a_{ij} 's satisfy $(a_{\pm})_{ij} \in C(D_{\pm})$. We define then the co-normal vector fields $\gamma_+(x) := a_+(x)\nu(x)$ and $\gamma_-(x) := -a_-(x)\nu(x)$, for $x \in \Gamma$, and introduce the following assumption.

ASSUMPTION 1.5. **(Γ)** : *The co-normal vector fields γ_+ and γ_- are of class C^4 .*

Note that under **(E)** it is clear that we have

$$(1.10) \quad \forall x \in \Gamma, \quad \langle \gamma_+(x), \nu(x) \rangle \geq \lambda > 0 \quad \text{and} \quad \langle \gamma_-(x), \nu(x) \rangle \leq -\lambda < 0.$$

Note that the notation γ for the trace operator follows the usual one ([33] for instance) and the notation γ_{\pm} for the co-normal vectors follows the one of the paper [4]. But it will be dealt with the trace operator only in Section 2 and in the Appendix, and with co-normal vectors only in Sections 4 and 5. So that these notations will cause no confusion.

To finish with we define the unbounded operator $A : \mathcal{D}(A) \subset L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ by

$$\mathcal{D}(A) = \left\{ u \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij} D_j u) \in L^2(\mathbb{R}^d) \right\}$$

and

$$\forall u \in \mathcal{D}(A), \quad Au = \sum_{i,j=1}^d D_i(a_{ij} D_j u).$$

Then we introduce the iterated domains defined recursively by

$$\mathcal{D}(A^k) = \{v \in \mathcal{D}(A^{k-1}) : Av \in \mathcal{D}(A^{k-1})\}, \quad k \geq 2.$$

2. The parabolic transmission problem. Let $0 < T < \infty$ a finite time horizon. Let us consider the transmission parabolic problem

$$(\mathcal{P}_T) \left\{ \begin{array}{ll} \partial_t u(t, x) - \mathcal{L}u(t, x) = 0 & \forall (t, x) \in (0, T] \times D \\ \langle a_+ \nabla_x u_+(t, y) - a_- \nabla_x u_-(t, y), \nu(y) \rangle = 0 & \forall (t, y) \in (0, T] \times \Gamma \quad (\star) \\ u(t, y_+) = u(t, y_-) & \forall (t, y) \in [0, T] \times \Gamma \\ u(0, x) = u_0(x) & \forall x \in \mathbb{R}^d. \end{array} \right.$$

We will say that $(t, x) \mapsto u(t, x)$ is *classical solution* to (\mathcal{P}_T) if it satisfies

$$(2.1) \quad u \in C([0, T]; C^2(\bar{D}_+) \cap C^2(\bar{D}_-)) \cap C^1([0, T]; C(\bar{D}_+) \cap C(\bar{D}_-)) \cap C([0, T]; C(\mathbb{R}^d))$$

and satisfies the following requisites. First, u satisfies the first line of (\mathcal{P}_T) , where the derivatives are understood in the classical sense. Second, for all $0 < t \leq T$ the limits $\lim_{z \rightarrow y, z \in D_\pm} \nabla_x u(t, z)$ satisfy the transmission condition (\star) for all $y \in \Gamma$. Note that these limits exist thanks to (2.1). Third, u is continuous across Γ (third line). Fourth, it satisfies the initial condition at the fourth line of (\mathcal{P}_T) .

The aim of this section is to prove the following result.

THEOREM 2.1. *Let $a = (a_{ij})_{1 \leq i, j \leq d}$ satisfy **(B)**, **(E)**.*

- Denote

$$(2.2) \quad k_0 = \begin{cases} \lfloor \frac{d}{4} \rfloor + 2 & \text{if } d \text{ is even;} \\ \lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor + 2 & \text{if } d \text{ is odd.} \end{cases}$$

Assume that the coefficients a_{ij} satisfy $(a_\pm)_{ij} \in C_b^{2k_0-3}(D_\pm)$ and Γ is bounded and of class C^{2k_0-2} . Then for $u_0 \in \mathcal{D}(A^{k_0})$ the parabolic transmission problem (\mathcal{P}_T) admits a classical solution.

- *Furthermore, if $u_0 \in \mathcal{D}(A^k)$ for $k \geq k_0$, the coefficients a_{ij} satisfy $(a_\pm)_{ij} \in C_b^{2k-1}(D_\pm)$ and Γ is bounded of class C^{2k} , this classical solution u is such that*

$$u \in C^{k-j} \left([0, T]; C^{n(j)}(\bar{D}_+) \cap C^{n(j)}(\bar{D}_-) \right), \quad \lfloor d/4 \rfloor \leq j \leq k$$

with $n(j) = \lfloor 2j - \frac{d}{2} \rfloor$.

The idea is to use the Hille-Yosida theorem. This requires to study in a first time the associated elliptic resolvent equation, in a weak sense. More precisely, for a source term $f \in L^2(\mathbb{R}^d)$ we will seek for a solution u in $\mathcal{D}(A)$ of

$$(2.3) \quad u - Au = f$$

(see Proposition 2.12 below). Then, by applying the Hille-Yosida theorem in $L^2(\mathbb{R}^d)$, we will get the existence of a solution to (\mathcal{P}_T) in a semi-weak sense (for which the time derivatives are understood in a classical sense and the space derivatives in a weak sense). Finally, using some Sobolev embedding arguments, we will get Theorem 2.1.

REMARK 2.2. *The result of Theorem 2.1 has to be compared with the classical results on parabolic PDE for smooth coefficients (and with no transmission condition) In this more classical situation, a unique classical solution to the parabolic PDE exists as soon as the a_{ij} 's are bounded and Hölder continuous and satisfy **(E)**, and u_0 is continuous and satisfies some growth condition (see for example [11] Chap. 1 or [25], Theorem 5.14). Here we ask additional smoothness on the coefficients $(a_{\pm})_{ij}$'s inside the domains D_{\pm} . This is because our technique of proof is very different: unlike the parametrix method in the classical case, this additional smoothness is required for the use of the Hille-Yosida theorem and the Sobolev embeddings. With this approach, we manage to study the global regularity of the classical solution of \mathcal{P}_T in the whole domains $[0, T] \times \overline{D_{\pm}}$ (for all times and all the way up to the transmission boundary Γ).*

2.1. *Study of the associated elliptic problem.* In this subsection we establish the existence of a solution to (2.3) belonging to $\mathcal{D}(A)$ and study its smoothness properties, together with the ones of functions belonging to the iterated domains $\mathcal{D}(A^k)$, for $k \geq 1$.

To that aim we will partly rely on the results in [33] Chap. 4, pp. 141-145. For the sake of clarity and completeness we have rewritten the proofs of some of these results (see also the Appendix).

We recall that the coefficients a_{ij} are assumed to be bounded by Λ_* so that we may define the following continuous bilinear and symmetric form, which will be used extensively in the sequel

$$(2.4) \quad \mathcal{E}(u, v) = \sum_{i,j=1}^d \langle a_{ij} D_j u, D_i v \rangle_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in H^1(\mathbb{R}^d).$$

Let $u \in \mathcal{D}(A)$. Using the definition of Au as a distribution acting on $C_c^\infty(\mathbb{R}^d; \mathbb{R})$, and the density of $C_c^\infty(\mathbb{R}^d; \mathbb{R})$ in $H^1(\mathbb{R}^d) = H_0^1(\mathbb{R}^d)$, one can establish the following relation, linking A and the form (2.4):

$$(2.5) \quad \mathcal{E}(u, v) = \langle -Au, v \rangle_{L^2(\mathbb{R}^d)}, \quad \forall v \in H^1(\mathbb{R}^d).$$

2.1.1. *Some preliminary results.* In the sequel, for $u \in L^2(\mathbb{R}^d)$, we frequently denote u_+ (resp. u_-) the restriction of u to D_+ (resp. D_-). It may happen that we use this notation for restricted distributions also.

We introduce the following notation for the jump across Γ of $u \in L^2(\mathbb{R}^d)$, with $u_+ \in H^1(D_+)$ and $u_- \in H^1(D_-)$:

$$[u]_{\Gamma} = \gamma(u_+) - \gamma(u_-).$$

If $[u]_{\Gamma} = 0$ we shall simply write $\gamma(u) = \gamma(u_+) = \gamma(u_-)$.

We start with two lemmas. The proofs are postponed to the Appendix.

LEMMA 2.3. *Let $v \in L^2(\mathbb{R}^d)$. Then, for any $1 \leq i \leq d$, the distribution $(D_i v)_{\pm}$ is equal to $D_i(v_{\pm})$. As a consequence, if $v \in H^1(\mathbb{R}^d)$, then $v_{\pm} \in H^1(D_{\pm})$.*

LEMMA 2.4 ([33], Exercise 4.5). *Suppose $u \in L^2(\mathbb{R}^d)$ with $u_{\pm} \in H^1(D_{\pm})$. Then $u \in H^1(\mathbb{R}^d)$ if and only if $[u]_{\Gamma} = 0$ a.e. on Γ .*

REMARK 2.5. *The result of Lemma 2.4 has to be compared with the fact that if $d = 1$, we know that any function in $H^1(\mathbb{R}^d)$ has a continuous version ([6] Theorem VIII.2).*

2.1.2. *Green identities.* We shall consider restricted operators and bilinear forms in the following sense. We define $A_+ : H^1(D_+) \rightarrow H^{-1}(D_+)$ by

$$\forall v \in H^1(D_+), \quad A_+ v = \sum_{i,j=1}^d D_i((a_+)_{ij} D_j v).$$

We define $A_- : H^1(D_-) \rightarrow H^{-1}(D_-)$ in the same manner (note that we do not specify here any domain $\mathcal{D}(A_\pm)$). Further, we define

$$\mathcal{E}_\pm(u, v) = \sum_{i,j=1}^d \int_{D_\pm} (a_\pm)_{ij} D_j u D_i v, \quad \forall u, v \in H^1(D_\pm).$$

In the same fashion as for Equation (2.5), we have, for $u_\pm \in H^1(D_\pm)$ with $A_\pm u_\pm \in L^2(D_\pm)$,

$$(2.6) \quad \mathcal{E}_\pm(u_\pm, v) = \int_{D_\pm} (-A_\pm u_\pm) v, \quad \forall v \in H_0^1(D_\pm).$$

Imagine now that in (2.6) we wish to take the test function in $H^1(D_\pm)$ instead of $H_0^1(D_\pm)$. There will still be a link between A_\pm and \mathcal{E}_\pm , but through Green type identities, involving conormal derivatives and boundary integrals.

We introduce a specific notation for the one-sided conormal derivatives on Γ of $u \in L^2(\mathbb{R}^d)$ with $u_\pm \in H^2(D_\pm)$. Provided the $(a_\pm)_{ij}$ are in $C_b^1(D_\pm; \mathbb{R})$ and Γ is bounded and Lipschitz we set

$$(2.7) \quad \mathcal{B}_\nu^\pm u = \nu^* \gamma(a_\pm \nabla u_\pm) = \sum_{i=1}^d \sum_{j=1}^d \nu_i \gamma((a_\pm)_{ij} D_j u_\pm) \quad \text{on } \Gamma.$$

For $g \in H^{-\frac{1}{2}}(\Gamma)$ and $f \in H^{\frac{1}{2}}(\Gamma)$ we denote by $(g, f)_\Gamma$ the action of g on f . If both f, g are in $H^{\frac{1}{2}}(\Gamma)$ the quantity $(g, f)_\Gamma$ coincides with the surface integral $\int_\Gamma g f d\zeta$.

We have for example the next result.

PROPOSITION 2.6 (First Green identity, first version; [33], Lemma 4.1). *Let $u \in L^2(\mathbb{R}^d)$ with $u_+ \in H^2(D_+)$ and $u_- \in H^2(D_-)$. Assume that the coefficients $(a_\pm)_{ij}$ are in $C_b^1(D_\pm; \mathbb{R})$. Then*

$$\mathcal{E}_+(u_+, v) = \int_{D_+} (-A_+ u_+) v - \left(\mathcal{B}_\nu^+ u, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(D_+)$$

and

$$\mathcal{E}_-(u_-, v) = \int_{D_-} (-A_- u_-) v + \left(\mathcal{B}_\nu^- u, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(D_-).$$

REMARK 2.7. *Note that the change of sign in front of the integral on Γ is due to the fact that $-\nu$ is the outward normal to D_+ , and ν is the outward normal to D_- .*

In fact, for our coming purpose (proof of Theorem 2.14), we need a version of the first Green identity that is valid for $u \in L^2(\mathbb{R}^d)$ with $u_+ \in H^1(D_+)$ and $u_- \in H^1(D_-)$ (and possibly non smooth coefficients $(a_\pm)_{ij}$). We thus need to extend the definition of $\mathcal{B}_\nu^\pm u$ to such functions, for which the trace in (2.7) is no more defined in $H^{1/2}(\Gamma)$ (the boundary Γ is assumed to be bounded and Lipschitz).

But thanks to Lemma 4.3 in [33], for any $f_{\pm} \in \tilde{H}^{-1}(D_{\pm})$ s.t.

$$(2.8) \quad -A_{\pm}u_{\pm} = f_{\pm} \quad \text{on } D_{\pm},$$

there exists $g_{\pm} \in H^{-\frac{1}{2}}(\Gamma)$, uniquely defined by u_{\pm} and f_{\pm} , and satisfying

$$\mathcal{E}_+(u_+, v) = \langle f_+, v \rangle_{\tilde{H}^{-1}(D_+), H^1(D_+)} - \left(g_+, \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_+),$$

$$\mathcal{E}_-(u_-, v) = \langle f_-, v \rangle_{\tilde{H}^{-1}(D_-), H^1(D_-)} + \left(g_-, \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_-).$$

and

$$\|g_{\pm}\|_{H^{-1/2}(\Gamma)} \leq C \left(\|u_{\pm}\|_{H^1(D_{\pm})} + \|f_{\pm}\|_{\tilde{H}^{-1}(D_{\pm})} \right)$$

REMARK 2.8. *Note that the meaning of equality (2.8) is that for any $\varphi \in C_c^{\infty}(D_{\pm}; \mathbb{R})$,*

$$\langle -A_{\pm}u_{\pm}, \varphi \rangle_{H^{-1}(D_{\pm}), H_0^1(D_{\pm})} = \langle f_{\pm}, \varphi \rangle_{\tilde{H}^{-1}(D_{\pm}), H^1(D_{\pm})}.$$

In the sequel the equality "on D_{\pm} " (for elements of $\tilde{H}^{-1}(D_{\pm})$) will have a similar meaning (as for example in Lemma 2.10 below). We will also use the same idea on some subsets of D_{\pm} (see for e.g. the proof of Proposition 2.15).

In particular if $A_{\pm}u_{\pm} \in L^2(D_{\pm})$ one makes the natural choice

$$f_+ = \begin{cases} -A_+u_+ & \text{on } D_+ \\ 0 & \text{on } \bar{D}_- \end{cases} \quad \text{and} \quad f_- = \begin{cases} 0 & \text{on } \bar{D}_+ \\ -A_-u_- & \text{on } D_- \end{cases}$$

and defines $\mathcal{B}_{\nu}^{\pm}u := g_{\pm}$. The notation $\mathcal{B}_{\nu}^{\pm}u$ comes from the fact that this new definition is consistent with the original one involved in Proposition 2.6 (see [33] p.117 for details). To sum up, we have, using the new sense of $\mathcal{B}_{\nu}^{\pm}u$, the following result.

PROPOSITION 2.9 (First Green identity, extended version; [33] Theorem 4.4, point i)). *Let $u \in L^2(\mathbb{R}^d)$ with $u_+ \in H^1(D_+)$ and $u_- \in H^1(D_-)$. Assume $A_+u_+ \in L^2(D_+)$, $A_-u_- \in L^2(D_-)$. Then*

$$\mathcal{E}_+(u_+, v) = \int_{D_+} (-A_+u_+)v - \left(\mathcal{B}_{\nu}^+u, \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_+)$$

and

$$\mathcal{E}_-(u_-, v) = \int_{D_-} (-A_-u_-)v + \left(\mathcal{B}_{\nu}^-u, \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_-).$$

Finally we introduce a notation for the jumps across Γ of the conormal derivative of a function u with $u_{\pm} \in H^1(D_{\pm})$:

$$[\mathcal{B}_{\nu}u]_{\Gamma} = \mathcal{B}_{\nu}^+u - \mathcal{B}_{\nu}^-u \in H^{-1/2}(\Gamma).$$

Using Lemma 2.3 and Proposition 2.9 it is possible to prove the following two-sided Green identity, that we state directly for a function $u \in H^1(\mathbb{R}^d)$ (not only with $u_{\pm} \in H^1(D_{\pm})$). This two-sided Green identity we will be thoroughly used in the sequel.

LEMMA 2.10 (Two-sided Green identity, [33] Lemma 4.19, Equation (4.33)). *Let $u \in H^1(\mathbb{R}^d)$. Let $f_+ \in L^2(D_+)$ and $f_- \in L^2(D_-)$ and assume*

$$(2.9) \quad u_{\pm} - A_{\pm}u_{\pm} = f_{\pm} \quad \text{on } D_{\pm}.$$

Set $f = f_+ + f_-$, then

$$(2.10) \quad \langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(u, v) = \langle f, v \rangle_{L^2(\mathbb{R}^d)} - \left([\mathcal{B}_{\nu}u]_{\Gamma}, \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(\mathbb{R}^d).$$

PROOF. Pick v in $H^1(\mathbb{R}^d)$. Notice that $u_{\pm} \in H^1(D_{\pm})$ and $v_{\pm} \in H^1(D_{\pm})$, thanks to Lemma 2.3. Thanks to (2.9) we get $A_{\pm}u_{\pm} \in L^2(D_{\pm})$, so that one can use Proposition 2.9 and get

$$(2.11) \quad \int_{D_+} u_+v_+ + \mathcal{E}_+(u_+, v_+) = \int_{D_+} (u_+ - A_+u_+)v_+ - \left(\mathcal{B}_{\nu}^+u, \gamma(v_+) \right)_{\Gamma}$$

and

$$(2.12) \quad \int_{D_-} u_-v_- + \mathcal{E}_-(u_-, v_-) = \int_{D_-} (u_- - A_-u_-)v_- + \left(\mathcal{B}_{\nu}^-u, \gamma(v_-) \right)_{\Gamma}.$$

But, as $v \in H^1(\mathbb{R}^d)$, we have $[v]_{\Gamma} = 0$ and thus $\gamma(v_+) = \gamma(v_-) = \gamma(v)$. Thus, using again (2.9), and the definition of f and $[\mathcal{B}_{\nu}u]_{\Gamma}$, we get, summing (2.11) and (2.12),

$$\langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}_+(u_+, v_+) + \mathcal{E}_-(u_-, v_-) = \langle f, v \rangle_{L^2(\mathbb{R}^d)} - \left([\mathcal{B}_{\nu}u]_{\Gamma}, \gamma(v) \right)_{\Gamma}.$$

To complete the proof it suffices to notice that, thanks to Lemma 2.3, we have

$$\mathcal{E}_+(u_+, v_+) + \mathcal{E}_-(u_-, v_-) = \sum_{i,j=1}^d \left\{ \int_{D_+} (a_+)_{ij} (D_j u)_+ (D_i v)_+ + \int_{D_-} (a_-)_{ij} (D_j u)_- (D_i v)_- \right\} = \mathcal{E}(u, v).$$

□

REMARK 2.11. *Note that in [33] Green identities are stated in the case of Γ bounded and Lipschitz. But we claim that these results are true for $D_{\pm} = \mathbb{R}_{\pm}^d$, as in the proofs one usually starts with $D_{\pm} = \mathbb{R}_{\pm}^d$ (or some hypograph type domain) and then turns to Γ bounded with the help of local change of coordinates.*

2.1.3. *Existence of a weak solution to the resolvent equation and immediate properties.* We have the next result.

PROPOSITION 2.12. *Assume **(B)**, **(E)**. Let $f \in L^2(\mathbb{R}^d)$. Then (2.3) has a unique solution in $\mathcal{D}(A)$.*

PROOF. Let us note that the symmetric bilinear form on $H^1(\mathbb{R}^d)$

$$(u, v) \mapsto \langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(u, v)$$

is continuous and, thanks to Assumption **(E)**, coercive. Thus the Lax-Milgram theorem ([6] Corollary V.8) immediately asserts the existence of a unique $u \in H^1(\mathbb{R}^d)$ such that

$$\forall v \in H^1(\mathbb{R}^d), \quad \langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(u, v) = \langle f, v \rangle_{L^2(\mathbb{R}^d)}.$$

In other words we have for any $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$,

$$\mathcal{E}(u, \varphi) = -\left\langle \sum_{i,j=1}^d D_i(a_{ij}D_j u), \varphi \right\rangle_{L^2(\mathbb{R}^d)} = \langle (f - u), \varphi \rangle_{L^2(\mathbb{R}^d)}.$$

Hence the distribution $\sum_{i,j=1}^d D_i(a_{ij}D_j u)$ belongs to $L^2(\mathbb{R}^d)$, and thus $u \in \mathcal{D}(A)$. Finally, from the above relations we deduce

$$\forall v \in H^1(\mathbb{R}^d), \quad \langle u - Au, v \rangle_{L^2(\mathbb{R}^d)} = \langle f, v \rangle_{L^2(\mathbb{R}^d)},$$

which implies (2.3). □

The proposition below gives properties of functions belonging to $\mathcal{D}(A)$. It indicates that the solution $u \in \mathcal{D}(A)$ of (2.3) encountered in Proposition 2.12 satisfies a continuity property and a transmission condition in a weak sense at the interface.

PROPOSITION 2.13. *Let $u \in \mathcal{D}(A)$. Then $[u]_\Gamma = [\mathcal{B}_\nu u]_\Gamma = 0$ a.e. on Γ .*

PROOF. Let $u \in \mathcal{D}(A)$. As $u \in H^1(\mathbb{R}^d)$ one gets by Lemma 2.4 that $[u]_\Gamma = 0$ a.e. on Γ . Set now $f = u - Au \in L^2(\mathbb{R}^d)$. According to Equation (2.5) we have

$$(2.13) \quad \forall v \in H^1(\mathbb{R}^d), \quad \langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(u, v) = \langle f, v \rangle_{L^2(\mathbb{R}^d)}.$$

Using repeatedly Lemma 2.3 one can see that $u_\pm - A_\pm u_\pm = f_\pm$ on D_\pm . Note that by construction $f = f_+ + f_-$. Using now Lemma 2.10, and comparing (2.10) and (2.13), one gets $([\mathcal{B}_\nu u]_\Gamma, \gamma(\varphi))_\Gamma = 0$ for any $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$, which completes the proof. □

2.1.4. Regularity of the weak solution of the elliptic problem and consequence on the iterated domains $\mathcal{D}(A^k)$. Here we will establish and use the following main result.

THEOREM 2.14. *Assume **(B)**, **(E)**.*

Let $r \in \mathbb{N}$. Assume that the coefficients $(a_\pm)_{ij}$ belong to $C_b^{r+1}(D_\pm; \mathbb{R})$. Assume Γ is bounded and of class C^{r+2} .

Let $f_\pm \in H^r(D_\pm)$. Let $u \in H^1(\mathbb{R}^d)$ satisfying

$$u_\pm - A_\pm u_\pm = f_\pm \quad \text{on } D_\pm$$

and $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}+r}(\Gamma)$. Then $u_\pm \in H^{2+r}(D_\pm)$.

Theorem 2.14 will follow from the two following propositions. The first one provides a local analysis of the regularity across the interface Γ , and its result is to be found in Theorem 4.20 in [33]. The second one asserts a classical result on the global regularity of the solution away from the interface.

PROPOSITION 2.15 ([33], Theorem 4.20). *Let G_1 and G_2 be bounded open connected subset of \mathbb{R}^d , such that $\overline{G_1} \subset G_2$ and G_1 intersects Γ , and put*

$$D_\pm^j = G_j \cap D_\pm \quad \text{and} \quad \Gamma_j = \Gamma \cap G_j \quad \text{for } j = 1, 2.$$

Assume that G_2 is constructed in such a way that there is a C^{r+2} -diffeomorphism between Γ_2 and a bounded portion of the hyperplan $x_d = 0$.

Assume **(B)**, **(E)**.

Let $r \in \mathbb{N}$. Assume that the coefficients $(a_{\pm})_{ij}$ belong to $C^{r+1}(\overline{D_{\pm}^2}; \mathbb{R})$.

Let $f_{\pm} \in L^2(D_{\pm})$ with $f_{\pm} \in H^r(D_{\pm}^2)$. Let $u \in L^2(\mathbb{R}^d)$ with $u \in H^1(G_2)$ satisfying

$$u_{\pm} - A_{\pm}u_{\pm} = f_{\pm} \quad \text{on } D_{\pm}^2$$

and $[\mathcal{B}_{\nu}u]_{\Gamma} \in H^{\frac{1}{2}+r}(\Gamma_2)$. Then $u_{\pm} \in H^{2+r}(D_{\pm}^1)$.

PROOF. As this result is not so classical we have found interesting to provide a detailed proof in the Appendix. \square

PROPOSITION 2.16 ([47], Theorem 8.10). Assume **(B)**, **(E)**.

Let $r \in \mathbb{N}$. Assume that the coefficients $(a_{\pm})_{ij}$ belong to $C_b^{r+1}(D_{\pm}; \mathbb{R})$. Assume Γ is bounded.

Let $f_{\pm} \in H^r(D_{\pm})$. Let $u \in H^1(\mathbb{R}^d)$ satisfying

$$u_{\pm} - A_{\pm}u_{\pm} = f_{\pm} \quad \text{on } D_{\pm}.$$

Let $D'_{\pm} \subset D_{\pm}$ open subsets with $\overline{D'_{\pm}} \subset D_{\pm}$ and denote $d'_{\pm} = \text{dist}(D'_{\pm}, \Gamma)$.

We have that $u_{\pm} \in H^{r+2}(D'_{\pm})$, with

$$\|u_{\pm}\|_{H^{r+2}(D'_{\pm})} \leq C_{\pm} (\|u_{\pm}\|_{H^1(D_{\pm})} + \|f\|_{H^r(D_{\pm})}),$$

where the constant C_{\pm} depends on d, λ, d'_{\pm} and $\max_{1 \leq i, j \leq d} \max_{|\alpha| \leq r+1} \sup_{x \in D_{\pm}} |\partial^{\alpha}(a_{\pm})_{ij}(x)|$.

PROOF. In [47] this result is asserted with the assumption that $\overline{D'_{\pm}} \subset D_{\pm}$, with $\overline{D'_{\pm}}$ compact. So that for the interior (bounded) domain D_+ the result is immediate. On the unbounded domain D_- we claim that the same result holds for non compact $\overline{D'_-}$, as in fact only the distance $d'_- = \text{dist}(D'_-, \Gamma)$ plays a role in the proof. We provide the proof in Appendix for the sake of completeness, inspired by the proof of Theorem 4.16 in [33], that we have found more coherent with our notations and setting. \square

PROOF OF THEOREM 2.14. As Γ is closed and bounded we will cover it by a finite number of balls and use the local regularity result of Proposition 2.15 inside each ball. We will combine this with the interior regularity result of Proposition 2.16, to finally get a result on the global regularity of the function u on D_+ and D_- .

As Γ is bounded and of class C^{r+2} one may cover it by a finite number of open balls $G_{2,k} \subset \mathbb{R}^d$, $1 \leq k \leq N$ (i.e. $\Gamma \subset \cup_{k=1}^N G_{2,k}$), that are chosen in such a way that there are a C^{r+2} diffeomorphisms between each $\Gamma_{2,k} = \Gamma \cap G_{2,k}$ and a bounded open portion of $\mathbb{R}^{d-1} \times \{0\}$ ([33] pp 89-90). Then one chooses $D'_{\pm} \subset D_{\pm}$ open subsets with $\overline{D'_{\pm}} \subset D_{\pm}$ and $D'_+ \cup D'_- \cup_{k=1}^N G_{2,k} = \mathbb{R}^d$. Note that $\overline{D'_-}$ is a closed but unbounded part of \mathbb{R}^d . It is convenient to choose D'_{\pm} so that we have the overlapping $D'_{\pm} \cap G_{2,k} \cap G_{2,k+1} \neq \emptyset$ for any $1 \leq k \leq N-1$ and $D'_+ \cap G_{2,N} \cap G_{2,1} \neq \emptyset$. Then one may choose arbitrary open connected sets $G_{1,k}$ with $\overline{G_{1,k}} \subset G_{2,k}$, and respecting $D'_+ \cup D'_- \cup_{k=1}^N G_{1,k} = \mathbb{R}^d$, $D'_{\pm} \cap G_{1,k} \cap G_{1,k+1} \neq \emptyset$ for any $1 \leq k \leq N-1$ and $D'_+ \cap G_{1,N} \cap G_{1,1} \neq \emptyset$. As in Proposition 2.15 we note $D_{\pm}^{j,k} = G_{j,k} \cap D_{\pm}$, $1 \leq k \leq N$, for $j = 1, 2$.

