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A transformed stochastic Euler scheme for multidimensional transmission PDE *

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Abstract

In this paper we consider multi-dimensional partial differential equations of parabolic type involving divergence form operators that possess a discontinuous coefficient matrix along some smooth interface. The solution of the equation is assumed to present a compatibility transmission condition of its conormal derivatives at this interface (multi-dimensional diffraction problem). We prove an existence and uniqueness result for the solution and construct a low complexity numerical Monte Carlo stochastic Euler scheme to approximate the solution of the parabolic partial differential equation in divergence form. In particular, we give new estimates for the partial derivatives of the solution. Using these estimates, we prove a convergence rate for our stochastic numerical method when the initial condition belongs to an iterated domain of the divergence form operator. Our method presents the same convergence rate as the stochastic numerical schemes elaborated for the same problem in the one-dimensional context. Finally, we compare our results to classical deterministic numerical approximations and illustrate the accuracy of our method.

Introduction

Statement of the problem

Given a finite time horizon T , a real valued function $x \mapsto f(x)$, and an elliptic symmetric matrix $x \mapsto a(x) \in \mathbb{R}^d \times \mathbb{R}^d$, which is smooth outside except at the interface surfaces Γ between subdomains of \mathbb{R}^d , we consider the parabolic transmission problem (or diffraction) problem : find u from $[0, T] \times \mathbb{R}^d$ to

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\mathbb{R} satisfying

$$\left\{ \begin{array}{l} \partial_t u(t, x) - \nabla \cdot (a(x) \nabla_x u(t, x)) = 0, \quad \forall (t, x) \in (0, T] \times (\mathbb{R}^d \setminus \Gamma) \\ u(0, x) = f(x), \quad \forall x \in \mathbb{R}^d \\ \text{Compatibility transmission conditions along the interfaces surfaces.} \end{array} \right. \quad (0.1)$$

The objective of this paper is to provide an efficient stochastic numerical resolution method for the solution of (0.1).

Parabolic PDEs in divergence form and their probabilistic representations

Parabolic equations involving $\mathcal{L} = \nabla \cdot (a(x) \nabla_x)$ have been a major preoccupation for mathematicians in the fifties and the sixties. We may cite the pioneering works of J.Nash [34, 33], E. De Giorgi [5], and J. Moser [30, 31, 32] that prove the continuity of the solution of the Cauchy problem attached to \mathcal{L} and also the celebrated paper by D.G. Aronson [1], which gives upper and lower Gaussian estimate bounds for the fundamental solution of the operator \mathcal{L} .

In the particular case where the matrix a is assumed to be discontinuous along the regular boundaries of some nice disjoint connected open sets in \mathbb{R}^d , a refined analysis of the parabolic equation may be found in the monograph [16], where the parabolic equation is interpreted as a diffraction problem with transmission conditions along the discontinuity boundaries (for a more modern perspective of parabolic transmission problems see also [25]).

Up to our knowledge, the first construction of a Markovian semi-group associated to $\mathcal{L} = \nabla \cdot (a \nabla)$ (in the general case where a is only supposed to be measurable) may be found in the seminal paper of D. Stroock [37]. To understand our difficulty, the conclusion of [37] is :

It should be obvious that the results obtained in this section can be used to construct a diffusion process on Ω corresponding to anyone of the semigroups discussed herein. In addition, the convergence results for the semigroups give rise to weak convergence of the corresponding measures on Ω .

It remains an open and challenging problem to provide a better probabilistic interpretation of these essentially analytic facts.

Since then, there have been many works that try to provide this 'better probabilistic interpretation' in a multidimensional framework.

The construction of the associated Dirichlet process X

In the general case where the symmetric matrix a is only supposed to be elliptic and measurable, the theory of Dirichlet forms as exposed in the monograph by Fukushima [10, 11] gives surely the best possible answer to this question under this general hypothesis. The symmetric operator \mathcal{L} is naturally attached to its corresponding symmetric Dirichlet form, giving rise to a stochastic Dirichlet process X that is described as the addition of a continuous martingale and a continuous additive functional of zero energy. The theory ensures the validity of a Feynman-Kac type formula linking the solution of the Cauchy problem and X .

Going further in the analysis, A. Roskosz [35, 36] proves that X satisfies a Lyons-Zheng decomposition, namely X may be written as the solution of a complex stochastic equation that is the addition of three processes : a martingale, an increment of a reversed time martingale (whose quadratic variations depend on the unknown process X), and an additive functional involving the logarithm derivative of the fundamental solution of the parabolic operator evaluated at X .

This description permits to retrieve some kind of Itô formula for $\phi(X_T)$ where ϕ belongs to some 'good' Sobolev space.

From a numerical perspective, it seems clear that the Lyons-Zheng decomposition provides a description of X that is so strongly nonlinear (time reversion and logarithm derivative of the density of the unknown process), that it seems quite impossible to propose a stochastic numerical scheme for X at this stage.

However one may hope to describe in more detail the behavior of the trajectories of X when the coefficient matrix a – instead of being assumed to be only elliptic measurable – is now assumed to be very smooth outside discontinuities that take place along nice and smooth surfaces Γ .

In this perspective, many papers go deeper and manage to apply the stochastic calculus tools developed by the theory of Dirichlet forms to give a more precise answer to the description of X (see for example [38], [18], [39] and also the results included in the exercises of the monograph [11]). It is possible to write the process X as the solution of some stochastic differential equation that is the addition of the expected martingale term driven by some Brownian motion and the expected additive drift term (both terms involving the unknown process X), and an additive functional A^Γ that captures the behavior of the process at the discontinuity boundaries Γ . The additive functional is rigorously constructed through its Revuz correspondence with some transformation of the natural surface measure of Γ involving the discontinuity jumps of a along Γ .

Numerical Monte-Carlo methods in the one-dimensional case

When the underlying space is one dimensional and the discontinuity is at zero (Γ then reduces to the single point 0), the theory allows to identify A^Γ with $\frac{a(0+) - a(0-)}{a(0+) + a(0-)} L_t^0(X)$ and gives rise to the following stochastic differential equation for the process X

$$dX_t = \sigma(X_t)dB_t + \frac{\sigma\sigma'_\pm}{2}(X_t)dt + \frac{a(0+) - a(0-)}{a(0+) + a(0-)}dL_t^0(X). \quad (0.2)$$

Here $(B_t)_{t \geq 0}$ stands for some standard one dimensional Brownian motion constructed on the probability space $(\Omega, \mathcal{F}, \mathbb{P}^{x_0})$ and we have $\sigma^2 = 2a$. The process $(L_t^0(X))_{t \geq 0}$ denotes the symmetric local time of the unknown process.

Under somewhat weaker conditions than those stated above, one can show that (0.2) has a unique strong solution $(X_t)_{t \geq 0}$, satisfying $X_0 = x_0$ \mathbb{P}^{x_0} - *a.s.*, which is moreover a strong Markov process : see Le Gall [17].

In this one dimensional context, the link between solutions of (0.2) and the solutions of parabolic PDEs with transmission conditions involving the operator \mathcal{L} has been thoroughly studied. One may refer to the overview [19], and the series of works [26, 27, 28], [20], and [6, 7] where stochastic numerical schemes are presented. Note that in most of these works the line-space is discretized and the scheme is in fact some rescaled random walk evolving on a space-time grid. The exception is the method proposed in [27], which is an Euler type scheme that does not require any discretization of the underlying one dimensional state space.

Though somewhat different, all these one dimensional numerical schemes are constructed using this explicit representation of A^Γ as a local time. In particular, all the tools related to the theory of one dimensional local times for semi martingales (Itô -Tanaka formula, occupation time formula) are used in force to construct these numerical schemes and prove that there is indeed convergence in some sense towards the solution of (0.2).

Contribution of this paper

However, when turning to the objective of constructing a stochastic numerical scheme for X in a multi-dimensional context (when $d \geq 2$), the description of A^Γ *via* its Revuz correspondence measure does not provide a direct natural way for the discretization of X (see however the Phd Thesis of L. Lenotre [21] and the walk on spheres algorithm in [2] in the special case of a diagonal coefficient matrix a constant outside the discontinuity boundary Γ). We also mention the work of [24], which attempts to interpret stochastically the deterministic Galerkin method using jump Markov Chains.

In this article, we propose to tackle the problem of the construction of a quite simple numerical scheme of Euler type for X from another perspective. Our starting point is the solution of the parabolic equation involving \mathcal{L} . Inspired by the proof of the convergence of the Euler scheme constructed in the one dimensional case in [28], we build a multidimensional Euler scheme that is purposely designed to capture the *multidimensional transmission conditions* of the parabolic PDE associated to \mathcal{L} .

The novelty of this approach is that we construct our scheme without being concerned at first sight by the description of the limiting process X . All our concern is to guarantee that the error between the expectation of our process – visualized through some very smooth arbitrary test function (belonging to the iterated domains of the operator \mathcal{L}) – and the corresponding solution of the parabolic equation (with the test function as initial condition) converges to zero.

The resulting numerical scheme may be viewed as an extension of the symmetrized Euler scheme of [3] for reflected diffusion with smooth coefficients to partially reflected diffusions with discontinuous coefficients, when no Skorokhod representation of the local time at the boundary Γ is available.

When turning to the proof of the convergence, we face the difficulty of getting global precise estimation bounds for the solution of the parabolic equation and its partial derivatives (up to order four in the space variable) outside the discontinuity boundary Γ for all strictly positive times (not just times after some $\epsilon > 0$) and all the way up to the boundary (not only interior estimates). Unfortunately, the analysis performed in [16], which is somewhat difficult to read and to understand in full detail, does not provide the refined estimations we need for our purpose (see also the recent work for the estimation of the gradient in [8] extending [22] to the parabolic case, but still for times after some $\epsilon > 0$ and estimates depending on ϵ).

Instead of trying to adapt and extend the results in [16], we preferred to look at the results stated in McLean [29], which are obtained for the solutions of *elliptic* divergence transmission problems involving \mathcal{L} . In this paper, we extend the results of [29] to the parabolic case by performing the classical Hille-Yosida theorem and we perform the analysis in order to get global estimates. The price to pay is to strengthen strongly our assumptions on the regularity of a outside its discontinuity boundary and to assume that the initial condition function (the test function) belongs to some order of the iterated domain of the operator \mathcal{L} . As expected, the orders of smoothness and iteration that we require increase strongly with the dimension d .

Then, using the classical Sobolev injections, we obtain all the estimates on the solution we need to prove the convergence of our stochastic numerical scheme.

Note that the analysis of the weak error, visualized through restricted test functions belonging to some large enough iteration of the domain of the underlying operator, seems quite natural. This is what is done for example in [3] for the symmetrized Euler scheme corresponding to reflected diffusions, where the spatial derivatives of the test functions are assumed to verify some compatibility conditions at the reflection boundary ; (whereas in [3] the compatibility conditions are fixed once and for all, the compatibility conditions we require depend crucially on the dimension d).

Finally, we prove that the weak error of our scheme is of order the square root of the time discretization step (improving slightly the results of [28] in the one dimensional case).

Organization of the paper

The paper is organized as follows.

After a first section where we present the notations of the paper and our main assumptions, we devote the longest section of this paper to the study of the parabolic transmission problem. Using classical stools of deterministic partial differential equation, we prove bounds on the solution and its partial derivatives when the initial condition is very smooth (i.e. belongs to iterated domains of the operator; see the result of Corollary 2.21). Up to our knowledge such results cannot be found in the existing literature on the subject. Since this section is quite long, the proofs of some technical lemmas needed to derive our results are transferred to an appendix section at the end of the paper. These proofs are technical and we believe they may be omitted at the first reading of the paper.

In the subsequent section, we introduce our transformed Euler type stochastic numerical scheme, which captures the transmission condition at the boundary interface Γ (see the explanation figure 2). Using in force the results obtained in the former section, we manage to prove a weak convergence result towards the solution of the parabolic transmission problem. We show that the convergence is of order the square root of the time discretization step. The precise result is stated in Theorem 4.1.

Finally, we conclude by presenting in a dedicated section some numerical studies where we compare results given by our procedure to numerical deterministic schemes.

1 General notations and assumptions

For two points $x, y \in \mathbb{R}^d$ we denote by $\langle x, y \rangle$ their scalar product $\langle x, y \rangle = x^*y = \sum_{i=1}^d x_i y_i$.

For a point $x \in \mathbb{R}^d$ we denote by $|x|$ its euclidean norm i.e. $|x|^2 = \sum_{i=1}^d x_i^2 = \langle x, x \rangle$.

We denote by (e_1, \dots, e_d) the usual orthonormal basis of \mathbb{R}^d .

For two metric spaces E, F we will denote by $C(E; F)$ the set of continuous functions from E to F and, for $1 \leq p \leq \infty$, by $C^p(E; F)$ the set of functions in $C(E; F)$ that are p times differentiable with continuous derivatives.

We will denote by $C_c^p(E; F)$ the set of functions in $C^p(E; F)$ that have a compact support.

We will denote by $C_b^p(E; F)$ the set of functions in $C^p(E; F)$ that are continuous with bounded p first derivatives ($C_b(E; F)$ denotes the set of functions in $C(E; F)$ that are bounded).

If $F = \mathbb{R}$, we will sometimes simply write for instance $C(E)$ for $C(E; \mathbb{R})$, for the sake of conciseness.

For any multi-index $\alpha = (i_1, \dots, i_d) \in \mathbb{N}^d$ and $x = (x_1, \dots, x_d) \in \mathbb{R}^d$, we note x^α the product $x_1^{i_1} \dots x_d^{i_d}$ and $|\alpha| = i_1 + \dots + i_d$. So that for $u \in C^{|\alpha|}(\mathbb{R}^d)$ we will denote $\frac{\partial^{|\alpha|} u}{\partial x^\alpha}$, or in short $\partial^\alpha u$, the partial derivative $\partial_{x_1}^{i_1} \dots \partial_{x_d}^{i_d} u$.

Let $U \subset \mathbb{R}^d$ an open subset. We will denote by $L^2(U)$ the set of square integrable functions from U to \mathbb{R} equipped with the usual norm and scalar product $\|\cdot\|_{L^2(U)}$ and $\langle \cdot, \cdot \rangle_{L^2(U)}$.

We denote $H^1(U)$ the usual Sobolev space $W^{1,2}(U)$, equipped with the norm

$$v \mapsto \left(\|v\|_{L^2(U)}^2 + \sum_{i=1}^d \|D_i v\|_{L^2(U)}^2 \right)^{1/2} =: \|v\|_{H^1(U)} \quad (1.1)$$

where $D_i v$ denotes the derivative in the distribution sense with respect to x_i of $v \in L^2(U)$. Note that for the sake of conciseness we will sometimes note $\nabla v = (D_1 v, \dots, D_d v)^*$ and thus

$$\|\nabla v\|_{L^2(U)}^2 = \sum_{i=1}^d \|D_i v\|_{L^2(U)}^2$$

for a function $v \in H^1(U)$.

We recall that the space $H_0^1(U) \subset H^1(U)$ can be defined as $H_0^1(U) = \overline{C_c^\infty(U; \mathbb{R})} = \overline{C_c^1(U; \mathbb{R})}$.

We denote $H^{-1}(U)$ the usual dual topological space of $H_0^1(U)$.

For $m \geq 2$, we denote $H^m(U)$ the usual Sobolev space $W^{m,2}(U) \subset L^2(U)$ of functions having m successive weak derivatives in $L^2(U)$ (and for a multi-index α with $|\alpha| \leq m$ and $u \in H^m(U)$ we denote $D^\alpha u$ such a weak derivative).

We will have to consider fractional Sobolev spaces - in fact $H^s(\Gamma)$, $s \in \{-\frac{1}{2}, \frac{1}{2}\}$, where Γ in some boundary of dimension $d-1$ between domains included in \mathbb{R}^d . We recall here the definition of $H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, based on Fourier transform. We denote $\mathcal{S}(\mathbb{R}^n)$ the Schwartz space of rapidly decreasing $C^\infty(\mathbb{R}^n)$ functions, and $\mathcal{S}'(\mathbb{R}^n)$ the space of *temperate distributions* (see [29] p72 for details). For $u \in \mathcal{S}(\mathbb{R}^n)$ we define $\hat{u}(\xi) = \int_{\mathbb{R}^n} e^{-i2\pi\xi^*x} u(x) dx$, $\xi \in \mathbb{R}^n$. Then for $u \in \mathcal{S}'(\mathbb{R}^n)$, \hat{u} is defined by extension, using

$$\langle \hat{u}, \varphi \rangle_{\mathcal{S}', \mathcal{S}} = \langle u, \hat{\varphi} \rangle_{\mathcal{S}', \mathcal{S}}, \quad \forall \varphi \in \mathcal{S}(\mathbb{R}^n). \quad (1.2)$$

We thus have

$$H^s(\mathbb{R}^n) = \left\{ u \in \mathcal{S}'(\mathbb{R}^n) : (1 + |\xi|^2)^{s/2} \hat{u}(\xi) \in L^2(\mathbb{R}^n) \right\}.$$

This space is equipped with the norm

$$v \mapsto \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi =: \|v\|_{H^s(\mathbb{R}^n)}.$$

In the case $s = 1$ the thus defined Sobolev space corresponds in fact exactly to our previous definition of $H^1(\mathbb{R}^n)$.

The notion of a Lipschitz domain $U \subset \mathbb{R}^d$ (resp. of class C^k) with bounded boundary $\Gamma = \partial U$ is defined with the help of a system of local Lipschitz change of coordinates (resp. of class C^k ; see [29] Chap.3 pp89-90).

This allows to define the space $H^s(\Gamma)$ from $H^s(\mathbb{R}^{d-1})$, $0 \leq s \leq 1$, in the case U is Lipschitz with bounded boundary Γ (see again [29] Chap. 3, pp98-99).

In the case where U is Lipschitz with bounded boundary, we denote $H^{-1/2}(\Gamma)$ and $\tilde{H}^{-1}(U)$ the dual spaces of respectively $H^{1/2}(\Gamma)$ and $H^1(U)$ (in coherence with respectively p98-99 and Theorem 3.30 p92 in [29]).

Consider $\mathbb{R}^d = \bar{D}_+ \cup D_-$ with D_+ and D_- two open connected subdomains separated by a transmission boundary Γ that is to say

$$\Gamma = \bar{D}_+ \cap \bar{D}_-.$$

By an assumption of type " Γ is bounded and Lipschitz (or of class C^k) " we will mean that both D_+ and D_- are Lipschitz (or C^k) domains, and that Γ is bounded. Note that in that case we shall consider D_+ (resp. D_-) as the interior (resp. exterior) domain. Note that D_- is then unbounded (although its boundary is bounded).

We shall encounter however the situation where D_{\pm} and Γ are unbounded but we will restrict ourselves to the case where $D_+ = \mathbb{R}_+^d = \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_d > 0\}$, $D_- = \mathbb{R}_-^d = \{x = (x', x_d) \in \mathbb{R}^{d-1} \times \mathbb{R}, x_d < 0\}$ and $\Gamma = \mathbb{R}^{d-1} \times \{0\}$. This will be convenient to lead some of our proofs. In fact when we assert a result mentioning the curve Γ without further precision this will mean that either $D_{\pm} = \mathbb{R}_{\pm}^d$, either Γ is bounded and Lipschitz (for example in the Green identities of Subsubsection 2.1.2; see also Remark 2.11).

We introduce the following assumption.

Assumption 1.1. (D) : *The transmission boundary Γ is bounded and of class C^5 .*

We denote $\gamma : H^1(D_{\pm}) \rightarrow H^{1/2}(\Gamma)$ the usual trace operator on Γ (Γ is supposed to be bounded and Lipschitz).

We denote

$$D = D_+ \cup D_- = \mathbb{R}^d \setminus \Gamma \subset \mathbb{R}^d.$$

For $u \in C(\mathbb{R}^d; \mathbb{R})$ we denote for a point $y \in \Gamma$

$$u(y_{\pm}) = \lim_{z \rightarrow y, z \in D_{\pm}} u(z).$$

For $u \in C(\mathbb{R}^d; \mathbb{R}) \cap C^1(D; \mathbb{R})$ we denote $\nabla_x u = (\frac{\partial u}{\partial x_1}, \dots, \frac{\partial u}{\partial x_d})^*$ and, for a point $y \in \Gamma$

$$\nabla_x u_{\pm}(y) = \lim_{z \rightarrow y, z \in D_{\pm}} \nabla_x u(z), \quad (1.3)$$

if this limit exists.

For a vector field $G \in C(\mathbb{R}^d; \mathbb{R}^d) \cap C^1(D; \mathbb{R}^d)$ we denote by $\nabla \cdot G$ its divergence i.e. $\nabla \cdot G = \sum_{i=1}^d \frac{\partial G_i}{\partial x_i}$. For $G \in C^2(D; \mathbb{R})$ and $x \in D$ we denote $\mathbf{H}[G](x)$ the Hessian matrix of G at point x .

Let $a(x)$ be a symmetric matrix valued and time homogeneous diffusion coefficient. In the whole paper the coefficients a_{ij} are always assumed to be measurable, and bounded by some constant $0 < \Lambda < \infty$.

If $a_{ij} \in C^1(D; \mathbb{R})$ for all $1 \leq i, j \leq d$ and $u \in C(\mathbb{R}^d; \mathbb{R}) \cap C^2(D; \mathbb{R})$ we denote

$$\mathcal{L}u(x) = \nabla \cdot (a(x) \nabla_x u(x)), \quad \forall x \in D. \quad (1.4)$$

In the following we will often make the below defined ellipticity assumption.

Assumption 1.2. (E) : *There exists $\lambda > 0$ such that*

$$\forall x \in \mathbb{R}^d, \forall \xi \in \mathbb{R}^d, \quad \lambda |\xi|^2 \leq \xi^* a(x) \xi. \quad (1.5)$$

Remark 1.3. *Under the ellipticity assumption (E) each diagonal term a_{ii} of the matrix-valued coefficient $a : \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ has a uniformly bounded inverse (see Lemma 4.17 in [29]).*

In the sequel we will frequently note a_{\pm} the restrictions of a to D_{\pm} .

Note that under (E) we can assert that for any $x \in \mathbb{R}^d$ we have

$$a_{\pm}(x) = P_{\pm}^*(x) D_{\pm}(x) P_{\pm}(x) \quad (1.6)$$

with $P_{\pm}(x)$ some orthogonal matrices and $D_{\pm}(x)$ some diagonal matrices with strictly positive eigenvalues.

For a point $x \in \Gamma$ we denote by $\nu(x) \in \mathbb{R}^d$ the unit normal to Γ at point x , pointing to D_+ . We define then the co-normal vector fields $x \mapsto \gamma_+(x) := a_+(x)\nu(x)$ and $x \mapsto \gamma_-(x) := -a_-(x)\nu(x)$, for $x \in \Gamma$, and introduce the following assumption.

Assumption 1.4. (Γ) : The co-normal vector fields $\gamma_+(x)$ and $\gamma_-(x)$ are of class C^4 .

Note that under **(E)** it is clear that we have

$$\forall x \in \Gamma, \quad \langle \gamma_+(x), \nu(x) \rangle \geq \lambda > 0 \quad \text{and} \quad \langle \gamma_-(x), \nu(x) \rangle \leq -\lambda < 0. \quad (1.7)$$

Note that the notation γ for the trace operator follows the usual one ([29] for instance) and the notation γ_\pm for the co-normal vectors follows the one of the paper [3]. But it will be dealt with the trace operator only in Section 2 and in the Appendix, and with co-normal vectors only in Sections 3 and 4. So that these notations will cause no confusion.

To finish with we define the unbounded operator $A : \mathcal{D}(A) \subset L^2(\mathbb{R}^d) \rightarrow L^2(\mathbb{R}^d)$ by

$$\mathcal{D}(A) = \left\{ u \in H^1(\mathbb{R}^d) \text{ with } \sum_{i,j=1}^d D_i(a_{ij}D_j u) \in L^2(\mathbb{R}^d) \right\}$$

and

$$\forall u \in \mathcal{D}(A), \quad Au = \sum_{i,j=1}^d D_i(a_{ij}D_j u).$$

Then we introduce the iterated domains defined recursively by

$$\mathcal{D}(A^k) = \{v \in \mathcal{D}(A^{k-1}) : Av \in \mathcal{D}(A^{k-1})\}, \quad k \geq 2.$$

2 The parabolic transmission problem

Let $0 < T < \infty$ a finite time horizon. Let us consider the transmission parabolic problem

$$(\mathcal{P}_T) \left\{ \begin{array}{ll} \partial_t u(t, x) - \mathcal{L}u(t, x) = 0 & \forall (t, x) \in (0, T] \times D \\ \langle a_+ \nabla_x u_+(t, y) - a_- \nabla_x u_-(t, y), \nu(y) \rangle = 0 & \forall (t, y) \in (0, T] \times \Gamma \quad (\star) \\ u(t, y+) = u(t, y-) & \forall (t, y) \in [0, T] \times \Gamma \\ u(0, x) = u_0(x) & \forall x \in \mathbb{R}^d. \end{array} \right.$$

We will say that $(t, x) \mapsto u(t, x)$ is *classical solution* to (\mathcal{P}_T) if it satisfies

$$u \in C([0, T]; C^2(\bar{D}_+) \cap C^2(\bar{D}_-)) \cap C^1([0, T]; C(\bar{D}_+) \cap C(\bar{D}_-)) \cap C([0, T]; C(\mathbb{R}^d))$$

and satisfies the following requisites. First, u satisfies the first line of (\mathcal{P}_T) , where the derivatives are understood in the classical sense. Second, for all $0 < t \leq T$ the function $x \mapsto u(t, x)$ is such that the limits $\lim_{z \rightarrow y, z \in D_\pm} \nabla_x u(t, z)$ exist for all $y \in \Gamma$ and satisfy the transmission condition (\star) . Third, u is continuous across Γ (third line). Fourth, it satisfies the initial condition at the fourth line of (\mathcal{P}_T) .

The aim of this section is to prove the following result.

Theorem 2.1. Let $a = (a_{ij})_{1 \leq i, j \leq d}$ satisfy **(E)**.

- Denote

$$k_0 = \begin{cases} \lfloor \frac{d}{4} \rfloor + 2 & \text{if } d \text{ is even;} \\ \lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor + 2 & \text{if } d \text{ is odd.} \end{cases} \quad (2.1)$$

Assume that the coefficients a_{ij} satisfy $(a_\pm)_{ij} \in C_b^{2k_0-3}(D_\pm)$ and Γ is bounded and of class C^{2k_0-2} . Then for $u_0 \in \mathcal{D}(A^{k_0})$ the parabolic transmission problem (\mathcal{P}_T) admits a classical solution.

- Furthermore, if $u_0 \in \mathcal{D}(A^k)$ for $k \geq k_0$, the coefficients a_{ij} satisfy $(a_{\pm})_{ij} \in C_b^{2k-1}(D_{\pm})$ and Γ is bounded of class C^{2k} , this classical solution u is such that

$$u \in C^{k-j} \left([0, T] ; C^{n(j)}(\bar{D}_+) \cap C^{n(j)}(\bar{D}_-) \right), \quad [d/4] \leq j \leq k$$

with $n(j) = \lfloor 2j - \frac{d}{2} \rfloor$.

The idea is to use the Hille-Yosida theorem. This requires to study in a first time the associated resolvent equation, of (elliptic) type

$$(\mathcal{E}_T^1) \left\{ \begin{array}{ll} u(x) - \mathcal{L}u(x) & = 0 \quad \forall x \in D \\ \langle a_+ \nabla_x u_+(y) - a_- \nabla u_-(y), \nu(y) \rangle & = 0 \quad \forall y \in \Gamma \\ u(y_+) & = u(y_-) \quad \forall y \in \Gamma \\ \lim_{|x| \rightarrow \infty} u(x) & = 0. \end{array} \right.$$

More precisely, we will study (\mathcal{E}_T^1) in a weak sense in $L^2(\mathbb{R}^d)$. In particular for a source term $f \in L^2(\mathbb{R}^d)$ we will seek for a solution u in $\mathcal{D}(A)$ of

$$u - Au = f \tag{2.2}$$

(see Proposition 2.12 below). Then, by applying the Hille-Yosida theorem in $L^2(\mathbb{R}^d)$, we will get the existence of a solution to (\mathcal{P}_T) in a semi-weak sense (for which the time derivatives are understood in a classical sense and the space derivatives in a weak sense). Finally, using some Sobolev embedding arguments, we will get Theorem 2.1.

Remark 2.2. *The result of Theorem 2.1 has to be compared with the classical results on parabolic PDE for smooth coefficients (and with no transmission condition) In this more classical situation, a unique classical solution to the parabolic PDE exists as soon as the a_{ij} 's are bounded and Hölder continuous and satisfy **(E)**, and u_0 is continuous and satisfies some growth condition (see for example [9] Chap. 1 or [23], Theorem 5.14). Here we can notice that we have asked additional smoothness on the $(a_{\pm})_{ij}$'s inside the D_{\pm} 's. This is because our technique of proof is very different: the use of the Hille-Yosida theorem and Sobolev embeddings requires this smoothness, unlike the use of the parametrix method in the classical case. But by doing so, we fully treat the transmission condition aspects in a modern way.*

2.1 Study of the associated elliptic problem

In this subsection we establish the existence of a solution to (2.2) belonging to $\mathcal{D}(A)$ and study its smoothness properties, together with the ones of functions belonging to the iterated domains $\mathcal{D}(A^k)$, for $k \geq 1$.

To that aim we will partly rely on the results in [29] Chap. 4, pp. 141-145. In this subsection we have rewritten the proofs of some of these results for the sake of clarity and completeness (however some proofs will be postponed to the Appendix).

We recall that the coefficients a_{ij} are assumed to be bounded by Λ so that we may define the following continuous bilinear and symmetric form, which will be used extensively in the sequel

$$\mathcal{E}(u, v) = \sum_{i,j=1}^d \langle a_{ij} D_j u, D_i v \rangle_{L^2(\mathbb{R}^d)}, \quad \forall u, v \in H^1(\mathbb{R}^d). \tag{2.3}$$

Let $u \in \mathcal{D}(A)$. Using the definition of Au as a distribution acting on $C_c^\infty(\mathbb{R}^d; \mathbb{R})$, and the density of $C_c^\infty(\mathbb{R}^d; \mathbb{R})$ in $H^1(\mathbb{R}^d) = H_0^1(\mathbb{R}^d)$, one can establish the following relation, linking A and the form (2.3):

$$\mathcal{E}(u, v) = \langle -Au, v \rangle_{L^2(\mathbb{R}^d)}, \quad \forall v \in H^1(\mathbb{R}^d). \tag{2.4}$$

2.1.1 Some preliminary results

In the sequel, for $u \in L^2(\mathbb{R}^d)$, we frequently denote u_+ (resp. u_-) the restriction of u to D_+ (resp. D_-). It may happen that we use this notation for restricted distributions also.