It is clear that the $(a_{\pm})_{ij}$ belong to $C^{r+1}(\overline{D_{\pm}^{2,k}}; \mathbb{R})$ for any $k \in \{1, \dots, N\}$. Besides $f_{\pm} \in H^r(D_{\pm}^{2,k})$ (by adapting the result of Lemma 2.3), and $u_{\pm} - A_{\pm}u_{\pm} = f_{\pm}$ on $D_{\pm}^{2,k}$. To finish with, we have

$[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}+r}(\Gamma_{2,k})$. Then, for each k one may then use Proposition 2.15 to conclude that $u_\pm \in H^{r+2}(D_\pm^{1,k})$. On another hand by Proposition 2.16 one gets that $u_\pm \in H^{r+2}(D'_\pm)$. So that for example $u_+ \in H^{r+2}(D'_+) \cap \left(\bigcap_{k=1}^N H^{r+2}(D_+^{1,k})\right)$. We claim that this implies that $u_+ \in H^{r+2}(D_+)$ - note that of course $D'_+ \cup \left(\bigcup_{k=1}^N D_+^{1,k}\right) = D_+$.

Let us start with $D'_+ \cup D_+^{1,1} = D'_+ \cup D_+^{1',1}$, where we have denoted $D_+^{1',1} = D_+^{1,1} \setminus D'_+$. Adapting again Lemma 2.3 we have that $u_+|_{D'_+} \in H^{r+2}(D'_+)$ and $u_+|_{D_+^{1',1}} \in H^{r+2}(D_+^{1',1})$. One may prove by induction that $u_+ \in H^{2+r}(D'_+ \cup D_+^{1',1})$ in the following way. First, notice that $u_+ \in H^1(D'_+ \cup D_+^{1',1})$, as $u \in H^1(\mathbb{R}^d)$ anyway (Lemma 2.3 again). Assume now $u_+ \in H^m(D'_+ \cup D_+^{1',1})$ for $m < 2+r$. Then for any multi-index $|\alpha| \leq m$, we have $D^\alpha u_+ \in L^2(D'_+ \cup D_+^{1',1})$. From the fact that $u_+|_{D'_+} \in H^{r+2}(D'_+)$ and $u_+|_{D_+^{1',1}} \in H^{r+2}(D_+^{1',1})$ we have that $(D^\alpha u_+)|_{D'_+} \in H^1(D'_+)$ and $(D^\alpha u_+)|_{D_+^{1',1}} \in H^1(D_+^{1',1})$ (we adapt again Lemma 2.3 for the restriction aspect). Adapting now the result of Lemma 2.4 we have that $D^\alpha u_+ \in H^1(D'_+ \cup D_+^{1',1})$, which is the desired result, if $[u_+]_{\Gamma''_{1,+}} = 0$ a.e. on $\Gamma''_{1,+} = \partial D'_+ \cap \overline{D_+^{1',1}}$. But $\Gamma''_{1,+} \subset D_+^{1,1}$ and $u_+ \in H^{r+2}(D_+^{1,1})$. Adapting this time the necessary part of Lemma 2.4 we get that indeed $[u_+]_{\Gamma''_{1,+}} = 0$ a.e. on $\Gamma''_{1,+}$. Thus indeed $D^\alpha u_+ \in H^1(D'_+ \cup D_+^{1',1})$ and more precisely $D_i D^\alpha u_+ = D_i(D^\alpha u_+)|_{D'_+} + D_i(D^\alpha u_+)|_{D_+^{1',1}} \in L^2(D'_+ \cup D_+^{1',1})$ for any $1 \leq i \leq d$. The proof by induction that $u_+ \in H^{2+r}(D'_+ \cup D_+^{1',1}) = H^{2+r}(D'_+ \cup D_+^{1,1})$ is completed.

Repeating this procedure one proves by induction that $u_+ \in H^{r+2}(D'_+ \cup \bigcup_{k=1}^N D_+^{1,k}) = H^{r+2}(D_+)$ (using in particular the fact that the $D_+^{1,k}$'s are in finite number). One proceeds in the same way on D_- . \square

Now that we have proved Theorem 2.14 we can get as a corollary the following result concerning the iterated domains $\mathcal{D}(A^k)$, $k \in \mathbb{N}^*$.

COROLLARY 2.17. *Let $k \in \mathbb{N}^*$ and $u \in \mathcal{D}(A^k)$. Assume that the coefficients $(a_\pm)_{ij} \in C_b^{2k-1}(D_\pm)$ and that Γ is bounded and of class C^{2k} . Then $u_\pm \in H^{2k}(D_\pm)$.*

PROOF. The proof proceeds by induction on k .

Let $u \in \mathcal{D}(A)$ (case $k = 1$). We have $[\mathcal{B}_\nu u]_\Gamma = 0$, according to Proposition 2.13. Thus in particular $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}}(\Gamma)$. As in the proof of Proposition 2.13 we set $f = u - Au$ and notice that we have $u_\pm - A_\pm u_\pm = f_\pm$ on D_\pm , with $f_\pm \in L^2(D_\pm)$.

Using Theorem 2.14 - remember that u is in $H^1(\mathbb{R}^d)$, $(a_\pm)_{ij} \in C_b^1(D_\pm; \mathbb{R})$ and Γ is bounded of class C^2 - we get that $u_\pm \in H^2(D_\pm)$.

Suppose now that the result is true at rank $k-1$ we prove its validity at rank k ($k \geq 2$). Let $u \in \mathcal{D}(A^k)$. As $u \in \mathcal{D}(A)$ we have $[\mathcal{B}_\nu u]_\Gamma = 0 \in H^{2k-\frac{3}{2}}(\Gamma)$. As $Au \in \mathcal{D}(A^{k-1})$ the quantity $u - Au =: f$ satisfies $f_\pm \in H^{2k-2}(D_\pm)$, using the induction hypothesis. But as we have $u_\pm - A_\pm u_\pm = f_\pm$ on D_\pm , one may use again the smoothness of $(a_\pm)_{ij}$ and Γ and Theorem 2.14 in order to conclude that $u_\pm \in H^{2k}(D_\pm)$. \square

2.2. The solution of the parabolic problem (\mathcal{P}_T) .

2.2.1. Application of the Hille-Yosida theorem. We now use the Hille-Yosida theorem ([6] Theorems VII.4 and VII.5) in order to prove the following proposition. Note that in Equation (2.14)

below, the time derivative is understood in the strong sense, while the space derivatives are understood in the weak sense. Besides, by convention $\mathcal{D}(A^0) = L^2(\mathbb{R}^d)$.

PROPOSITION 2.18. *Let $u_0 \in \mathcal{D}(A)$. Then there exists a unique function*

$$u \in C^1([0, T]; L^2(\mathbb{R}^d)) \cap C([0, T]; \mathcal{D}(A))$$

satisfying

$$(2.14) \quad \frac{du}{dt} = Au, \quad u(0) = u_0.$$

Furthermore, let $u_0 \in \mathcal{D}(A^k)$, $k \geq 2$. Then,

$$u \in C^{k-j}([0, T]; \mathcal{D}(A^j)), \quad 0 \leq j \leq k.$$

PROOF. According to [6] it suffices to show that the operator $(-A, \mathcal{D}(A))$ is maximal monotone. But thanks to Assumption **(E)** we immediately see that $\langle -Av, v \rangle_{L^2(\mathbb{R}^d)} = \mathcal{E}(v, v) \geq 0$, for any $v \in \mathcal{D}(A)$, and thanks to Proposition 2.12 we have that for any $f \in L^2(\mathbb{R}^d)$ there exists $u \in \mathcal{D}(A)$ solving (2.3). \square

Using now Propositions 2.13, Corollary 2.17 and Proposition 2.18 together with some Sobolev embedding theorems, we show Theorem 2.1.

2.2.2. Proof of Theorem 2.1.

PROOF. Assume d is even. Apply the result of Proposition 2.18 with $k = k_0 = \lfloor \frac{d}{4} \rfloor + 2$ and consider u solution of (2.14). We have that

$$u \in C^1\left([0, T]; \mathcal{D}\left(A^{k_0-1}\right)\right)$$

with $k_0 - 1 = \lfloor \frac{d}{4} \rfloor + 1$. Using the result of Corollary 2.17 and combining Corollary IX.13 p. 168 with Theorem IX.7 p. 157 in [6], we see that for any $t \in [0, T]$

$$(2.15) \quad u_{\pm}(t, \cdot) \in H^{4+2\lfloor \frac{d}{4} \rfloor}(D_{\pm}) \subset H^{2+\frac{d}{2}}(D_{\pm}) \hookrightarrow C^2(\bar{D}_{\pm}).$$

Assume now that d is odd. Apply the result of Proposition 2.18 with $k = k_0 = \lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor + 2$ and consider u solution of (2.14). We have that

$$u \in C^1\left([0, T]; \mathcal{D}\left(A^{k_0-1}\right)\right)$$

with $k_0 - 1 = \lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor + 1$. Using the result of Corollary 2.17 and combining Corollary IX.13 p. 168 with Theorem IX.7 p. 157 in [6], we see that for any $t \in [0, T]$

$$(2.16) \quad u_{\pm}(t, \cdot) \in H^{2+2\lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor}(D_{\pm}) \hookrightarrow C^2(\bar{D}_{\pm})$$

since

$$\left[2 + 2\left\lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \right\rfloor - \frac{d}{2} \right] \geq \left[2 + 2\left(\frac{1}{2} + \frac{\lfloor d/2 \rfloor}{2}\right) - \frac{d}{2} \right] \geq \left[3 + \lfloor \frac{d}{2} \rfloor - \frac{d}{2} \right] \geq 2.$$

Let us now show that u solution of (2.14) (for the corresponding k_0) is a classical solution of (\mathcal{P}_T) .

First, it is clear that $\mathcal{L}u$ coincides with Au on any bounded part of D_\pm (the derivatives in the distributional sense coincide with the classical derivatives thanks to the established smoothness of u). This shows the first line of (\mathcal{P}_T) .

Second, as for any $t \in [0, T]$ the function $u(t, \cdot)$ belongs to $\mathcal{D}(A)$, we have using the result of Proposition 2.13 that

$$(2.17) \quad [u(t, \cdot)]_\Gamma = 0 \quad \text{a.e. on } \Gamma; \quad [\mathcal{B}_\nu u(t, \cdot)]_\Gamma = 0 \quad \text{a.e. on } \Gamma.$$

Note that $u(t, \cdot) \in \mathcal{D}(A)$ implies that $u_\pm(t, \cdot)$ are in $H^2(D_\pm)$. So that the second part of (2.17) reads

$$\nu^* (\gamma (a_+ \nabla u_+(t, \cdot)) - \gamma (a_- \nabla u_-(t, \cdot))) = 0 \quad \text{a.e.}$$

But as $(a_\pm \nabla u_\pm) \in C^1(\bar{D}_\pm; \mathbb{R}^d)$, we get

$$\langle (a_+ \nabla_x u_+(t, \cdot))(y) - (a_- \nabla_x u_-(t, \cdot))(y), \nu(y) \rangle = 0$$

for almost every $y \in \Gamma$, and consequently for every $y \in \Gamma$ by continuity. The same argument applies to the first part of (2.17) and the second and third lines of (\mathcal{P}_T) are satisfied. Note that the constructed solution satisfies $u(t, \cdot) \in C(\mathbb{R}^d)$ for any time $t \in [0, T]$.

Now let $k \geq k_0$. For $\lceil \frac{d}{4} \rceil \leq j \leq k$, we have $2j - \frac{d}{2} > 0$. Thus, for $v \in \mathcal{D}(A^j)$ we have from Corollary 2.17,

$$v_\pm \in H^{2j}(D_\pm) \hookrightarrow C^{n(j)}(\bar{D}_\pm)$$

with $n(j) = \lfloor 2j - \frac{d}{2} \rfloor$. Using again the result of Proposition 2.18, we get the announced result. \square

2.3. Conclusion and consequences: boundedness of the partial derivatives. Going a bit further in the analysis, and using additional Sobolev embedding arguments, we can state the following result.

PROPOSITION 2.19. *Let $p, q \in \mathbb{N}$ with $p + \lfloor q/2 \rfloor \geq 2$. Let $m = \lceil \frac{q}{2} + \frac{d}{4} \rceil$, $m' = m + 1$ and $k = m' + p$. Assume that the coefficients a_{ij} satisfy $(a_\pm)_{ij} \in C_b^{2m'-1}(D_\pm)$, that Γ is bounded and of class $C^{2m'}$, and that $u_0 \in \mathcal{D}(A^k)$.*

Then the classical solution $u(t, x)$ of (\mathcal{P}_T) constructed in Theorem 2.1 satisfies

$$u \in C^p([0, T]; C_b^q(\bar{D}_+) \cap C_b^q(\bar{D}_-)).$$

PROOF. First, notice that it is easy to check that k is greater than the k_0 defined in Theorem 2.1, so that it makes sense speaking of the classical solution of (\mathcal{P}_T) , for $u_0 \in \mathcal{D}(A^k)$.

This solution is constructed in the same way as in Theorem 2.1, in particular by the mean of Proposition 2.18. So that one can assert that

$$u \in C^p([0, T]; \mathcal{D}(A^{m'})).$$

It remains to check that if $v \in \mathcal{D}(A^{m'})$, then $v_\pm \in C_b^q(\bar{D}_\pm)$. First, note that $m \geq \lceil \frac{d}{4} \rceil$, and that one may easily check

$$\lfloor 2m - \frac{d}{2} \rfloor \geq q$$

(using in particular $\lfloor 2a \rfloor \leq 2\lceil a \rceil$). So that if $v \in \mathcal{D}(A^{m'}) \subset \mathcal{D}(A^m)$, we have, as for the second part of Theorem 2.1,

$$v_\pm \in H^{2m}(D_\pm) \hookrightarrow C^{\lfloor 2m - \frac{d}{2} \rfloor}(\bar{D}_\pm) \subset C^q(\bar{D}_\pm).$$

We claim that for any multi-index α , $|\alpha| \leq q$, the partial derivatives $\partial^\alpha v_\pm$ are bounded. Indeed, using again Corollary 2.17, we get

$$v_\pm \in H^{2m'}(D_\pm),$$

so that for α , $|\alpha| \leq q$,

$$\partial^\alpha v_\pm \in H^{2\lceil \frac{q}{2} + \frac{d}{4} \rceil - q + 2}(D_\pm) \subset H^{\frac{d}{2} + 2}(D_\pm) \hookrightarrow L^\infty(D_\pm).$$

Here we have used the fact $\frac{1}{2} - \frac{1}{2} - \frac{2}{d} < 0$, so that one can use the third embedding result of Corollary IX.13 in [6] (and again Theorem IX.7 for the projection argument). The result is proved. \square

From the above proposition we get the following control on the partial derivatives of the solution to (\mathcal{P}_T) .

COROLLARY 2.20. *In the context of Proposition 2.19 we have*

$$\sup_{t \in [0, T]} \sup_{x \in \bar{D}_\pm} |\partial_t^j \partial^\alpha u_\pm(t, x)| < \infty$$

for any $j \leq p$ and any multi-index α , with $|\alpha| \leq q$.

PROOF. By Proposition 2.19 any of the considered partial derivatives of u_\pm belongs to the space

$$C([0, T]; C_b(\bar{D}_\pm)).$$

Let for example $v \in C([0, T]; C_b(\bar{D}_+))$. We prove the continuity of the map $t \mapsto \sup_{x \in \bar{D}_+} |v(t, x)|$, $t \in [0, T]$. Let $t_0 \in [0, T]$. Using the reverse triangle inequality we get for any $t \neq t_0$,

$$\left| \sup_{x \in \bar{D}_+} |v(t, x)| - \sup_{x \in \bar{D}_+} |v(t_0, x)| \right| \leq \sup_{x \in \bar{D}_+} |v(t, x) - v(t_0, x)|,$$

and we get the continuity at t_0 , as v is continuous from $[0, T]$ to $C_b(\bar{D}_+)$ (equipped with the supreme norm). Thus the desired continuity is proved, and from this we can assert that

$$\sup_{t \in [0, T]} \sup_{x \in \bar{D}_+} |v(t, x)| = \sup_{x \in \bar{D}_+} |v(t^*, x)|$$

for some $t^* \in [0, T]$. As $v(t^*, \cdot) \in C_b(\bar{D}_+)$ we have that

$$\sup_{t \in [0, T]} \sup_{x \in \bar{D}_+} |v(t, x)| < \infty.$$

The result is proved. \square

In the analysis of the convergence of our Euler scheme, we will use the above corollary with p up to 2 and q up to 4.

3. Stochastic processes associated to multidimensional parabolic transmission problems in divergence form. The aim of this section is to define and study the stochastic process X naturally in link with the already encountered operator $(A, \mathcal{D}(A))$, via the theory of Dirichlet forms as exposed in [12].

The starting point of the construction is some closed symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$, which is in link with $(A, \mathcal{D}(A))$ and its corresponding semigroup (T_t) on $L^2(\mathbb{R}^d)$. In Subsection 3.1 we define these objects and study the regularity in t of $\mathcal{E}(T_t f, g)$, $f \in L^2(\mathbb{R}^d), g \in \mathcal{D}[\mathcal{E}]$, using the spectral resolution of the identity associated to $(A, \mathcal{D}(A))$ (Subsubsection 3.1.1). This study allows to establish the relation with the results of D.W. Stroock in [45] (Subsubsection 3.1.2) which are exposed in the $C_b(\mathbb{R}^d)$ setting (Feller semigroup).

Then in Subsection 3.2 the Hunt process X associated to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is defined and studied. In particular we aim at writing down its Skorokhod representation as precisely as possible (Subsubsection 3.2.2).

3.1. *Dirichlet form and Markovian semigroup associated to general elliptic divergence form operators.*

3.1.1. *Construction.* Let $a : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ a symmetric coefficient matrix satisfying the ellipticity condition **(E)** and the uniform boundedness condition **(B)**.

To the coefficient matrix a , we may associate a closed symmetric Dirichlet form $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ defined on $L^2(\mathbb{R}^d)$ by

$$\begin{cases} \mathcal{D}[\mathcal{E}] &= H^1(\mathbb{R}^d), \\ \mathcal{E}(u, v) &= \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j u D_i v, \quad u, v \in \mathcal{D}[\mathcal{E}] \end{cases}$$

(see [12], p111). This will be the starting point of our construction. Note that of course \mathcal{E} is nothing else than the symmetric bilinear form already defined in (2.4).

On the underlying Hilbert space $L^2(\mathbb{R}^d)$, we denote within this subsection by $(A, \mathcal{D}(A))$ the (unique) self-adjoint operator associated to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ and characterized by

$$\begin{cases} \mathcal{D}(A) \subset \mathcal{D}[\mathcal{E}], \\ \mathcal{E}(u, v) = -\langle Au, v \rangle_{L^2(\mathbb{R}^d)}, \quad u \in \mathcal{D}(A), v \in \mathcal{D}[\mathcal{E}] \end{cases}$$

([12], Theorem 1.3.1 and Corollary 1.3.1 p.21). We aim at identifying this operator - as expected it will turn out that $(A, \mathcal{D}(A))$ is nothing else than the operator defined at the end of Section 1, therefore the common notation.

By the very definition of $(A, \mathcal{D}(A))$, we have for any $f \in \mathcal{D}(A)$ and any $g \in C_c^\infty(\mathbb{R}^d)$

$$-\langle Af, g \rangle_{L^2(\mathbb{R}^d)} = \mathcal{E}(f, g) = \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j f D_i g = -\left\langle \sum_{i,j=1}^d D_i(a_{ij} D_j f), g \right\rangle_{H^{-1}(\mathbb{R}^d), H^1(\mathbb{R}^d)}$$

where $\sum_{i,j=1}^d D_i(a_{ij} D_j f)$ is understood in the distributional sense as an element of $H^{-1}(\mathbb{R}^d)$. But as $Af \in L^2(\mathbb{R}^d)$ by the definition of $\mathcal{D}(A)$ the above equality shows that $\sum_{i,j=1}^d D_i(a_{ij} D_j f) \in L^2(\mathbb{R}^d)$ (for any $f \in \mathcal{D}(A)$).

Thus, it is proved that $\mathcal{D}(A) \subseteq \{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij} D_j f) \in L^2(\mathbb{R}^d)\}$.

In turn (by the density of $C_c^\infty(\mathbb{R}^d)$ in $L^2(\mathbb{R}^d)$) the equality permits to identify for any $f \in \mathcal{D}(A)$,

$$Af = \sum_{i,j=1}^d D_i(a_{ij}D_jf).$$

Let us now prove the reverse inclusion $\{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij}D_jf) \in L^2(\mathbb{R}^d)\} \subseteq \mathcal{D}(A)$.

Let $f \in \{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij}D_jf) \in L^2(\mathbb{R}^d)\}$. By the symmetry of the coefficient matrix a and integration by parts, it is not hard to prove that for any $v \in \mathcal{D}(A)$,

$$\langle Av, f \rangle_{L^2(\mathbb{R}^d)} = -\mathcal{E}(v, f) = - \sum_{j,i=1}^d \int_{\mathbb{R}^d} a_{ji} D_i f D_j v = \left\langle \sum_{j,i} D_j(a_{ji} D_i f), v \right\rangle_{L^2(\mathbb{R}^d)}$$

and in particular $f \in \mathcal{D}(A^*) \stackrel{\text{def}}{=} \{g \in L^2(\mathbb{R}^d) \mid \exists h_g \in L^2(\mathbb{R}^d) \text{ s.t. } (Av, g) = (v, h_g), \forall v \in \mathcal{D}(A)\}$ (see [40]). So that we get the reverse inclusion

$$\{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij}D_jf) \in L^2(\mathbb{R}^d)\} \subseteq \mathcal{D}(A^*) = \mathcal{D}(A)$$

where the equality comes from the fact that $(A, \mathcal{D}(A))$ is self-adjoint. Finally, we have proved

$$(3.1) \quad \mathcal{D}(A) = \{f \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij}D_jf) \in L^2(\mathbb{R}^d)\}$$

and $(A, \mathcal{D}(A))$ is fully identified as being the same operator as at the end of Section 1.

Note that since a is only assumed to be measurable, $C_c^\infty(\mathbb{R}^d)$ - which is a core for the Dirichlet form $(\mathcal{E}, \mathcal{D}(\mathcal{E}))$ - is not even a subset of $\mathcal{D}(A)$.

We now turn to the study of the spectral resolution and the semigroup associated to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ and $(A, \mathcal{D}(A))$. For the sake of conciseness we denote $(\cdot, \cdot) = \langle \cdot, \cdot \rangle_{L^2(\mathbb{R}^d)}$ and $\|\cdot\| = \|\cdot\|_{L^2(\mathbb{R}^d)}$ till the end of the section.

Since $(-A, \mathcal{D}(A))$ is a self-adjoint operator on the Hilbert space $L^2(\mathbb{R}^d)$ that is non-negative definite, it admits a spectral resolution of the identity $\{E_\gamma : \gamma \in [0, \infty)\}$. For any $\gamma \geq 0$ the operator $E_\gamma : L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ is a self-adjoint projection operator with $(E_\gamma f, f) \geq 0$, $f \in L^2(\mathbb{R}^d)$, and the E_γ 's form a spectral family with in particular $E_\mu E_\gamma = E_{\mu \wedge \gamma}$, (see [12] p18 for a list of properties). The link with $(-A, \mathcal{D}(A))$ is through

$$(-Af, g) = \int_{[0, \infty)} \gamma d(E_\gamma f, g) \quad \forall f \in \mathcal{D}(A), g \in L^2(\mathbb{R}^d)$$

and $\mathcal{D}(A) = \left\{ f \in L^2(\mathbb{R}^d) : \int_{[0, \infty)} \gamma^2 d(E_\gamma f, f) < \infty \right\}$ (see [12] paragraph 1.3.4 p.18).

Consequently, the family of operators $\{T_t \stackrel{\text{def}}{=} e^{tA} : t > 0\}$ is a strongly continuous semigroup of self-adjoint contractions acting on $L^2(\mathbb{R}^d)$ ([12] Lemma 1.3.2 p.19) and

$$(T_t f, g) = \int_{[0, \infty)} e^{-\gamma t} d(E_\gamma f, g) \quad \forall f \in L^2(\mathbb{R}^d), g \in L^2(\mathbb{R}^d).$$

Note that for any $\gamma \geq 0$, $t > 0$, and any functions $f \in L^2(\mathbb{R}^d)$ and $g \in L^2(\mathbb{R}^d)$, we have the commutation property

$$\begin{aligned}
(T_t E_\gamma f, g) &= (E_\gamma f, T_t g) \\
&= \int_{[0, \infty)} e^{-\xi t} d_\xi(E_\gamma f, E_\xi g) = \int_{[0, \gamma]} e^{-\xi t} d_\xi(E_\xi E_\gamma f, g) + \int_{[\gamma, \infty)} e^{-\xi t} d_\xi(E_\xi E_\gamma f, g) \\
&= \int_{[0, \gamma]} e^{-\xi t} d_\xi(E_\xi f, g) \\
&= \int_{[0, \infty)} e^{-\xi t} d_\xi(E_\xi f, E_\gamma g) \\
&= (T_t f, E_\gamma g) = (E_\gamma T_t f, g).
\end{aligned}$$

Note also that for any $f \in L^2(\mathbb{R}^d)$ and any $t > 0$,

$$\begin{aligned}
\int_{[0, \infty)} \gamma^2 d(E_\gamma T_t f, T_t f) &= \int_{[0, \infty)} \gamma^2 d_\gamma \left(\int_{[0, \infty)} e^{-\xi t} d_\xi(E_\gamma E_\xi f, T_t f) \right) \\
&= \int_{[0, \infty)} \gamma^2 d_\gamma \left(\int_{[0, \infty)} e^{-\xi t} d_\xi \left(\int_{[0, \infty)} e^{-\theta t} d_\theta(E_\gamma E_\xi f, E_\theta f) \right) \right) \\
&= \int_{[0, \infty)} \gamma^2 e^{-2\gamma t} d_\gamma(E_\gamma f, f) \\
&\leq \frac{4}{t^2} e^{-2} \int_{[0, \infty)} e^{-\gamma t} d_\gamma(E_\gamma f, f) = \frac{4}{t^2} e^{-2} (T_t f, f) \leq \frac{4}{t^2} e^{-2} \|f\|^2 < +\infty,
\end{aligned}$$

where we have used the spectral family property, the associativity of the Stieltjes integral and the inequality $\gamma^2 e^{-\gamma t} \leq 4e^{-2}/t^2$. The above inequality ensures that $T_t f \in \mathcal{D}(A)$ for any $t > 0$.

From the fact that $|\frac{d}{dt} e^{-\gamma t}| \leq \gamma$ is integrable w.r.t. $d(E_\gamma h, g)$ whenever $h \in \mathcal{D}(A)$, we deduce from the commutation property that for any $f, g \in L^2(\mathbb{R}^d)$ and for any $s > 0$

$$-\frac{d}{dt} (T_t f, T_s g) = \int_{[0, \infty)} \gamma e^{-\gamma t} d(E_\gamma f, T_s g) \xrightarrow[t \searrow 0^+]{\quad} \int_{[0, \infty)} \gamma d(E_\gamma T_s f, g) = (-AT_s f, g)$$

where the limit exists and is well defined (since we have shown that $T_s f \in \mathcal{D}(A)$).

If moreover $g \in \mathcal{D}[\mathcal{E}]$ then

$$\begin{aligned}
-\frac{d}{ds} (T_s f, g) &= -\frac{d}{dt} (T_{s+t} f, g)|_{t=0^+} = -\frac{d}{dt} (T_t f, T_s g)|_{t=0^+} \\
(3.2) \quad &= (-AT_s f, g) = \mathcal{E}(T_s f, g).
\end{aligned}$$

And since $-\frac{d}{ds} (T_s f, g) = -\frac{d}{ds} (T_s g, f)$ by the symmetry property of T_s , we deduce $\mathcal{E}(T_s f, g) = \mathcal{E}(f, T_s g)$ for any $f, g \in \mathcal{D}[\mathcal{E}]$.