We introduce the following notation for the jump across Γ of $u \in L^2(\mathbb{R}^d)$, with $u_+ \in H^1(D_+)$ and $u_- \in H^1(D_-)$:

$$[u]_\Gamma = \gamma(u_+) - \gamma(u_-).$$

If $[u]_\Gamma = 0$ we shall simply write $\gamma(u) = \gamma(u_+) = \gamma(u_-)$.

We start with two lemmas whose proof is postponed to the Appendix.

Lemma 2.3. *Let $v \in L^2(\mathbb{R}^d)$. Then, for any $1 \leq i \leq d$, the distribution $(D_i v)_\pm$ is equal to $D_i(v_\pm)$. As a consequence, if $v \in H^1(\mathbb{R}^d)$, then $v_\pm \in H^1(D_\pm)$.*

Lemma 2.4 ([29], Exercise 4.5). *Suppose $u \in L^2(\mathbb{R}^d)$ with $u_\pm \in H^1(D_\pm)$. Then $u \in H^1(\mathbb{R}^d)$ if and only if $[u]_\Gamma = 0$ a.e. on Γ .*

Remark 2.5. *The result of Lemma 2.4 has to be compared with the fact that if $d = 1$, we know that any function in $H^1(\mathbb{R}^d)$ as a continuous version ([4] Theorem VIII.2).*

2.1.2 Green identities

We shall consider restricted operators and bilinear forms in the following sense. We define $A_+ : H^1(D_+) \rightarrow H^{-1}(D_+)$ by

$$\forall v \in H^1(D_+), \quad A_+ v = \sum_{i,j=1}^d D_i((a_+)_{ij} D_j v).$$

We define $A_- : H^1(D_-) \rightarrow H^{-1}(D_-)$ in the same manner (note that we do not specify here any domain $\mathcal{D}(A_\pm)$). Further, we define

$$\mathcal{E}_\pm(u, v) = \sum_{i,j=1}^d \int_{D_\pm} (a_\pm)_{ij} D_j u D_i v, \quad \forall u, v \in H^1(D_\pm).$$

In the same fashion as for Equation (2.4), we have, for $u_\pm \in H^1(D_\pm)$ with $A_\pm u_\pm \in L^2(D_\pm)$,

$$\mathcal{E}_\pm(u_\pm, v) = \int_{D_\pm} (-A_\pm u_\pm) v, \quad \forall v \in H_0^1(D_\pm). \quad (2.5)$$

Imagine now that in (2.5) we wish to take the test function in $H^1(D_\pm)$ instead of $H_0^1(D_\pm)$ (note that for instance $H^1(D_+) \neq H_0^1(D_+)$). There will still be a link between A_\pm and \mathcal{E}_\pm , but through Green type identities, involving conormal derivatives and boundary integrals.

We introduce a specific notation for the one-sided conormal derivatives on Γ of $u \in L^2(\mathbb{R}^d)$ with $u_\pm \in H^2(D_\pm)$. Provided the $(a_\pm)_{ij}$ are in $C_b^1(D_\pm; R)$ and Γ is bounded and Lipschitz we set

$$\mathcal{B}_\nu^\pm u = \nu^* \gamma(a_\pm \nabla u_\pm) = \sum_{i=1}^d \sum_{j=1}^d \nu_i \gamma((a_\pm)_{ij} D_j u_\pm) \quad \text{on } \Gamma. \quad (2.6)$$

For $g \in H^{-\frac{1}{2}}(\Gamma)$ and $f \in H^{\frac{1}{2}}(\Gamma)$ we denote by $(g, f)_\Gamma$ the action of g on f . If both f, g are in $H^{\frac{1}{2}}(\Gamma)$ the quantity $(g, f)_\Gamma$ coincides with the surface integral $\int_\Gamma g f d\sigma$. We have for example the next result.

Proposition 2.6 (First Green identity, first version; [29], Lemma 4.1). *Let $u \in L^2(\mathbb{R}^d)$ with $u_+ \in H^2(D_+)$ and $u_- \in H^2(D_-)$. Assume that the coefficients $(a_\pm)_{ij}$ are in $C_b^1(D_\pm; R)$. Then*

$$\mathcal{E}_+(u_+, v) = \int_{D_+} (-A_+ u_+) v - \left(\mathcal{B}_\nu^+ u, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(D_+)$$

and

$$\mathcal{E}_-(u_-, v) = \int_{D_-} (-A_- u_-) v + \left(\mathcal{B}_\nu^- u, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(D_-).$$

Remark 2.7. Note that the change of sign in front of the integral on Γ is due to the fact that $-\nu$ is the outward normal to D_+ , and ν is the outward normal to D_- .

In fact, for our coming purpose (proof of Theorem 2.14), we need a version of the first Green identity that is valid for $u \in L^2(\mathbb{R}^d)$ with $u_+ \in H^1(D_+)$ and $u_- \in H^1(D_-)$ (and possibly non smooth coefficients $(a_{\pm})_{ij}$). We thus need to extend the definition of $\mathcal{B}_\nu^\pm u$ to such functions, for which the trace in (2.6) is no more defined in $H^{1/2}(\Gamma)$ (the boundary Γ is assumed to be bounded and Lipschitz).

But thanks to Lemma 4.3 in [29], for any $f_\pm \in \tilde{H}^{-1}(D_\pm)$ s.t.

$$-A_\pm u_\pm = f_\pm \quad \text{on } D_\pm, \quad (2.7)$$

there exists $g_\pm \in H^{-\frac{1}{2}}(\Gamma)$, uniquely defined by u_\pm and f_\pm , and satisfying

$$\mathcal{E}_+(u_+, v) = \langle f_+, v \rangle_{\tilde{H}^{-1}(D_+), H^1(D_+)} - \left(g_+, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(D_+),$$

$$\mathcal{E}_-(u_-, v) = \langle f_-, v \rangle_{\tilde{H}^{-1}(D_-), H^1(D_-)} + \left(g_-, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(D_-).$$

and

$$\|g_\pm\|_{H^{-1/2}(\Gamma)} \leq C \left(\|u_\pm\|_{H^1(D_\pm)} + \|f_\pm\|_{\tilde{H}^{-1}(D_\pm)} \right)$$

Remark 2.8. Note that the meaning of equality (2.7) is that for any $\varphi \in C_c^\infty(D_\pm; \mathbb{R})$,

$$\langle -A_\pm u_\pm, \varphi \rangle_{H^{-1}(D_\pm), H_0^1(D_\pm)} = \langle f_\pm, \varphi \rangle_{\tilde{H}^{-1}(D_\pm), H^1(D_\pm)}.$$

In the sequel the equality "on D_\pm " (for elements of $\tilde{H}^{-1}(D_\pm)$) will often have this kind of sense, as for example in Lemma 2.10 below, or in the proof of Lemma 6.5 in the Appendix. We will also use the same idea on some subsets of D_\pm (proof of Proposition 2.17).

In particular if $A_\pm u_\pm \in L^2(D_\pm)$ one makes the natural choice

$$f_+ = \begin{cases} -A_+ u_+ & \text{on } D_+ \\ 0 & \text{on } \bar{D}_- \end{cases} \quad \text{and} \quad f_- = \begin{cases} 0 & \text{on } \bar{D}_+ \\ -A_- u_- & \text{on } D_- \end{cases}$$

and defines $\mathcal{B}_\nu^\pm u := g_\pm$. The notation $\mathcal{B}_\nu^\pm u$ comes from the fact that this new definition is consistent with the original one involved in Proposition 2.6 (see [29] p.117 for details). To sum up, we have, using the new sense of $\mathcal{B}_\nu^\pm u$, the following result.

Proposition 2.9 (First Green identity, extended version; [29] Theorem 4.4, point i)). *Let $u \in L^2(\mathbb{R}^d)$ with $u_+ \in H^1(D_+)$ and $u_- \in H^1(D_-)$. Assume $A_+ u_+ \in L^2(D_+)$, $A_- u_- \in L^2(D_-)$. Then*

$$\mathcal{E}_+(u_+, v) = \int_{D_+} (-A_+ u_+) v - \left(\mathcal{B}_\nu^+ u, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(D_+)$$

and

$$\mathcal{E}_-(u_-, v) = \int_{D_-} (-A_- u_-) v + \left(\mathcal{B}_\nu^- u, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(D_-).$$

Finally we introduce a notation for the jumps across Γ of the conormal derivative of a function u with $u_\pm \in H^1(D_\pm)$:

$$[\mathcal{B}_\nu u]_\Gamma = \mathcal{B}_\nu^+ u - \mathcal{B}_\nu^- u \in H^{-1/2}(\Gamma).$$

Using Lemma 2.3 and Proposition 2.9 it is possible to prove the following two-sided Green identity, that we state directly for a function $u \in H^1(\mathbb{R}^d)$ (not only with $u_\pm \in H^1(D_\pm)$), as this is what we will use in the sequel.

Lemma 2.10 (Two-sided Green identity, [29] Lemma 4.19, Equation (4.33)). *Let $u \in H^1(\mathbb{R}^d)$. Let $f_+ \in L^2(D_+)$ and $f_- \in L^2(D_-)$ and assume*

$$u_\pm - A_\pm u_\pm = f_\pm \quad \text{on } D_\pm. \quad (2.8)$$

Set $f = f_+ + f_-$, then

$$\langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(u, v) = \langle f, v \rangle_{L^2(\mathbb{R}^d)} - \left([\mathcal{B}_\nu u]_\Gamma, \gamma(v) \right)_\Gamma, \quad \forall v \in H^1(\mathbb{R}^d). \quad (2.9)$$

Proof. Pick v in $H^1(\mathbb{R}^d)$. Notice that $u_{\pm} \in H^1(D_{\pm})$ and $v_{\pm} \in H^1(D_{\pm})$, thanks to Lemma 2.3. Thanks to (2.8) we get $A_{\pm}u_{\pm} \in L^2(D_{\pm})$, so that one can use Proposition 2.9 and get

$$\int_{D_+} u_+ v_+ + \mathcal{E}_+(u_+, v_+) = \int_{D_+} (u_+ - A_+ u_+) v_+ - \left(\mathcal{B}_\nu^+ u, \gamma(v_+) \right)_\Gamma \quad (2.10)$$

and

$$\int_{D_-} u_- v_- + \mathcal{E}_-(u_-, v_-) = \int_{D_-} (u_- - A_- u_-) v_- + \left(\mathcal{B}_\nu^- u, \gamma(v_-) \right)_\Gamma. \quad (2.11)$$

But, as $v \in H^1(\mathbb{R}^d)$, we have $[v]_\Gamma = 0$ and thus $\gamma(v_+) = \gamma(v_-) = \gamma(v)$. Thus, using again (2.8), and the definition of f and $[\mathcal{B}_\nu u]_\Gamma$, we get, summing (2.10) and (2.11),

$$\langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}_+(u_+, v_+) + \mathcal{E}_-(u_-, v_-) = \langle f, v \rangle_{L^2(\mathbb{R}^d)} - \left([\mathcal{B}_\nu u]_\Gamma, \gamma(v) \right)_\Gamma.$$

To complete the proof it suffices to notice that, thanks to Lemma 2.3, we have

$$\mathcal{E}_+(u_+, v_+) + \mathcal{E}_-(u_-, v_-) = \sum_{i,j=1}^d \left\{ \int_{D_+} (a_+)_{ij} (D_j u)_+ (D_i v)_+ + \int_{D_-} (a_-)_{ij} (D_j u)_- (D_i v)_- \right\} = \mathcal{E}(u, v).$$

□

Remark 2.11. Note that in [29] Green identities are stated in the case of Γ bounded and Lipschitz. But we claim that these results are true for $D_{\pm} = \mathbb{R}_{\pm}^d$, as in the proofs one usually starts with $D_{\pm} = \mathbb{R}_{\pm}^d$ (or some hypograph type domain) and then turn to Γ bounded with the help of local change of coordinates.

2.1.3 Existence of a weak solution to the resolvent equation and immediate properties

We have the next result.

Proposition 2.12. Assume **(E)**. Let $f \in L^2(\mathbb{R}^d)$. Then (2.2) has a unique solution in $\mathcal{D}(A)$.

Proof. Let us note that the symmetric bilinear form on $H^1(\mathbb{R}^d)$

$$(u, v) \mapsto \langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(u, v)$$

is continuous and, thanks to Assumption **(E)**, coercive. Thus the Lax-Milgram theorem ([4] Corollary V.8) immediately asserts the existence of a unique $u \in H^1(\mathbb{R}^d)$ such that

$$\forall v \in H^1(\mathbb{R}^d), \quad \langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(u, v) = \langle f, v \rangle_{L^2(\mathbb{R}^d)}.$$

In other words we have for any $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$,

$$\mathcal{E}(u, \varphi) = - \left\langle \sum_{i,j=1}^d D_i (a_{ij} D_j u), \varphi \right\rangle_{L^2(\mathbb{R}^d)} = \langle (f - u), \varphi \rangle_{L^2(\mathbb{R}^d)}.$$

Hence the distribution $\sum_{i,j=1}^d D_i (a_{ij} D_j u)$ belongs to $L^2(\mathbb{R}^d)$, and thus $u \in \mathcal{D}(A)$. Finally, from the above relations we deduce

$$\forall v \in H^1(\mathbb{R}^d), \quad \langle u - Au, v \rangle_{L^2(\mathbb{R}^d)} = \langle f, v \rangle_{L^2(\mathbb{R}^d)},$$

which implies (2.2). □

The proposition below gives properties of functions belonging to $\mathcal{D}(A)$, and thus indicates that the solution $u \in \mathcal{D}(A)$ of (2.2) encountered in Proposition 2.12 satisfies the second and third lines of (\mathcal{E}_Γ^1) in a weak sense (note that u satisfies the fourth line as it belongs to $H^1(\mathbb{R}^d) = H_0^1(\mathbb{R}^d)$).

Proposition 2.13. Let $u \in \mathcal{D}(A)$. Then $[u]_\Gamma = [\mathcal{B}_\nu u]_\Gamma = 0$ a.e. on Γ .

Proof. Let $u \in \mathcal{D}(A)$. As $u \in H^1(\mathbb{R}^d)$ one gets by Lemma 2.4 that $[u]_\Gamma = 0$ a.e. on Γ . Set now $f = u - Au \in L^2(\mathbb{R}^d)$. According to Equation (2.4) we have

$$\forall v \in H^1(\mathbb{R}^d), \quad \langle u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(u, v) = \langle f, v \rangle_{L^2(\mathbb{R}^d)}. \quad (2.12)$$

Using repeatedly Lemma 2.3 one can see that $u_{\pm} - A_{\pm}u_{\pm} = f_{\pm}$ on D_{\pm} . Note that by construction $f = f_+ + f_-$. Using now Lemma 2.10, and comparing (2.9) and (2.12), one gets $([\mathcal{B}_\nu u]_\Gamma, \gamma(\varphi))_\Gamma = 0$ for any $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$, which completes the proof. □

2.1.4 Regularity of the weak solution of the elliptic problem and consequence on the iterated domains $\mathcal{D}(A^k)$

Here we will establish and use the following main result.

Theorem 2.14. *Assume the ellipticity assumption (E).*

Let $r \in \mathbb{N}$. Assume that the coefficients $(a_{\pm})_{ij}$ belong to $C_b^{r+1}(D_{\pm}; \mathbb{R})$. Assume Γ is bounded and of class C^{r+2} .

Let $f_{\pm} \in H^r(D_{\pm})$. Let $u \in H^1(\mathbb{R}^d)$ satisfying

$$u_{\pm} - A_{\pm}u_{\pm} = f_{\pm} \quad \text{on } D_{\pm}$$

and $[\mathcal{B}_{\nu}u]_{\Gamma} \in H^{\frac{1}{2}+r}(\Gamma)$. Then $u_{\pm} \in H^{2+r}(D_{\pm})$.

In order to prove Theorem 2.14 we need a set of technical results.

For $1 \leq l \leq d$ we introduce the l -th partial difference quotient

$$\Delta_{l,h}u(x) = \frac{u(x + he_l) - u(x)}{h}, \quad h \in \mathbb{R}.$$

We gather in the following lemma the results we will need about difference quotients.

Lemma 2.15. *Let $u, v \in L^2(\mathbb{R}^d)$ and $1 \leq l \leq d$.*

i) If $D_l u \in L^2(\mathbb{R}^d)$ then $\|\Delta_{l,h}u\|_{L^2(\mathbb{R}^d)} \leq \|D_l u\|_{L^2(\mathbb{R}^d)}$ for all $h \in \mathbb{R}$.

ii) If there is a constant M such that $\|\Delta_{l,h}u\|_{L^2(\mathbb{R}^d)} \leq M$ for all h sufficiently small, then $D_l u$ is in $L^2(\mathbb{R}^d)$, with $\|D_l u\|_{L^2(\mathbb{R}^d)} \leq M$.

iii) For all $h \in \mathbb{R}$ we have $\Delta_{l,h}(uv) = \Delta_{l,h}u v(\cdot + he_l) + u \Delta_{l,h}v$.

iv) For all $h \in \mathbb{R}$ we have $\int_{\mathbb{R}^d} u \Delta_{l,h}v = - \int_{\mathbb{R}^d} v \Delta_{l,-h}u$.

Proof. Point iii) follows from elementary computations and Point iv) from a change of variable in the integral $\int_{\mathbb{R}^d} u \Delta_{l,h}v$.

For Points i) and ii) see the proof of Lemma 4.13 in [29]. □

Then Theorem 2.14 will follow from the two following propositions. The first one asserts a classical result on the global regularity of the solution away from the interface. The second one provides a local analysis of the regularity across the interface Γ , and is to be found in Theorem 4.20 in [29]. As it is less classical we have found interesting to detail its proof here.

Proposition 2.16 ([12], Theorem 8.10). *Assume the ellipticity assumption (E).*

Let $r \in \mathbb{N}$. Assume that the coefficients $(a_{\pm})_{ij}$ belong to $C_b^{r+1}(D_{\pm}; \mathbb{R})$. Assume Γ is bounded.

Let $f_{\pm} \in H^r(D_{\pm})$. Let $u \in H^1(\mathbb{R}^d)$ satisfying

$$u_{\pm} - A_{\pm}u_{\pm} = f_{\pm} \quad \text{on } D_{\pm}.$$

Let $D'_{\pm} \subset D_{\pm}$ open subsets with $\overline{D'_{\pm}} \subset D_{\pm}$ and denote $d'_{\pm} = \text{dist}(D'_{\pm}, \Gamma)$.

We have that $u_{\pm} \in H^{r+2}(D'_{\pm})$, with

$$\|u_{\pm}\|_{H^{r+2}(D'_{\pm})} \leq C_{\pm} (\|u_{\pm}\|_{H^1(D_{\pm})} + \|f\|_{H^r(D_{\pm})}),$$

where the constant C_{\pm} depends on d, λ, d'_{\pm} and $\max_{1 \leq i, j \leq d} \max_{|\alpha| \leq r+1} \sup_{x \in D_{\pm}} |\partial^{\alpha}(a_{\pm})_{ij}(x)|$.

Proof. In [12] this result is asserted with the assumption that $\overline{D'_{\pm}} \subset D_{\pm}$, with $\overline{D'_{\pm}}$ compact. So that for the interior (bounded) domain D_+ the result is immediate. On the unbounded domain D_- we claim that the same result holds for non compact $\overline{D'_-}$, as in fact only the distance $d'_- = \text{dist}(D'_-, \Gamma)$ plays a role in the proof. We provide the proof in Appendix for the sake of completeness, inspired by the proof Theorem 4.16 in [29], that we have found more coherent with our notations and setting. □

Proposition 2.17 ([29], Theorem 4.20). *Let G_1 and G_2 be bounded open connected subset of \mathbb{R}^d , such that $\overline{G_1} \subset G_2$ and G_1 intersects Γ , and put*

$$D_{\pm}^j = G_j \cap D_{\pm} \quad \text{and} \quad \Gamma_j = \Gamma \cap G_j \quad \text{for } j = 1, 2.$$

Assume that G_2 is constructed in such a way that there is a C^{r+2} -diffeomorphism between Γ_2 and a bounded portion of the hyperplan $x_d = 0$.

*Assume the ellipticity assumption **(E)**.*

Let $r \in \mathbb{N}$. Assume that the coefficients $(a_{\pm})_{ij}$ belong to $C^{r+1}(\overline{D_{\pm}^2}; \mathbb{R})$.

Let $f_{\pm} \in L^2(D_{\pm})$ with $f_{\pm} \in H^r(D_{\pm}^2)$. Let $u \in L^2(\mathbb{R}^d)$ with $u \in H^1(G_2)$ satisfying

$$u_{\pm} - A_{\pm}u_{\pm} = f_{\pm} \quad \text{on } D_{\pm}^2$$

and $[\mathcal{B}_{\nu}u]_{\Gamma} \in H^{\frac{1}{2}+r}(\Gamma_2)$. Then $u_{\pm} \in H^{2+r}(D_{\pm}^1)$.

Proof. We only treat the case $D_{\pm} = \mathbb{R}_{\pm}^d$. Indeed if this is not the case one may use change of coordinates of class C^{r+2} to change the equation in a new one, on new domains D_{\pm}^{\prime} , with new coefficients of class $C^{r+1}(\overline{D_{\pm}^{\prime}}; \mathbb{R})$, and $\Gamma_2^{\prime} = \overline{D_{+}^{\prime}} \cap \overline{D_{-}^{\prime}}$ a portion of the hyperplan $x_d = 0$ (see Exercise 4.2 in [29]). And in that case the value of the functions outside $G_2^{\prime} = D_{+}^{\prime} \cup \Gamma \cup D_{-}^{\prime}$ is of no importance, as in the sequel we will multiply everything by a cutoff function with compact support in G_2^{\prime} . Thus indeed things can be considered on $D_{\pm} = \mathbb{R}_{\pm}^d$.

The proof proceeds by induction on $r \in \mathbb{N}$.

STEP1. We establish the result for $r = 0$. Note that by convention $H^0(D_{\pm}^j) = L^2(D_{\pm}^j)$, $j = 1, 2$.

We fix a cutoff function $\chi \in C_c^{\infty}(G_2; \mathbb{R})$, $0 \leq \chi \leq 1$, with $\chi \equiv 1$ on G_1 (the derivatives of χ are bounded by some constant depending on $\text{dist}(G_1, \partial \text{supp}(\chi))$).

SUBSTEP a) We first prove that $D_l D_i(\chi u) \in L^2(\mathbb{R}^d)$ for any $1 \leq l \leq d-1$, $1 \leq i \leq d$. As $u \in H^1(G_2)$ and $\chi \in C_c^{\infty}(G_2; \mathbb{R})$ and we have already $D_i(\chi u) \in L^2(\mathbb{R}^d)$ for any $1 \leq i \leq d$. Thus, using Lemma 2.15-ii), we are done if we find M s.t. $\|\Delta_{l,h}(D_i \chi u)\|_{L^2(\mathbb{R}^d)} \leq M$ for all h sufficiently small, and all $1 \leq l \leq d-1$, $1 \leq i \leq d$.

As $u_+ - A_+ u_+ = f_+$ on D_+^2 we have for any $\varphi \in C_c^{\infty}(D_+^2; \mathbb{R})$

$$\langle u_+ - A_+ u_+, \chi_+ \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} = \langle f_+, \chi_+ \varphi \rangle_{L^2(D_+^2)}.$$

Thus, for any φ in $C_c^{\infty}(D_+^2; \mathbb{R})$ we have

$$\langle (\chi u)_+ - A_+(\chi u)_+, \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} = \langle \bar{f}_+, \varphi \rangle_{L^2(D_+^2)},$$

with

$$\bar{f}_+ = (\chi f)_+ + \sum_{i,j=1}^d (a_+)_{ij} D_i \chi_+ D_j u_+ + D_i((a_+)_{ij} (D_j \chi_+) u_+) \in L^2(D_+^2)$$

(here we have used successively (2.5), with D_+^2 replacing D_+ , and the facts that for any $1 \leq i, j \leq d$, $D_i(\chi_+ \varphi) = \varphi D_i \chi_+ + \chi_+ D_i \varphi$, $\chi_+ D_j u_+ = D_j(\chi u)_+ - (D_j \chi_+) u_+$ and $\int_{D_+^2} (a_+)_{ij} (D_j \chi_+) u_+ D_i \varphi = - \int_{D_+^2} D_i[(a_+)_{ij} (D_j \chi_+) u_+] \varphi$; note that $(a_+)_{ij} (D_j \chi_+) u_+$ is in $H^1(D_+^2)$ as $u_+ \in H^1(D_+)$, χ_+ is smooth and $(a_+)_{ij}$ is of class $C^1(\overline{D_+^2}; \mathbb{R})$).

One may proceed in the same manner on D_-^2 . But as $(\chi u)_{\pm}$ and \bar{f}_{\pm} have support in D_{\pm}^2 we have in fact that

$$(\chi u)_{\pm} - A_{\pm}(\chi u)_{\pm} = \bar{f}_{\pm} \quad \text{on } D_{\pm}. \quad (2.13)$$

We set $\bar{f} = \bar{f}_+ + \bar{f}_-$. According to Lemma 2.10 we have

$$\langle \chi u, v \rangle_{L^2(\mathbb{R}^d)} + \mathcal{E}(\chi u, v) = \langle \bar{f}, v \rangle_{L^2(\mathbb{R}^d)} - \left([\mathcal{B}_{\nu}(\chi u)]_{\Gamma}, \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(\mathbb{R}^d). \quad (2.14)$$

Let $1 \leq l \leq d-1$ and $h \in \mathbb{R}$. We take $v = -\Delta_{l,-h}(\Delta_{l,h}(\chi u))$ in (2.14). Using successively Points iv) and iii) in Lemma 2.15, together with linearity arguments, we get

$$\begin{aligned}
& \left([\mathcal{B}_\nu(\chi u)]_\Gamma, \gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u))) \right)_\Gamma - \int_{\mathbb{R}^d} \bar{f} \Delta_{l,-h}(\Delta_{l,h}(\chi u)) \\
&= \int_{\mathbb{R}^d} |\Delta_{l,h}(\chi u)|^2 + \sum_{i,j=1}^d \int_{\mathbb{R}^d} \Delta_{l,h}(a_{ij} D_j(\chi u)) \Delta_{l,h}(D_i(\chi u)) \\
&= \int_{\mathbb{R}^d} |\Delta_{l,h}(\chi u)|^2 + \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j(\Delta_{l,h}(\chi u)) D_i(\Delta_{l,h}(\chi u)) \\
&\quad + \sum_{i,j=1}^d \int_{\mathbb{R}^d} (\Delta_{l,h} a_{ij}) D_i(\chi u) (\cdot + h e_l) \Delta_{l,h}(D_i(\chi u))
\end{aligned} \tag{2.15}$$

But thanks to **(E)** we have

$$\begin{aligned}
& \|\Delta_{l,h}(\chi u)\|_{L^2(\mathbb{R}^d)}^2 + \lambda \sum_{i=1}^d \|D_i(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 \\
&\leq \int_{\mathbb{R}^d} |\Delta_{l,h}(\chi u)|^2 + \sum_{i,j=1}^d \int_{\mathbb{R}^d} a_{ij} D_j(\Delta_{l,h}(\chi u)) D_i(\Delta_{l,h}(\chi u)),
\end{aligned}$$

which combined with (2.15) leads to

$$\begin{aligned}
c \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 &\leq \int_{\mathbb{R}^d} |\bar{f}| \cdot |\Delta_{l,-h}(\Delta_{l,h}(\chi u))| + \sum_{i,j=1}^d |\Delta_{l,h} a_{ij}| \cdot |D_j(\chi u) (\cdot + h e_l)| \cdot |\Delta_{l,h}(D_i(\chi u))| \\
&\quad + \left| \left([\mathcal{B}_\nu(\chi u)]_\Gamma, \gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u))) \right)_\Gamma \right|.
\end{aligned}$$

We first focus on the third RHS term. We have

$$\left| \left([\mathcal{B}_\nu(\chi u)]_\Gamma, \gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u))) \right)_\Gamma \right| \leq \| [\mathcal{B}_\nu(\chi u)]_\Gamma \|_{H^{1/2}(\Gamma_2)} \| \gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u))) \|_{H^{-1/2}(\Gamma_2)}.$$

Indeed remember that by assumption $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}}(\Gamma_2)$, which implies

$$\| [\mathcal{B}_\nu(\chi u)]_\Gamma \|_{H^{1/2}(\Gamma_2)} =: \tilde{K} < \infty$$

(see Lemma 6.4 in the Appendix). Besides, one can see that

$$\begin{aligned}
\| \gamma(\Delta_{l,-h}(\Delta_{l,h}(\chi u))) \|_{H^{-1/2}(\Gamma_2)} &= \| \Delta_{l,-h} \gamma(\Delta_{l,h}(\chi u)) \|_{H^{-1/2}(\Gamma)} \leq \| D_l \gamma(\Delta_{l,h}(\chi u)) \|_{H^{-1/2}(\Gamma)} \\
&\leq k \| \gamma(\Delta_{l,h}(\chi u)) \|_{H^{1/2}(\Gamma)} \leq k \| \Delta_{l,h}(\chi u) \|_{H^1(D_\pm)} \leq k \| \Delta_{l,h}(\chi u) \|_{H^1(\mathbb{R}^d)}
\end{aligned}$$

(here we have used for the first inequality, a version of Exercise 4.4 in [29], adapted to \mathbb{R}_\pm^d and tangential derivatives; for the second inequality we have used Lemma 6.1 in the Appendix). In the sequel we set $K = k\tilde{K}$.

Note now that as $a_{ij} \in C^1(\bar{D}_+^2; \mathbb{R}) \cap C^1(\bar{D}_-^2; \mathbb{R})$, for all $1 \leq i, j \leq d$ we have that $|\Delta_{l,h} a_{ij}| \leq C$ for some constant C not depending on h . Using now Young's inequality and Point i) of Lemma 2.15 we get

$$\begin{aligned}
c \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 &\leq \frac{1}{2\varepsilon} \|\bar{f}\|_{L^2(\mathbb{R}^d)}^2 + \frac{\varepsilon}{2} \|\nabla(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 + \frac{Cd^2\delta}{2} \|\nabla(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad + \frac{Cd^2}{2\delta} \|\nabla(\chi u)\|_{L^2(\mathbb{R}^d)}^2 + K^2 \frac{\eta}{2} + \frac{1}{2\eta} \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2
\end{aligned}$$

for any $\varepsilon, \delta, \eta > 0$, and thus

$$\begin{aligned}
c \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 &\leq \frac{1}{2\varepsilon} \|\bar{f}\|_{L^2(\mathbb{R}^d)}^2 + \frac{\varepsilon}{2} \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 + \frac{Cd^2}{2\delta} \|\nabla(\chi u)\|_{L^2(\mathbb{R}^d)}^2 \\
&\quad + \frac{Cd^2\delta}{2} \|\nabla(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 + K^2 \frac{\eta}{2} + \frac{1}{2\eta} \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2.
\end{aligned}$$

Adjusting now ε, δ and η we get constants $c', C' > 0$, not depending on h s.t.

$$c' \|\Delta_{l,h}(\chi u)\|_{H^1(\mathbb{R}^d)}^2 \leq C' (\|\bar{f}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla(\chi u)\|_{L^2(\mathbb{R}^d)}^2 + K^2)$$

and thus, considering $M = C' (\|\bar{f}\|_{L^2(\mathbb{R}^d)}^2 + \|\nabla(\chi u)\|_{L^2(\mathbb{R}^d)}^2 + K^2) / c'$ we have

$$\forall 1 \leq i \leq d, \quad \|D_i(\Delta_{l,h}(\chi u))\|_{L^2(\mathbb{R}^d)}^2 = \|\Delta_{l,h}(D_i(\chi u))\|_{L^2(\mathbb{R}^d)}^2 \leq M.$$

As M does not depend on h we have indeed proved that $D_l D_i(\chi u) \in L^2(\mathbb{R}^d)$ for any $1 \leq l \leq d-1$, $1 \leq i \leq d$. Remembering that $\chi \equiv 1$ on G_1 this proves $D_l D_i u \in L^2(G_1)$ for any $1 \leq l \leq d-1$, $1 \leq i \leq d$.