Consequently, for any $f \in \mathcal{D}[\mathcal{E}]$ and using the ellipticity of the coefficient matrix a ,

$$\begin{aligned}
\lambda \|\nabla T_s f\|^2 &\leq \mathcal{E}(T_s f, T_s f) = \mathcal{E}(T_{2s} f, f) = (-AT_{2s} f, f) \\
&= \int_{[0, \infty)} \gamma e^{-2\gamma s} d(E_\gamma f, f) \leq \frac{e^{-1}}{s} \int_{[0, \infty)} e^{-\gamma s} d(E_\gamma f, f) = \frac{e^{-1}}{s} (T_s f, f) \leq \frac{\|f\|^2}{s},
\end{aligned}$$

from which we deduce the fundamental estimate

$$(3.3) \quad \|\nabla T_s f\| \leq \frac{\|f\|}{\sqrt{\lambda s}}, \quad \forall s > 0.$$

In turn this estimate implies that for any $f \in L^2(\mathbb{R}^d)$, $g \in \mathcal{D}[\mathcal{E}]$, the function

$$s \mapsto \mathcal{E}(T_s f, g) \text{ is integrable on } (0, t],$$

and from (3.2) and the right continuity of $s \mapsto T_s f$ at time $s = 0+$ (one may extend $T_0 f = f$ as long as no differentiation of $s \mapsto T_s f$ is implied at $s = 0+$ when $f \notin \mathcal{D}(A)$), we deduce the integrated version of (3.2) namely

$$(3.4) \quad \forall f \in L^2(\mathbb{R}^d), \forall g \in \mathcal{D}[\mathcal{E}],$$

$$(T_t f, g) - (f, g) = - \int_0^t \sum_{i,j=1}^d (a_{ij} D_j T_s f, D_i g) ds = - \int_0^t \mathcal{E}(T_s f, g) ds, \quad t \in (0, \infty).$$

3.1.2. *Link with the results of D.W. Stroock [45].* In his celebrated article *Diffusion semigroups corresponding to uniformly elliptic divergence form operators* D.W. Stroock constructs via a regularization procedure a Feller continuous semigroup $\{P_t : t > 0\}$ associated to a with the properties that (with our notations)

1. the map $t \in [0, \infty) \mapsto P_t \phi \in H^1(\mathbb{R}^d)$ is a weakly continuous map for each $\phi \in C_c^\infty(\mathbb{R}^d)$.
2. $\forall \phi, \psi \in C_c^\infty(\mathbb{R}^d)$,

$$(3.5) \quad (P_t \phi, \psi) - (\phi, \psi) = - \int_0^t (a \nabla P_s \phi, \nabla \psi) ds = - \int_0^t \mathcal{E}(P_s \phi, \psi) ds, \quad t \in (0, \infty).$$

(Nota : please note that there is a sign error in the original version of [45]).

In fact, $\{P_t : t > 0\}$ determines a unique strongly continuous semigroup $\{\bar{P}_t : t > 0\}$ of self-adjoint contractions on $L^2(\mathbb{R}^d)$. The semigroup $\{\bar{P}_t : t > 0\}$ is strongly continuous on $H^1(\mathbb{R}^d)$. Moreover, for each $t > 0$, \bar{P}_t maps $L^2(\mathbb{R}^d)$ into $H^1(\mathbb{R}^d)$ and for each $f \in H^1(\mathbb{R}^d) = \mathcal{D}[\mathcal{E}]$, we have the fundamental estimate

$$(3.6) \quad \|\nabla \bar{P}_s f\| \leq \frac{1}{\sqrt{\lambda}} \left(\frac{\|f\|}{\sqrt{s}} \right) \wedge \|\nabla f\|, \quad \forall s > 0.$$

(See [45] Theorem II.3.1. p.341).

This estimate implies that for for each $f, g \in \mathcal{D}[\mathcal{E}]$ and any $t, s > 0$,

$$|\mathcal{E}(\bar{P}_t f, g) - \mathcal{E}(\bar{P}_s f, g)| \leq \Lambda \|\nabla g\| \frac{\|\bar{P}_{t \vee s - t \wedge s} f - f\|}{\sqrt{\lambda(t \wedge s)}} \xrightarrow{s \rightarrow t} 0,$$

which ensures the continuity of $s \mapsto \mathcal{E}(\bar{P}_s \phi, \psi)$ for any $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$. Since $(\bar{P}_t)_{t>0}$ and $(P_t)_{t>0}$ coincide on $C_c^\infty(\mathbb{R}^d)$, we may differentiate in (3.5) (as long as $t > 0$) to find that

$$(3.7) \quad \frac{d}{dt} (\bar{P}_t \phi, \psi) = -\mathcal{E}(\bar{P}_t \phi, \psi), \quad t \in (0, \infty).$$

This has to be compared to (3.2).

Let us now justify rigorously that for any $t > 0$, $s \in (0, t)$ and $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$,

$$(3.8) \quad \frac{d}{ds}(T_s\phi, \bar{P}_{t-s}\psi) = \frac{d}{du}(T_u\phi, \bar{P}_{t-s}\psi)|_{u=s} - \frac{d}{du}(T_s\phi, \bar{P}_{t-u}\psi)|_{u=s}.$$

We have for sufficiently small $0 \leq h < t - s$ and using the strong continuity of $(\bar{P}_t)_{t>0}$

$$\begin{aligned} |(T_{s+h}\phi - T_s\phi, \bar{P}_{t-s+h}\psi - \bar{P}_{t-s}\psi)| &\leq \|\bar{P}_{t-s+h}\psi - \bar{P}_{t-s}\psi\| \|T_{s+h}\phi - T_s\phi\| \\ &\leq \varepsilon_\psi(h) \left(\int_{[0,\infty)} e^{-2\gamma s} (e^{-\gamma h} - 1)^2 d(E_\gamma\phi, \phi) \right)^{1/2} \\ &\leq \varepsilon_\psi(h) \left(\int_{[0,\infty)} e^{-2\gamma s} (\gamma h)^2 d(E_\gamma\phi, \phi) \right)^{1/2} \\ &\leq h\varepsilon_\psi(h) \left(\int_{[0,\infty)} e^{-\gamma s} (\gamma^2 e^{-\gamma s}) d(E_\gamma\phi, \phi) \right)^{1/2} \\ &\leq h\varepsilon_\psi(h) \frac{2e^{-1}}{s} \|\phi\|, \end{aligned}$$

where as usual $\varepsilon_\psi(\cdot)$ denotes some positive continuous function vanishing at zero. We deduce that

$$\frac{1}{h} |(T_{s+h}\phi - T_s\phi, \bar{P}_{t-s+h}\psi - \bar{P}_{t-s}\psi)| \xrightarrow{h \rightarrow 0} 0,$$

implying (3.8).

Hence, from (3.8) and applying (3.2) and (3.7), we have that

$$(3.9) \quad \frac{d}{ds}(T_s\phi, \bar{P}_{t-s}\psi) = 0, \quad s \in (0, t).$$

Integrating the identity (3.9) on $(0, t)$ and using the time continuity of both semigroups (T_t) and (\bar{P}_t) up to time $s = 0+$ gives

$$(3.10) \quad (T_t\phi, \psi) = (\phi, \bar{P}_t\psi) = (\bar{P}_t\phi, \psi)$$

which holds for any $\phi, \psi \in C_c^\infty(\mathbb{R}^d)$. Since $C_c^\infty(\mathbb{R}^d)$ is dense in $L^2(\mathbb{R}^d)$, using the strong continuity of both semigroups (T_t) and (\bar{P}_t) , we finally deduce from (3.10) the identification

$$(3.11) \quad \{\bar{P}_t : t > 0\} = \{T_t : t > 0\} \quad \text{on } L^2(\mathbb{R}^d).$$

Consequently, all results in [45] that are valid for $\{\bar{P}_t : t > 0\}$ are true for $\{T_t : t > 0\}$. For example, identifying abusively $\{T_t : t > 0\}$ with its Feller restriction $\{P_t : t > 0\}$ on $C_c^\infty(\mathbb{R}^d)$, we deduce that there is a $p \in C((0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d)$ such that

$$(3.12) \quad [T_t\phi](x) = \int_{\mathbb{R}^d} \phi(y)p(t, x, y)dy, \quad \ell(dx) - \text{a.e.}, \quad \phi \in C_c^\infty(\mathbb{R}^d).$$

Moreover, the fundamental function p satisfies the well-known Aronson's estimates for the fundamental solutions of elliptic divergence form operators, namely there exists a constant $M(\lambda, \Lambda, d) \in [1, \infty)$ such that

$$(3.13) \quad \frac{1}{Mt^{d/2}} \exp(-M|x-y|^2/t) \leq p(t, x, y) \leq \frac{M}{t^{d/2}} \exp(-|x-y|^2/Mt).$$

Finally, we have the convergence result of [45] (Theorem II.3.1. p.341) that we state roughly without introducing the necessary notations (see [45] for details) : if $\{a_n\}_1^\infty \subset \mathcal{A}(\lambda, \Lambda)$ and $a_n \rightarrow a$ almost everywhere, then $p^n(t, x, y) \rightarrow p(t, x, y)$ uniformly on compacts (in $(0, \infty) \times \mathbb{R}^d \times \mathbb{R}^d$) and for each $t \in [0, \infty)$ and $\phi \in C_c^\infty(\mathbb{R}^d)$, $T_t^n \phi \rightarrow T_t \phi$ in $H^1(\mathbb{R}^d)$.

3.2. Stochastic representation of transmission operators in divergence form.

3.2.1. *Representation of the Hunt process associated to elliptic divergence form operators using the Revuz correspondence for additive functionals.* Since $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is a regular Dirichlet form (with the space $\mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d)$ or $C_c^\infty(\mathbb{R}^d)$ as a special standard core, see e.g. Exercice 1.4.1 in [12]), we are in position to apply Theorem 7.2.1 p. 380 of [12].

We may associate to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ and its corresponding semigroup (T_t) a Hunt process, symmetric w.r.t the Lebesgue measure $\ell(dx)$ on \mathbb{R}^d . We shall denote by $\mathbb{M} = (\Omega, (\mathcal{F}_t)_{t \geq 0}, \mathcal{F}, (X_t)_{t \geq 0}, (\mathbb{P}^x)_{x \in \mathbb{R}^d})$ this Hunt process, with $X = (X^1, \dots, X_d)$. The correspondence with $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ and (T_t) is through

$$(3.14) \quad \mathbb{E}^x[f(X_t)] = T_t f(x), \quad \forall f \in L^2(\mathbb{R}^d), \forall t \geq 0, \forall x \in \mathbb{R}^d$$

(see the discussion p160 in [12], at the beginning of Section 4.2).

Let us also denote by $\{R_\alpha : \alpha > 0\}$ the Markovian resolvent kernel of the Markovian transition function $\{\hat{p}(t, x, dy) := p(t, x, y)dy : t > 0\}$. Then, for any $\alpha > 0$, $f \in \mathcal{B}_b(\mathbb{R}^d)$ and $x \in \mathbb{R}^d$, $R_\alpha f(x) = \int_{\mathbb{R}^d} r_\alpha(x, y) f(y) dy$ with $r_\alpha(x, y) = \int_0^\infty e^{-\alpha t} p(t, x, y) dt$.

Denote by S the set of positive Radon measures on $(\mathbb{R}^d, \mathcal{B}(\mathbb{R}^d))$. For $\mu \in S$ define $R_1 \mu(x) = \int_{\mathbb{R}^d} r_1(x, y) \mu(dy)$ ($x \in \mathbb{R}^d$) and introduce the subset of *finite energy measures*

$$\begin{aligned} S_0 &:= \left\{ \mu \in S : \exists C > 0, \forall v \in \mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d), \int_{\mathbb{R}^d} |v(x)| \mu(dx) \leq C (\mathcal{E}(v, v) + (v, v))^{1/2} \right\} \\ &= \left\{ \mu \in S : \sup_{v \in \mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d)} \int_{\mathbb{R}^d} \frac{|v(x)|}{\|v\|_{\mathcal{E}_1}} \mu(dx) < \infty \right\}, \quad \text{where we follow the notations of [12].} \end{aligned}$$

Finally, introduce

$$S_{00} := \{ \mu \in S_0 : \mu(\mathbb{R}^d) < \infty, \|R_1 \mu(\cdot)\|_\infty < \infty \}.$$

Let us denote respectively by \mathbf{A}_c^+ and $\mathbf{A}_{c,1}^+$ the families of all Positive Continuous Additive Functionals (PCAF in short) (resp. the family of all PCAF in the strict sense) associated to \mathbb{M} (for the distinction between \mathbf{A}_c^+ and $\mathbf{A}_{c,1}^+$, see [12] the introduction of Section 5.1).

The *Revuz correspondence* asserts that there is a one-to-one correspondence (up to equivalence of processes) between \mathbf{A}_c^+ and S . This correspondence permits to construct for any $\mu \in S_{00}$ a unique PCAF in the strict sense $A \in \mathbf{A}_{c,1}^+$ such that

$$(3.15) \quad \forall x \in \mathbb{R}^d, \quad \mathbb{E}^x \int_0^\infty e^{-t} dA_t = R_1 \mu(x).$$

(see for e.g. Theorem 5.1.4 in [12]).

In order to get a bijective map, introduce a new subset S_1 of S defined by $\mu \in S_1$ if there exists a sequence $(E_n)_{n \geq 0}$ of Borel finely open sets increasing to \mathbb{R}^d satisfying that $\mathbb{I}_{E_n} \cdot \mu \in S_{00}$ for each n . Then, there is a one-to-one correspondence between S_1 and $\mathbf{A}_{c,1}^+$ (up to equivalence) which is given by relation (3.15) whenever $\mu \in S_{00}$. The set of measures S_1 is called the set of *smooth measures (in the strict sense)*.

Let us introduce $\mathcal{D}[\mathcal{E}]_{b,\text{loc}}$ the space of essentially bounded functions belonging locally to $\mathcal{D}[\mathcal{E}]$ ($u \in \mathcal{D}[\mathcal{E}]_{b,\text{loc}}$ if for any compact set G , there exists a bounded function ω such that $u = \omega$, $\ell(dx)$ -a.e. on G).

For $u \in \mathcal{D}[\mathcal{E}]_b$, we may associate a unique positive Radon measure $\mu_{\langle u \rangle} \in S$, satisfying

$$(3.16) \quad \int_{\mathbb{R}^d} f(x) \mu_{\langle u \rangle}(dx) = 2\mathcal{E}(uf, u) - \mathcal{E}(u^2, f), \quad \forall f \in \mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d).$$

If $\mathcal{D}[\mathcal{E}]_{b,\text{loc}}$, we may construct $\mu_{\langle u \rangle} \in S$ with the help of a sequence $(G_n)_{n \geq 0}$ of relatively compact open sets such that $\overline{G_n} \subset G_{n+1}$ and $\bigcup_{n \geq 0} G_n = \mathbb{R}^d$. Let $(u_n)_{n \geq 0}$ a sequence of functions in $\mathcal{D}[\mathcal{E}]_b$ satisfying $u_n = u$ on G_n . There is no ambiguity in defining $\mu_{\langle u \rangle} = \mu_{\langle u_n \rangle}$ on G_n because the construction is consistent (since $\mu_{\langle u_n \rangle} = \mu_{\langle u_{n+1} \rangle}$ on G_n). For an account on the above assertions, please refer to [12] Section 3.2.

Note that obviously $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is strong local, so we may apply Theorem 5.5.5 in [12].

Suppose that a function u satisfies the following conditions :

1. $u \in \mathcal{D}[\mathcal{E}]_{b,\text{loc}}$, u is finely continuous on \mathbb{R}^d .
2. $\mathbb{I}_G \cdot \mu_{\langle u \rangle} \in S_{00}$ for any relatively compact open set G .
3. $\exists \varrho = \varrho^{(1)} - \varrho^{(2)}$ with $\mathbb{I}_G \cdot \varrho^{(1)}, \mathbb{I}_G \cdot \varrho^{(2)} \in S_{00}$ for any relatively compact open set G and

$$\mathcal{E}(u, v) = (\varrho, v), \quad \forall v \in C_c^\infty(\mathbb{R}^d).$$

(Note that even though u is not formally in $\mathcal{D}[\mathcal{E}]$, the quantity $\mathcal{E}(u, v)$ is well-defined because v has compact support and $u \in \mathcal{D}[\mathcal{E}]_{b,\text{loc}}$).

Let $A^{(1)}$, $A^{(2)}$, and B be PCAF's in the strict sense with Revuz measures $\varrho^{(1)}$, $\varrho^{(2)}$, and $\mu_{\langle u \rangle}$ respectively. Then, Theorem 5.5.5 in [12] asserts that

$$(3.17) \quad u(X_t) - u(X_0) = M_t^{[u]} + N_t^{[u]}, \quad \mathbb{P}^x - a.s., \quad \forall x \in \mathbb{R}^d.$$

Here,

$$(3.18) \quad N^{[u]} = -A^{(1)} + A^{(2)}, \quad \mathbb{P}^x - a.s., \quad \forall x \in \mathbb{R}^d$$

and $M^{[u]}$ is a local Additive Functional in the strict sense such that for any relatively compact set G ,

$$\mathbb{E}^x M_{t \wedge \tau_G}^{[u]} = 0, \quad \forall x \in G$$

and

$$\mathbb{E}^x \left[(M_{t \wedge \tau_G}^{[u]})^2 \right] = \mathbb{E}^x B_{t \wedge \tau_G}, \quad \forall x \in G,$$

where $\tau_G = \inf(s > 0 : X_s \notin G)$ stands for the first leaving time from G (with the convention $\inf \emptyset = \infty$) and B denotes the PCAF in the strict sense with Revuz measure $\mu_{\langle u \rangle}$.

3.2.2. Skorokhod representation of the Hunt process associated to a transmission operator in divergence form. Consider $\mathbb{R}^d = \bar{D}_+ \cup D_-$ with D_+ and D_- two open connected subdomains separated by a transmission boundary Γ that is to say

$$\Gamma = \bar{D}_+ \cap \bar{D}_-.$$

We denote

$$D = D_+ \cup D_- = \mathbb{R}^d \setminus \Gamma \subset \mathbb{R}^d.$$

We apply the results of Theorem 5.5.5 in [12] in this context for the coordinate functions

$$p_k(x_1, \dots, x_d) := x_k \quad (k \in \{1, \dots, d\})$$

THEOREM 3.1. *Assume that Assumptions **(B)** **(E)** are fulfilled and that $(a_{\pm})_{ij} \in C_b^1(D_{\pm}; \mathbb{R})$ for all $1 \leq i, j \leq d$ with a_{ij} possessing a possible discontinuity on Γ . Then, the Hunt process \mathbb{M} associated to $(\mathcal{E}, \mathcal{D}[\mathcal{E}])$ is a diffusion which possesses the following Skorokhod decomposition : for any $k \in \{1, \dots, d\}$,*

$$(3.19) \quad \begin{aligned} X_t^k &= x_k + \int_0^t \sum_{j=1}^d \sqrt{2} \sigma_{kj}(X_s) dW_s^j + \int_0^t \sum_{j=1}^d \partial_j a_{kj}(X_s) \mathbb{1}_{X_s \in D} ds \\ &- \frac{1}{2} \int_0^t \gamma_{+,k}(X_s) dK_s + \frac{1}{2} \int_0^t \gamma_{-,k}(X_s) dK_s, \quad t \geq 0, \mathbb{P}^x - a.s., \forall x = (x_1, \dots, x_d) \in \mathbb{R}^d. \end{aligned}$$

In the above equality $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ denotes the positive square-root of coefficient $2a$ i.e. the positive matrix real valued coefficient satisfying

$$\sigma \sigma^*(x) = 2a(x), \quad \forall x \in D.$$

(Note that this coefficient exists because $a(x)$ is non-negative definite for all $x \in D$). The process $W = (W^1, \dots, W^d)$ is a d -dimensional standard Brownian motion starting from zero and $(K_t)_{t \geq 0}$ denotes the unique PCAF associated to the surface measure $\varsigma(d\xi) \in S$ on Γ through the Revuz correspondence. The process (K_t) increases only at times where X lies on Γ ,

$$\int_0^t \mathbb{1}_{X_s \in \Gamma} dK_s = K_t, \quad t \geq 0.$$

PROOF. We follow the ideas of [48] Theorem 5.2. Of course $p_k \in \mathcal{D}[\mathcal{E}]_{b, \text{loc}}$ and p_k is finely continuous on \mathbb{R}^d . Let G a relatively compact open set containing Γ and a function $f_k \in \mathcal{D}[\mathcal{E}]_b$ such that $p_k = f_k$ on G . Let $\langle M^{[f_k]} \rangle$ the square bracket of $M^{[f_k]}$. Then, an easy computation from (3.16) shows that the energy measure of $M^{[f_k]}$ (the Revuz measure of $\langle M^{[f_k]} \rangle$) is

$$\mu_{\langle f_k \rangle}(dy) = \mu_{\langle M^{[f_k]} \rangle}(dy) = \langle a(y) \nabla f_k(y), \nabla f_k(y) \rangle \ell(dy)$$

and we know that $\mu_{\langle f_k \rangle} = \mu_{\langle p_k \rangle}$ on G . It is easy to show that $\mathbb{1}_G \cdot \mu_{\langle p_k \rangle}$ is a finite Radon measure belonging to S_{00} and that $\mu_{\langle p_k \rangle}$ is a smooth measure. Then, an easy computation from (3.15) shows that

$$\langle M^{[f_k]} \rangle_t = \int_0^t \langle a(X_s) \nabla f_k(X_s), \nabla f_k(X_s) \rangle ds, \quad k \in \{1, \dots, d\}$$

and by the well-known results on stochastic representation of martingales, there exists a d dimensional Brownian motion $W = (W^1, \dots, W^d)$ such that

$$M_t^{[f_k]} = \int_0^t [\sigma(X_s) \nabla f_k(X_s)]^* dW_s, \quad \mathbb{P}^x - a.s. \quad \forall x \in \mathbb{R}^d, \quad k \in \{1, \dots, d\}$$

(see for e.g. [42] Chapter V Theorem 3.9 and the remark following its proof).

Moreover, for any $v \in C_c^\infty(\mathbb{R}^d)$, using the First Green Identities of Proposition 2.6 and taking into account that v is of compact support, we have :

$$\begin{aligned}
\mathcal{E}(f_k, v) &= \mathcal{E}_+(f_{k,+}, v) + \mathcal{E}_-(f_{k,-}, v) \\
&= \int_{D_+} (-A_+ f_{k,+})v - \left(\mathcal{B}_\nu^+ f_{k,+}, \gamma(v) \right)_\Gamma + \int_{D_-} (-A_- f_{k,-})v + \left(\mathcal{B}_\nu^- f_{k,-}, \gamma(v) \right)_\Gamma \\
&= \int_D \sum_{i,j=1}^d D_i (a_{ij}(y) D_j f_k(y)) v(y) \mathbb{I}_{y \in D} \ell(dy) - \int_\Gamma \nu^* [\gamma(a_+ \nabla f_{k,+}) - \gamma(a_- \nabla f_{k,-})] \gamma(v) d\zeta \\
&= \int_D \sum_{j=1}^d \partial_j a_{kj}(y) v(y) \mathbb{I}_{y \in D} \ell(dy) - \int_\Gamma [\gamma((a_+ \nu)_k) - \gamma(a_- \nu)_k] v d\zeta \\
&= \int_D \sum_{j=1}^d \partial_j a_{kj}(y) v(y) \mathbb{I}_{y \in D} \ell(dy) - \int_\Gamma [(\tilde{a}_+ \nu)_k - (\tilde{a}_- \nu)_k] v d\zeta \\
&= (\varrho_k^+, v) - (\varrho_k^-, v)
\end{aligned}$$

with

$$\varrho_k^\pm(dy) := \sum_{j=1}^d [\partial_j a_{kj}(y)]^\pm \mathbb{I}_{y \in D} \ell(dy) + [(\gamma_-)_k - (\gamma_+)_k]^\pm(y) \mathbb{I}_{y \in \Gamma} \zeta(dy).$$

(here, the notation $[a]^+$ (resp. $[a]^-$) stands for the positive (resp. negative) part of some real number a).

Let us now proceed to show that the measures $\mathbb{I}_G \cdot \varrho_k^\pm$ belong to S_{00} .

Note that $\|\partial_j a_{kj} \mathbb{I}_{D \cap G}\|_\infty < \infty$ and from the definition of S_{00} and the Revuz correspondence (3.15), it is not difficult to prove that the measures $[\partial_j a_{kj}]_\pm(y) \mathbb{I}_{y \in D} \ell(dy)$ are smooth with their corresponding additive functional writing as $\left(\int_0^t [\partial_j a_{kj}]_\pm(X_s) \mathbb{I}_{X_s \in D} ds \right)_{t \geq 0}$.

We now turn to the surface measures $\zeta_k^\pm(dy) := [(\gamma_-)_k - (\gamma_+)_k]^\pm(y) \mathbb{I}_{y \in \Gamma} \zeta(dy)$. It is well-known (see e.g. [10] p.134 3. (**), (***)) that there exists a universal constant $C_0 > 0$, depending only on the Lipschitz domain D_+ , such that for all $h \in C^1(\overline{D}_+)$,

$$\int_\Gamma |h(y)| \zeta(dy) \leq C_0 \int_{D_+} (|\nabla h(x)| + |h(x)|) \ell(dx).$$

Thus, for all $h \in \mathcal{D}[\mathcal{E}] \cap C_c(\mathbb{R}^d)$, we have

$$\begin{aligned}
\int_\Gamma |h(y)| \zeta(dy) &\leq C_0 \int_{D_+} (|\nabla h(x)| + |h(x)|) \ell(dx) \\
&\leq C_0 \ell(D_+)^{1/2} \left(\int_{D_+} (|\nabla h(x)| + |h(x)|)^2 \ell(dx) \right)^{1/2} \\
&\leq C_0 (2\ell(D_+))^{1/2} \left(\int_{\mathbb{R}^d} (|\nabla h(x)|^2 + |h(x)|^2) \ell(dx) \right)^{1/2} \\
&\leq C_0 \sqrt{\frac{(2\ell(D_+))}{\lambda}} (\mathcal{E}(h, h) + (h, h))^{1/2}
\end{aligned}$$

so that the surface measure $\zeta(dy)$ belongs to S_0 . Since

$$\forall y \in \Gamma, \quad |[(\gamma_-)_k - (\gamma_+)_k]^\pm(y)| \leq 2|\tilde{a}_\pm(y)\nu(y)| \leq 2\Lambda^*,$$

the surface measures $\zeta_k^\pm(dy) := [(\gamma_-)_k - (\gamma_+)_k]^\pm(y) \mathbb{I}_{y \in \Gamma} \zeta(dy)$ belong also to S_0 .

Note that from Aronson's estimates (3.13) we retrieve the following estimations

$$r_1(x, y) \leq C|x - y|^{-(d-2)} \quad \text{if } d > 2; \quad r_1(x, y) \leq C(\ln(1/|x - y|) \vee 1) \quad \text{if } d = 2.$$

Then, using the same arguments as in [12] (Example 5.2.2 p.255), we can assert that the measures $\zeta_k^\pm(dy)$ belong to S_{00} . Moreover, let $(K_t)_{t \geq 0}$ denote the PCAF associated to $\zeta(dy)$; in regard of the results stated in the original article of D. Revuz (cf. [41] p.507) we may assert that $\left(\int_0^t [(\gamma_-)_k - (\gamma_+)_k]^\pm(X_s) \mathbb{I}_{X_s \in \Gamma} dK_s\right)_{t \geq 0}$ is the PCAF associated to $\zeta_k^\pm(dy)$ via the Revuz correspondence.

By application of Theorem 5.5.5 in [12] and since all the necessary hypothesis are fulfilled, we get the decomposition (3.19) on the set $\{t \geq 0 : t \leq \tau_{G_q}\}$ where $G_q := \{x \in \mathbb{R}^d : |x| < q\}$. The identification of the process for all times follows by letting q tend to infinity. \square

The following corollary enhances the link between the solution of (3.19) and the PDE results of Section 2.

COROLLARY 3.2. *Let $0 < T < \infty$. Under the conditions of Theorem 3.1, for any $u_0 \in \mathcal{D}(A)$, we have*

$$(3.20) \quad \mathbb{E}^x[u_0(X_t)] = u(t, x), \quad \forall t \in [0, T], \quad \forall x \in \mathbb{R}^d,$$

where X is the diffusion considered in Theorem 3.1 and u is the solution of (2.14) considered in Proposition 2.18.

PROOF. On the one hand we have $\mathbb{E}^x[u_0(X_t)] = T_t u_0(x)$ thanks to (3.14). On the other hand we have $\frac{d}{dt} T_t u_0 = A T_t u_0$ (see [40] Thm 2.4-c), i.e. $T_t u_0$ solves (2.14) whose solution is unique (Prop. 2.18). Thus $T_t u_0$ and u are equal in the space $C^1([0, T]; L^2(\mathbb{R}^d)) \cap C([0, T]; \mathcal{D}(A))$, and finally $T_t u_0(x) = u(t, x)$ for any t, x (where we have used the fact that $\mathcal{D}(A) \subset H^1(\mathbb{R}^d)$ and elements of $H^1(\mathbb{R}^d)$ are identified with their continuous versions). \square

In the light of (3.20) and in order to compute an approximated value of $u(t, x)$, one could think of producing a Monte Carlo method that uses (possibly approximated) paths of X . Unfortunately, it is not clear how to simulate X from (3.19), even in an approximated way. We believe the question is challenging and could be the subject of future research. In the following section we propose to circumvent this difficulty by proposing our modified Euler scheme. We do not try to discretize directly the paths of X from (3.19), but we provide a Monte Carlo method that approaches the value of $u(t, x)$ (as long as the initial condition is sufficiently regular) by taking crucially into account the transmission condition (\star) of (\mathcal{P}_T) .

4. Euler scheme.

4.1. *Recalls on the projection and the distance to the transmission boundary and further notations and preliminaries.* In this subsection we adopt the notations from [4]. We have the following set of geometric results.

PROPOSITION 4.1 ([4], Proposition 1; see also [15]). *Assume (\mathbf{D}) and (Γ) . There is constant $R > 0$ such that:*

1. (a) for any $x \in V_\Gamma^-(R)$, there are unique $s = \pi_\Gamma^{\gamma^+}(x) \in \Gamma$ and $F^{\gamma^+}(x) \leq 0$ such that :

$$(4.1) \quad x = \pi_\Gamma^{\gamma^+}(x) + F^{\gamma^+}(x)\gamma_+(\pi_\Gamma^{\gamma^+}(x)) ;$$

- (b) for any $x \in V_\Gamma^+(R)$, there are unique $s = \pi_\Gamma^{\gamma^-}(x) \in \Gamma$ and $F^{\gamma^-}(x) \leq 0$ such that :

$$(4.2) \quad x = \pi_\Gamma^{\gamma^-}(x) + F^{\gamma^-}(x)\gamma_-(\pi_\Gamma^{\gamma^-}(x)) ;$$

2. (a) the function $x \mapsto \pi_\Gamma^{\gamma^+}(x)$ is called the projection of x on Γ parallel to γ_+ : this is a C^4 function on $V_\Gamma^-(R)$;

- (b) the function $x \mapsto \pi_\Gamma^{\gamma^-}(x)$ is called the projection of x on Γ parallel to γ_- : this is a C^4 function on $V_\Gamma^+(R)$;

3. Let us set $\tilde{F}^{\gamma^\pm}(x) = F^{\gamma^\pm}(x)|\gamma_\pm(\pi_\Gamma^{\gamma^\pm}(x))|$ the normalized version of F^{γ^\pm} corresponding to the unit vector field $\tilde{\gamma}_\pm : x \mapsto \frac{\gamma_\pm(x)}{|\gamma_\pm(x)|}$.

- (a) the functions $x \mapsto \tilde{F}^{\gamma^\pm}(x)$ are called the algebraic distance of x to Γ parallel to γ_\pm (to $\tilde{\gamma}_\pm$) : these are C^4 functions on $V_\Gamma^\mp(R)$. One has $F^{\gamma^+}, \tilde{F}^{\gamma^+} \leq 0$ on $V_\Gamma^-(R)$ and $F^{\gamma^-}, \tilde{F}^{\gamma^-} \leq 0$ on $V_\Gamma^+(R)$.

- (b) It is possible to extend $F^{\gamma^+}, \tilde{F}^{\gamma^+}$ and $F^{\gamma^-}, \tilde{F}^{\gamma^-}$ to $C_b^4(\mathbb{R}^d, \mathbb{R})$ functions, with the conditions $F^{\gamma^\pm}, \tilde{F}^{\gamma^\pm} > 0$ on D_\pm and $F^{\gamma^\pm}, \tilde{F}^{\gamma^\pm} < 0$ on D_\mp .

4. The above extensions for \tilde{F}^{γ^\pm} and F^ν can be performed in a way such that the functions \tilde{F}^{γ^\pm} and F^ν are equivalent in the sense that

$$(4.3) \quad \frac{1}{c_1}d(x, \Gamma) = \frac{1}{c_1}|F^\nu(x)| \leq \left| \tilde{F}^{\gamma^\pm}(x) \right| \leq c_1|F^\nu(x)| = c_1d(x, \Gamma) \quad \text{for all } x \in \mathbb{R}^d$$

for some constant $c_1 > 1$.

5. For $x \in \Gamma$,

$$(4.4) \quad \nabla \tilde{F}^{\gamma^\pm}(x) = \frac{\nu^*}{\langle \nu, \tilde{\gamma}_\pm \rangle}(x).$$

We sometimes use the notation $\nu(x)$ or $\gamma_\pm(x)$ even if $x \notin \Gamma$. For $x \in V_\Gamma^\pm(R)$, we set $\nu(x) = \nu(\pi_\Gamma^{\gamma^\pm}(x))$ and $\gamma_\pm(x) = \gamma_\pm(\pi_\Gamma^{\gamma^\pm}(x))$ and for $x \notin V_\Gamma^\pm(R)$, arbitrary values are given.

Note that if u is a classical solution to the transmission parabolic problem (\mathcal{P}_T) defined in Section 2, the transmission condition (\star) can be expressed as

$$(4.5) \quad \langle \gamma_+(y), \nabla_x u_+(t, y) \rangle = -\langle \gamma_-(y), \nabla_x u_-(t, y) \rangle, \quad \forall (t, y) \in (0, T] \times \Gamma.$$

This in fact will be the crux of our approach (see Subsubsection 5.5.2).

In the sequel, we will need the following result.

PROPOSITION 4.2. *Assume (\mathbf{B}) , (\mathbf{E}) , (\mathbf{D}) and (Γ) . Let $\hat{x} \in V_\Gamma^\mp(R)$ and $\bar{x} \in V_\Gamma^\mp(R)$ be linked by the following relation :*

$$(4.6) \quad \bar{x} = \pi_\Gamma^{\gamma^\pm}(\hat{x}) - F^{\gamma^\pm}(\hat{x})\gamma_\mp(\pi_\Gamma^{\gamma^\pm}(\hat{x})).$$

Then, there exists $c_2 > 1$ such that

$$(4.7) \quad \frac{1}{c_2}d(\bar{x}, \Gamma) \leq d(\hat{x}, \Gamma) \leq c_2d(\bar{x}, \Gamma).$$

PROOF. Without loss of generality, assume for example that $\bar{x} \in V_\Gamma^-(R)$ and $\hat{x} \in V_\Gamma^-(R)$ are related by (4.6). Then we have

$$(4.8) \quad \bar{x} - \pi_\Gamma^{\gamma^+}(\hat{x}) = -F^{\gamma^+}(\hat{x})\gamma_-(\pi_\Gamma^{\gamma^+}(\hat{x})).$$

and by uniqueness of the projection $\pi_\Gamma^{-\gamma^-}(\bar{x})$, we see that $\pi_\Gamma^{\gamma^+}(\hat{x}) = \pi_\Gamma^{-\gamma^-}(\bar{x})$ (note that $F^{-\gamma^-}(\bar{x}) = F^{\gamma^+}(\hat{x})$).

We deduce that

$$\frac{1}{c_1}d(\bar{x}, \Gamma) \leq |\tilde{F}^{-\gamma^-}(\bar{x})| = |F^{-\gamma^-}(\bar{x})| \times |\gamma_-(\pi_\Gamma^{-\gamma^-}(\bar{x}))| = |\bar{x} - \pi_\Gamma^{\gamma^+}(\hat{x})| = |\bar{x} - \pi_\Gamma^{-\gamma^-}(\bar{x})| \leq c_1 d(\bar{x}, \Gamma)$$

due to the same kind of relation as (4.3), but written for $-\gamma_-$ instead of γ_- . Returning back to (4.8), we see that

$$\frac{1}{c_1}d(\bar{x}, \Gamma) \leq |F^{\gamma^+}(\hat{x})| \times |\gamma_-(\pi_\Gamma(\hat{x}))| = |\tilde{F}^{\gamma^+}(\hat{x})| \frac{|\gamma_-(\pi_\Gamma(\hat{x}))|}{|\gamma_+(\pi_\Gamma(\hat{x}))|} \leq c_1 d(\bar{x}, \Gamma).$$

So that in view of (4.3) written for \hat{x} and γ_+ ,

$$\frac{1}{c_1^2} \frac{|\gamma_+(\pi_\Gamma(\hat{x}))|}{|\gamma_-(\pi_\Gamma(\hat{x}))|} d(\bar{x}, \Gamma) \leq d(\hat{x}, \Gamma) \leq c_1^2 \frac{|\gamma_+(\pi_\Gamma(\hat{x}))|}{|\gamma_-(\pi_\Gamma(\hat{x}))|} d(\bar{x}, \Gamma).$$

But using (1.7), (1.8) and (1.9), it easy to see that for any $z \in \Gamma$,

$$\frac{\lambda^2}{\Lambda^*} \leq \frac{|\gamma_+(z)|^2}{|\gamma_-(z)|^2} \leq \frac{\Lambda^*}{\lambda^2}$$

from which we deduce the result of the proposition. \square

4.2. *Our transformed Euler scheme.* We are now in position to introduce our transformed Euler scheme.

Let us denote from now on $\Delta t = h_n = \frac{T}{n}$ the time step (where $n \in \mathbb{N}^*$) and fix a starting point $x_0 \in \mathbb{R}^d$.

The time grid is given by $(t_k^n)_{k=0}^n$ with $t_k^n = \frac{Tk}{n}$ for $0 \leq k \leq n$.

We denote by $(\Delta W_{k+1})_{k=0}^n$ the i.i.d. sequence of brownian increments constructed on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ and defined by

$$\Delta W_{k+1} = W_{t_{k+1}} - W_{t_k}, \quad \forall 0 \leq k \leq n.$$

Recall that $\sigma : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ stands for a matrix valued coefficient satisfying

$$\sigma \sigma^*(x) = 2a(x), \quad \forall x \in D.$$

Set $(\partial a(x))_j = \text{div}(x \mapsto (a_{1j}(x), \dots, a_{nj}(x)))$.

Our stochastic numerical scheme $(\bar{X}_{t_k}^n)_{k=0}^n$ is defined as follows (we omit the superscript n)

$$\bar{X}_0 = x_0$$

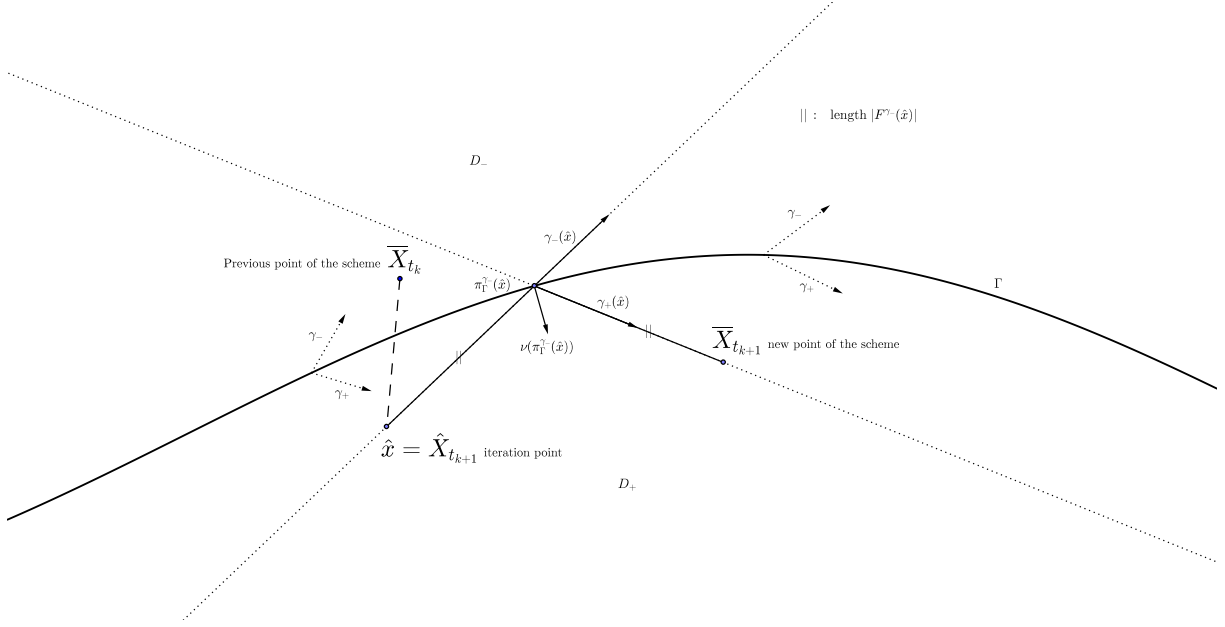


FIG 1. Correction of our scheme when the path crosses the boundary Γ .

and for $t \in (t_k, t_{k+1}]$, we set

$$(4.9) \quad \left\{ \begin{array}{l} \hat{X}_t = \bar{X}_{t_k} + \sigma(\bar{X}_{t_k})(W_t - W_{t_k}) + \partial a(\bar{X}_{t_k})(t - t_k) \quad (\text{standard Euler incrementation}) \\ \bar{X}_{t_{k+1}} = \hat{X}_{t_{k+1}} \quad \text{if } \left(\bar{X}_{t_k} \in \bar{D}_+ \text{ and } \hat{X}_{t_{k+1}} \in D_+ \right) \\ \quad \text{or } \left(\bar{X}_{t_k} \in \bar{D}_- \text{ and } \hat{X}_{t_{k+1}} \in D_- \right) ; \\ \bar{X}_{t_{k+1}} = \pi_{\Gamma}^{\gamma^+}(\hat{X}_{t_{k+1}}) - F^{\gamma^+}(\hat{X}_{t_{k+1}})\gamma_-(\pi_{\Gamma}^{\gamma^+}(\hat{X}_{t_{k+1}})) \quad \text{if } \bar{X}_{t_k} \in \bar{D}_+ \text{ and } \hat{X}_{t_{k+1}} \in D_- ; \\ \bar{X}_{t_{k+1}} = \pi_{\Gamma}^{\gamma^-}(\hat{X}_{t_{k+1}}) - F^{\gamma^-}(\hat{X}_{t_{k+1}})\gamma_+(\pi_{\Gamma}^{\gamma^-}(\hat{X}_{t_{k+1}})) \quad \text{if } \bar{X}_{t_k} \in \bar{D}_- \text{ and } \hat{X}_{t_{k+1}} \in D_+ . \end{array} \right.$$

REMARK 4.3. When the dimension d is reduced to 1 (one dimensional problem), the discontinuity surface reduces to a single point (say 0). In this case and when the coefficient $a = a_+ \mathbb{1}_{y>0} + a_- \mathbb{1}_{y<0}$ is constant on both sides of the discontinuity, it is remarkable that our Euler Scheme is exactly the same as the one described in [32].

Indeed, in this one-dimensional context, let $\varphi(y) = (a_- \mathbb{1}_{y>0} + a_+ \mathbb{1}_{y<0})y$. Note that φ is a bijective map from \mathbb{R} to \mathbb{R} . The Euler Scheme constructed in [32] is then defined by $\bar{X}_0 = x_0$ and for all $k \in \{0, \dots, n\}$,

$$\bar{X}_{t_k} = \varphi^{-1}(\bar{Y}_{t_k})$$

where $\bar{Y}_0 = \varphi(x_0)$ and for all $k \in \{0, \dots, n-1\}$

$$\bar{Y}_{t_{k+1}} = \bar{Y}_{t_k} + \left(a_- \sigma_+ \mathbb{1}_{\bar{Y}_{t_k} > 0} + a_+ \sigma_- \mathbb{1}_{\bar{Y}_{t_k} < 0} \right) (W_{t_{k+1}} - W_{t_k});$$

(see [32] for details - please take care that [32] is written for the right-hand sided local time; the above computation is valid for the symmetric local time). For example if $\varphi(\bar{X}_{t_k}) < 0$ and $\bar{Y}_{t_{k+1}} \geq 0$,

we get (because $\varphi^{-1}(0) = 0$ and φ^{-1} is continuous at 0 and also because \bar{X} and \bar{Y} share the same sign),

$$\begin{aligned}
\bar{X}_{t_{k+1}} &= \varphi^{-1} \left(\varphi(\bar{X}_{t_k}) + \left(a_- \sigma_+ \mathbb{I}_{\bar{Y}_{t_k} > 0} + a_+ \sigma_- \mathbb{I}_{\bar{Y}_{t_k} < 0} \right) \Delta W_k^{k+1} \right) \\
&= \bar{X}_{t_k} + \int_{\varphi(\bar{X}_{t_k})}^0 (\varphi^{-1})'(z) dz + \int_0^{\underbrace{\varphi(\bar{X}_{t_k}) + (a_- \sigma_+ \mathbb{I}_{\bar{Y}_{t_k} > 0} + a_+ \sigma_- \mathbb{I}_{\bar{Y}_{t_k} < 0}) \Delta W_k^{k+1}}_{=0}} (\varphi^{-1})'(z) dz \\
&= \bar{X}_{t_k} - \varphi(\bar{X}_{t_k}) \frac{1}{a_+} + \left(\varphi(\bar{X}_{t_k}) + a_+ \sigma_- \Delta W_k^{k+1} \right) \frac{1}{a_-} \\
&= \frac{a_+}{a_-} \bar{X}_{t_k} + \frac{a_+}{a_-} \sigma_- \Delta W_k^{k+1},
\end{aligned}$$

which turns out to be the corresponding case in (4.9) in this one-dimensional context. This correspondence is valid in all cases and our transformed Euler Scheme may be viewed as some kind of generalization of the Euler Scheme presented in [32].

5. Convergence rate of our Euler scheme. The purpose of this section is to prove the following result.

THEOREM 5.1. *Let $0 < T < \infty$. Assume (B), (E), (D) and Γ . Let $m' = \lceil 2 + \frac{d}{4} \rceil + 1$ and $k = m' + 2$. Assume that the coefficients a_{ij} satisfy $(a_{\pm})_{ij} \in C_b^{2m'-1}(D_{\pm})$ and that Γ is of class $C^{2m'}$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be in the space $\mathcal{D}(A^k)$. Let u be the classical solution of (\mathcal{P}_T) .*

We have that for all n large enough, and all x_0 in \mathbb{R}^d ,

$$(5.1) \quad \left| u(T, x_0) - \mathbb{E}^{x_0} f(\bar{X}_T^n) \right| \leq K \sqrt{h_n},$$

where the constant K depends on d, λ, Λ_*, f and T .

REMARK 5.2. *In this theorem the assumptions on $a(x)$ and Γ involving the integers m' and k are here in order to use Corollary 2.20, which ensures that we will have $\sup_{t \in [0, T], x \in \bar{D}_{\pm}} |\partial_t^j \partial^\alpha u_{\pm}(t, x)| < \infty$ for any $j \leq 2$ and any $|\alpha| \leq 4$. This control on the derivatives on u is what we need in order to lead our convergence proof. In fact if there is a way to get this control under weaker assumptions on $a(x)$ and Γ , this will lead to a convergence theorem stated under these weaker assumptions.*

Assume we are under the assumptions of Theorem 5.1. Remember that the classical solution u of (\mathcal{P}_T) is in fact constructed starting from the semi-weak solution of (\mathcal{P}_T) , it is the unique solution of (2.14) (Proposition 2.18). So that in the light of Corollary 3.2 and Theorem 5.1 we get the following Corollary.

COROLLARY 5.3. *Under the assumptions of Theorem 5.1 - recall in particular that $f \in \mathcal{D}(A^k)$ - consider the scheme \bar{X}^n and the solution X of (3.19). We have for any $x_0 \in \mathbb{R}^d$*

$$\left| \mathbb{E}^{x_0} f(X_T) - \mathbb{E}^{x_0} f(\bar{X}_T^n) \right| \leq K \sqrt{h_n},$$

where the constant K depends on d, λ, Λ_*, f and T .

REMARK 5.4. Note that f has to be taken in some $\mathcal{D}(A^k)$ and the above corollary does not even state the convergence in law of \bar{X}_T^n to X_T .

However and loosely speaking, let us emphasize that having to take very regular initial conditions is very often the case for Monte Carlo methods dealing with problems involving a boundary. For e.g. in [4] that treats the case of reflected diffusions, the authors have to choose an initial condition satisfying some kind of compatibility condition involving directional partial derivatives of the initial condition up to order three on the reflection boundary in order to ensure the convergence of their symmetrized Euler scheme. Again we think that these questions could be addressed in future research.

5.1. Preliminary results.

LEMMA 5.5. (see [4]) Consider an Itô process with uniformly bounded coefficients $dU_t = b_t dt + \sigma_t dW_t$ on $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$. There exist some constants $c > 0$ and K (depending on $p \geq 1$, T and the bounds on σ , b) such that, for any stopping times S and S' (with $0 \leq S \leq S' \leq \delta \leq T$) and any $\eta \geq 0$,

$$(5.2) \quad \mathbb{P}^{x_0} \left[\sup_{t \in [S, S']} |U_t - U_s| \geq \eta \right] \leq K \exp \left(-c \frac{\eta^2}{\delta} \right) ;$$

$$(5.3) \quad \mathbb{E}^{x_0} \left[\sup_{t \in [S, S']} |U_t - U_s|^p \right] \leq K \delta^{p/2}.$$

We have when $\bar{X}_{t_k} \in D_+$

$$\bar{X}_{t_{k+1}} = \hat{X}_{t_{k+1}} + \left[F^{\gamma_+}(\hat{X}_{t_{k+1}}) \right]^- \left(\gamma_+(\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}})) + \gamma_-(\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}})) \right)$$

and when $X_{t_k} \in D_-$

$$\bar{X}_{t_{k+1}} = \hat{X}_{t_{k+1}} + \left[F^{\gamma_-}(\hat{X}_{t_{k+1}}) \right]^- \left(\gamma_+(\pi_{\Gamma}^{\gamma_-}(\hat{X}_{t_{k+1}})) + \gamma_-(\pi_{\Gamma}^{\gamma_-}(\hat{X}_{t_{k+1}})) \right)$$

This shows that $(\bar{X}_t)_{0 \leq t \leq T}$ behaves like a continuous semimartingale on each of the intervals $[t_k, t_{k+1})$. Using Tanaka's formula, we have – for example for $\bar{X}_{t_k} \in D_+$ – that for any $t \in [t_k, t_{k+1})$,

$$(5.4) \quad \begin{aligned} d\bar{X}_t &= d\hat{X}_t + \frac{1}{2} (\gamma_+ + \gamma_-)(\hat{X}_t) dL_t^0(F^{\gamma_+}(\hat{X})) \\ &+ \left[F^{\gamma_+}(\hat{X}_t) \right]^- \left(\nabla(\gamma_+ + \gamma_-)(\hat{X}_t) d\hat{X}_t + \frac{1}{2} \text{Tr} \left[\mathbf{H}[\gamma_+ + \gamma_-](\hat{X}_t) a(\bar{X}_{t_k}) \right] dt \right) \\ &- \mathbb{I}_{F^{\gamma_+}(\hat{X}_t) < 0} \left[\nabla(\gamma_+ + \gamma_-)(\hat{X}_t) a(\bar{X}_{t_k}) \left(\nabla F^{\gamma_+}(\hat{X}_t) \right)^* dt + (\gamma_+ + \gamma_-)(\hat{X}_t) \nabla F^{\gamma_+}(\hat{X}_t) d\hat{X}_t \right] \\ &- \mathbb{I}_{F^{\gamma_+}(\hat{X}_t) < 0} (\gamma_+ + \gamma_-)(\hat{X}_t) \frac{1}{2} \text{Tr} \left[\mathbf{H}[F^{\gamma_+}](\hat{X}_t) a(\bar{X}_{t_k}) \right] dt. \end{aligned}$$

LEMMA 5.6. Under Assumptions **(B)**, **(E)**, **(D)** and (Γ) , for all $c > 0$, there exists a constant $K(T)$ such that

$$(5.5) \quad h_n \mathbb{E}^{x_0} \sum_{i=0}^{n-1} \left[\exp \left(-c \frac{d^2(\bar{X}_{t_i}^n, \Gamma)}{h_n} \right) \right] \leq K(T) \sqrt{h_n}.$$

PROOF. The idea is to use the occupation times formula. Using successively (4.3) and the inequality (4.7) of Proposition 4.2, we have $d(\bar{x}, \Gamma) \geq \frac{1}{c_2} d(\hat{x}, \Gamma) \geq \frac{1}{c_1 c_2} |\tilde{F}^{\gamma \pm}(\hat{x})|$ so that

$$\begin{aligned}
\mathcal{A}_{i+1} &:= \mathbb{E}^{x_0} \left[\exp \left(-c \frac{d^2(\bar{X}_{t_{i+1}}^n, \Gamma)}{h_n} \right) \right] \\
&\leq \mathbb{E}^{x_0} \left[\exp \left(-c \frac{|\tilde{F}^{\gamma+}(\hat{X}_{t_{i+1}}^n)|^2}{|c_1 c_2|^2 h_n} \right) \mathbb{I}_{\bar{X}_{t_{i+1}}^n \in D_-} \right] + \mathbb{E}^{x_0} \left[\exp \left(-c \frac{|\tilde{F}^{\gamma-}(\hat{X}_{t_{i+1}}^n)|^2}{|c_1 c_2|^2 h_n} \right) \mathbb{I}_{\bar{X}_{t_{i+1}}^n \in D_+} \right] \\
(5.6) \quad &:= \mathcal{A}_{i+1}^+ + \mathcal{A}_{i+1}^-.
\end{aligned}$$

We concentrate on term \mathcal{A}_{i+1}^+ as both terms are treated in a similar manner.

Set $c' = c/2c_1^2 c_2^2 > 0$ and $g(x) = \exp(-2c'x^2/h)$; it is easy to check that $|g(x)| + \sqrt{h}|g'(x)| + h|g''(x)| \leq K(T) \exp(-c'x^2/h)$. Hence, for $t \in [t_i, t_{i+1}]$, Itô's formula yields that

$$\begin{aligned}
&\mathbb{E}^{x_0} \exp \left(-2c' \frac{|\tilde{F}^{\gamma+}(\hat{X}_{t_{i+1}}^n)|^2}{h_n} \right) \\
&\leq K(T) \left[\mathbb{E}^{x_0} \exp \left(-c' \frac{|\tilde{F}^{\gamma+}(\hat{X}_t^n)|^2}{h_n} \right) + \frac{1}{h_n} \int_t^{t_{i+1}} ds \mathbb{E}^{x_0} \exp \left(-c' \frac{|\tilde{F}^{\gamma+}(\hat{X}_s^n)|^2}{h_n} \right) \right].
\end{aligned}$$

We integrate this inequality with respect to t over $[t_i, t_{i+1}]$ to get

$$(5.7) \quad h_n \mathcal{A}_{i+1}^+ \leq K(T) \int_{t_i}^{t_{i+1}} ds \mathbb{E}^{x_0} \exp \left(-c' \frac{|\tilde{F}^{\gamma+}(\hat{X}_s^n)|^2}{h_n} \right).$$

(for possibly some new constant $K(T)$).

Observe that from (4.4),

$$(5.8) \quad d\langle \tilde{F}^{\gamma+}(\hat{X}^n), \tilde{F}^{\gamma+}(\hat{X}^n) \rangle_s = \nabla \tilde{F}^{\gamma+}(\hat{X}_s^n) a(\bar{X}_{t_i}^n) \left[\nabla \tilde{F}^{\gamma+}(\hat{X}_s^n) \right]^* ds \geq \lambda ds.$$

Indeed, using the Cauchy-Schwarz inequality and $|\nu(\hat{x})| = 1$, we have that

$$\begin{aligned}
\nabla \tilde{F}^{\gamma+}(\hat{x}) a(\bar{x}) \left[\nabla \tilde{F}^{\gamma+}(\hat{x}) \right]^* &= \frac{\nu^*(\hat{x}) a(\bar{x}) \nu(\hat{x})}{\langle \nu(\hat{x}), \tilde{\gamma}_+(\hat{x}) \rangle^2} \\
&= \frac{\langle \nu(\hat{x}), a(\bar{x}) \nu(\hat{x}) \rangle}{\langle \nu(\hat{x}), \frac{a(\hat{x}) \nu(\hat{x})}{|a(\hat{x}) \nu(\hat{x})|} \rangle^2} \\
&\geq \frac{\langle \nu(\hat{x}), a(\bar{x}) \nu(\hat{x}) \rangle}{|\nu(\hat{x})|^2 |a(\hat{x}) \nu(\hat{x})|^2} |a(\hat{x}) \nu(\hat{x})|^2 = \langle \nu(\hat{x}), a(\bar{x}) \nu(\hat{x}) \rangle \geq \lambda
\end{aligned}$$

which justifies (5.8).