SUBSTEP b) We consider the restriction $(\chi u)_+$ of (χu) to D_+ . We already have $D_d(\chi u)_+ \in L^2(D_+)$, as $(\chi u) \in H^1(\mathbb{R}^d)$ (and using again Lemma 2.3). We will show that $D_{dd}^2(\chi u)_+ \in L^2(D_+^2)$. One may prove in the same manner that $D_{dd}^2(\chi u)_- \in L^2(D_-^2)$.

Using $(\chi u)_+ - A_+(\chi u)_+ = \bar{f}_+$ and (2.5) we have for any $\varphi \in C_c^\infty(D_+; \mathbb{R})$

$$\int_{D_+} (a_+)_{dd} D_d(\chi u)_+ D_d \varphi = \int_{D_+} (\bar{f}_+ - (\chi u)_+) \varphi - \sum_{(i,j) \neq (d,d)} \int_{D_+} (a_+)_{ij} D_j(\chi u)_+ D_i \varphi.$$

From now on we drop the subscript $+$ on functions, in order to lighten the notations. Reinterpreting immediately the preceding equation we have for any $\varphi \in C_c^\infty(D_+^2; \mathbb{R})$

$$-\langle D_d(a_{dd} D_d(\chi u)), \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} = \int_{D_+^2} (\bar{f} - (\chi u)) \varphi + \sum_{(i,j) \neq (d,d)} \int_{D_+^2} D_i(a_{ij} D_j(\chi u)) \varphi. \quad (2.16)$$

Note that in the above expression we have used the fact that for any $(i, j) \neq (d, d)$, the weak derivative $D_i(a_{ij} D_j(\chi u))$ is in $L^2(D_+^2)$. Indeed remember that $D_i D_j(\chi u) \in L^2(D_+^2)$ thanks to SUBSTEP a) and that a_{ij} is in $C^1(\overline{D_+^2}; \mathbb{R})$.

We now use the fact that for any distribution $v' \in H^{-1}(D_+^2)$ and any $w \in C^1(\overline{D_+^2}; \mathbb{R})$ the distribution wv' is simply defined by $\langle wv', \varphi \rangle = \langle v', w\varphi \rangle$ for any $\varphi \in C_c^\infty(D_+^2; \mathbb{R})$ (and if $v' \in L^2(D_+^2)$ and $w \in C(\overline{D_+^2}; \mathbb{R})$ the distribution wv' is simply defined by $\langle wv', \varphi \rangle = \int_{D_+^2} wv'\varphi$ for any $\varphi \in C_c^\infty(D_+^2; \mathbb{R})$). Thus if $w \in C^1(\overline{D_+^2}; \mathbb{R})$ and $v' \in L^2(D_+^2)$ we have $D_d(wv') = (D_d w)v' + w D_d v'$ and using this in (2.16) we get

$$\begin{aligned} & -\langle a_{dd} D_{dd}^2(\chi u), \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} \\ &= \int_{D_+^2} (\bar{f} - (\chi u)) \varphi + \int_{D_+^2} (D_d a_{dd}) D_d(\chi u) \varphi + \sum_{(i,j) \neq (d,d)} \int_{D_+^2} D_i(a_{ij} D_j(\chi u)) \varphi, \end{aligned}$$

that is to say we have finally for any $\varphi \in C_c^\infty(D_+^2; \mathbb{R})$

$$-\langle D_{dd}^2(\chi u), \varphi \rangle_{H^{-1}(D_+^2), H_0^1(D_+^2)} = \int_{D_+^2} \frac{1}{a_{dd}} \left\{ (\bar{f} - (\chi u)) + (D_d a_{dd}) D_d(\chi u) + \sum_{(i,j) \neq (d,d)} D_i(a_{ij} D_j(\chi u)) \right\} \varphi. \quad (2.17)$$

Using now Remark 1.3 and the fact that $a_{dd} \in C^1(\overline{D_+^2}; \mathbb{R})$, one may conclude that indeed $D_{dd}(\chi u)$ is in $L^2(D_+^2)$ (note that we have already stressed that for any $(i, j) \neq (d, d)$, the weak derivative $D_i(a_{ij} D_j(\chi u))$ is in $L^2(D_+^2)$). This implies $D_{dd} u_+ \in L^2(D_+^1)$ (we use the subscript again). Taking in account SUBSTEP a) we have indeed proved that $u_\pm \in H^2(D_\pm^1)$.

STEP2. Take $r \in \mathbb{N}^*$. Assuming the result is true at $r-1$ we prove its validity at rank r .

Let $f_\pm \in L^2(D_\pm)$ with f_\pm in $H^r(D_\pm^2)$ and assume that $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}+r}(\Gamma_2)$ and the coefficients $(a_\pm)_{ij}$ belong to $C^{r+1}(\overline{D_\pm^2}; \mathbb{R})$. Thus, in particular, f_\pm are in $H^{r-1}(D_\pm^2)$ and $[\mathcal{B}_\nu u]_\Gamma \in H^{r-\frac{1}{2}}(\Gamma_2)$. Thus, by induction hypothesis, we have $u_\pm \in H^{r+1}(D_\pm^1)$, writing $D_\pm^1 = G_{1'} \cap D_\pm$ for any open connected subset $G_{1'}$ with $\overline{G_1} \subset G_{1'} \subset \overline{G_1} \subset G_2$.

SUBSTEP a) Let $1 \leq i_0 \leq d-1$. To lighten notation we denote $D := D_{i_0}$. We wish to show that $Du_{\pm} \in H^{1+r}(D_{\pm}^1)$. Note that $Du_{\pm} = (Du)_{\pm}$ and that we already know by STEP1-a) that $D_i Du$ is in $L^2(G_{1'})$ for any $1 \leq i \leq d$, so that Du is in $H^1(G_{1'})$. So the idea is now to use the induction hypothesis on Du .

Remember that $u_+ - A_+ u_+ = f_+$ on D_+^2 . Let $\varphi \in C_c^\infty(D_+^{1'}; \mathbb{R})$, note that $D\varphi \in C_c^\infty(D_+^{1'}; \mathbb{R})$, and use $D\varphi$ as a test function in (2.5). This writes (again we drop the subscript $+$ on functions)

$$\int_{D_+^{1'}} u D\varphi + \sum_{i,j=1}^d \int_{D_+^{1'}} a_{ij} D_j u D_i (D\varphi) = \int_{D_+^{1'}} f D\varphi,$$

which we immediately rewrite in

$$\begin{aligned} - \int_{D_+^{1'}} Df \varphi &= - \int_{D_+^{1'}} Du \varphi + \sum_{i,j=1}^d \int_{D_+^{1'}} a_{ij} D_j u D_i (D\varphi) \\ &= - \int_{D_+^{1'}} Du \varphi - \sum_{i,j=1}^d \langle D(a_{ij} D_j u), D_i \varphi \rangle_{H^{-1}(D_+^{1'}), H_0^1(D_+^{1'})} \\ &= - \int_{D_+^{1'}} Du \varphi - \sum_{i,j=1}^d \int_{D_+^{1'}} a_{ij} D(D_j u) D_i \varphi - \sum_{i,j=1}^d \int_{D_+^{1'}} D(a_{ij}) D_j u D_i \varphi \\ &= - \int_{D_+^{1'}} Du \varphi - \sum_{i,j=1}^d \int_{D_+^{1'}} a_{ij} D_j (Du) D_i \varphi + \sum_{i,j=1}^d \int_{D_+^{1'}} D_i (D(a_{ij}) D_j u) \varphi. \end{aligned}$$

Here we have first used the facts that $u, f \in H^1(D_+^{1'})$ and $\varphi \in C_c^\infty(D_+^{1'}; \mathbb{R})$. Second we have used $DD_j u \in L^2(D_+^{1'})$ and $a_{ij} \in C^1(\overline{D_+^2}; \mathbb{R})$, $1 \leq i, j \leq d$, so that one can easily check that the distribution $D(a_{ij} D_j u)$ is equal to $a_{ij} D_j (Du) + D(a_{ij}) D_j u \in L^2(D_+^{1'})$. Third we have used the fact that $D_i (D(a_{ij}) D_j u) \in L^2(D_+^{1'})$, thanks to the smoothness of $(a_+)_{ij}$ and u_+ .

As φ was arbitrarily chosen in $C_c^\infty(D_+^{1'}; \mathbb{R})$ we have in fact proved that

$$\forall \varphi \in C_c^\infty(D_+^{1'}; \mathbb{R}), \quad \int_{D_+^{1'}} Du \varphi + \mathcal{E}_+(Du, \varphi) = \int_{D_+^{1'}} \left(Df + \sum_{i,j=1}^d D_i (D(a_{ij}) D_j u) \right) \varphi.$$

This means that (we use the subscript $+$ again)

$$(Du)_+ - A_+(Du)_+ = \tilde{f}_+ \quad \text{on } D_+^{1'}$$

with

$$\tilde{f}_+ = Df_+ + \sum_{i,j=1}^d D_i (D((a_+)_{ij}) D_j u_+).$$

Notice now that thanks to $f_+ \in H^r(D_+^2)$, $u_+ \in H^{r+1}(D_+^{1'})$ and the smoothness of a we can claim that $\tilde{f}_+ \in H^{r-1}(D_+^{1'})$. Proceeding in the same way on the domain D_- we prove that

$$(Du)_- - A_-(Du)_- = \tilde{f}_- \quad \text{on } D_-^{1'}$$

with $\tilde{f}_- \in H^{r-1}(D_-)$. In order to use the induction hypothesis it remains now to check that

$$[\mathcal{B}_\nu(Du)]_\Gamma \in H^{r-\frac{1}{2}}(\Gamma_{1'}) \quad (2.18)$$

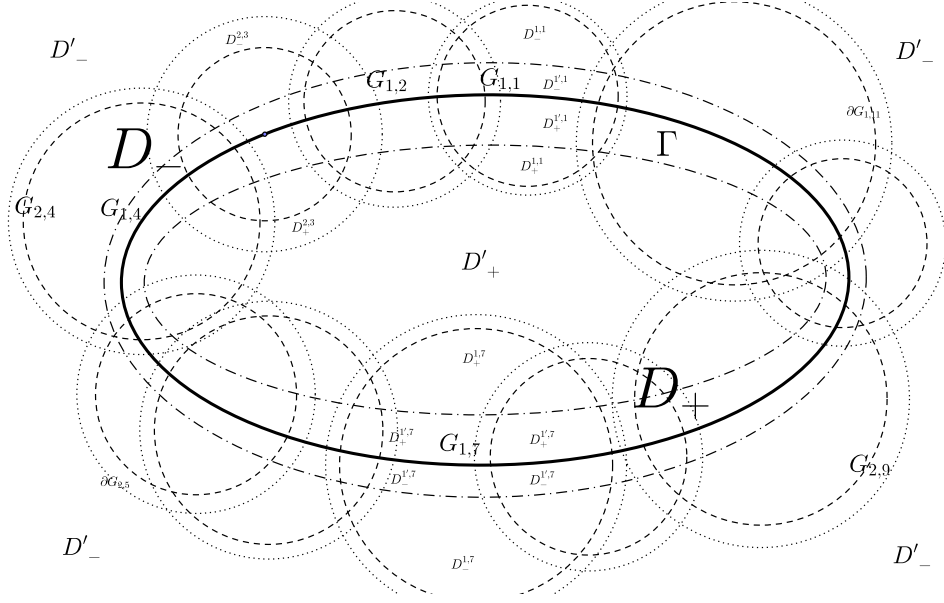


Figure 1: The sets used in the proof of Theorem 2.14 (here $N = 11$).

(here we have denoted $\Gamma_{1'} = \Gamma \cap G_{1'}$). Using the technical Lemma 6.5 in the Appendix (note that $u_{\pm} \in H^2(D_{\pm}^1)$, as $u_{\pm} \in H^{r+1}(D_{\pm}^1)$ with $r > 0$) we get,

$$\begin{aligned}
[\mathcal{B}_{\nu}(Du)]_{\Gamma} &= \mathcal{B}_{\nu}^{+}(Du) - \mathcal{B}_{\nu}^{-}(Du) \in H^{-1/2}(\Gamma_{1'}) \\
&= D\mathcal{B}_{\nu}^{+}(u) - \sum_{j=1}^d \gamma[D((a_{+})_{dj})D_j u_{+}] - D\mathcal{B}_{\nu}^{-}(u) + \sum_{j=1}^d \gamma[D((a_{-})_{dj})D_j u_{-}] \\
&= D[\mathcal{B}_{\nu}u]_{\Gamma} - \sum_{j=1}^d \gamma[D((a_{+})_{dj})D_j u_{+} - D((a_{-})_{dj})D_j u_{-}].
\end{aligned}$$

Remember now that by assumption $[\mathcal{B}_{\nu}u]_{\Gamma} \in H^{\frac{1}{2}+r}(\Gamma_2)$, and, again, that $u_{\pm} \in H^{r+1}(D_{\pm}^1)$, so that $D_j u_{\pm} \in H^r(D_{\pm}^1)$ for any $1 \leq j \leq d$. Using in addition the smoothness of a_{\pm} it is clear that we have (2.18).

Using now the induction hypothesis we conclude that $(Du)_{\pm}$ are in $H^{r+1}(D_{\pm}^1)$. Thus we have proved that

$$\forall 1 \leq i \leq d-1, \quad D_i u_{+} \in H^{1+r}(D_{+}^1) \quad \text{and} \quad D_i u_{-} \in H^{1+r}(D_{-}^1).$$

SUBSTEP b) To conclude that u_{\pm} are in $H^{2+r}(D_{\pm}^1)$ it remains to prove that $D_d u_{\pm}$ are in $H^{1+r}(D_{\pm}^1)$. Let us consider $D_d u_{+}$. As we have already, thanks to SUBSTEP a), that $D_i D_d u_{+} = D_d D_i u_{+}$ is in $H^r(D_{+}^1)$ for any $1 \leq i \leq d-1$, it remains to show that $D_{dd}^2 u_{+}$ is in $H^r(D_{+}^1)$. But Equation (2.17) shows that

$$D_{dd}^2 u_{+} = -\frac{1}{(a_{+})_{dd}} \left\{ (f_{+} - u_{+}) + (D_d(a_{+})_{dd})D_d u_{+} + \sum_{(i,j) \neq (d,d)} D_i((a_{+})_{ij})D_j u_{+} \right\} \quad \text{on } D_{+}^1.$$

But, using $f_{+}, u_{+} \in H^{1+r}(D_{+}^1)$ and SUBSTEP a), the RHS term is easily seen to be in $H^r(D_{+}^1)$. One can proceed in the same way on D_{-}^1 , and thus the proof by induction is completed. \square

Proof of Theorem 2.14. As Γ is bounded we will cover it by a finite number of balls and use the local regularity result of Theorem 2.17 inside each ball. We will combine this with the interior regularity result of Proposition 2.16, to finally get a result on the global regularity of the function u on D_{+} and D_{-} .