It readily follows from the occupation times formula that

$$(5.9) \quad h_n \mathcal{A}_{i+1}^+ \leq K(T) \int_{-R}^R dy \exp \left(-c' \frac{y^2}{h_n} \right) \mathbb{E}^{x_0} \left[L_{t_{i+1}}^y \left(\tilde{F}^{\gamma+}(\hat{X}^n) \right) - L_{t_i}^y \left(\tilde{F}^{\gamma+}(\hat{X}^n) \right) \right].$$

Now,

$$\begin{aligned}
& \mathbb{E}^{x_0} \left[L_{t_{i+1}}^y \left(\tilde{F}^{\gamma+}(\hat{X}^n) \right) - L_{t_i}^y \left(\tilde{F}^{\gamma+}(\hat{X}^n) \right) \right] \\
&= 2\mathbb{E}^{x_0} \left[\left(\tilde{F}^{\gamma+}(\hat{X}_{t_{i+1}}^n) - y \right)^+ - \left(\tilde{F}^{\gamma+}(\hat{X}_{t_i}^n) - y \right)^+ - \int_{t_i}^{t_{i+1}} \mathbb{I}_{\tilde{F}^{\gamma+}(\hat{X}_s^n) \geq y} d \left(\tilde{F}^{\gamma+}(\hat{X}_s^n) \right) \right] \\
&\leq 2\mathbb{E}^{x_0} \left[\left(\tilde{F}^{\gamma+}(\hat{X}_{t_{i+1}}^n) - y \right)^+ - \left(\tilde{F}^{\gamma+}(\hat{X}_{t_i}^n) - y \right)^+ \right] + K(T)h_n.
\end{aligned}$$

Therefore, $\sum_{i=0}^{n-1} \mathbb{E}^{x_0} \left[L_{t_{i+1}}^y \left(\tilde{F}^{\gamma+}(\hat{X}^n) \right) - L_{t_i}^y \left(\tilde{F}^{\gamma+}(\hat{X}^n) \right) \right] \leq K(T)$ uniformly in $|y| \leq R$ since the sum is telescoping. We can thus conclude that $h_n \sum_{i=0}^{n-1} \mathcal{A}_{i+1}^+ \leq K(T)\sqrt{h_n}$.

The sum $h_n \sum_{i=0}^{n-1} \mathcal{A}_{i+1}^-$ is treated similarly. The proof of the Lemma is complete. \square

5.2. *Error decomposition.* In all the sequel x_0 is arbitrarily fixed.

For all $0 \leq k \leq n$ set

$$\theta_k^n := T - t_k^n.$$

The proof of Theorem 5.1 proceeds as follows (we omit the superscript n). Since $u(0, x) = f(x)$ for all $x \in \mathbb{R}^d$ and $u(T, x_0) = \mathbb{E}^{x_0} u(T, \bar{X}_0)$, the discretization error at time T can be decomposed as follows:

$$\begin{aligned}
(5.10) \quad \epsilon_T^{x_0} &= \left| u(T, x_0) - \mathbb{E}^{x_0} f(\bar{X}_T) \right| \\
&= \left| \sum_{k=0}^{n-1} \mathbb{E}^{x_0} u(T - t_k, \bar{X}_{t_k}) - \mathbb{E}^{x_0} u(T - t_{k+1}, \bar{X}_{t_{k+1}}) \right|,
\end{aligned}$$

and thus

$$\begin{aligned}
(5.11) \quad \epsilon_T^{x_0} &\leq \left| \sum_{k=0}^{n-1} \mathbb{E}^{x_0} \left\{ u(\theta_k, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_k}) \right. \right. \\
&\quad \left. \left. + u(\theta_{k+1}, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_{k+1}}) \right\} \right|.
\end{aligned}$$

The rest of this section is devoted to the analysis of

$$\left| \sum_{k=0}^{n-1} \mathbb{E}^{x_0} (T_k - S_k) \right|,$$

where the time increment T_k is defined as

$$(5.12) \quad T_k := u(\theta_k, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_k})$$

and the space increment is defined as

$$(5.13) \quad S_k := u(\theta_{k+1}, \bar{X}_{t_{k+1}}) - u(\theta_{k+1}, \bar{X}_{t_k}).$$

5.3. *Estimate for the time increment T_k .* Remember the definition (5.12) of T_k and that $\theta_k = T - t_k$. We have

$$\begin{aligned} & \{u(\theta_k, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_k})\} \mathbb{I}_{\bar{X}_{t_k} \in D_+} \\ &= h_n \partial_t u(\theta_{k+1}, \bar{Y}_{t_k}) \mathbb{I}_{\bar{X}_{t_k} \in D_+} + h_n^2 \int_{[0,1]^2} \partial_{tt}^2 u(\theta_{k+1} + \alpha_1 \alpha_2 h_n, \bar{X}_{t_k}) \alpha_1 d\alpha_1 d\alpha_2 \mathbb{I}_{\bar{X}_{t_k} \in D_+} \\ &=: T_k^+ + R_k^+. \end{aligned}$$

Similarly,

$$\begin{aligned} & \{u(\theta_k, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_k})\} \mathbb{I}_{\bar{X}_{t_k} \in D_-} \\ &= h_n \partial_t u(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\bar{X}_{t_k} \in D_-} + h_n^2 \int_{[0,1]^2} \partial_{tt}^2 u(\theta_{k+1} + \alpha_1 \alpha_2 h_n, \bar{X}_{t_k}) \alpha_1 d\alpha_1 d\alpha_2 \mathbb{I}_{\bar{X}_{t_k} \in D_-} \\ &=: T_k^- + R_k^-. \end{aligned}$$

In view of Corollary 2.20 and Remark 5.2 we have

$$\mathbb{E}^{x_0} |R_k^+ + R_k^-| \leq C h_n^2.$$

From the preceding we deduce

$$(5.14) \quad \mathbb{E}^{x_0} T_k = \mathbb{E}^{x_0} \partial_t u(\theta_{k+1}, \bar{X}_{t_k}) h_n + O(h_n^2).$$

5.4. *Expansion of the space increment S_k .* Let S_k be defined as in (5.13). Set

$$\begin{aligned} \Delta_{k+1} \bar{X} &:= \bar{X}_{t_{k+1}} - \bar{X}_{t_k} \\ \Delta_{k+1}^\# \bar{X} &:= \hat{X}_{t_{k+1}} - \bar{X}_{t_k}. \end{aligned}$$

and recall that $\Delta_{k+1} W = W_{t_{k+1}} - W_{t_k}$.

PROPOSITION 5.7.

$$(5.15) \quad \mathbb{E}^{x_0} |(\Delta_{k+1}^\# \bar{X})^\alpha| \leq C(\alpha) h_n^{|\alpha|/2}.$$

PROOF. This is a consequence of the result of Lemma 5.5 combined with the fact that $|(x)^\alpha| \leq |x|^{|\alpha|}$ for any $x \in \mathbb{R}^d$. \square

We emphasize that, due to the definition of our stochastic numerical scheme, $\Delta_{k+1}^\# \bar{X}$ does not coincide with $\bar{X}_{t_{k+1}} - \bar{X}_{t_k}$ when $\bar{X}_{t_{k+1}}$ and \bar{X}_{t_k} do not belong to the same region, which explains the two notations Δ and $\Delta^\#$.

We need to introduce the four following events:

$$(5.16) \quad \begin{cases} \Omega_k^{++} & := [\bar{X}_{t_k} \in \bar{D}_+ \text{ and } \hat{X}_{t_{k+1}} \in D_+], \\ \Omega_k^{--} & := [\bar{X}_{t_k} \in D_- \text{ and } \hat{X}_{t_{k+1}} \in \bar{D}_-], \\ \Omega_k^{+-} & := [\bar{X}_{t_k} \in D_+ \text{ and } \hat{X}_{t_{k+1}} \in \bar{D}_-], \\ \Omega_k^{-+} & := [\bar{X}_{t_k} \in \bar{D}_- \text{ and } \hat{X}_{t_{k+1}} \in D_+]. \end{cases}$$

In view of the definition of our stochastic numerical scheme we have

$$\text{On } \Omega_k^{++}, \Delta_{k+1}\bar{X} = \Delta_{k+1}^\sharp \bar{X}.$$

Therefore

$$\begin{aligned} S_k \mathbb{I}_{\Omega_k^{++}} &= \langle \Delta_{k+1}\bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \mathbb{I}_{\Omega_k^{++}} + \frac{1}{2} (\Delta_{k+1}\bar{X})^* \mathbf{H}[u](\theta_{k+1}, \bar{X}_{t_k}) \Delta_{k+1}\bar{X} \mathbb{I}_{\Omega_k^{++}} \\ &\quad + \sum_{|\alpha|=3} \frac{1}{\alpha!} (\Delta_{k+1}\bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\Omega_k^{++}} \\ &\quad + \int_0^1 d\xi \sum_{|\alpha|=4} \frac{(1-\xi)^4}{\alpha!} (\Delta_{k+1}\bar{X})^\alpha \frac{\partial^4 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k} + \xi \Delta_{k+1}\bar{X}) \mathbb{I}_{\Omega_k^{++}} \\ &=: S_k^{+++} + S_k^{++2} + S_k^{++3} + S_k^{++4}. \end{aligned}$$

Similarly,

$$\begin{aligned} S_k \mathbb{I}_{\Omega_k^{--}} &= \langle \Delta_{k+1}\bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \mathbb{I}_{\Omega_k^{--}} + \frac{1}{2} (\Delta_{k+1}\bar{X})^* \mathbf{H}[u](\theta_{k+1}, \bar{X}_{t_k}) \Delta_{k+1}\bar{X} \mathbb{I}_{\Omega_k^{--}} \\ &\quad + \sum_{|\alpha|=3} \frac{1}{\alpha!} (\Delta_{k+1}\bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\Omega_k^{--}} \\ &\quad + \int_0^1 d\xi \sum_{|\alpha|=4} \frac{(1-\xi)^4}{\alpha!} (\Delta_{k+1}\bar{X})^\alpha \frac{\partial^4 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k} + \xi \Delta_{k+1}\bar{X}) \mathbb{I}_{\Omega_k^{--}} \\ &=: S_k^{--1} + S_k^{--2} + S_k^{--3} + S_k^{--4}. \end{aligned}$$

We now use that $\Omega_k^{++} \cup \Omega_k^{--} = \Omega - (\Omega_k^{+-} \cup \Omega_k^{-+})$. Notice that $\Omega_k^{+-} \cup \Omega_k^{-+}$ belongs to the σ -field generated by (W_t) up to time t_{k+1} . In view of the first line of (4.9) and the fact that $\mathbb{E}^{\mathcal{F}_{t_k}} \Delta W_{k+1} = 0$, we get

$$\begin{aligned} \mathbb{E}^{x_0}(S_k^{+++} + S_k^{--1}) &= \frac{h_n}{2} \mathbb{E}^{x_0} [\langle \partial a(\bar{X}_{t_k}), \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle] \\ &\quad - \mathbb{E}^{x_0} [\langle \Delta_{k+1}^\sharp \bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}}] \end{aligned}$$

Proceeding similarly and conditioning $(\Delta_{k+1}^\sharp \bar{X})^2$ w.r.t. the past of (W_t) up to time t_k , we obtain

$$\begin{aligned} \mathbb{E}^{x_0}(S_k^{++2} + S_k^{--2}) &= \frac{1}{2} \mathbb{E}^{x_0} [\text{Tr}[\sigma \mathbf{H}[u] \sigma^*](\theta_{k+1}, \bar{X}_{t_k})] h_n \\ &\quad - \frac{1}{2} \mathbb{E}^{x_0} [(\Delta_{k+1}^\sharp \bar{X})^* \mathbf{H}[u](\theta_{k+1}, \bar{X}_{t_k}) \Delta_{k+1}^\sharp \bar{X} \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}}], \end{aligned}$$

and, since $\mathbb{E}^{x_0}(\Delta_{k+1} W)^\alpha = 0$ whenever $|\alpha| = 3$,

$$\begin{aligned} \mathbb{E}^{x_0}(S_k^{++3} + S_k^{--3}) &= \sum_{|\alpha|=3} \frac{1}{\alpha!} \mathbb{E}^{x_0} \left[(\Delta_{k+1}^\sharp \bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \right] \\ &\quad - \sum_{|\alpha|=3} \frac{1}{\alpha!} \mathbb{E}^{x_0} \left[(\Delta_{k+1}^\sharp \bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} \right]. \end{aligned}$$

We have, combining the results of Corollary 2.20 and Proposition 5.7,

$$(5.17) \quad \left| \sum_{|\alpha|=3} \frac{1}{\alpha!} \mathbb{E}^{x_0} \left[(\Delta_{k+1}^\# \bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \right] \right| \leq C h_n^{3/2}.$$

In addition, and for the same reasons, we have

$$\mathbb{E}^{x_0} |S_k^{+++4} + S_k^{---4}| \leq C h_n^2.$$

To summarize the calculations of this subsection, we have obtained

$$(5.18) \quad \begin{aligned} \mathbb{E}^{x_0} S_k &= \\ \mathbb{E}^{x_0} \mathcal{L}u(\theta_{k+1}, \bar{X}_{t_k}) h_n &+ \mathbb{E}^{x_0} \left[\left(S_k - \langle \Delta_{k+1}^\# \bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \right) \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} \right] \\ - \mathbb{E}^{x_0} \left[\left(\frac{1}{2} (\Delta_{k+1}^\# \bar{X})^* \mathbf{H}[u](\theta_{k+1}, \bar{X}_{t_k}) \Delta_{k+1}^\# \bar{X} + \sum_{|\alpha|=3} \frac{1}{\alpha!} (\Delta_{k+1}^\# \bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \right) \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} \right] \\ &+ O(h_n^{3/2}) \\ &=: \mathbb{E}^{x_0} \mathcal{L}u(\theta_{k+1}, \bar{X}_{t_k}) h_n + \mathbb{E}^{x_0} \mathcal{R}_k^{(1)} - \mathbb{E}^{x_0} \mathcal{R}_k^{(2)} + O(h_n^{3/2}). \end{aligned}$$

We now estimate the remaining terms $\mathbb{E}^{x_0} \mathcal{R}_k^{(1)}$ and $\mathbb{E}^{x_0} \mathcal{R}_k^{(2)}$.

5.5. *Control of the term $\mathbb{E}^{x_0} \mathcal{R}_k^{(1)}$. Expansion around a well chosen point in Γ .* On the event Ω_k^{+-} we have that $\bar{X}_{t_{k+1}}$ and \bar{X}_{t_k} are close to Γ . On this event, we also have that $\hat{X}_{t_{k+1}} \in D_-$ and $\bar{X}_{t_k} \in D_+$. Remember our definition of $(F^{\gamma+}(x), \pi_\Gamma^{\gamma+}(x))$ for $x \in D_-$.

5.5.1. *Decomposition of $\mathbb{E}^{x_0} \mathcal{R}_k^{(1)}$.* As the function u is continuous across the surface Γ at point $\pi_\Gamma^{\gamma+}(x)$, we get

$$\begin{aligned} &\mathbb{E}^{x_0} \left(\left(S_k - \langle \Delta_{k+1}^\# \bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \right) \mathbb{I}_{\Omega_k^{+-}} \right) \\ &= \mathbb{E}^{x_0} \left[\left(\left(u(\theta_{k+1}, \bar{X}_{t_{k+1}}) - u(\theta_{k+1}, \pi_\Gamma^{\gamma+}(\hat{X}_{t_{k+1}})) \right) + \left(u(\theta_{k+1}, \pi_\Gamma^{\gamma+}(\hat{X}_{t_{k+1}})) - u(\theta_{k+1}, \bar{X}_{t_k}) \right) \right) \mathbb{I}_{\Omega_k^{+-}} \right] \\ &\quad - \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^\# \bar{X}, \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \\ &\quad - \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^\# \bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) - \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \end{aligned}$$

$$\begin{aligned}
&= \left\{ \mathbb{E}^{x_0} \left[\langle \bar{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}), \nabla_x u_-(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \right. \\
&\quad - \mathbb{E}^{x_0} \left[\langle \bar{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}), \nabla_x u_+(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \\
&\quad \left. - \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^{\sharp} \bar{X}, \nabla_x u_+(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \right\} := L_k^{+-1} \\
&\quad + \left\{ \int_0^1 d\xi \sum_{|\alpha|=2} \frac{(1-\xi)^2}{\alpha!} \mathbb{E}^{x_0} \left[(\hat{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}))^{\alpha} \right. \right. \\
&\quad \quad \times \frac{\partial^{\alpha} u}{\partial x^{\alpha}}(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) + \xi(\hat{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \left. \right] \mathbb{I}_{\Omega_k^{+-}} \left. \right] \\
&\quad - \int_0^1 d\xi \sum_{|\alpha|=2} \frac{(1-\xi)^2}{\alpha!} \mathbb{E}^{x_0} \left[(\bar{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}))^{\alpha} \right. \\
&\quad \quad \times \frac{\partial^{\alpha} u}{\partial x^{\alpha}}(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) + \xi(\bar{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \left. \right] \mathbb{I}_{\Omega_k^{+-}} \left. \right\} := L_k^{+-2} \\
&\quad - \left\{ \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^{\sharp} \bar{X}, \nabla_x u_+(\theta_{k+1}, \bar{X}_{t_k}) - \nabla_x u_+(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \right\} := L_k^{+-3} \\
&= L_k^{+-1} + L_k^{+-2} + L_k^{+-3}.
\end{aligned}$$

5.5.2. *Canceling the term L_k^{+-1} using the transmission condition.* Observe that due to the fact that

$$\left(\hat{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}) \right) + \left(\hat{X}_{t_{k+1}} - \hat{X}_{t_k} \right) = \hat{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}).$$

we have that

$$\begin{aligned}
L_k^{+-1} &= \mathbb{E}^{x_0} \left[\left(\langle \bar{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}), \nabla_x u_-(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \right. \right. \\
&\quad \left. \left. - \langle \hat{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}), \nabla_x u_+(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \right) \mathbb{I}_{\Omega_k^{+-}} \right] \\
&= \mathbb{E}^{x_0} \left[F^{\gamma+}(\hat{X}_{t_{k+1}}) \left(\langle -\gamma_-(\pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})), \nabla_x u_-(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \right. \right. \\
&\quad \left. \left. - \langle \gamma_+(\pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})), \nabla_x u_+(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})) \rangle \right) \mathbb{I}_{\Omega_k^{+-}} \right] \\
&= 0,
\end{aligned}$$

where we have used the vector problem solved by $(F^{\gamma+}, \pi_{\Gamma}^{\gamma+})$ and Equation (4.5) (i.e. the transmission condition (\star) and the definition of $\gamma_{\pm}(x)$).

5.5.3. *The term L_k^{+-2} .* We now turn to the term L_k^{+-2} .

The term L_k^{+-2} is the sum of two terms. These two terms are treated similarly, so we concentrate only on the first. Let α such that $|\alpha| = 2$. We have that

$$\mathbb{E}^{x_0} \left[\left| (\bar{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}))^{\alpha} \right| \mathbb{I}_{\Omega_k^{+-}} \right] \leq c_1 \mathbb{E}^{x_0} \left[|\bar{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})|^2 \mathbb{I}_{\Omega_k^{+-}} \right] \leq c_2 \mathbb{E}^{x_0} \left[|\Delta_{k+1}^{\sharp} X|^2 \mathbb{I}_{\Omega_k^{+-}} \right]$$

The same kind of treatment can be performed for the second term of L_k^{+-2} . Conditioning w.r.t \mathcal{F}_{t_k} and applying the Cauchy-Schwarz inequality in the conditionnal expectation, we find using the

result of Lemma 5.5,

$$|L_k^{+-2}| \leq C \mathbb{E}^{x_0} \left[\mathbb{E}^{\mathcal{F}_{t_k}} \left[|\Delta_{k+1}^\# X|^4 \right]^{1/2} \mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{+-})^{1/2} \right] \leq C h_n \mathbb{E}^{x_0} \mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{+-})^{1/2}.$$

5.5.4. *The term L_k^{+-3} .* For the term L_k^{+-3} , we may perform a Taylor's expansion to the term

$$\nabla_x u_+(\theta_{k+1}, \bar{X}_{t_k}) - \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})).$$

Using Corollary 2.20 and the Cauchy-Schwarz inequality, we find that

$$(5.19) \quad |L_k^{+-3}| \leq C \mathbb{E}^{x_0} \left[|\Delta_{k+1}^\# \bar{X}| |\bar{X}_{t_k} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})| \mathbb{I}_{\Omega_k^{+-}} \right] \leq C \mathbb{E}^{x_0} \left[|\Delta_{k+1}^\# X|^2 \mathbb{I}_{\Omega_k^{+-}} \right].$$

Finally, as for the term L_k^{+-2} , we find that

$$|L_k^{+-3}| \leq C h_n \mathbb{E}^{x_0} \mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{+-})^{1/2}.$$

Using the same method for the other side Ω_k^{-+} , we find that

$$\mathbb{E}^{x_0} \mathcal{R}_k^{(1)} \leq C h_n \mathbb{E}^{x_0} \left(\mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{+-})^{1/2} + \mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{-+})^{1/2} \right).$$

5.6. *Summing up.* The term $\mathbb{E}^{x_0} \mathcal{R}_k^{(2)}$ can be estimated using the same techniques used in the previous section and we omit the details.

Using now the fact that $\partial_t u - \mathcal{L}u = 0$, we finally find that

$$(5.20) \quad \epsilon_T^{x_0} \leq C h_n \mathbb{E}^{x_0} \sum_{k=0}^{n-1} \left(\mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{+-})^{1/2} + \mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{-+})^{1/2} \right) + C \sqrt{h_n}.$$

Observe – using the result of Lemma 5.5 – that

$$\begin{aligned} \mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{+-})^{1/2} &= \mathbb{P}^{\mathcal{F}_{t_k}} \left(\bar{X}_{t_k} \in D_+, \hat{X}_{t_{k+1}} \in D_- \right)^{1/2} \\ &\leq \mathbb{P}^{\mathcal{F}_{t_k}} \left(\|\hat{X}_{t_{k+1}} - \bar{X}_{t_k}\| \geq d(\bar{X}_{t_k}, \Gamma) \right)^{1/2} \\ &\leq K(T) \exp \left(-\frac{1}{2} \frac{d^2(\bar{X}_{t_k}, \Gamma)}{h_n} \right) \end{aligned}$$

and the same kind of inequality holds true for $\mathbb{P}^{\mathcal{F}_{t_k}} (\Omega_k^{-+})^{1/2}$.

Finally,

$$\epsilon_T^{x_0} \leq K(T) h_n \mathbb{E}^{x_0} \sum_{k=0}^{n-1} \exp \left(-\frac{1}{2} \frac{d^2(\bar{X}_{t_k}, \Gamma)}{h_n} \right) + C \sqrt{h_n},$$

and we conclude the proof of Theorem 5.1 using the result of Lemma 5.6 (note that if we sum up all the dependancies of our constants, we indeed have that K in (5.1) depends on d, λ, Λ_*, f and T).

6. Numerical experiments. In these examples $d = 2$ and the domain D is the open unit disc, i.e.,

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}.$$

Note that the boundary of D is the unit circle $\partial D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$.

The subdomains D_+ and D_- are defined by

$$D_+ = \{(x_1, x_2) \in D \text{ with } x_2 > 0\} \quad \text{and} \quad D_- = \{(x_1, x_2) \in D \text{ with } x_2 < 0\},$$

so that the interface is $\Gamma = \{(x_1, 0) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1\}$.

The diffusion matrix is defined by

$$a(x) = a_+(x)\mathbb{I}_{x \in D_+} + a_-(x)\mathbb{I}_{x \in \bar{D}_-},$$

with

$$a_{\pm}(x) = P_{\pm}^* D_{\pm}(x) P_{\pm}$$

where P_{\pm} are rotation (therefore orthogonal) matrices

$$P_{\pm} = \begin{pmatrix} \cos(\theta_{\pm}) & -\sin(\theta_{\pm}) \\ \sin(\theta_{\pm}) & \cos(\theta_{\pm}) \end{pmatrix}$$

(for $\theta_{\pm} \in [0, 2\pi)$), and $D_{\pm}(x)$ are diagonal matrix-valued functions

$$D_{\pm}(x) = \begin{pmatrix} \lambda_{\pm}^1 + \epsilon_{\pm} x_2 & 0 \\ 0 & \lambda_{\pm}^2 + \epsilon_{\pm} x_2 \end{pmatrix}$$

where $\lambda_{\pm}^1, \lambda_{\pm}^2 > 0$ and $\epsilon_{\pm} < \lambda_{\pm}^i$ for $i = 1, 2$. Note that this ensures that $a(x)$ satisfies the uniform ellipticity assumption **(E)**.

We take $\theta_+ = \frac{\pi}{4}$, $\theta_- = \frac{\pi}{3}$, $\lambda_+^1 = 1$, $\lambda_+^2 = 9$, $\lambda_-^1 = 2$, $\lambda_-^2 = 3$, $\epsilon_+ = 0.5$ and $\epsilon_- = 1.9$. This gives

$$a_+(x) = \frac{1}{2} \begin{pmatrix} 5 + 0.5x_2 & 4 \\ 4 & 5 + 0.5x_2 \end{pmatrix} \quad \text{and} \quad a_-(x) = \frac{1}{2} \begin{pmatrix} \frac{11}{4} + 1.9x_2 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{9}{4} + 1.9x_2 \end{pmatrix}.$$

Performing our Transformed Euler Scheme.

We have the Cholesky decompositions $2a_{\pm}(x) = \sigma_{\pm}\sigma_{\pm}^*(x)$, with

$$\sigma_+(x) = \begin{pmatrix} \sqrt{5 + 0.5x_2} & 0 \\ 4/\sqrt{5 + 0.5x_2} & \sqrt{5 + 0.5x_2 - 16/(5 + 0.5x_2)} \end{pmatrix}$$

and

$$\sigma_-(x) = \begin{pmatrix} \sqrt{\frac{11}{4} + 1.9x_2} & 0 \\ \frac{\sqrt{3}}{4}/\sqrt{\frac{11}{4} + 1.9x_2} & \sqrt{\frac{9}{4} + 1.9x_2 - 3/(44 + 30.4x_2)} \end{pmatrix},$$

so that $2a(x) = \sigma\sigma^*(x)$ with $\sigma(x) = \sigma_+(x)\mathbb{I}_{x \in D_+} + \sigma_-(x)\mathbb{I}_{x \in \bar{D}_-}$. Besides we have

$$\partial a(x) = \begin{pmatrix} 0 \\ 0.25 \end{pmatrix} \mathbb{I}_{x \in D_+} + \begin{pmatrix} 0 \\ 0.95 \end{pmatrix} \mathbb{I}_{x \in \bar{D}_-}.$$

Note that when the scheme crosses the interface Γ , we compute the quantities $\pi_{\Gamma}^{\gamma_{\pm}}(\hat{X}_{t_{k+1}})$ and $F^{\gamma_{\pm}}(\hat{X}_{t_{k+1}})$ in the following way (we will detail the procedure for $\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}})$ and $F^{\gamma_+}(\hat{X}_{t_{k+1}})$). Recall that we have

$$\hat{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}}) = F^{\gamma_+}(\hat{X}_{t_{k+1}})\gamma_+(\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}})).$$

But here $\nu = (0, 1)^*$ so that for any $x \in \Gamma$

$$\gamma_+(x) = \frac{1}{2} \begin{pmatrix} 4 \\ 5 + 0.5x_2 \end{pmatrix}$$

and $(\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}}))_2 = 0$ so that $(\hat{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}}))_2 = (\hat{X}_{t_{k+1}})_2$. This yields

$$F^{\gamma_+}(\hat{X}_{t_{k+1}}) = \frac{(\hat{X}_{t_{k+1}})_2}{2.5},$$

and then

$$\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}}) = \begin{pmatrix} (\hat{X}_{t_{k+1}})_1 - F^{\gamma_+}(\hat{X}_{t_{k+1}}) \times 2 \\ 0 \end{pmatrix}.$$

Then we have everything in hand to perform our Transformed Euler Scheme \bar{X} .