As Γ is bounded and of class C^{r+2} one may cover it by a finite number of open balls $G_{2,k} \subset \mathbb{R}^d$, $1 \leq k \leq N$ (i.e. $\Gamma \subset \cup_{k=1}^N G_{2,k}$), that are chosen in such a way that there is a C^{r+2} diffeomorphism between each $\Gamma_{2,k} = \Gamma \cap G_{2,k}$ and a bounded open portion of $\mathbb{R}^{d-1} \times \{0\}$ ([29] pp 89-90). Then one

chooses $D'_\pm \subset D_\pm$ open subsets with $\overline{D'_\pm} \subset D_\pm$ and $D'_+ \cup D'_- \cup_{k=1}^N G_{2,k} = \mathbb{R}^d$. Note that $\overline{D'_-}$ is a closed but unbounded part of \mathbb{R}^d . It is convenient to choose D'_\pm so that we have the overlapping $D'_\pm \cap G_{2,k} \cap G_{2,k+1} \neq \emptyset$ for any $1 \leq k \leq N-1$ and $D'_\pm \cap G_{2,N} \cap G_{2,1} \neq \emptyset$. Then one may choose arbitrary open connected sets $G_{1,k}$ with $\overline{G_{1,k}} \subset G_{2,k}$, and respecting $D'_+ \cup D'_- \cup_{k=1}^N G_{1,k} = \mathbb{R}^d$, $D'_\pm \cap G_{1,k} \cap G_{1,k+1} \neq \emptyset$ for any $1 \leq k \leq N-1$ and $D'_\pm \cap G_{1,N} \cap G_{1,1} \neq \emptyset$. As in Proposition 2.17 we note $D_\pm^{j,k} = G_{j,k} \cap D_\pm$, $1 \leq k \leq N$, for $j = 1, 2$ (on all these sets see Figure 1).

For any k It is clear that the $(a_\pm)_{ij}$ belong to $C^{r+1}(\overline{D_\pm^{2,k}}; \mathbb{R})$. Besides $f_\pm \in H^r(D_\pm^{2,k})$ (by adapting the result of Lemma 2.3), and $u_\pm - A_\pm u_\pm = f_\pm$ on $D_\pm^{2,k}$. To finish with, we have $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}+r}(\Gamma_{2,k})$. For each k one may then use Proposition 2.17 to conclude that $u_\pm \in H^{r+2}(D_\pm^{1,k})$. On another hand by Proposition 2.16 one gets that $u_\pm \in H^{r+2}(D'_\pm)$. So that for example $u_+ \in H^{r+2}(D'_+) \cap \left(\cap_{k=1}^N H^{r+2}(D_+^{1,k}) \right)$. We claim that this implies that $u_+ \in H^{r+2}(D_+)$ - note that of course $D'_+ \cup \left(\cup_{k=1}^N D_+^{1,k} \right) = D_+$.

Let us start with $D'_+ \cup D_+^{1,1} = D'_+ \cup D_+^{1',1}$, where we have denoted $D_+^{1',1} = D_+^{1,1} \setminus D'_+$. Adapting again Lemma 2.3 we have that $u_+|_{D'_+} \in H^{r+2}(D'_+)$ and $u_+|_{D_+^{1',1}} \in H^{r+2}(D_+^{1',1})$. One may prove by induction that $u_+ \in H^{2+r}(D'_+ \cup D_+^{1',1})$ in the following way. First, notice that $u_+ \in H^1(D'_+ \cup D_+^{1',1})$, as $u \in H^1(\mathbb{R}^d)$ anyway (Lemma 2.3 again). Assume now $u_+ \in H^m(D'_+ \cup D_+^{1',1})$ for $m < 2+r$. Then for any multi-index $|\alpha| \leq m$, we have $D^\alpha u_+ \in L^2(D'_+ \cup D_+^{1',1})$. From the fact that $u_+|_{D'_+} \in H^{r+2}(D'_+)$ and $u_+|_{D_+^{1',1}} \in H^{r+2}(D_+^{1',1})$ we have that $(D^\alpha u_+)|_{D'_+} \in H^1(D'_+)$ and $(D^\alpha u_+)|_{D_+^{1',1}} \in H^1(D_+^{1',1})$ (we adapt again Lemma 2.3 for the restriction aspect). Adapting now the result of Lemma 2.4 we have that $D^\alpha u_+ \in H^1(D'_+ \cup D_+^{1',1})$, which is the desired result, if $[u_+]_{\Gamma'_{1,+}} = 0$ a.e. on $\Gamma'_{1,+} = \partial D'_+ \cap \overline{D_+^{1',1}}$. But $\Gamma'_{1,+} \subset D_+^{1,1}$ and $u_+ \in H^{r+2}(D_+^{1,1})$. Adapting this time the necessary part of Lemma 2.4 we get that indeed $[u_+]_{\Gamma'_{1,+}} = 0$ a.e. on $\Gamma'_{1,+}$. Thus indeed $D^\alpha u_+ \in H^1(D'_+ \cup D_+^{1',1})$ and more precisely $D_i D^\alpha u_+ = D_i(D^\alpha u_+)|_{D'_+} + D_i(D^\alpha u_+)|_{D_+^{1',1}} \in L^2(D'_+ \cup D_+^{1',1})$ for any $1 \leq i \leq d$. The proof by induction that $u_+ \in H^{2+r}(D'_+ \cup D_+^{1',1}) = H^{2+r}(D'_+ \cup D_+^{1,1})$ is completed.

Repeating this procedure one proves by induction that $u_+ \in H^{r+2}(D'_+ \cup (\cup_{k=1}^N D_+^{1,k})) = H^{r+2}(D_+)$ (using in particular the fact that the $D_+^{1,k}$'s are in finite number). One proceeds in the same way on D_- . \square

Now that we have proved Theorem 2.14 we can get as a corollary the following result concerning the iterated domains $\mathcal{D}(A^k)$, $k \in \mathbb{N}^*$.

Corollary 2.18. *Let $k \in \mathbb{N}^*$ and $u \in \mathcal{D}(A^k)$. Assume that the coefficients $(a_\pm)_{ij} \in C_b^{2k-1}(D_\pm)$ and that Γ is bounded and of class C^{2k} . Then $u_\pm \in H^{2k}(D_\pm)$.*

Proof. The proof proceeds by induction on k .

Let $u \in \mathcal{D}(A)$ (case $k = 1$). We have $[\mathcal{B}_\nu u]_\Gamma = 0$, according to Proposition 2.13. Thus in particular $[\mathcal{B}_\nu u]_\Gamma \in H^{\frac{1}{2}}(\Gamma)$. As in the proof of Proposition 2.13 we set $f = u - Au$ and notice that we have $u_\pm - A_\pm u_\pm = f_\pm$ on D_\pm , with $f_\pm \in L^2(D_\pm)$.

Using Theorem 2.14 - remember that u is in $H^1(\mathbb{R}^d)$, $(a_\pm)_{ij} \in C_b^1(D_\pm; \mathbb{R})$ and Γ is bounded of class C^2 - we get that $u_\pm \in H^2(D_\pm)$.

Suppose now that the result is true at rank $k-1$ we prove its validity at rank k ($k \geq 2$). Let $u \in \mathcal{D}(A^k)$. As $u \in \mathcal{D}(A)$ we have $[\mathcal{B}_\nu u]_\Gamma = 0 \in H^{2k-\frac{3}{2}}(\Gamma)$. As $Au \in \mathcal{D}(A^{k-1})$ the quantity $u - Au =: f$ satisfies $f_\pm \in H^{2k-2}(D_\pm)$, using the induction hypothesis. But as we have $u_\pm - A_\pm u_\pm = f_\pm$ on D_\pm , one may use again the smoothness of $(a_\pm)_{ij}$ and Γ and Theorem 2.14 in order to conclude that $u_\pm \in H^{2k}(D_\pm)$. \square

2.2 The solution of the parabolic problem (\mathcal{P}_T)

2.2.1 Application of the Hille-Yosida theorem

We now use the Hille-Yosida theorem ([4] Theorems VII.4 and VII.5) in order to prove the following proposition. Note that in Equation (2.19) below, the time derivative is understood in the strong sense, while the space derivatives are understood in the weak sense. Besides, by convention $\mathcal{D}(A^0) = L^2(\mathbb{R}^d)$.

Proposition 2.19. *Let $u_0 \in \mathcal{D}(A)$. Then there exists a unique function*

$$u \in C^1([0, T]; L^2(\mathbb{R}^d)) \cap C([0, T]; \mathcal{D}(A))$$

satisfying

$$\frac{du}{dt} = Au, \quad u(0) = u_0. \quad (2.19)$$

Furthermore, let $u_0 \in \mathcal{D}(A^k)$, $k \geq 2$. Then,

$$u \in C^{k-j}([0, T]; \mathcal{D}(A^j)), \quad 0 \leq j \leq k.$$

Proof. According to [4] it suffices to show that the operator $(-A, \mathcal{D}(A))$ is maximal monotone. But thanks to Assumption **(E)** we immediately see that $\langle -Av, v \rangle_{L^2(\mathbb{R}^d)} = \mathcal{E}(v, v) \geq 0$, for any $v \in \mathcal{D}(A)$, and thanks to Proposition 2.12 we have that for any $f \in L^2(\mathbb{R}^d)$ there exists $u \in \mathcal{D}(A)$ solving (2.2). \square

Using now Propositions 2.13, Corollary 2.18 and Proposition 2.19 together with some Sobolev embedding theorems, we show Theorem 2.1.

2.2.2 Proof of Theorem 2.1

Proof. Assume d is even. Apply the result of Proposition 2.19 with $k = k_0 = \lfloor \frac{d}{4} \rfloor + 2$ and consider u solution of (2.19). We have that

$$u \in C^1([0, T]; \mathcal{D}(A^{k_0-1}))$$

with $k_0 - 1 = \lfloor \frac{d}{4} \rfloor + 1$. Using the result of Corollary 2.18 and combining Corollary IX.13 p. 168 with Theorem IX.7 p. 157 in [4], we see that for any $t \in [0, T]$

$$u_{\pm}(t, \cdot) \in H^{4+2\lfloor \frac{d}{4} \rfloor}(D_{\pm}) \subset H^{2+\frac{d}{2}}(D_{\pm}) \hookrightarrow C^2(\bar{D}_{\pm}). \quad (2.20)$$

Assume now that d is odd. Apply the result of Proposition 2.19 with $k = k_0 = \lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor + 2$ and consider u solution of (2.19). We have that

$$u \in C^1([0, T]; \mathcal{D}(A^{k_0-1}))$$

with $k_0 - 1 = \lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor + 1$. Using the result of Corollary 2.18 and combining Corollary IX.13 p. 168 with Theorem IX.7 p. 157 in [4], we see that for any $t \in [0, T]$

$$u_{\pm}(t, \cdot) \in H^{2+2\lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor}(D_{\pm}) \hookrightarrow C^2(\bar{D}_{\pm}) \quad (2.21)$$

since

$$\lfloor 2 + 2\lfloor \frac{3}{2} + \frac{\lfloor d/2 \rfloor}{2} \rfloor - \frac{d}{2} \rfloor \geq \lfloor 2 + 2\left(\frac{1}{2} + \frac{\lfloor d/2 \rfloor}{2}\right) - \frac{d}{2} \rfloor \geq \lfloor 3 + \lfloor \frac{d}{2} \rfloor - \frac{d}{2} \rfloor \geq 2.$$

Let us now show that u solution of (2.19) (for the corresponding k_0) is a classical solution of (\mathcal{P}_T) .

First, it is clear that $\mathcal{L}u$ coincides with Au on any bounded part of D_{\pm} (the derivatives in the distributional sense coincide with the classical derivatives thanks to the established smoothness of u). This shows the first line of (\mathcal{P}_T) .

Second, as for any $t \in [0, T]$ the function $u(t, \cdot)$ belongs to $\mathcal{D}(A)$, we have using the result of Proposition 2.13 that

$$[u(t, \cdot)]_{\Gamma} = 0 \quad \text{a.e. on } \Gamma; \quad [\mathcal{B}_{\nu}u(t, \cdot)]_{\Gamma} = 0 \quad \text{a.e. on } \Gamma. \quad (2.22)$$

Note that $u(t, \cdot) \in \mathcal{D}(A)$ implies that $u_{\pm}(t, \cdot)$ are in $H^2(D_{\pm})$. So that the second part of (2.22) reads

$$\nu^* (\gamma (a_+ \nabla u_+(t, \cdot)) - \gamma (a_- \nabla u_-(t, \cdot))) = 0 \quad \text{a.e.}$$

But as $(a_{\pm} \nabla u_{\pm}) \in C^1(\bar{D}_{\pm}; \mathbb{R}^d)$, we get

$$\langle (a_+ \nabla_x u_+(t, \cdot))(y) - (a_- \nabla_x u_-(t, \cdot))(y), \nu(y) \rangle = 0$$

for almost every $y \in \Gamma$, and consequently for every $y \in \Gamma$ by continuity. The same argument applies to the first part of (2.22) and the second and third lines of (\mathcal{P}_{Γ}) are satisfied. Note that the constructed solution satisfies $u(t, \cdot) \in C(\mathbb{R}^d)$ for any time $t \in [0, T]$.

Now let $k \geq k_0$. For $\lceil \frac{d}{4} \rceil \leq j \leq k$, we have $2j - \frac{d}{2} > 0$. Thus, for $v \in \mathcal{D}(A^j)$ we have from Corollary 2.18,

$$v_{\pm} \in H^{2j}(D_{\pm}) \hookrightarrow C^{m(j)}(\bar{D}_{\pm})$$

with $n(j) = \lfloor 2j - \frac{d}{2} \rfloor$. Using again the result of Proposition 2.19, we get the announced result. \square

2.3 Conclusion and consequences: boundedness of the partial derivatives

Going a bit further in the analysis, and using additional Sobolev embedding arguments, we can state the following result.

Proposition 2.20. *Let $p, q \in \mathbb{N}$ with $p + \lfloor q/2 \rfloor \geq 2$. Let $m = \lceil \frac{q}{2} + \frac{d}{4} \rceil$, $m' = m + 1$ and $k = m' + p$. Assume that the coefficients a_{ij} satisfy $(a_{\pm})_{ij} \in C_b^{2m'-1}(D_{\pm})$, that Γ is bounded and of class $C^{2m'}$, and that $u_0 \in \mathcal{D}(A^k)$.*

Then the classical solution $u(t, x)$ of (\mathcal{P}_{Γ}) constructed in Theorem 2.1 satisfies

$$u \in C^p([0, T]; C_b^q(\bar{D}_+) \cap C_b^q(\bar{D}_-)).$$

Proof. First, notice that it is easy to check that k is greater than the k_0 defined in Theorem 2.1, so that it makes sense speaking of the classical solution of (\mathcal{P}_{Γ}) , for $u_0 \in \mathcal{D}(A^k)$.