Comparing with an Euler scheme applied on regularised coefficients. A natural method with which to compare our transformed scheme is to regularise first the coefficients and then to perform a standard (i.e. not transformed) Euler scheme. More precisely consider the operator

$$C^2(\mathbb{R}^d; \mathbb{R}) \ni f \mapsto \mathcal{L}^{\varepsilon} f = \nabla \cdot (a^{\varepsilon} \nabla_x f) = \text{Tr}[\mathbf{H}[f]a^{\varepsilon}] + (\partial a^{\varepsilon})^* \nabla_x f$$

where a^{ε} is some smoothed version of a (ε is the regularisation step, see the following discussion about its choice). Then $\mathcal{L}^{\varepsilon}$ is the generator of the solution of the SDE

$$(6.1) \quad dX_t^{\varepsilon} = \sigma^{\varepsilon}(X_t^{\varepsilon}) dW_t + [\partial a^{\varepsilon}](X_t^{\varepsilon}) dt,$$

where $\sigma^{\varepsilon}(\sigma^{\varepsilon})^* = 2a^{\varepsilon}$. The process X^{ε} may be approached by a standard (i.e. not transformed) Euler scheme \bar{X}^{ε} , with time step h_n .

Let h_n be fixed. In fact ε will be chosen in function of h_n . We are first inspired by the random walk approach proposed in [46]. In this later paper Equation (3.11) indicated that ε has to be proportional to the square root of the space discretisation step. Then, using a scaling argument we choose $\varepsilon = h_n^{1/4}$.

Then we set

$$a^{\varepsilon}(x) = a(x)\mathbf{1}_{|x_2| > \varepsilon} + A^{\varepsilon}(x)\mathbf{1}_{|x_2| \leq \varepsilon}$$

where

$$A^{\varepsilon}(x) = \frac{1}{2} \begin{pmatrix} \frac{31}{8} - 0.7\varepsilon + x_2(\frac{9}{8\varepsilon} + 1.2) & \frac{\sqrt{3}}{8} + 2 + x_2(\frac{2}{\varepsilon} - \frac{\sqrt{3}}{8\varepsilon}) \\ \frac{\sqrt{3}}{8} + 2 + x_2(\frac{2}{\varepsilon} - \frac{\sqrt{3}}{8\varepsilon}) & \frac{29}{8} - 0.7\varepsilon + x_2(\frac{11}{8\varepsilon} + 1.2) \end{pmatrix}.$$

Note that the thus defined coefficient a^{ε} is continuous and piecewise differentiable. Then we have $\partial a^{\varepsilon} = \partial a(x)\mathbf{1}_{|x_2| > \varepsilon} + \partial A^{\varepsilon}(x)\mathbf{1}_{|x_2| \leq \varepsilon}$ where

$$\partial A^{\varepsilon}(x) = \begin{pmatrix} \frac{1}{16\varepsilon} - \frac{\sqrt{3}}{16\varepsilon} \\ \frac{\varepsilon}{16\varepsilon} + 0.6 \end{pmatrix},$$

and $2a^\varepsilon(x) = \sigma^\varepsilon[\sigma^\varepsilon]^*(x)$ with $\sigma^\varepsilon(x) = \sigma(x)\mathbf{1}_{|x_2|>\varepsilon} + \Sigma^\varepsilon(x)\mathbf{1}_{|x_2|\leq\varepsilon}$ and

$$\Sigma^\varepsilon(x) = \begin{pmatrix} \sqrt{\frac{31}{8} - 0.7\varepsilon + x_2(\frac{9}{8\varepsilon} + 1.2)} & 0 \\ \frac{\frac{\sqrt{3}}{8} + 2 + x_2(\frac{2}{\varepsilon} - \frac{\sqrt{3}}{8\varepsilon})}{\sqrt{\frac{31}{8} - 0.7\varepsilon + x_2(\frac{9}{8\varepsilon} + 1.2)}} & \sqrt{\frac{29}{8} - 0.7\varepsilon + x_2(\frac{11}{8\varepsilon} + 1.2) - \frac{(\frac{\sqrt{3}}{8} + 2 + x_2(\frac{2}{\varepsilon} - \frac{\sqrt{3}}{8\varepsilon}))^2}{\frac{31}{8} - 0.7\varepsilon + x_2(\frac{9}{8\varepsilon} + 1.2)}} \end{pmatrix}.$$

With these coefficients it is easy to perform a standard Euler Scheme on the SDE (6.1).

We will compare both methods on the two following examples. Benchmarks will be provided by a deterministic approximation of the solutions of the PDE of interest.

Example 1. We wish here to treat the elliptic transmission problem

$$(\mathcal{E}_{\text{T,bounded } D}^0) \left\{ \begin{array}{ll} \mathcal{L}u(x) = 0 & \forall x \in D \\ \langle a_+ \nabla_x u_+(y) - a_- \nabla u_-(y), \nu(y) \rangle = 0 & \forall y \in \Gamma \\ u(y+) = u(y-) & \forall y \in \Gamma \\ u(x) = f(x) & \forall x \in \partial D. \end{array} \right.$$

We take the function f to be

$$f(x) = \sin(3x_1) + \cos(4x_2).$$

Consider then on one side our study of the convergence in the parabolic case, and on the other side the Feynman-Kac representation for elliptic PDEs available in the smooth case (see for instance Theorem 5.7.2 in [16]). One can hope that

$$\mathbb{E}^x[f(\overline{X}_{\overline{\tau}})] \xrightarrow{h_n \rightarrow 0} u(x),$$

where \overline{X} denotes our scheme and $\overline{\tau} = \inf\{t \geq 0 : \overline{X}_t \notin D\}$.

We thus compute a Monte Carlo approximation of $\mathbb{E}^x[f(\overline{X}_{\overline{\tau}})]$ on one side (with $N = 10^6$ paths). Note that in this Monte Carlo procedure we have used a boundary shifting method, on order to reduce the bias introduced by the approximation of the exit time $\tau = \inf\{t \geq 0 : X_t \notin D\}$ by $\overline{\tau}$ (see [14] Subsection 5.4.3, and the references therein).

On the other side $\mathbb{E}^x[f(\overline{X}_{\overline{\tau}^\varepsilon})]$, with $\overline{\tau}^\varepsilon = \inf\{t \geq 0 : \overline{X}_t^\varepsilon \notin D\}$, provides another approximation of $u(x)$ (note that we use again a boundary shifting method).

Benchmarks are provided by the software FREEFEM with which we compute an approximation of $u(x)$ by a finite element method, using around 1.5×10^6 triangles and 7×10^5 vertices (finite elements basis consists of polynomial functions of order 1).

Table 1 shows the results. It seems that our Transformed Euler scheme converges quicker to the benchmark than the standard Euler scheme applied on regularized coefficients.

Example 2. We now turn to some parabolic example (but we keep the same matrix-valued coef-

Point x	Finite Element ($7 \cdot 10^5$ vertices)	Euler Scheme on regularized coefficients ($h_n = 10^{-n}$, $n = 4, 5, 6$)	Transformed Euler Scheme ($h_n = 10^{-n}$, $n = 2, 4, 5, 6$)
$x = (0, 0.5)^*$	-0.1207	-	-0.136356
		-0.115913	-0.121001
		-0.117946	-0.121299
		-0.118792	-0.120821
$x = (0.9, 0.05)^*$	0.92527	-	0.824901
		0.915937	0.924759
		0.922813	0.925370
		0.923853	0.925389
$x = (-0.3, -0.5)^*$	-0.745461	-	-0.737754
		-0.738184	-0.746226
		-0.739099	-0.745676
		-0.742611	-0.745829

TABLE 1

Approximated values of the solution $u(x)$ of $(\mathcal{E}_{T, \text{bounded } D}^0)$ at points $x = (0, 0.5)^*$, $(0.9, 0.05)^*$, $(-0.3, -0.5)^*$ computed with a finite element method ($7 \cdot 10^5$ vertices), a standard Euler scheme applied on a regularisation a^ε of a , and our transformed Euler scheme (with $N = 10^6$ Monte Carlo sample, and different values of h_n).

ficient a):

$$(\mathcal{P}_{T, \text{bounded } D}) \left\{ \begin{array}{ll} \partial_t u(t, x) - \mathcal{L}u(t, x) = 0 & \forall (t, x) \in (0, T] \times D \\ \langle a_+ \nabla_x u_+(t, y) - a_- \nabla_x u_-(t, y), \nu(y) \rangle = 0 & \forall (t, y) \in (0, T] \times \Gamma \quad (\star) \\ u(t, y+) = u(t, y-) & \forall (t, y) \in [0, T] \times \Gamma \\ u(t, x) = 0 & \forall (t, x) \in (0, T] \times \partial D \\ u(0, x) = u_0(x) & \forall x \in \mathbb{R}^d. \end{array} \right.$$

Here we will take $T = 0.1$ and

$$u_0(x) = 10 * (1 - |x|^2).$$

Note that the parabolic problem $(\mathcal{P}_{T, \text{bounded } D})$ is posed in a bounded domain, unlike in our theoretical study. But we have found that convenient for numerical purposes.

Note also that u_0 belongs to $H_0^1(D)$ and is therefore compatible with the uniform Dirichlet boundary condition in $(\mathcal{P}_{T, \text{bounded } D})$. But it does not belong to the domain $\mathcal{D}(A)$, as it does not satisfy the transmission condition (\star) .

Nevertheless one can hope that

$$\mathbb{E}^x [u_0(\overline{X}_t) \mathbf{1}_{t \leq \bar{\tau}}] \xrightarrow{h_n \rightarrow 0} u(t, x)$$

(here we use for example 4.4.5 in [14] and use again the notation $\bar{\tau}$ of Example 1).

Again we compute a Monte Carlo approximation of the expectations $\mathbb{E}^x [u_0(\overline{X}_t) \mathbf{1}_{t \leq \bar{\tau}}]$ on one side and of $\mathbb{E}^x [u_0(\overline{X}_t^\varepsilon) \mathbf{1}_{t \leq \bar{\tau}^\varepsilon}]$ on the other side (with $N = 10^6$ paths and using again the boundary shifting method).

We use FREEFEM to compute an approximation of $u(t, x)$ by a finite element method (discretization in space) and a Crank-Nicholson scheme (discretization in time), using around 9×10^9 triangles and 4.5×10^5 vertices, and 300 time steps.

Point x	Finite Element / Crank-Nicholson (4.5×10^5 vertices, 300 time steps)	Euler Scheme on regularised coefficients ($h_n = 10^{-n}$, $n = 5, 6, 7$)	Transformed Euler Scheme ($h_n = 10^{-n}$, $n = 4, 5, 6, 7$)
$x = (0, 0.5)^*$	2.26288	-	-
		2.26766	2.28299
		2.26332	2.27562
		2.26233	2.26621
$x = (0.9, 0.05)^*$	0.2564	-	-
		-	-
		0.269654	0.263835
		0.263029	0.258807
$x = (-0.3, -0.5)^*$	4.24525	-	-
		-	-
		4.23472	4.23862
		4.23981	4.24483
$x = (0, 0.05)^*$	4.02857	-	-
		4.03452	4.0381
		4.02488	4.03255
		4.02483	4.02936

TABLE 2

Approximated values of the solution $u(T = 0.1, x)$ of $(\mathcal{P}_{T, \text{bounded } D})$ at points $x = (0, 0.5)^$, $(0.9, 0.05)^*$, $(-0.3, -0.5)^*$, $(0, 0.05)^*$ computed with a finite element / Crank-Nicholson scheme method (4.5×10^5 vertices, 300 times steps), a standard Euler scheme applied on a regularisation a^ε of a , and our transformed Euler scheme (with $N = 10^6$ Monte Carlo sample, and different values of h_n).*

Table 2 shows the results, for $t = T$. Again it seems that our tranformed Euler scheme converges quicker to the benchmark, even if for some reason it is less obvious at point $x = (0, 0.5)^*$.

Remark on the not corrected scheme applied to discontinuous coefficients. Note that if we suppress the correction at the interface step in our scheme we get wrong results (see Tables 3 and 4; it seems that this procedure converges to a wrong value).

This is because by doing so we simulate paths of

$$dX_t = \sigma(X_t) dW_t + [\partial a](X_t) dt,$$

whose generator is

$$\text{Tr}[\mathbf{H}[f]a] + (\partial a)^* \nabla_x f = \sum_{i,j=1}^d a_{ij} \partial_{x_i x_j}^2 + \sum_{j=1}^d \left(\sum_{i=1}^d \partial_{x_i} a_{ij} \right) \partial_{x_j}$$

and not $\nabla \cdot (a \nabla)$ (note that these two operators are the same in the case of a smooth coefficient a - at least C^1 ; but for a discontinuous coefficient a this is not the case). In fact not correcting the scheme does not allow to capture the divergence structure of the operator.

Point x	Finite Element (7.10^5 vertices)	Not Transformed Euler Scheme ($h_n = 10^{-n}$, $n = 5, 6$)	Transformed Euler Scheme ($h_n = 10^{-n}$, $n = 2, 4, 5, 6$)
$x = (0.9, 0.05)^*$	0.92527	-	0.824901
		-	0.924759
		0.815699	0.925370
		0.816116	0.925389

TABLE 3

Approximated values of the solution $u(x)$ of $(\mathcal{E}_{\Gamma, \text{bounded } D}^0)$ at point $x = (0.9, 0.2)^*$ computed with a finite element method (7.10^5 vertices), a not-transformed Euler scheme directly applied on the discontinuous coefficients and our transformed Euler scheme (with $N = 10^6$ Monte Carlo sample, and different values of h_n).

Point x	Finite Element / Crank-Nicholson (4.5×10^5 vertices, 300 time steps)	Not Transformed Euler Scheme ($h_n = 10^{-n}$, $n = 5, 6, 7$)	Transformed Euler Scheme ($h_n = 10^{-n}$, $n = 4, 5, 6, 7$)
$x = (0, 0.5)^*$	2.26288	-	-
		2.10057	2.28299
		2.08179	2.27562
		2.08290	2.26621

TABLE 4

Approximated values of the solution $u(T = 0.1, x)$ of $(\mathcal{P}_{\Gamma, \text{bounded } D})$ at point $x = (0, 0.5)^*$ computed with a finite element / Crank-Nicholson scheme method (4.5×10^5 vertices, 300 times steps), a not-transformed Euler scheme directly applied on the discontinuous coefficients and our transformed Euler scheme (with $N = 10^6$ Monte Carlo sample, and different values of h_n).

APPENDIX A: APPENDIX

LEMMA A.1. Let $\Gamma = \mathbb{R}^{d-1} \times \{0\}$ ($d > 1$). We have

$$\forall v \in H^{\frac{1}{2}}(\Gamma), \quad \forall 1 \leq j \leq d-1, \quad D_j v \in H^{-\frac{1}{2}}(\Gamma) \text{ with } \|D_j v\|_{H^{-1/2}(\Gamma)} \leq C \|v\|_{H^{1/2}(\Gamma)},$$

where C is a universal constant.

PROOF. **STEP1.** We prove that for any $v \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $n \in \mathbb{N}^*$, we have $D_j v \in H^{s-1}(\mathbb{R}^n)$, for any $1 \leq j \leq n$, with $\|D_j v\|_{H^{s-1}(\mathbb{R}^n)} \leq C \|v\|_{H^s(\mathbb{R}^n)}$.

We denote $\mathcal{F} : v \mapsto \hat{v}$, $v \in \mathcal{S}(\mathbb{R}^n)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $1 \leq j \leq n$, we have $\mathcal{F}\{(-i2\pi x_j)\varphi(x)\}(\xi) = D_j \hat{\varphi}(\xi)$ (see [33] p72). From this and (1.2) it is an exercise to check that we get

$$\forall v \in \mathcal{S}'(\mathbb{R}^n), \quad \widehat{D_j v}(\xi) = \mathcal{F}\{D_j v\}(\xi) = (i2\pi \xi_j) \hat{v}(\xi).$$

Then we get, for any $v \in H^s(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |(1 + |\xi|^2)^{\frac{s-1}{2}} \widehat{D_j v}(\xi)|^2 d\xi \leq 4\pi^2 \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s-1} |\xi|^2 |\hat{v}(\xi)|^2 d\xi \leq 4\pi^2 \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi < \infty.$$

Step 1 is proved.

STEP2. It suffices to notice that $\|v\|_{H^{1/2}(\Gamma)} = \|v_\zeta\|_{H^{1/2}(\mathbb{R}^{d-1})}$, with $v_\zeta(x') = v(x', 0)$, $x' \in \mathbb{R}^{d-1}$, that $(D_j v)_\zeta = D_j v_\zeta$, and to use Step 1 with $s = \frac{1}{2}$ and $\|D_j v\|_{H^{-1/2}(\Gamma)} = \|(D_j v)_\zeta\|_{H^{-1/2}(\mathbb{R}^{d-1})}$ (cf [33] p98). \square

PROOF OF LEMMA 2.3. To show the result for instance on D_+ , it suffices to notice that for any $\varphi \in C_c^\infty(D_+; \mathbb{R})$ we have

$$\begin{aligned} \langle (D_i v)_+, \varphi \rangle_{H^{-1}(D_+), H_0^1(D_+)} &= \langle D_i v, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = - \int_{\mathbb{R}^d} v D_i \varphi = - \int_{D_+} v_+ D_i \varphi \\ &= \langle D_i v_+, \varphi \rangle_{H^{-1}(D_+), H_0^1(D_+)}. \end{aligned}$$

One may of course proceed in the same fashion on D_- . \square

In order to prove Lemma 2.4 we first need a lemma.

LEMMA A.2. *Let $u \in L^2(\mathbb{R}^d)$ with $u_\pm \in H^1(\mathbb{R}_\pm^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. We have*

$$(A.1) \quad - \int_{\mathbb{R}_\pm^d} u_\pm D_j \varphi = \int_{\mathbb{R}_\pm^d} (D_j u_\pm) \varphi, \quad \forall 1 \leq j \leq d-1,$$

and

$$(A.2) \quad - \int_{\mathbb{R}_\pm^d} u_\pm D_d \varphi = \pm \int_{\Gamma} \gamma(u_\pm) \gamma(\varphi) d\sigma + \int_{\mathbb{R}_\pm^d} (D_d u_\pm) \varphi$$

(we recall that here $\Gamma = \mathbb{R}^{d-1} \times \{0\}$).

PROOF. We establish the formulae on $D_+ = \mathbb{R}_+^d$, as the case \mathbb{R}_-^d can be treated in a similar manner. Let (u_n) a sequence in $C_c^\infty(\mathbb{R}^d; \mathbb{R})$, s.t. $\|u_+ - u_n|_{D_+}\|_{H^1(\mathbb{R}_+^d)} \rightarrow 0$, $n \rightarrow \infty$, where $\|\cdot\|_{H^1(\mathbb{R}_+^d)}$ denotes the usual Sobolev norm. Let $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Let $1 \leq j \leq d-1$. Using integration by parts with respect to the j -th variable for smooth functions vanishing at infinity in the e_j -direction, we have

$$- \int_{\mathbb{R}_+^d} u_+ D_j \varphi = \lim_{n \rightarrow \infty} \left(- \int_{\mathbb{R}_+^d} u_n D_j \varphi \right) = \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}_+^d} (D_j u_n) \varphi \right) = \int_{\mathbb{R}_+^d} (D_j u_+) \varphi,$$

which proves (A.1). Further, we have

$$\begin{aligned} - \int_{\mathbb{R}_+^d} u_\pm D_d \varphi &= \lim_{n \rightarrow \infty} \left(- \int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} u_n D_d \varphi \right) \\ &= \lim_{n \rightarrow \infty} \left(- \int_{\mathbb{R}^{d-1}} \left(\lim_{x_d \rightarrow \infty} (u_n \varphi)(\cdot, x_d) \right. \right. \\ &\quad \left. \left. - \lim_{x_d \rightarrow 0, x_d > 0} (u_n \varphi)(\cdot, x_d) \right) dx_1 \dots dx_{d-1} + \int_{\mathbb{R}_+^d} (D_d u_n) \varphi \right) \\ &= \lim_{n \rightarrow \infty} \left(\int_{\Gamma} \gamma(u_n) \gamma(\varphi) d\varsigma + \int_{\mathbb{R}_+^d} (D_d u_n) \varphi \right) \\ &= \int_{\Gamma} \gamma(u_+) \gamma(\varphi) d\varsigma + \int_{\mathbb{R}_+^d} (D_d u_+) \varphi, \end{aligned}$$

where we have used the continuity of the trace operator at the last line. Equation (A.2) is proved. \square

PROOF OF LEMMA 2.4. We first treat the case $D_{\pm} = \mathbb{R}_{\pm}^d$. Let $u \in L^2(\mathbb{R}^d)$ with $u_{\pm} \in H^1(\mathbb{R}_{\pm}^d)$. We will prove that for any $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ and any $1 \leq j \leq d$, we have

$$(A.3) \quad \langle D_j u, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = \langle D_j u_+, \varphi \rangle_{L^2(\mathbb{R}_+^d)} + \langle D_j u_-, \varphi \rangle_{L^2(\mathbb{R}_-^d)} + \delta_{jd} \int_{\Gamma} [u]_{\Gamma} \gamma(\varphi) d\varsigma$$

(δ_{jd} stands for the Kronecker symbol). This is sufficient to provide the result.

Let then $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Let $1 \leq j \leq d-1$. We have

$$\langle D_j u, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = - \int_{\mathbb{R}^d} u D_j \varphi = - \int_{\mathbb{R}_+^d} u_+ D_j \varphi - \int_{\mathbb{R}_-^d} u_- D_j \varphi.$$

Using then $u_{\pm} \in H^1(\mathbb{R}_{\pm}^d)$ and the fact that φ vanishes at infinity in the e_j -direction, we get, using an integration by parts formula (see Lemma A.2),

$$\langle D_j u, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = \int_{\mathbb{R}_+^d} (D_j u_+) \varphi + \int_{\mathbb{R}_-^d} (D_j u_-) \varphi.$$

Further, we have in the same manner

$$\langle D_d u, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = - \int_{\mathbb{R}^d} u D_d \varphi = - \int_{\mathbb{R}_+^d} u_+ D_d \varphi - \int_{\mathbb{R}_-^d} u_- D_d \varphi,$$

but here integration by parts will provide different results, as φ_{\pm} vanish at $\mathbb{R}^{d-1} \times \{\pm\infty\}$ but not at $\mathbb{R}^{d-1} \times \{0\}$. Indeed we have from Lemma A.2,

$$(A.4) \quad - \int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} u_+ D_d \varphi = \int_{\Gamma} \gamma(u_+) \gamma(\varphi) d\varsigma + \int_{\mathbb{R}_+^d} (D_d u_+) \varphi$$

In the same manner we have

$$(A.5) \quad - \int_{\mathbb{R}_-^d} u_- D_d \varphi = - \int_{\Gamma} \gamma(u_-) \gamma(\varphi) d\varsigma + \int_{\mathbb{R}_-^d} (D_d u_-) \varphi.$$

Summing (A.4) and (A.5) we get (A.3). If Γ is bounded and smooth one may cover it by a finite number of balls, use change of coordinates, cutoff functions and several times the result for $D_{\pm} = \mathbb{R}_{\pm}^d$. We will then get the desired result. \square

In order to prove Propositions 2.15 and 2.16 we need a set of technical results.

For $1 \leq l \leq d$ we introduce the l -th partial difference quotient

$$\Delta_{l,h} u(x) = \frac{u(x + h e_l) - u(x)}{h}, \quad h \in \mathbb{R}.$$

We gather in the following lemma the results we will need about difference quotients.

LEMMA A.3. *Let $u, v \in L^2(\mathbb{R}^d)$ and $1 \leq l \leq d$.*

i) If $D_l u \in L^2(\mathbb{R}^d)$ then $\|\Delta_{l,h} u\|_{L^2(\mathbb{R}^d)} \leq \|D_l u\|_{L^2(\mathbb{R}^d)}$ for all $h \in \mathbb{R}$.

ii) If there is a constant M such that $\|\Delta_{l,h} u\|_{L^2(\mathbb{R}^d)} \leq M$ for all h sufficiently small, then $D_l u$ is in $L^2(\mathbb{R}^d)$, with $\|D_l u\|_{L^2(\mathbb{R}^d)} \leq M$.

iii) For all $h \in \mathbb{R}$ we have $\Delta_{l,h}(uv) = \Delta_{l,h} u v(\cdot + h e_l) + u \Delta_{l,h} v$.

iv) For all $h \in \mathbb{R}$ we have $\int_{\mathbb{R}^d} u \Delta_{l,h} v = - \int_{\mathbb{R}^d} v \Delta_{l,-h} u$.

PROOF. Point iii) follows from elementary computations and Point iv) from a change of variable in the integral $\int_{\mathbb{R}^d} u \Delta_{l,h} v$.

For Points i) and ii) see the proof of Lemma 4.13 in [33]. \square

The following lemma allows to weaken the assumptions on the coefficients in Proposition 2.6, under the condition that the function under investigation is smooth. It allows to prove the forthcoming Lemma A.5.

LEMMA A.4. *Let G_2 an open bounded subset of \mathbb{R}^d with $G_2 \cap \Gamma \neq \emptyset$. Let $\chi \in C_c^\infty(G_2; \mathbb{R})$. Let some coefficients a_{ij}^u satisfy $(a_{\pm}^u)_{ij} \in H^1(D_{\pm}^2)$ (where $D_{\pm}^2 = G_2 \cap D_{\pm}$; note that $\chi_{\pm} \in H^2(D_{\pm}^2)$). Then, using Einstein's convention for summation over repeated indexes,*

$$\int_{D_+} (a_+^u)_{ij} D_j \chi_+ D_i v = - \int_{D_+} (D_i((a_+^u)_{ij} D_j \chi_+) v - (\nu_i \gamma[(a_+^u)_{ij} D_j \chi_+], \gamma(v))_{\Gamma})_{\Gamma}, \quad \forall v \in H^1(D_+)$$

and

$$\int_{D_-} (a_-^u)_{ij} D_j \chi_- D_i v = - \int_{D_-} (D_i((a_-^u)_{ij} D_j \chi_-) v + (\nu_i \gamma[(a_-^u)_{ij} D_j \chi_-], \gamma(v))_{\Gamma})_{\Gamma}, \quad \forall v \in H^1(D_-).$$

PROOF. We prove the result on D_+^2 , w.l.g. For each $1 \leq i, j \leq d$ one may construct a sequence $((a_+^{u,n})_{ij})_n$ in $C_c^\infty(\mathbb{R}^d; \mathbb{R})$ such that $\|(a_+^{u,n})_{ij} D_j^2 - (a^u)_{ij} D_j^2\|_{H^1(D_+^2)} \rightarrow 0$ as $n \rightarrow \infty$. Then the $(a_+^{u,n})_{ij}$'s are obviously Lipschitz in D_+^2 and from Proposition 2.6 we have

$$\int_{D_+} (a_+^{u,n})_{ij} D_j \chi_+ D_i v = - \int_{D_+} (D_i((a_+^{u,n})_{ij} D_j \chi_+) v - (\nu_i \gamma[(a_+^{u,n})_{ij} D_j \chi_+], \gamma(v))_{\Gamma})_{\Gamma}, \quad \forall v \in H^1(D_+).$$

We will now pass to the limit in the above equality when $n \rightarrow \infty$. We drop the subscript $+$ on functions in order to lighten notations. To start with, as the $D_j \chi$ are bounded, it is obvious that the $(a_+^{u,n})_{ij} D_j \chi$ converge to $(a^u)_{ij} D_j \chi$ in $H^1(D_+^2)$. So that it is immediate that $\int_{D_+} (a_+^{u,n})_{ij} D_j \chi_+ D_i v$ converges to $\int_{D_+} (a^u)_{ij} D_j \chi_+ D_i v$. Further, using the continuity of the trace operator $\gamma : H^1(D_+) \rightarrow H^{1/2}(\Gamma)$ it is clear that $\gamma[(a_+^{u,n})_{ij} D_j \chi]$ converges to $\gamma[(a^u)_{ij} D_j \chi]$ in $H^{1/2}(\Gamma)$. Therefore by Cauchy-Schwarz, $(\nu_i \gamma[(a_+^{u,n})_{ij} D_j \chi_+], \gamma(v))_{\Gamma}$ converges to $(\nu_i \gamma[(a^u)_{ij} D_j \chi_+], \gamma(v))_{\Gamma}$. To treat the third and last term it suffices to develop $(D_i((a_+^{u,n})_{ij} D_j \chi))$ into

$$D_i((a_+^{u,n})_{ij} D_j \chi) + (a_+^{u,n})_{ij} D_i D_j \chi.$$

Using again the boundedness of the $D_i \chi$'s and $D_i D_j \chi$'s, this is easily seen to be converging in $L^2(D_+^2)$ to $D_i((a^u)_{ij} D_j \chi) + (a^u)_{ij} D_i D_j \chi = D_i((a^u)_{ij} D_j \chi)$ (note that the $(a^u)_{ij} D_j \chi$ are in $H^1(D_+^2)$). The proof is completed. \square

LEMMA A.5. *Assume we are under the assumptions of Proposition 2.15. Let a cutoff function $\chi \in C_c^\infty(G_2; \mathbb{R})$, $0 \leq \chi \leq 1$, with $\chi \equiv 1$ on G_1 (the derivatives of χ are bounded by some constant depending on $\text{dist}(G_1, \partial \text{supp}(\chi))$).*

Then the assumption $[\mathcal{B}_\nu u]_{\Gamma} \in H^{\frac{1}{2}}(\Gamma_2)$ implies that $[\mathcal{B}_\nu(\chi u)]_{\Gamma} \in H^{\frac{1}{2}}(\Gamma_2)$.