This solution is constructed in the same way as in Theorem 2.1, in particular by the mean of Proposition 2.19. So that one can assert that

$$u \in C^p([0, T]; \mathcal{D}(A^{m'})).$$

It remains to check that if $v \in \mathcal{D}(A^{m'})$, then $v_{\pm} \in C_b^q(\bar{D}_{\pm})$. First, note that $m \geq \lceil \frac{d}{4} \rceil$, and that one may easily check

$$\lfloor 2m - \frac{d}{2} \rfloor \geq q$$

(using in particular $\lceil 2a \rceil \leq 2\lceil a \rceil$). So that if $v \in \mathcal{D}(A^{m'}) \subset \mathcal{D}(A^m)$, we have, as for the second part of Theorem 2.1,

$$v_{\pm} \in H^{2m}(D_{\pm}) \hookrightarrow C^{\lfloor 2m - \frac{d}{2} \rfloor}(\bar{D}_{\pm}) \subset C^q(\bar{D}_{\pm}).$$

We claim that for any multi-index α , $|\alpha| \leq q$, the partial derivatives $\partial^{\alpha} v_{\pm}$ are bounded. Indeed, using again Corollary 2.18, we get

$$v_{\pm} \in H^{2m'}(D_{\pm}),$$

so that for α , $|\alpha| \leq q$,

$$\partial^{\alpha} v_{\pm} \in H^{2\lceil \frac{q}{2} + \frac{d}{4} \rceil - q + 2}(D_{\pm}) \subset H^{\frac{d}{2} + 2}(D_{\pm}) \hookrightarrow L^{\infty}(D_{\pm}).$$

Here we have used the fact $\frac{1}{2} - \frac{1}{2} - \frac{2}{d} < 0$, so that one can use the third embedding result of Corollary IX.13 in [4] (and again Theorem IX.7 for the projection argument). The result is proved. \square

From the above proposition we get the following control on the partial derivatives of the solution to (\mathcal{P}_{Γ}) .

Corollary 2.21. *In the context of Proposition 2.20 we have*

$$\sup_{t \in [0, T]} \sup_{x \in \bar{D}_\pm} |\partial_t^j \partial^\alpha u_\pm(t, x)| < \infty$$

for any $j \leq p$ and any multi-index α , with $|\alpha| \leq q$.

Proof. By Proposition 2.20 any of the considered partial derivatives of u_\pm belongs to the space

$$C([0, T]; C_b(\bar{D}_\pm)).$$

Let for example $v \in C([0, T]; C_b(\bar{D}_+))$. We prove the continuity of the map $t \mapsto \sup_{x \in \bar{D}_+} |v(t, x)|$, $t \in [0, T]$. Let $t_0 \in [0, T]$. Using the reverse triangle inequality we get for any $t \neq t_0$,

$$\left| \sup_{x \in \bar{D}_+} |v(t, x)| - \sup_{x \in \bar{D}_+} |v(t_0, x)| \right| \leq \sup_{x \in \bar{D}_+} |v(t, x) - v(t_0, x)|,$$

and we get the continuity at t_0 , as v is continuous from $[0, T]$ to $C_b(\bar{D}_+)$ (equipped with the supreme norm). Thus the desired continuity is proved, and from this we can assert that

$$\sup_{t \in [0, T]} \sup_{x \in \bar{D}_+} |v(t, x)| = \sup_{x \in \bar{D}_+} |v(t^*, x)|$$

for some $t^* \in [0, T]$. As $v(t^*, \cdot) \in C_b(\bar{D}_+)$ we have that

$$\sup_{t \in [0, T]} \sup_{x \in \bar{D}_+} |v(t, x)| < \infty.$$

The result is proved. \square

In the analysis of the convergence of our Euler scheme, we will use the above corollary with p up to 2 and q up to 4.

3 Euler scheme

3.1 Recalls on the projection and the distance to the transmission boundary and further notations and preliminaries

In this subsection we adopt the notations from [3]. We have the following set of fundamental results.

Proposition 3.1 ([3], Proposition 1; see also [13]). *Assume (\mathbf{D}) and (Γ) . There is constant $R > 0$ such that:*

1. (a) for any $x \in V_\Gamma^-(R)$, there are unique $s = \pi_\Gamma^{\gamma^+}(x) \in \Gamma$ and $F^{\gamma^+}(x) \leq 0$ such that :

$$x = \pi_\Gamma^{\gamma^+}(x) + F^{\gamma^+}(x) \gamma_+(\pi_\Gamma^{\gamma^+}(x)); \quad (3.1)$$

- (b) for any $x \in V_\Gamma^+(R)$, there are unique $s = \pi_\Gamma^{\gamma^-}(x) \in \Gamma$ and $F^{\gamma^-}(x) \leq 0$ such that :

$$x = \pi_\Gamma^{\gamma^-}(x) + F^{\gamma^-}(x) \gamma_-(\pi_\Gamma^{\gamma^-}(x)); \quad (3.2)$$

2. (a) the function $x \mapsto \pi_\Gamma^{\gamma^+}(x)$ is called the projection of x on Γ parallel to γ_+ : this is a C^4 function on $V_\Gamma^-(R)$;
(b) the function $x \mapsto \pi_\Gamma^{\gamma^-}(x)$ is called the projection of x on Γ parallel to γ_- : this is a C^4 function on $V_\Gamma^+(R)$;
3. Let us set $\tilde{F}^{\gamma^\pm}(x) = F^{\gamma^\pm}(x) |\gamma_\pm(\pi_\Gamma^{\gamma^\pm}(x))|$ the normalized version of F^{γ^\pm} corresponding to the unit vector field $\tilde{\gamma}_\pm : x \mapsto \frac{\gamma_\pm(x)}{|\gamma_\pm(x)|}$.

(a) the functions $x \mapsto \tilde{F}^{\gamma_{\pm}}(x)$ are called the algebraic distance of x to Γ parallel to γ_{\pm} (to $\tilde{\gamma}_{\pm}$): these are C^4 functions on $V_{\Gamma}^{\mp}(R)$. One has $F^{\gamma_+}, \tilde{F}^{\gamma_+} \leq 0$ on $V_{\Gamma}^-(R)$ and $F^{\gamma_-}, \tilde{F}^{\gamma_-} \leq 0$ on $V_{\Gamma}^+(R)$.

(b) It is possible to extend $F^{\gamma_+}, \tilde{F}^{\gamma_+}$ and $F^{\gamma_-}, \tilde{F}^{\gamma_-}$ to $C_b^4(\mathbb{R}^d, \mathbb{R})$ functions, with the conditions $F^{\gamma_{\pm}}, \tilde{F}^{\gamma_{\pm}} > 0$ on D_{\pm} and $F^{\gamma_{\pm}}, \tilde{F}^{\gamma_{\pm}} < 0$ on D_{\mp} .

4. The above extensions for $\tilde{F}^{\gamma_{\pm}}$ and F^{ν} can be performed in a way such that the functions $\tilde{F}^{\gamma_{\pm}}$ and F^{ν} are equivalent in the sense that

$$\frac{1}{c_1}d(x, \Gamma) = \frac{1}{c_1}|F^{\nu}(x)| \leq |\tilde{F}^{\gamma_{\pm}}(x)| \leq c_1|F^{\nu}(x)| = c_1d(x, \Gamma) \quad \text{for all } x \in \mathbb{R}^d \quad (3.3)$$

for some constant $c_1 > 1$.

5. For $x \in \Gamma$,

$$\nabla \tilde{F}^{\gamma_{\pm}}(x) = \frac{\nu^*}{\langle \nu, \tilde{\gamma}_{\pm} \rangle}(x). \quad (3.4)$$

We sometimes use the notation $\nu(x)$ or $\gamma_{\pm}(x)$ even if $x \notin \Gamma$. For $x \in V_{\Gamma}^{\pm}(R)$, we set $\nu(x) = \nu(\pi_{\Gamma}^{\pm}(x))$ and $\gamma_{\pm}(x) = \gamma_{\pm}(\pi_{\Gamma}^{\pm}(x))$ and for $x \notin V_{\Gamma}^{\pm}(R)$, arbitrary values are given.

Note that if u is a classical solution to the transmission parabolic problem (\mathcal{P}_{Γ}) defined in Section 2, the transmission condition (\star) can be expressed as

$$\langle \gamma_+(y), \nabla_x u_+(t, y) \rangle = -\langle \gamma_-(y), \nabla_x u_-(t, y) \rangle, \quad \forall (t, y) \in (0, T] \times \Gamma. \quad (3.5)$$

This in fact will be the crux of our approach (see Subsubsection 4.5.2).

In the sequel, we will need the following result.

Proposition 3.2. Assume **(E)**, **(D)** and **(\Gamma)**. Let $\hat{x} \in V_{\Gamma}^{\mp}(R)$ and $\bar{x} \in V_{\Gamma}^{\mp}(R)$ be linked by the following relation :

$$\bar{x} = \pi_{\Gamma}^{\gamma_{\pm}}(\hat{x}) - F^{\gamma_{\pm}}(\hat{x})\gamma_{\mp}(\pi_{\Gamma}^{\gamma_{\pm}}(\hat{x})). \quad (3.6)$$

Then, there exists $c_2 > 1$ such that

$$\frac{1}{c_2}d(\bar{x}, \Gamma) \leq d(\hat{x}, \Gamma) \leq c_2d(\bar{x}, \Gamma). \quad (3.7)$$

Proof. Without loss of generality, assume for example that $\bar{x} \in V_{\Gamma}^-(R)$ and $\hat{x} \in V_{\Gamma}^-(R)$ are related by (3.6). Then we have

$$\bar{x} - \pi_{\Gamma}^{\gamma_+}(\hat{x}) = -F^{\gamma_+}(\hat{x})\gamma_-(\pi_{\Gamma}^{\gamma_+}(\hat{x})). \quad (3.8)$$

and by uniqueness of the projection $\pi_{\Gamma}^{-\gamma_-}(\bar{x})$, we see that $\pi_{\Gamma}^{\gamma_+}(\hat{x}) = \pi_{\Gamma}^{-\gamma_-}(\bar{x})$ (note that $F^{-\gamma_-}(\bar{x}) = F^{\gamma_+}(\hat{x})$).

We deduce that

$$\frac{1}{c_1}d(\bar{x}, \Gamma) \leq |\tilde{F}^{-\gamma_-}(\bar{x})| = |F^{-\gamma_-}(\bar{x})| \times |\gamma_-(\pi_{\Gamma}^{-\gamma_-}(\bar{x}))| = |\bar{x} - \pi_{\Gamma}^{\gamma_+}(\hat{x})| = |\bar{x} - \pi_{\Gamma}^{-\gamma_-}(\bar{x})| \leq c_1d(\bar{x}, \Gamma)$$

due to the same kind of relation as (3.3), but written for $-\gamma_-$ instead of γ_- . Returning back to (3.8), we see that

$$\frac{1}{c_1}d(\bar{x}, \Gamma) \leq |F^{\gamma_+}(\hat{x})| \times |\gamma_-(\pi_{\Gamma}(\hat{x}))| = |\tilde{F}^{\gamma_+}(\hat{x})| \frac{|\gamma_-(\pi_{\Gamma}(\hat{x}))|}{|\gamma_+(\pi_{\Gamma}(\hat{x}))|} \leq c_1d(\bar{x}, \Gamma).$$

So that in view of (3.3) written for \hat{x} and γ_+ ,

$$\frac{1}{c_1^2} \frac{|\gamma_+(\pi_{\Gamma}(\hat{x}))|}{|\gamma_-(\pi_{\Gamma}(\hat{x}))|} d(\bar{x}, \Gamma) \leq d(\hat{x}, \Gamma) \leq c_1^2 \frac{|\gamma_+(\pi_{\Gamma}(\hat{x}))|}{|\gamma_-(\pi_{\Gamma}(\hat{x}))|} d(\bar{x}, \Gamma).$$

But using (1.5) and (1.6), it is easy to see that for any $z \in \Gamma$,

$$\frac{\lambda^2}{\Lambda^2} \leq \frac{|\gamma_+(z)|^2}{|\gamma_-(z)|^2} \leq \frac{\Lambda^2}{\lambda^2}$$

from which we deduce the result of the proposition. \square

4 Convergence rate of our Euler scheme

The purpose of this section is to prove the following result.

Theorem 4.1. *Let $0 < T < \infty$. Assume **(E)**, **(D)** and **(Γ)**. Let $m' = \lceil 2 + \frac{d}{4} \rceil + 1$ and $k = m' + 2$. Assume that the coefficients a_{ij} satisfy $(a_{\pm})_{ij} \in C_b^{2m'-1}(D_{\pm})$ and that Γ is of class $C^{2m'}$. Let $f : \mathbb{R}^d \rightarrow \mathbb{R}$ be in the space $\mathcal{D}(A^k)$. Let u be the classical solution of (\mathcal{P}_{Γ}) .*

We have that for all n large enough, and all x_0 in \mathbb{R}^d ,

$$\left| u(T, x_0) - \mathbb{E}^{x_0} f(\bar{X}_T^n) \right| \leq K \sqrt{h_n}, \quad (4.1)$$

where the constant K depends on d, λ, Λ, f and T .

Remark 4.2. *In this theorem the assumptions on $a(x)$ and Γ involving the integers m' and k are here in order to use Corollary 2.21, which ensures that we will have $\sup_{t \in [0, T]} \sup_{x \in \bar{D}_{\pm}} |\partial_t^j \partial^\alpha u_{\pm}(t, x)| < \infty$ for any $j \leq 2$ and any $|\alpha| \leq 4$. This control on the derivatives on u is what we need in order to lead our convergence proof. In fact if there is a way to get this control under weaker assumptions on $a(x)$ and Γ , this will lead to a convergence theorem stated under these weaker assumptions.*

4.1 Preliminary results

Lemma 4.3. *(see [3]) Consider an Itô process with uniformly bounded coefficients $dU_t = b_t dt + \sigma_t dW_t$. There exist some constants $c > 0$ and K (depending on $p \geq 1, T$ and the bounds on σ, b) such that, for any stopping times S and S' (with $0 \leq S \leq S' \leq \delta \leq T$) and any $\eta \geq 0$,*

$$\mathbb{P} \left[\sup_{t \in [S, S']} |U_t - U_s| \geq \eta \right] \leq K \exp \left(-c \frac{\eta^2}{\delta} \right); \quad (4.2)$$

$$\mathbb{E} \left[\sup_{t \in [S, S']} |U_t - U_s|^p \right] \leq K \delta^{p/2}. \quad (4.3)$$

We have when $\bar{X}_{t_k} \in D_+$

$$\bar{X}_{t_{k+1}} = \hat{X}_{t_{k+1}} + \left[F^{\gamma_+}(\hat{X}_{t_{k+1}}) \right]^- \left(\gamma_+(\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}})) + \gamma_-(\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}})) \right)$$

and when $X_{t_k} \in D_-$

$$\bar{X}_{t_{k+1}} = \hat{X}_{t_{k+1}} + \left[F^{\gamma_-}(\hat{X}_{t_{k+1}}) \right]^- \left(\gamma_+(\pi_{\Gamma}^{\gamma_-}(\hat{X}_{t_{k+1}})) + \gamma_-(\pi_{\Gamma}^{\gamma_-}(\hat{X}_{t_{k+1}})) \right)$$