PROOF. We aim at proving that

$$(A.6) \quad [\mathcal{B}_\nu(\chi u)]_{\Gamma} = \chi [\mathcal{B}_\nu u]_{\Gamma} + \sum_{j=1}^d \{ \gamma[(D_j \chi)_+ (a_+)_{dj} u_+] - \gamma[(D_j \chi)_- (a_-)_{dj} u_-] \} \quad \text{in } H^{-1/2}(\Gamma).$$

As $0 \leq \chi \leq 1$ with bounded derivatives, the $(a_{\pm})_{ij}$'s are of class $C^1(\overline{D_{\pm}^2}; \mathbb{R})$, and we have $[\mathcal{B}_{\nu}u]_{\Gamma} \in H^{\frac{1}{2}}(\Gamma_2)$ and $u_{\pm} \in H^1(D_{\pm})$, we will get the desired result. In fact (A.6) will follow simply from

$$(A.7) \quad \mathcal{B}_{\nu}^{\pm}u = \chi \mathcal{B}_{\nu}^{\pm}u + \sum_{j=1}^d \gamma[(D_j \chi)_{\pm}(a_{\pm})_{dj}u_{\pm}] \quad \text{in } H^{-1/2}(\Gamma)$$

(note that in (A.7) for example $\chi \mathcal{B}_{\nu}^+u$ is the element of $H^{-1/2}(\Gamma)$ defined by

$$\langle \chi \mathcal{B}_{\nu}^+u, \phi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \langle \mathcal{B}_{\nu}^+u, \gamma(\chi)\phi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$$

for any $\phi \in H^{1/2}(\Gamma)$).

Thus we aim at proving (A.7) on $D_+ = \mathbb{R}_+^d$ (D_- is treated in the same manner). Let $v \in C_c^{\infty}(\mathbb{R}^d; \mathbb{R})$ and consider its restriction v_+ on D_+ . From now on we drop the subscript $+$ on functions and conormal derivatives. Using Proposition 2.9 we have

$$(\mathcal{B}_{\nu}(\chi u), \gamma(v))_{\Gamma} = -\mathcal{E}_+(\chi u, v) - \int_{D_+} A_+(\chi u) v$$

(note that we know that $A_+(\chi u) \in L^2(D_+)$ thanks to (A.9), $\bar{f} \in L^2(D_+)$ and $u \in H^1(\mathbb{R}^d)$). But, using Einstein's convention for summation over repeated indexes, $D_j(\chi u) = (D_j \chi)u + \chi D_j u$ and $\chi D_i v = D_i(\chi v) - (D_i \chi)v$, we get

$$(\mathcal{B}_{\nu}(\chi u), \gamma(v))_{\Gamma} = -\mathcal{E}_+(u, \chi v) - \int_{D_+} a_{ij}u D_j \chi D_i v + \int_{D_+} a_{ij}D_j u D_i \chi v - \int_{D_+} A_+(\chi u) v.$$

Using again Proposition 2.9 one gets

$$\begin{aligned} & (\mathcal{B}_{\nu}(\chi u), \gamma(v))_{\Gamma} = \\ & (\mathcal{B}_{\nu}u, \gamma(\chi)\gamma(v))_{\Gamma} + \int_{D_+} (A_+u)\chi v - \int_{D_+} a_{ij}u D_j \chi D_i v + \int_{D_+} a_{ij}D_j u D_i \chi v - \int_{D_+} A_+(\chi u) v. \end{aligned}$$

Then using surjection and density arguments we will get (A.7) if we prove that

$$(A.8) \quad \int_{D_+} (A_+u)\chi v - \int_{D_+} a_{ij}u D_j \chi D_i v + \int_{D_+} a_{ij}D_j u D_i \chi v - \int_{D_+} A_+(\chi u) v = \left(\gamma[(D_j \chi)a_{dj}u], \gamma(v) \right)_{\Gamma}.$$

But using now $\nu = (0, \dots, 0, 1)^*$, $\chi \in C_c^{\infty}(G_2; \mathbb{R})$, $a_{ij}u \in H^1(D_+^2)$, and Lemma A.4 one gets

$$- \int_{D_+} a_{ij}u D_j \chi D_i v - \int_{D_+} (A_+^u \chi) v = \left(\gamma[(D_j \chi)a_{dj}u], \gamma(v) \right)_{\Gamma}$$

where we have denoted $A_+^u \chi = D_i(a_{ij}u D_j \chi) \in L^2(D_+)$. We now claim that, as an element of $\tilde{H}^{-1}(D_+)$, the form $H^1(D_+) \ni \varphi \mapsto \int_{D_+} (A_+^u \chi) \varphi$ coincides on D_+ with

$$H^1(D_+) \ni \varphi \mapsto - \int_{D_+} (A_+u)\chi \varphi + \int_{D_+} A_+(\chi u) \varphi - \int_{D_+} a_{ij}D_j u D_i \chi \varphi,$$

which will provide (A.8). Let then $\varphi \in C_c^\infty(D_+; \mathbb{R})$ (see Remark 2.8). We have

$$\begin{aligned}
\int_{D_+} (A_+^u \chi) \varphi &= - \int_{D_+} a_{ij} u D_j \chi D_i \varphi \\
&= - \int_{D_+} a_{ij} D_j (\chi u) D_i \varphi + \int_{D_+} a_{ij} \chi D_j u D_i \varphi \\
&= - \int_{D_+} a_{ij} D_j (\chi u) D_i \varphi + \int_{D_+} a_{ij} D_j u D_i (\chi \varphi) - \int_{D_+} a_{ij} D_j u (D_i \chi) \varphi \\
&= \int_{D_+} A_+ (\chi u) \varphi - \int_{D_+} (A_+ u) \chi \varphi - \int_{D_+} a_{ij} D_j u (D_i \chi) \varphi,
\end{aligned}$$

where we have used successively $u D_j \chi = D_j (\chi u) - \chi D_j (u)$ and $\chi D_i \varphi = D_i (\chi \varphi) - (D_i \chi) \varphi$. The proof is completed. \square

PROOF OF PROPOSITION 2.15. We only treat the case $D_\pm = \mathbb{R}_\pm^d$. Indeed if this is not the case one may use change of coordinates of class C^{r+2} to change the equation in a new one, on new domains $D_\pm^{2'}$, with new coefficients of class $C^{r+1}(\overline{D_\pm^{2'}}; \mathbb{R})$, and $\Gamma'_2 = \overline{D_+^{2'}} \cap \overline{D_-^{2'}}$ a portion of the hyperplan $x_d = 0$ (see Exercise 4.2 in [33]). And in that case the value of the functions outside $G'_2 = D_+^{2'} \cup \Gamma \cup D_-^{2'}$ is of no importance, as in the sequel we will multiply everything by a cutoff function with compact support in G'_2 . For this proof, we will only consider $D_\pm = \mathbb{R}_\pm^d$.

The proof proceeds by induction on $r \in \mathbb{N}$.

STEP1. We establish the result for $r = 0$. Note that by convention $H^0(D_\pm^j) = L^2(D_\pm^j)$, $j = 1, 2$.

We fix a cutoff function $\chi \in C_c^\infty(G_2; \mathbb{R})$, $0 \leq \chi \leq 1$, with $\chi \equiv 1$ on G_1 (the derivatives of χ are bounded by some constant depending on $\text{dist}(G_1, \partial \text{supp}(\chi))$).

SUBSTEP a) We first prove that $D_l D_i (\chi u) \in L^2(\mathbb{R}^d)$ for any $1 \leq l \leq d-1$, $1 \leq i \leq d$. As $u \in H^1(G_2)$ and $\chi \in C_c^\infty(G_2; \mathbb{R})$ and we have already $D_i (\chi u) \in L^2(\mathbb{R}^d)$ for any $1 \leq i \leq d$. Thus, using Lemma A.3-ii), we are done if we find M s.t. $\|\Delta_{l,h} (D_i \chi u)\|_{L^2(\mathbb{R}^d)} \leq M$ for all h sufficiently small, and all $1 \leq l \leq d-1$, $1 \leq i \leq d$.

As $u_+ - A_+ u_+ = f_+$ on D_+^2 we have for any φ in $C_c^\infty(D_+^2; \mathbb{R})$

$$\langle u_+ - A_+ u_+, \chi_+ \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} = \langle f_+, \chi_+ \varphi \rangle_{L^2(D_+^2)}.$$

Thus, for any φ in $C_c^\infty(D_+^2; \mathbb{R})$ we have

$$\langle (\chi u)_+ - A_+ (\chi u)_+, \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} = \langle \bar{f}_+, \varphi \rangle_{L^2(D_+^2)},$$

with

$$\bar{f}_+ = (\chi f)_+ + \sum_{i,j=1}^d (a_+)_{ij} D_i \chi_+ D_j u_+ + D_i [(a_+)_{ij} (D_j \chi_+) u_+] \in L^2(D_+^2)$$

(here we have used successively (2.6), with D_+^2 replacing D_+ , and the facts that for any $1 \leq i, j \leq d$, $D_i (\chi_+ \varphi) = \varphi D_i \chi_+ + \chi_+ D_i \varphi$, $\chi_+ D_j u_+ = D_j (\chi u)_+ - (D_j \chi_+) u_+$ and

$$\int_{D_+^2} (a_+)_{ij} (D_j \chi_+) u_+ D_i \varphi = - \int_{D_+^2} D_i [(a_+)_{ij} (D_j \chi_+) u_+] \varphi ;$$

note that $(a_+)_{ij} (D_j \chi_+) u_+$ is in $H^1(D_+^2)$ as $u_+ \in H^1(D_+)$, χ_+ is smooth and $(a_+)_{ij}$ is of class $C^1(\overline{D_+^2}; \mathbb{R})$.

One may proceed in the same manner on D_-^2 . But as $(\chi u)_\pm$ and \bar{f}_\pm have support in D_\pm^2 we have in fact that

$$(A.9) \quad (\chi u)_\pm - A_\pm(\chi u)_\pm = \bar{f}_\pm \quad \text{on } D_\pm.$$

We set $\bar{f} = \bar{f}_+ + \bar{f}_-$. According to Lemma 2.10 we have

$$(A.10) \quad \langle \chi u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(\chi u, v) = \langle \bar{f}, v \rangle_{L^2(\mathbb{R}^d)} - \left([\mathcal{B}_\nu(\chi u)]_\Gamma, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(\mathbb{R}^d).$$

Let $1 \leq l \leq d-1$ and $h \in \mathbb{R}$. We take $v = -\Delta_{l,-h}(\Delta_{l,h}(\chi u))$ in (A.10). Using successively Points iv) and iii) in Lemma A.3, together with linearity arguments, we get

$$(A.11) \quad \begin{aligned} & \left([\mathcal{B}_\nu(\chi u)]_\Gamma, \gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u))) \right)_\Gamma - \int_{\mathbb{R}^d} \bar{f} \Delta_{l,-h}(\Delta_{l,h}(\chi u)) \\ &= \int_{\mathbb{R}^d} |\Delta_{l,h}(\chi u)|^2 + \sum_{i,j=1}^d \int_{\mathbb{R}^d} \Delta_{l,h}(a_{ij} D_j(\chi u)) \Delta_{l,h}(D_i(\chi u)) \\ &= \int_{\mathbb{R}^d} |\Delta_{l,h}(\chi u)|^2 + \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j(\Delta_{l,h}(\chi u)) D_i(\Delta_{l,h}(\chi u)) \\ & \quad + \sum_{i,j=1}^d \int_{\mathbb{R}^d} (\Delta_{l,h} a_{ij}) D_i(\chi u) (\cdot + h e_l) \Delta_{l,h}(D_i(\chi u)) \end{aligned}$$

But thanks to (E) we have

$$\begin{aligned} & \|\Delta_{l,h}(\chi u)\|_{L^2(\mathbb{R}^d)}^2 + \lambda \sum_{i=1}^d \|D_i(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 \\ & \leq \int_{\mathbb{R}^d} |\Delta_{l,h}(\chi u)|^2 + \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j(\Delta_{l,h}(\chi u)) D_i(\Delta_{l,h}(\chi u)), \end{aligned}$$

which combined with (A.11) leads to

$$\begin{aligned} c \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 & \leq \int_{\mathbb{R}^d} |\bar{f}| \cdot |\Delta_{l,-h}(\Delta_{l,h}(\chi u))| \\ & \quad + \sum_{i,j=1}^d |\Delta_{l,h} a_{ij}| \cdot |D_j(\chi u)(\cdot + h e_l)| \cdot |\Delta_{l,h}(D_i(\chi u))| \\ & \quad + \left| \left([\mathcal{B}_\nu(\chi u)]_\Gamma, \gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u))) \right)_\Gamma \right|. \end{aligned}$$

We first focus on the third RHS term. We have

$$\left| \left([\mathcal{B}_\nu(\chi u)]_\Gamma, \gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u))) \right)_\Gamma \right| \leq \| [\mathcal{B}_\nu(\chi u)]_\Gamma \|_{H^{1/2}(\Gamma_2)} \|\gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u)))\|_{H^{-1/2}(\Gamma_2)}.$$

Indeed remember that by assumption $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}}(\Gamma_2)$, which implies

$$\| [\mathcal{B}_\nu(\chi u)]_\Gamma \|_{H^{1/2}(\Gamma_2)} =: \tilde{K} < \infty$$

thanks to Lemma A.5. Besides, one can see that

$$\begin{aligned} \|\gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u)))\|_{H^{-1/2}(\Gamma_2)} &= \|\Delta_{l,-h} \gamma(\Delta_{l,h}(\chi u))\|_{H^{-1/2}(\Gamma)} \leq \|D_l \gamma(\Delta_{l,h}(\chi u))\|_{H^{-1/2}(\Gamma)} \\ &\leq k \|\gamma(\Delta_{l,h}(\chi u))\|_{H^{1/2}(\Gamma)} \leq k \|\Delta_{l,h}(\chi u)\|_{H^1(D_\pm)} \leq k \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)} \end{aligned}$$

(here we have used for the first inequality, a version of Exercise 4.4 in [33], adapted to \mathbb{R}_\pm^d and tangential derivatives; for the second inequality we have used Lemma A.1 in the Appendix). In the sequel we set $K = k\tilde{K}$.

Note now that as $a_{ij} \in C^1(\bar{D}_+^2; \mathbb{R}) \cap C^1(\bar{D}_-^2; \mathbb{R})$, for all $1 \leq i, j \leq d$ we have that $|\Delta_{l,h} a_{ij}| \leq C$ for some constant C not depending on h . Using now Young's inequality and Point i) of Lemma A.3 we get

$$\begin{aligned} c \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 &\leq \frac{1}{2\varepsilon} \|\bar{f}\|_{L^2(\mathbb{R}^d)}^2 + \frac{\varepsilon}{2} \|\nabla(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 + \frac{Cd^2\delta}{2} \|\nabla(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 \\ &\quad + \frac{Cd^2}{2\delta} \|\nabla(\chi u)\|_{L^2(\mathbb{R}^d)}^2 + K^2 \frac{\eta}{2} + \frac{1}{2\eta} \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 \end{aligned}$$

for any $\varepsilon, \delta, \eta > 0$, and thus

$$\begin{aligned} c \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 &\leq \frac{1}{2\varepsilon} \|\bar{f}\|_{L^2(\mathbb{R}^d)}^2 + \frac{\varepsilon}{2} \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 + \frac{Cd^2}{2\delta} \|\nabla(\chi u)\|_{L^2(\mathbb{R}^d)}^2 \\ &\quad + \frac{Cd^2\delta}{2} \|\nabla(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 + K^2 \frac{\eta}{2} + \frac{1}{2\eta} \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2. \end{aligned}$$

Adjusting now ε, δ and η we get constants $c', C' > 0$, not depending on h s.t.

$$c' \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 \leq C' (\|\bar{f}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla(\chi u)\|_{L^2(\mathbb{R}^d)}^2 + K^2)$$

and thus, considering $M = C' (\|\bar{f}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla(\chi u)\|_{L^2(\mathbb{R}^d)}^2 + K^2) / c'$ we have

$$\forall 1 \leq i \leq d, \quad \|D_i(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 = \|\Delta_{l,h}(D_i(\chi u))\|_{L^2(\mathbb{R}^d)}^2 \leq M.$$

As M does not depend on h we have indeed proved that $D_l D_i(\chi u) \in L^2(\mathbb{R}^d)$ for any $1 \leq l \leq d-1$, $1 \leq i \leq d$. Remembering that $\chi \equiv 1$ on G_1 this proves $D_l D_i u \in L^2(G_1)$ for any $1 \leq l \leq d-1$, $1 \leq i \leq d$.

SUBSTEP b) We consider the restriction $(\chi u)_+$ of (χu) to D_+ . We already have $D_d(\chi u)_+ \in L^2(D_+)$, as $(\chi u) \in H^1(\mathbb{R}^d)$ (and using again Lemma 2.3). We will show that $D_{dd}^2(\chi u)_+ \in L^2(D_+^2)$. One may prove in the same manner that $D_{dd}^2(\chi u)_- \in L^2(D_-^2)$.

Using $(\chi u)_+ - A_+(\chi u)_+ = \bar{f}_+$ and (2.6) we have for any $\varphi \in C_c^\infty(D_+; \mathbb{R})$

$$\int_{D_+} (a_+)_{dd} D_d(\chi u)_+ D_d \varphi = \int_{D_+} (\bar{f}_+ - (\chi u)_+) \varphi - \sum_{(i,j) \neq (d,d)} \int_{D_+} (a_+)_{ij} D_j(\chi u)_+ D_i \varphi.$$

From now on we drop the subscript $+$ on functions, in order to lighten the notations. Reinterpreting immediately the preceding equation we have for any $\varphi \in C_c^\infty(D_+^2; \mathbb{R})$

$$(A.12) \quad -\langle D_d(a_{dd} D_d(\chi u)), \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} = \int_{D_+^2} (\bar{f} - (\chi u)) \varphi + \sum_{(i,j) \neq (d,d)} \int_{D_+^2} D_i(a_{ij} D_j(\chi u)) \varphi.$$

Note that in the above expression we have used the fact that for any $(i, j) \neq (d, d)$, the weak derivative $D_i(a_{ij} D_j(\chi u))$ is in $L^2(D_+^2)$. Indeed remember that $D_i D_j(\chi u) \in L^2(D_+^2)$ thanks to SUBSTEP a) and that a_{ij} is in $C^1(\bar{D}_+^2; \mathbb{R})$.

We now use the fact that for any distribution $v' \in H^{-1}(D_+^2)$ and any $w \in C^1(\overline{D_+^2}; \mathbb{R})$ the distribution wv' is simply defined by $\langle wv', \varphi \rangle = \langle v', w\varphi \rangle$ for any $\varphi \in C_c^\infty(D_+^2; \mathbb{R})$ (and if $v' \in L^2(D_+^2)$ and $w \in C(\overline{D_+^2}; \mathbb{R})$ the distribution wv' is simply defined by $\langle wv', \varphi \rangle = \int_{D_+} wv'\varphi$ for any $\varphi \in C_c^\infty(D_+^2; \mathbb{R})$). Thus if $w \in C^1(\overline{D_+^2}; \mathbb{R})$ and $v' \in L^2(D_+^2)$ we have $D_d(wv') = (D_d w)v' + w D_d v'$ and using this in (A.12) we get

$$\begin{aligned} & - \langle a_{dd} D_{dd}^2(\chi u), \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} \\ &= \int_{D_+^2} (\bar{f} - (\chi u))\varphi + \int_{D_+^2} (D_d a_{dd}) D_d(\chi u) \varphi + \sum_{(i,j) \neq (d,d)} \int_{D_+} D_i(a_{ij} D_j(\chi u)) \varphi, \end{aligned}$$

that is to say we have finally for any $\varphi \in C_c^\infty(D_+^2; \mathbb{R})$

$$(A.13) \quad - \langle D_{dd}^2(\chi u), \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} = \int_{D_+^2} \frac{1}{a_{dd}} \left\{ (\bar{f} - (\chi u)) + (D_d a_{dd}) D_d(\chi u) + \sum_{(i,j) \neq (d,d)} D_i(a_{ij} D_j(\chi u)) \right\} \varphi.$$

Using now Remark 1.4 and the fact that $a_{dd} \in C^1(\overline{D_+^2}; \mathbb{R})$, one may conclude that indeed $D_{dd}(\chi u)$ is in $L^2(D_+^2)$ (note that we have already stressed that for any $(i, j) \neq (d, d)$, the weak derivative $D_i(a_{ij} D_j(\chi u))$ is in $L^2(D_+^2)$). This implies $D_{dd} u_+ \in L^2(D_+^1)$ (we use the subscript again). Taking in account SUBSTEP a) we have indeed proved that $u_\pm \in H^2(D_\pm^1)$.

STEP2. Take $r \in \mathbb{N}^*$. Assuming the result is true at $r - 1$ we prove its validity at rank r .

Let $f_\pm \in L^2(D_\pm)$ with f_\pm in $H^r(D_\pm^2)$ and assume that $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}+r}(\Gamma_2)$ and the coefficients $(a_\pm)_{ij}$ belong to $C^{r+1}(\overline{D_\pm^2}; \mathbb{R})$. Thus, in particular, f_\pm are in $H^{r-1}(D_\pm^2)$ and $[\mathcal{B}_\nu u]_\Gamma \in H^{r-\frac{1}{2}}(\Gamma_2)$. Thus, by induction hypothesis, we have $u_\pm \in H^{r+1}(D_\pm^1)$, writing $D_\pm^1 = G_{1'} \cap D_\pm$ for any open connected subset $G_{1'}$ with $\overline{G_1} \subset G_{1'} \subset \overline{G_{1'}} \subset G_2$.

SUBSTEP a) Let $1 \leq i_0 \leq d - 1$. To lighten notation we denote $D := D_{i_0}$. We wish to show that $Du_\pm \in H^{1+r}(D_\pm^1)$. Note that $Du_\pm = (Du)_\pm$ and that we already know by STEP1-a) that $D_i D u$ is in $L^2(G_{1'})$ for any $1 \leq i \leq d$, so that Du is in $H^1(G_{1'})$. So the idea is now to use the induction hypothesis on Du .

Remember that $u_+ - A_+ u_+ = f_+$ on D_+^2 . Let $\varphi \in C_c^\infty(D_+^1; \mathbb{R})$, note that $D\varphi \in C_c^\infty(D_+^1; \mathbb{R})$, and use $D\varphi$ as a test function in (2.6). This writes (again we drop the subscript $+$ on functions)

$$\int_{D_+^1} u D\varphi + \sum_{i,j=1}^d \int_{D_+^1} a_{ij} D_j u D_i(D\varphi) = \int_{D_+^1} f D\varphi,$$

which we immediately rewrite in

$$\begin{aligned}
-\int_{D_+^{1'}} Df \varphi &= -\int_{D_+^{1'}} Du \varphi + \sum_{i,j=1}^d \int_{D_+^{1'}} a_{ij} D_j u D(D_i \varphi) \\
&= -\int_{D_+^{1'}} Du \varphi - \sum_{i,j=1}^d \langle D(a_{ij} D_j u), D_i \varphi \rangle_{H^{-1}(D_+^{1'}), H_0^1(D_+^{1'})} \\
&= -\int_{D_+^{1'}} Du \varphi - \sum_{i,j=1}^d \int_{D_+^{1'}} a_{ij} D(D_j u) D_i \varphi - \sum_{i,j=1}^d \int_{D_+^{1'}} D(a_{ij}) D_j u D_i \varphi \\
&= -\int_{D_+^{1'}} Du \varphi - \sum_{i,j=1}^d \int_{D_+^{1'}} a_{ij} D_j (Du) D_i \varphi + \sum_{i,j=1}^d \int_{D_+^{1'}} D_i (D(a_{ij}) D_j u) \varphi.
\end{aligned}$$

Here we have first used the facts that $u, f \in H^1(D_+^{1'})$ and $\varphi \in C_c^\infty(D_+^{1'}; \mathbb{R})$. Second we have used $DD_j u \in L^2(D_+^{1'})$ and $a_{ij} \in C^1(\overline{D_+^2}; \mathbb{R})$, $1 \leq i, j \leq d$, so that one can easily check that the distribution $D(a_{ij} D_j u)$ is equal to $a_{ij} D_j (Du) + D(a_{ij}) D_j u \in L^2(D_+^{1'})$. Third we have used the fact that $D_i (D(a_{ij}) D_j u) \in L^2(D_+^{1'})$, thanks to the smoothness of $(a_+)_{ij}$ and u_+ .

As φ was arbitrarily chosen in $C_c^\infty(D_+^{1'}; \mathbb{R})$ we have in fact proved that

$$\forall \varphi \in C_c^\infty(D_+^{1'}; \mathbb{R}), \quad \int_{D_+^{1'}} Du \varphi + \mathcal{E}_+(Du, \varphi) = \int_{D_+^{1'}} \left(Df + \sum_{i,j=1}^d D_i (D(a_{ij}) D_j u) \right) \varphi.$$

This means that (we use the subscript $+$ again)

$$(Du)_+ - A_+(Du)_+ = \tilde{f}_+ \quad \text{on } D_+^{1'}$$

with

$$\tilde{f}_+ = Df_+ + \sum_{i,j=1}^d D_i (D((a_+)_{ij}) D_j u_+).$$

Notice now that thanks to $f_+ \in H^r(D_+^2)$, $u_+ \in H^{r+1}(D_+^{1'})$ and the smoothness of a we can claim that $\tilde{f}_+ \in H^{r-1}(D_+^{1'})$. Proceeding in the same way on the domain D_- we prove that

$$(Du)_- - A_-(Du)_- = \tilde{f}_- \quad \text{on } D_-^{1'}$$

with $\tilde{f}_- \in H^{r-1}(D_-)$. In order to use the induction hypothesis it remains now to check that

$$(A.14) \quad [\mathcal{B}_\nu(Du)]_\Gamma \in H^{r-\frac{1}{2}}(\Gamma_{1'})$$

(here we have denoted $\Gamma_{1'} = \Gamma \cap G_{1'}$). Using the technical Lemma A.6 (it is stated just after this proof; note that $u_\pm \in H^2(D_\pm^{1'})$, as $u_\pm \in H^{r+1}(D_\pm^{1'})$ with $r > 0$) we get,

$$\begin{aligned}
[\mathcal{B}_\nu(Du)]_\Gamma &= \mathcal{B}_\nu^+(Du) - \mathcal{B}_\nu^-(Du) \in H^{-1/2}(\Gamma_{1'}) \\
&= D\mathcal{B}_\nu^+(u) - \sum_{j=1}^d \gamma [D((a_+)_{dj}) D_j u_+] - D\mathcal{B}_\nu^-(u) + \sum_{j=1}^d \gamma [D((a_-)_{dj}) D_j u_-] \\
&= D[\mathcal{B}_\nu u]_\Gamma - \sum_{j=1}^d \gamma [D((a_+)_{dj}) D_j u_+ - D((a_-)_{dj}) D_j u_-].
\end{aligned}$$

Remember now that by assumption $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}+r}(\Gamma_2)$, and, again, that $u_\pm \in H^{r+1}(D_\pm^1)$, so that $D_j u_\pm \in H^r(D_\pm^1)$ for any $1 \leq j \leq d$. Using in addition the smoothness of a_\pm it is clear that we have (A.14).

Using now the induction hypothesis we conclude that $(Du)_\pm$ are in $H^{r+1}(D_\pm^1)$. Thus we have proved that

$$\forall 1 \leq i \leq d-1, \quad D_i u_+ \in H^{1+r}(D_+^1) \quad \text{and} \quad D_i u_- \in H^{1+r}(D_-^1).$$

SUBSTEP b) To conclude that u_\pm are in $H^{2+r}(D_\pm^1)$ it remains to prove that $D_d u_\pm$ are in $H^{1+r}(D_\pm^1)$. Let us consider $D_d u_+$. As we have already, thanks to SUBSTEP a), that $D_i D_d u_+ = D_d D_i u_+$ is in $H^r(D_+^1)$ for any $1 \leq i \leq d-1$, it remains to show that $D_{dd}^2 u_+$ is in $H^r(D_+^1)$. But Equation (A.13) shows that

$$D_{dd}^2 u_+ = -\frac{1}{(a_+)_{dd}} \left\{ (f_+ - u_+) + (D_d(a_+)_{dd}) D_d u_+ + \sum_{(i,j) \neq (d,d)} D_i((a_+)_{ij}) D_j u_+ \right\} \quad \text{on } D_+^1.$$

But, using $f_+, u_+ \in H^{1+r}(D_+^1)$ and SUBSTEP a), the RHS term is easily seen to be in $H^r(D_+^1)$. One can proceed in the same way on D_-^1 , and thus the proof by induction is completed. \square

LEMMA A.6. *In the context of the proof of Proposition 2.15, STEP2-a), let $u \in L^2(\mathbb{R}^d)$ with $u_\pm \in H^2(D_\pm^1)$. Let $1 \leq i_0 \leq d-1$. Then*

$$\mathcal{B}_\nu^\pm(D_{i_0} u) = D_{i_0} \mathcal{B}_\nu^\pm(u) - \sum_{j=1}^d \gamma[D_{i_0}((a_\pm)_{dj}) D_j u_\pm] \quad \text{in } H^{-1/2}(\Gamma_1').$$

PROOF. Let $1 \leq i_0 \leq d-1$. We denote $D = D_{i_0}$ and prove the result on $D_+ = \mathbb{R}_+^d$. Let $v \in C_c^\infty(G_1'; \mathbb{R})$ and consider its restriction v_+ on D_+^1 . From now on we drop the subscript $+$ on functions and conormal derivatives and use Einstein's convention for summation over repeated indexes, when possible.

Note that since $u \in H^2(D_+^1)$ and the a_{ij} 's are in $C^2(\overline{D_+^2}; \mathbb{R})$, we have, using Proposition 2.6,

$$(\mathcal{B}_\nu u, \gamma(Dv))_\Gamma = -\mathcal{E}_+(u, Dv) - \int_{D_+} A_+ u Dv.$$

Now observe that $v \in H^2(D_+)$ so that $\gamma(Dv) = D\gamma(v)$. Note also that

$$(D\mathcal{B}_\nu(u), \gamma(v))_\Gamma = -(\mathcal{B}_\nu(u), D\gamma(v))_\Gamma,$$

since $\gamma(v) \in C_c^\infty(\Gamma_1'; \mathbb{R})$. So that

$$(A.15) \quad -(D\mathcal{B}_\nu u, \gamma(v))_\Gamma = -\mathcal{E}_+(u, Dv) - \int_{D_+} A_+ u Dv.$$

Let us consider the term $\mathcal{E}_+(u, Dv)$ in (A). Let us define $A_{k\ell}^j = a_{j\ell} \delta_{ki_0}$ where δ_{ki_0} is the Kronecker symbol. We have,

$$\mathcal{E}_+(u, Dv) = \int_{D_+} a_{ji} D_i u D_j (D_{i_0} v) = \int_{D_+} a_{ji} D_i u D_{i_0} (D_j v) = \int_{D_+} A_{k\ell}^j D_\ell u D_k (D_j v).$$

Applying the first Green identity of Proposition of 2.6 with the j dependent matrix $(A_{k\ell}^j)_{1 \leq k, \ell \leq d}$ instead of $(a_{ij})_{1 \leq i, j \leq d}$, we see that for any fixed $j \in \{1, \dots, d\}$

$$\sum_{1 \leq k, \ell \leq d} \int_{D_+} A_{k\ell}^j D_\ell u D_k (D_j v) = - \left\langle \sum_{1 \leq k, \ell \leq d} D_k (A_{k\ell}^j D_\ell u), D_j v \right\rangle_{L^2(D_+)} - \left(\sum_{\ell=1}^d \gamma (A_{d\ell}^j D_\ell u), \gamma (D_j v) \right)_\Gamma.$$

But, we see that from the definition of $A_{k\ell}^j$, the surface integral $\left(\sum_{\ell=1}^d \gamma (A_{d\ell}^j D_\ell u), \gamma (D_j v) \right)_\Gamma$ is null.

Further,

$$(A.16) \quad - \left\langle \sum_{1 \leq k, \ell \leq d} D_k (A_{k\ell}^j D_\ell u), D_j v \right\rangle_{L^2(D_+)} \\ = - \sum_{1 \leq k, \ell \leq d} \int_{D_+} D_k (A_{k\ell}^j) D_\ell u D_j v - \sum_{1 \leq k, \ell \leq d} \int_{D_+} A_{k\ell}^j D_\ell (D_k u) D_j v.$$

And from the definition of $A_{k\ell}^j$,

$$- \left\langle \sum_{1 \leq k, \ell \leq d} D_k (A_{k\ell}^j D_\ell u), D_j v \right\rangle_{L^2(D_+)} = - \sum_{\ell=1}^d \int_{D_+} D (a_{j\ell}) D_\ell u D_j v - \sum_{\ell=1}^d \int_{D_+} a_{j\ell} D_\ell (D u) D_j v.$$

Finally, summing over $1 \leq j \leq d$, we get

$$\mathcal{E}_+(u, Dv) = -\mathcal{E}_+(Du, v) - \int_{D_+} D(a_{j\ell}) D_\ell u D_j v.$$

Using this and Proposition 2.9 (note that Du is in $H^1(D_+^{1'})$) in (A) we get

$$(\mathcal{B}_\nu(Du), \gamma(v))_\Gamma = (D\mathcal{B}_\nu(u), \gamma(v))_\Gamma - \int_{D_+} A_+ Du v - \int_{D_+} A_+ u Dv + \int_{D_+} D(a_{j\ell}) D_\ell u D_j v.$$

So that, similarly to the proof of Lemma A.5, we will be done if we prove that

$$(A.17) \quad - \int_{D_+} A_+ Du v - \int_{D_+} A_+ u Dv + \int_{D_+} D(a_{j\ell}) D_\ell u D_j v = -(\gamma[D(a_{dj}) D_j u], \gamma(v))_\Gamma.$$

But, using $\nu = (0, \dots, 0, 1)^*$ and again Proposition 2.6 (this time with the matrix $D(a_{ij})_{1 \leq i, j \leq d}$ instead of $(a_{ij})_{1 \leq i, j \leq d}$), we see that

$$\int_{D_+} D(a_{j\ell}) D_\ell u D_j v + \int_{D_+} D_j (D(a_{j\ell}) D_\ell u) v = -(\gamma(D(a_{dj}) D_j u), \gamma(v))_\Gamma$$

(note that $D_j(D(a_{j\ell}) D_\ell u) \in L^2(D_+^{1'})$, thanks to the smoothness of the $a_{j\ell}$'s and of u). So that we will get (A.17) if we see that, as an element of $\tilde{H}^{-1}(D_+^{1'})$, the form $H^1(D_+^{1'}) \ni \varphi \mapsto \int_{D_+^{1'}} D_j (D(a_{j\ell}) D_\ell u) \varphi$ coincides on $D_+^{1'}$ with $H^1(D_+^{1'}) \ni \varphi \mapsto - \int_{D_+^{1'}} A_+(Du) \varphi - \int_{D_+^{1'}} A_+ u D\varphi$. Let then $\varphi \in C_c^\infty(D_+^{1'}; \mathbb{R})$.

We have

$$\begin{aligned}
-\int_{D_+^{1'}} A_+ u D\varphi &= \int_{D_+^{1'}} a_{j\ell} D_{\ell u} D_j D\varphi = \int_{D_+^{1'}} a_{j\ell} D_{\ell u} D D_j \varphi = -\int_{D_+^{1'}} D(a_{j\ell} D_{\ell u}) D_j \varphi \\
&= -\int_{D_+^{1'}} D(a_{j\ell}) D_{\ell u} D_j \varphi - \int_{D_+^{1'}} a_{j\ell} D(D_{\ell u}) D_j \varphi \\
&= -\int_{D_+^{1'}} D(a_{j\ell}) D_{\ell u} D_j \varphi - \int_{D_+^{1'}} a_{j\ell} D_{\ell}(Du) D_j \varphi \\
&= \int_{D_+^{1'}} D_j(D(a_{j\ell}) D_{\ell u}) \varphi + \int_{D_+^{1'}} A_+(Du) \varphi,
\end{aligned}$$

where we have used the fact that $D(a_{j\ell} D_{\ell u}) = D(a_{j\ell}) D_{\ell u} + a_{j\ell} D(D_{\ell u})$ in $L^2(D_+^{1'})$. The proof is complete. \square

PROOF OF PROPOSITION 2.16. We prove the result for $r = 0$. Again we fix a cutoff function $\chi \in C^\infty(D_-; \mathbb{R})$, $0 \leq \chi \leq 1$, with $\chi \equiv 1$ on D'_- and $\text{supp}(\chi) \subset D_-$. Note that again the derivatives of χ are bounded by some constant depending on $\text{dist}(D'_-, \partial \text{supp}(\chi))$, thus on d'_- .

Proceeding as at the beginning of STEP 1 - SUBSTEP a) of the proof of Proposition 2.15 one may show that $\chi u_- - A_-(\chi u_-) = \bar{f}_-$ on D_- with

$$\bar{f}_- = (\chi f)_- + \sum_{i,j=1}^d (a_-)_{ij} D_i \chi D_j u_- + D_i((a_-)_{ij} (D_j \chi) u_-) \in L^2(D_-).$$

Thus using (2.6) one gets

$$\langle \chi u_-, v \rangle_{L^2(D_-)} + \mathcal{E}_-(\chi u_-, v) = \langle \bar{f}_-, v \rangle_{L^2(D_-)}, \quad \forall v \in H_0^1(D_-).$$

In the computations below we drop for a moment the subscript $-$ on the functions. For any $1 \leq l \leq d$, one may now take $v = -\Delta_{l,-h}(\Delta_{l,h}(\chi u))$ in the above equation, with h sufficiently small, namely $|h| < \text{dist}(\text{supp}(\chi), \Gamma)$. Thus in the same manner than for (A.11) one gets

$$\begin{aligned}
-\int_{D_-} \bar{f} \Delta_{l,-h}(\Delta_{l,h}(\chi u)) &= \int_{D_-} |\Delta_{l,h}(\chi u)|^2 + \sum_{i,j=1}^d \int_{D_-} a_{ij} D_j(\Delta_{l,h}(\chi u)) D_i(\Delta_{l,h}(\chi u)) \\
&\quad + \sum_{i,j=1}^d \int_{D_-} (\Delta_{l,h} a_{ij}) D_i(\chi u) (\cdot + h e_l) \Delta_{l,h}(D_i(\chi u)).
\end{aligned}$$

(Note that this time there is no boundary term). So that combining again **(E)**, Young's inequality and the fact that $|\Delta_{l,h} a_{ij}| \leq C$ (for some constant C depending on $\max_{1 \leq i,j,k \leq d} \sup_{x \in D_-} |\partial_{x_k} a_{ij}(x)|$ but not on h) we get

$$\begin{aligned}
c \|\Delta_{l,h}(\chi u)\|_{H^1(D_-)}^2 &\leq \frac{1}{2\varepsilon} \|\bar{f}\|_{L^2(D_-)}^2 + \frac{\varepsilon}{2} \|\Delta_{l,h}(\chi u)\|_{H^1(D_-)}^2 \\
&\quad + \frac{C d^2}{2\delta} \|\nabla(\chi u)\|_{L^2(D_-)}^2 + \frac{C d^2 \delta}{2} \|\nabla(\Delta_{l,h}(\chi u))\|_{L^2(D_-)}^2
\end{aligned}$$

for any $\varepsilon, \delta > 0$ (the constant c depends on λ). Adjusting now ε and δ we get constants $c', C' > 0$, depending d, λ, C , but not on h s.t.

$$c' \|\Delta_{l,h}(\chi u)\|_{H^1(D_-)}^2 \leq C' (\|\bar{f}\|_{L^2(D_-)}^2 + \|\nabla(\chi u)\|_{L^2(D_-)}^2)$$

and thus, considering $M = C'(\|\bar{f}\|_{L^2(D_-)}^2 + \|\nabla(\chi u)\|_{L^2(D_-)}^2)/c'$ we have

$$\forall 1 \leq i \leq d, \quad \|D_i(\Delta_{l,h}(\chi u))\|_{L^2(D_-)}^2 = \|\Delta_{l,h}(D_i(\chi u))\|_{L^2(D_-)}^2 \leq M.$$

Obviously $\|\bar{f}\|_{L^2(D_-)}^2 \leq C''(\|u_-\|_{H^1(D_-)}^2 + \|f_-\|_{L^2(D_-)}^2)$ with C'' depending on d'_- .

Thus (Lemma A.3-ii) again) we have proved that $D_l D_i(\chi u) \in L^2(D_-)$ for any $1 \leq l \leq d$, $1 \leq i \leq d$ with

$$\|D_l D_i(\chi u_-\|_{L^2(D_-)}^2 \leq C_-(\|u_-\|_{H^1(D_-)}^2 + \|f\|_{L^2(D_-)}^2),$$

with a constant C_- depending on the announced quantities. Remembering that $\chi \equiv 1$ on D'_- this shows the result for u_- and $r = 0$. An induction argument in the spirit of STEP 2 of the proof of Proposition 2.15 takes care of $r > 0$. \square

REFERENCES

- [1] ARONSON, D. G., (1967). Bounds for the fundamental solution of a parabolic equation, *Bull. Amer. Math. Soc.*, Vol. 73, p. 890–896, [MR0217444](#), DOI 10.1090/S0002-9904-1967-11830-5, <https://doi.org/10.1090/S0002-9904-1967-11830-5>.
- [2] BASS, RICHARD F., (1998). *Diffusions and elliptic operators*, Probability and its Applications (New York), Springer-Verlag, New York, p. xiv+232, [MR1483890](#).
- [3] BORSUK, MIKHAIL, Transmission problems for elliptic second-order equations in non-smooth domains, (2010). *Frontiers in Mathematics*, Birkhäuser/Springer Basel AG, Basel, p. xii+218, [MR2676605](#), DOI 10.1007/978-3-0346-0477-2, <https://doi.org/10.1007/978-3-0346-0477-2>.
- [4] BOSSY, MIREILLE AND GOBET, EMMANUEL AND TALAY, DENIS, A symmetrized Euler scheme for an efficient approximation of reflected diffusions, (2004). *Journal of Applied Probability*, Vol. 41, **3**, p. 877–889, [MR2074829](#).
- [5] BOSSY, MIREILLE AND CHAMPAGNAT, NICOLAS AND MAIRE, SYLVAIN AND TALAY, DENIS, Probabilistic interpretation and random walk on spheres algorithms for the Poisson-Boltzmann equation in molecular dynamics (2010), *M2AN Math. Model. Numer. Anal.*, **44**, p. 997–1048, [MR2731401](#), DOI = 10.1051/m2an/2010050, URL = <http://dx.doi.org/10.1051/m2an/2010050>.
- [6] BREZIS, HAÏM, *Analyse fonctionnelle*, (1983). Collection Mathématiques Appliquées pour la Maîtrise. [Collection of Applied Mathematics for the Master's Degree], Masson Paris, p. xiv+234, [MR697382](#) ([85a:46001](#)).
- [7] DE GIORGI, ENNIO, Sulla differenziabilità e l'analiticità delle estremali degli integrali multipli regolari, (1957). *Mem. Accad. Sci. Torino. Cl. Sci. Fis. Mat. Nat. (3)*, Vol. 3, p. 25–43, [MR0093649](#).
- [8] ÉTORÉ, PIERRE, On random walk simulation of one-dimensional diffusion processes with discontinuous coefficients, (2006). *Electron. J. Probab.*, Vol. 11, no. 9, p. 249–275, [MR2217816](#), DOI 10.1214/EJP.v11-311, <http://dx.doi.org/10.1214/EJP.v11-311>.
- [9] ÉTORÉ, PIERRE AND LEJAY, ANTOINE, A Donsker theorem to simulate one-dimensional processes with measurable coefficients, (2007). *ESAIM Probab. Stat.*, Vol. 11, p. 301–326, [MR2339295](#), DOI 10.1051/ps:2007021, <http://dx.doi.org/10.1051/ps:2007021>.
- [10] EVANS, LAWRENCE C. AND GARIEPY, RONALD F., Measure theory and fine properties of functions, (2015). *Textbooks in Mathematics*, p. xiv+299, [MR3409135](#).
- [11] FRIEDMAN, AVNER, *Partial differential equations of parabolic type*, (1964). Prentice-Hall Inc., p. xiv+347, [MR0181836](#) ([31 #6062](#)).
- [12] MASATOSHI AND OSHIMA, YOICHI AND TAKEDA, MASAYOSHI, *Dirichlet forms and symmetric Markov processes*, (2011). De Gruyter Studies in Mathematics, Vol. 19, Walter de Gruyter & Co., Berlin, p. x+489, [MR2778606](#).
- [13] FUKUSHIMA, MASATOSHI, *Dirichlet forms and Markov processes*, (1980). North-Holland Mathematical Library, Vol. 23, North-Holland Publishing Co., Amsterdam-New York; Kodansha, Ltd., Tokyo, p. x+196, [MR569058](#).
- [14] GOBET, EMMANUEL, *Monte-Carlo methods and stochastic processes*, (2016). CRC Press, Boca Raton, FL, p. xxv+309, [MR3558839](#).
- [15] GOBET, EMMANUEL, Euler schemes and half-space approximation for the simulation of diffusion in a domain, (2001). *ESAIM Probab. Statist., European Series in Applied and Industrial Mathematics. Probability and Statistics*, **5**, p. 261–297, [MR1889160](#), DOI 10.1051/ps:2001112, <https://doi.org/10.1051/ps:2001112>.
- [16] KARATZAS, IOANNIS AND SHREVE, STEVEN E., Brownian motion and stochastic calculus, (1991). *Graduate Texts in Mathematics*, Vol. 113, Second edition Springer-Verlag, New York, p. xxiv+470, [MR1121940](#), DOI 10.1007/978-1-4612-0949-2, <http://dx.doi.org/10.1007/978-1-4612-0949-2>.

- [17] LADYŽENSKAJA, O. A. AND SOLONNIKOV, V. A. AND URALCEVA, N. N., Linear and Quasi Linear Equations of Parabolic type, (1967). *Izdat. "Nauka", Moscow*, p. 736, [MR0241821](#).
- [18] LE GALL, J.-F., Applications du temps local aux équations différentielles stochastiques unidimensionnelles, (1983). *Seminar on probability, XVII*, Lecture Notes in Math., Vol. 986, p. 15–31, Springer, Berlin, [MR770393](#), DOI 10.1007/BFb0068296, <http://dx.doi.org/10.1007/BFb0068296>.
- [19] LE GALL, J.-F., One-dimensional stochastic differential equations involving the local times of the unknown process (1984). *Stochastic analysis and applications (Swansea, 1983)*, Lecture Notes in Math., Vol. 1095, p. 51–82, Springer, Berlin, [MR777514](#), DOI 10.1007/BFb0099122, <https://dx-doi-org.fennec.u-pem.fr/10.1007/BFb0099122>.
- [20] LEJAY, ANTOINE, Monte Carlo methods for fissured porous media: a gridless approach, (2004). *Monte Carlo Methods Appl.*, Vol. 10, **3-4**, p. 385–392, [MR2105066](#), DOI 10.1515/mcma.2004.10.3-4.385, <https://doi.org/10.1515/mcma.2004.10.3-4.385>.
- [21] LEJAY, ANTOINE, On the constructions of the skew Brownian motion, (2006). *Probab. Surv.*, Vol. 3, p. 413–466, [MR2280299](#), DOI 10.1214/154957807000000013, <http://dx.doi.org/10.1214/154957807000000013>.
- [22] LEJAY, ANTOINE, A probabilistic representation of the solution of some quasi-linear PDE with a divergence form operator. Application to existence of weak solutions of FBSDE, (2004). *Stochastic Process. Appl.*, Vol. 110, **1**, p. 145–176, [MR2052140](#), DOI 10.1016/j.spa.2003.09.012, <https://doi.org/10.1016/j.spa.2003.09.012>.
- [23] LEJAY, ANTOINE AND MARTINEZ, MIGUEL, A scheme for simulating one-dimensional diffusion processes with discontinuous coefficients, (2006). *Ann. Appl. Probab.*, Vol. 16, **1**, p. 107–139, [MR2209338](#), DOI 10.1214/105051605000000656, <https://dx-doi-org.fennec.u-pem.fr/10.1214/105051605000000656>.
- [24] LENÔTRE, LIONEL, Étude et simulation des processus de diffusion biaisés, (2015). *Ph. D Thesis, Université Rennes 1, France*.
- [25] LIEBERMAN, G.M., Second Order Parabolic Differential Equations, (1996). World Scientific, <https://books.google.fr/books?id=s9Guiwylm3cC>.
- [26] LIMIC, N., Markov Jump Processes Approximating a Non-Symmetric Generalized Diffusion, (2011). *Applied Mathematics and Optimization*, Vol. 64, p. 101-133, DOI 10.1007/s00245-011-9133-1, <http://dx.doi.org/10.1007/s00245-011-9133-1>.
- [27] LI, YAN YAN AND VOGELIUS, MICHAEL, Gradient estimates for solutions to divergence form elliptic equations with discontinuous coefficients, (2000). *Archive for Rational Mechanics and Analysis*, Vol. 153, **2**, p. 91–151, [MR1770682](#), DOI 10.1007/s002050000082, <https://doi.org/10.1007/s002050000082>.
- [28] LIONS, J.-L. AND MAGENES, E., Non-homogeneous boundary value problems and applications. Vol. I, (1972). *Springer-Verlag, New York-Heidelberg*, p. xvi+357, [MR0350177 \(50 #2670\)](#).
- [29] MARTINEZ, MIGUEL, Interprétations probabilistes d’opérateurs sous forme divergence et analyse des méthodes numériques probabilistes associées, (2004). *Ph. D Thesis, Université de Provence, Marseille, France*.
- [30] MARTINEZ, MIGUEL AND TALAY, DENIS, Discrétisation d’équations différentielles stochastiques unidimensionnelles à générateur sous forme divergence avec coefficient discontinu, (2006). *C. R. Math. Acad. Sci. Paris*, Vol. 342, **1**, p.51–56, [MR2193396](#), DOI 10.1016/j.crma.2005.10.025, <https://dx-doi-org.fennec.u-pem.fr/10.1016/j.crma.2005.10.025>.
- [31] MARTINEZ, MIGUEL AND RUBENTHALER, SYLVAIN AND TANRÉ, ETIENNE, Approximations of a continuous time filter. Application to optimal allocation problems in finance, (2009). *Stoch. Anal. Appl.*, Vol. 27, **2**, p. 270–296, [MR2503294](#), DOI 10.1080/07362990802678846, <http://dx.doi.org/10.1080/07362990802678846>.
- [32] MARTINEZ, MIGUEL AND TALAY, DENIS, One-dimensional parabolic diffraction equations: pointwise estimates and discretization of related stochastic differential equations with weighted local times, (2012). *Electron. J. Probab.*, Vol. 17, p. 27-30, [MR2912504](#), DOI 10.1214/EJP.v17-1905, <https://dx-doi-org.fennec.u-pem.fr/10.1214/EJP.v17-1905>.
- [33] MCLEAN, W.C.H., *Strongly Elliptic Systems and Boundary Integral Equations*, (2000). Cambridge University Press, <https://books.google.ca/books?id=RILqjEeMfK0C>.
- [34] MOSER, JÜRGEN, Correction to: “A Harnack inequality for parabolic differential equations”, (1967). *Comm. Pure Appl. Math.*, Vol. 20, p. 231–236 [MR0203268](#), DOI 10.1002/cpa.3160200107, <https://doi.org/10.1002/cpa.3160200107>.
- [35] MOSER, JÜRGEN, A Harnack inequality for parabolic differential equations, (1964). *Comm. Pure Appl. Math.*, Vol. 17, p. 101–134, [MR0159139](#), DOI 10.1002/cpa.3160170106, <https://doi.org/10.1002/cpa.3160170106>.
- [36] MOSER, J., A Harnack inequality for parabolic differential equations, *Outlines Joint Sympos. Partial Differential Equations*, (Novosibirsk, 1963). Acad. Sci. USSR Siberian Branch, Moscow, [MR0206469](#).
- [37] FAN, JISHAN AND KIM, KYOUNGSUN AND NAGAYASU, SEI AND NAKAMURA, GEN, A gradient estimate for solutions to parabolic equations with discontinuous coefficients, (2013). *Electron. J. Differential Equations*, p. 93, 24, [MR3065046](#).

- [38] NASH, JOHN, Parabolic equations, (1957) *Proc. Nat. Acad. Sci. U.S.A.*, Vol. 43, p. 754–758, [MR0089986](#).
- [39] NASH, J., (1958). Continuity of solutions of parabolic and elliptic equations, *Amer. J. Math.*, Vol. 80, p. 931–954, [MR0100158](#), URL = <https://doi.org/10.2307/2372841>.
- [40] PAZY, A., *Semigroups of Linear Operators and Applications to Partial Differential Equations*, (1992). Applied Mathematical Sciences, <https://books.google.fr/books?id=sIAyOgM4R3kC>.
- [41] REVUZ, D., Mesures associées aux fonctionnelles additives de Markov. I, (1970). *Trans. Amer. Math. Soc.*, Vol. 148, p. 501–531, [MR0279890](#), DOI 10.2307/1995386, <https://doi.org/10.2307/1995386>.
- [42] REVUZ, DANIEL AND YOR, MARC, *Continuous martingales and Brownian motion*, (1999). Grundlehren der Mathematischen Wissenschaften [Fundamental Principles of Mathematical Sciences], Vol. 293, p.xiv+602, Third edition, Springer-Verlag, Berlin, [MR1725357](#), DOI 10.1007/978-3-662-06400-9, <https://dx-doi-org.fennec.u-pem.fr/10.1007/978-3-662-06400-9>.
- [43] ROZKOSZ, ANDRZEJ, (1996). Stochastic representation of diffusions corresponding to divergence form operators, *Stochastic Process. Appl.*, Vol. 63, 1, p. 11–33, [MR1411187](#), DOI 10.1016/0304-4149(96)00059-2, [http://dx.doi.org/10.1016/0304-4149\(96\)00059-2](http://dx.doi.org/10.1016/0304-4149(96)00059-2).
- [44] ROZKOSZ, ANDRZEJ, (1996). Weak convergence of diffusions corresponding to divergence form operators, *Stochastics Stochastics Rep.*, Vol. 57, p. 129–157, [MR1407951](#).
- [45] STROOCK, DANIEL W., Diffusion semigroups corresponding to uniformly elliptic divergence form operators, (1988). *Séminaire de Probabilités, XXII*, Lecture Notes in Math., Vol. 1321, p. 316–347, Springer, Berlin, [MR960535](#), 10.1007/BFb0084145, <http://dx.doi.org/10.1007/BFb0084145>.
- [46] STROOCK, DANIEL W. AND ZHENG, WEIAN, Markov chain approximations to symmetric diffusions, (1997). *Annales de l'I.H.P. Probabilités et statistiques*, Vol. 33, 5, p. 619-649, <http://eudml.org/doc/77584>.
- [47] GILBARG, DAVID AND TRUDINGER, NEIL S., *Elliptic partial differential equations of second order* (1983). Grundlehren der mathematischen Wissenschaften, Springer-Verlag, <http://opac.inria.fr/record=b1091868>.
- [48] TRUTNAU, GERALD, Multidimensional skew reflected diffusions, (2005). *Stochastic analysis: classical and quantum*, p. 228–244, World Sci. Publ., Hackensack, NJ, [MR2233163](#).
- [49] ZAITSEVA, LUDMILA L., On a multidimensional Brownian motion with partly reflecting membrane on a hyperplane, (1999). *Theory Stoch. Process.*, 5, p. 258-262.

PIERRE ÉTORÉ
LJK - BÂTIMENT IMAG
UMR 5224
700, AVENUE CENTRALE
38401 ST MARTIN D'HÈRES, FRANCE

MIGUEL MARTINEZ
LABORATOIRE D'ANALYSE ET DE MATHÉMATIQUES APPLIQUÉES
UMR 8050
UNIVERSITÉ PARIS-EST – MARNE-LA-VALLÉE
5, BOULEVARD DESCARTES
CITÉ DESCARTES – CHAMPS-SUR-MARNE
77454 MARNE-LA-VALLÉE CEDEX 2, FRANCE

E-MAIL: Pierre.Etore@univ-grenoble-alpes.fr
Miguel.Martinez@u-pem.fr