This shows that $(\bar{X}_t)_{0 \leq t \leq T}$ behaves like a continuous semimartingale on each of the intervals $[t_k, t_{k+1})$. Using Tanaka's formula, we have – for example for $\bar{X}_{t_k} \in D_+$ – that for any $t \in [t_k, t_{k+1})$,

$$\begin{aligned} d\bar{X}_t &= d\hat{X}_t + \frac{1}{2} (\gamma_+ + \gamma_-) (\hat{X}_t) dL_t^0(F^{\gamma_+}(\hat{X})) \\ &+ \left[F^{\gamma_+}(\hat{X}_t) \right]^- \left(\nabla (\gamma_+ + \gamma_-) (\hat{X}_t) d\hat{X}_t + \frac{1}{2} \text{Tr} \left[\mathbf{H}[\gamma_+ + \gamma_-](\hat{X}_t) a(\bar{X}_{t_k}) \right] dt \right) \\ &- \mathbb{I}_{F^{\gamma_+}(\hat{X}_t) < 0} \left[\nabla (\gamma_+ + \gamma_-) (\hat{X}_t) a(\bar{X}_{t_k}) \left(\nabla F^{\gamma_+}(\hat{X}_t) \right)^* dt + (\gamma_+ + \gamma_-) (\hat{X}_t) \nabla F^{\gamma_+}(\hat{X}_t) d\hat{X}_t \right] \\ &- \mathbb{I}_{F^{\gamma_+}(\hat{X}_t) < 0} (\gamma_+ + \gamma_-) (\hat{X}_t) \frac{1}{2} \text{Tr} \left[\mathbf{H}[F^{\gamma_+}](\hat{X}_t) a(\bar{X}_{t_k}) \right] dt. \end{aligned} \quad (4.4)$$

Lemma 4.4. *Under Assumptions **(E)**, **(D)** and **(Γ)**, for all $c > 0$, there exists a constant $K(T)$ such that*

$$h_n \mathbb{E}^{x_0} \sum_{i=0}^{n-1} \left[\exp \left(-c \frac{d^2(\bar{X}_{t_i}^n, \Gamma)}{h_n} \right) \right] \leq K(T) \sqrt{h_n}. \quad (4.5)$$

Proof. The idea is to use the occupation times formula. Using successively (3.3) and the inequality (3.7) of Proposition 3.2, we have $d(\bar{x}, \Gamma) \geq \frac{1}{c_2} d(\hat{x}, \Gamma) \geq \frac{1}{c_1 c_2} |\tilde{F}^{\gamma_{\pm}}(\hat{x})|$ so that

$$\begin{aligned} \mathcal{A}_{i+1} &:= \mathbb{E}^{x_0} \left[\exp \left(-c \frac{d^2(\bar{X}_{t_{i+1}}^n, \Gamma)}{h_n} \right) \right] \\ &\leq \mathbb{E}^{x_0} \left[\exp \left(-c \frac{|\tilde{F}^{\gamma_+}(\hat{X}_{t_{i+1}}^n)|^2}{|c_1 c_2|^2 h_n} \right) \mathbb{I}_{\bar{X}_{t_{i+1}}^n \in D_-} \right] + \mathbb{E}^{x_0} \left[\exp \left(-c \frac{|\tilde{F}^{\gamma_-}(\hat{X}_{t_{i+1}}^n)|^2}{|c_1 c_2|^2 h_n} \right) \mathbb{I}_{\bar{X}_{t_{i+1}}^n \in D_+} \right] \\ &:= \mathcal{A}_{i+1}^+ + \mathcal{A}_{i+1}^-. \end{aligned} \quad (4.6)$$

We concentrate on term \mathcal{A}_{i+1}^+ as both terms are treated in a similar manner.

Set $c' = c/2c_1^2 c_2^2 > 0$ and $g(x) = \exp(-2c'x^2/h)$; it is easy to check that $|g(x)| + \sqrt{h}|g'(x)| + h|g''(x)| \leq K(T) \exp(-c'x^2/h)$. Hence, for $t \in [t_i, t_{i+1}]$, Itô's formula yields that

$$\begin{aligned} &\mathbb{E}^{x_0} \exp \left(-2c' \frac{|\tilde{F}^{\gamma_+}(\hat{X}_{t_{i+1}}^n)|^2}{h_n} \right) \\ &\leq K(T) \left[\mathbb{E}^{x_0} \exp \left(-c' \frac{|\tilde{F}^{\gamma_+}(\hat{X}_t^n)|^2}{h_n} \right) + \frac{1}{h_n} \int_t^{t_{i+1}} ds \mathbb{E}^{x_0} \exp \left(-c' \frac{|\tilde{F}^{\gamma_+}(\hat{X}_s^n)|^2}{h_n} \right) \right]. \end{aligned}$$

We integrate this inequality with respect to t over $[t_i, t_{i+1}]$ to get

$$h_n \mathcal{A}_{i+1}^+ \leq K(T) \int_{t_i}^{t_{i+1}} ds \mathbb{E}^{x_0} \exp \left(-c' \frac{|\tilde{F}^{\gamma_+}(\hat{X}_s^n)|^2}{h_n} \right). \quad (4.7)$$

(for possibly some new constant $K(T)$).

Observe that from (3.4),

$$d\langle \tilde{F}^{\gamma_+}(\hat{X}^n), \tilde{F}^{\gamma_+}(\hat{X}^n) \rangle_s = \nabla \tilde{F}^{\gamma_+}(\hat{X}_s^n) a(\bar{X}_{t_i}^n) \left[\nabla \tilde{F}^{\gamma_+}(\hat{X}_s^n) \right]^* ds \geq \lambda ds. \quad (4.8)$$

Indeed, using the Cauchy-Schwarz inequality and $|\nu(\hat{x})| = 1$, we have that

$$\begin{aligned} \nabla \tilde{F}^{\gamma_+}(\hat{x}) a(\bar{x}) \left[\nabla \tilde{F}^{\gamma_+}(\hat{x}) \right]^* &= \frac{\nu^*(\hat{x}) a(\bar{x}) \nu(\hat{x})}{\langle \nu(\hat{x}), \tilde{\gamma}_+(\hat{x}) \rangle^2} \\ &= \frac{\langle \nu(\hat{x}), a(\bar{x}) \nu(\hat{x}) \rangle}{\langle \nu(\hat{x}), \frac{a(\hat{x}) \nu(\hat{x})}{|a(\hat{x}) \nu(\hat{x})|} \rangle^2} \\ &\geq \frac{\langle \nu(\hat{x}), a(\bar{x}) \nu(\hat{x}) \rangle}{|\nu(\hat{x})|^2 |a(\hat{x}) \nu(\hat{x})|^2} |a(\hat{x}) \nu(\hat{x})|^2 = \langle \nu(\hat{x}), a(\bar{x}) \nu(\hat{x}) \rangle \geq \lambda \end{aligned}$$

which justifies (4.8).

It readily follows from the occupation times formula that

$$h_n \mathcal{A}_{i+1}^+ \leq K(T) \int_{-R}^R dy \exp \left(-c' \frac{y^2}{h_n} \right) \mathbb{E}^{x_0} \left[L_{t_{i+1}}^y \left(\tilde{F}^{\gamma_+}(\hat{X}^n) \right) - L_{t_i}^y \left(\tilde{F}^{\gamma_+}(\hat{X}^n) \right) \right]. \quad (4.9)$$

Now,

$$\begin{aligned} &\mathbb{E}^{x_0} \left[L_{t_{i+1}}^y \left(\tilde{F}^{\gamma_+}(\hat{X}^n) \right) - L_{t_i}^y \left(\tilde{F}^{\gamma_+}(\hat{X}^n) \right) \right] \\ &= 2\mathbb{E}^{x_0} \left[\left(\tilde{F}^{\gamma_+}(\hat{X}_{t_{i+1}}^n) - y \right)^+ - \left(\tilde{F}^{\gamma_+}(\hat{X}_{t_i}^n) - y \right)^+ - \int_{t_i}^{t_{i+1}} \mathbb{I}_{\tilde{F}^{\gamma_+}(\hat{X}_s^n) \geq y} d \left(\tilde{F}^{\gamma_+}(\hat{X}_s^n) \right) \right] \\ &\leq 2\mathbb{E}^{x_0} \left[\left(\tilde{F}^{\gamma_+}(\hat{X}_{t_{i+1}}^n) - y \right)^+ - \left(\tilde{F}^{\gamma_+}(\hat{X}_{t_i}^n) - y \right)^+ \right] + K(T) h_n. \end{aligned}$$

Therefore, $\sum_{i=0}^{n-1} \mathbb{E}^{x_0} \left[L_{t_{i+1}}^y \left(\tilde{F}^{\gamma+}(\hat{X}^n) \right) - L_{t_i}^y \left(\tilde{F}^{\gamma+}(\hat{X}^n) \right) \right] \leq K(T)$ uniformly in $|y| \leq R$ since the sum is telescoping. We can thus conclude that $h_n \sum_{i=0}^{n-1} \mathcal{A}_{i+1}^+ \leq K(T) \sqrt{h_n}$. The sum $h_n \sum_{i=0}^{n-1} \mathcal{A}_{i+1}^-$ is treated similarly. The proof of the Lemma is complete. \square

4.2 Error decomposition

In all the sequel x_0 is arbitrarily fixed.

For all $0 \leq k \leq n$ set

$$\theta_k^n := T - t_k^n.$$

The proof of Theorem 4.1 proceeds as follows (we omit the superscript n). Since $u(0, x) = f(x)$ for all $x \in \mathbb{R}^d$ and $u(T, x_0) = \mathbb{E}^{x_0} u(T, \bar{X}_0)$, the discretization error at time T can be decomposed as follows:

$$\begin{aligned} \epsilon_T^{x_0} &= \left| u(T, x_0) - \mathbb{E}^{x_0} f(\bar{X}_T) \right| \\ &= \left| \sum_{k=0}^{n-1} \mathbb{E}^{x_0} u(T - t_k, \bar{X}_{t_k}) - \mathbb{E}^{x_0} u(T - t_{k+1}, \bar{X}_{t_{k+1}}) \right|, \end{aligned} \quad (4.10)$$

and thus

$$\begin{aligned} \epsilon_T^{x_0} &\leq \left| \sum_{k=0}^{n-1} \mathbb{E}^{x_0} \left\{ u(\theta_k, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_k}) \right. \right. \\ &\quad \left. \left. + u(\theta_{k+1}, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_{k+1}}) \right\} \right|. \end{aligned} \quad (4.11)$$

The rest of this section is devoted to the analysis of

$$\left| \sum_{k=0}^{n-1} \mathbb{E}^{x_0} (T_k - S_k) \right|,$$

where the time increment T_k is defined as

$$T_k := u(\theta_k, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_k}) \quad (4.12)$$

and the space increment is defined as

$$S_k := u(\theta_{k+1}, \bar{X}_{t_{k+1}}) - u(\theta_{k+1}, \bar{X}_{t_k}). \quad (4.13)$$

4.3 Estimate for the time increment T_k

Remember the definition (4.12) of T_k and that $\theta_k = T - t_k$. We have

$$\begin{aligned} &\left\{ u(\theta_k, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_k}) \right\} \mathbb{I}_{\bar{X}_{t_k} \in D_+} \\ &= h_n \partial_t u(\theta_{k+1}, \bar{Y}_{t_k}) \mathbb{I}_{\bar{X}_{t_k} \in D_+} + h_n^2 \int_{[0,1]^2} \partial_{tt}^2 u(\theta_{k+1} + \alpha_1 \alpha_2 h_n, \bar{X}_{t_k}) \alpha_1 d\alpha_1 d\alpha_2 \mathbb{I}_{\bar{X}_{t_k} \in D_+} \\ &=: T_k^+ + R_k^+. \end{aligned}$$

Similarly,

$$\begin{aligned} &\left\{ u(\theta_k, \bar{X}_{t_k}) - u(\theta_{k+1}, \bar{X}_{t_k}) \right\} \mathbb{I}_{\bar{X}_{t_k} \in D_-} \\ &= h_n \partial_t u(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\bar{X}_{t_k} \in D_-} + h_n^2 \int_{[0,1]^2} \partial_{tt}^2 u(\theta_{k+1} + \alpha_1 \alpha_2 h_n, \bar{X}_{t_k}) \alpha_1 d\alpha_1 d\alpha_2 \mathbb{I}_{\bar{X}_{t_k} \in D_-} \\ &=: T_k^- + R_k^-. \end{aligned}$$

In view of Corollary 2.21 and Remark 4.2 we have

$$\mathbb{E}^{x_0} |R_k^+ + R_k^-| \leq C h_n^2.$$

From the preceding we deduce

$$\mathbb{E}^{x_0} T_k = \mathbb{E}^{x_0} \partial_t u(\theta_{k+1}, \bar{X}_{t_k}) h_n + O(h_n^2). \quad (4.14)$$

4.4 Expansion of the space increment S_k

Let S_k be defined as in (4.13). Set

$$\begin{aligned}\Delta_{k+1}\bar{X} &:= \bar{X}_{t_{k+1}} - \bar{X}_{t_k} \\ \Delta_{k+1}^\sharp\bar{X} &:= \hat{X}_{t_{k+1}} - \bar{X}_{t_k}.\end{aligned}$$

and recall that $\Delta_{k+1}W = W_{t_{k+1}} - W_{t_k}$.

Proposition 4.5.

$$\mathbb{E}^{x_0} |(\Delta_{k+1}^\sharp\bar{X})^\alpha| \leq C(\alpha)h_n^{|\alpha|/2}. \quad (4.15)$$

Proof. This is a consequence of the result of Lemma 4.3 combined with the fact that $|(x)^\alpha| \leq |x|^{|\alpha|}$ for any $x \in \mathbb{R}^d$. \square

We emphasize that, due to the definition of our stochastic numerical scheme, $\Delta_{k+1}^\sharp\bar{X}$ does not coincide with $\bar{X}_{t_{k+1}} - \bar{X}_{t_k}$ when $\bar{X}_{t_{k+1}}$ and \bar{X}_{t_k} do not belong to the same region, which explains the two notations Δ and Δ^\sharp .

We need to introduce the four following events:

$$\begin{cases} \Omega_k^{++} & := [\bar{X}_{t_k} \in \bar{D}_+ \text{ and } \hat{X}_{t_{k+1}} \in D_+], \\ \Omega_k^{--} & := [\bar{X}_{t_k} \in D_- \text{ and } \hat{X}_{t_{k+1}} \in \bar{D}_-], \\ \Omega_k^{+-} & := [\bar{X}_{t_k} \in D_+ \text{ and } \hat{X}_{t_{k+1}} \in \bar{D}_-], \\ \Omega_k^{-+} & := [\bar{X}_{t_k} \in \bar{D}_- \text{ and } \hat{X}_{t_{k+1}} \in D_+]. \end{cases} \quad (4.16)$$

In view of the definition of our stochastic numerical scheme we have

$$\text{On } \Omega_k^{++}, \Delta_{k+1}\bar{X} = \Delta_{k+1}^\sharp\bar{X}.$$

Therefore

$$\begin{aligned}S_k \mathbb{I}_{\Omega_k^{++}} &= \langle \Delta_{k+1}\bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \mathbb{I}_{\Omega_k^{++}} + \frac{1}{2} (\Delta_{k+1}\bar{X})^* \mathbf{H}[u](\theta_{k+1}, \bar{X}_{t_k}) \Delta_{k+1}\bar{X} \mathbb{I}_{\Omega_k^{++}} \\ &\quad + \sum_{|\alpha|=3} \frac{1}{\alpha!} (\Delta_{k+1}\bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\Omega_k^{++}} \\ &\quad + \int_0^1 d\xi \sum_{|\alpha|=4} \frac{(1-\xi)^4}{\alpha!} (\Delta_{k+1}\bar{X})^\alpha \frac{\partial^4 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k} + \xi \Delta_{k+1}\bar{X}) \mathbb{I}_{\Omega_k^{++}} \\ &=: S_k^{++1} + S_k^{++2} + S_k^{++3} + S_k^{++4}.\end{aligned}$$

Similarly,

$$\begin{aligned}S_k \mathbb{I}_{\Omega_k^{--}} &= \langle \Delta_{k+1}\bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \mathbb{I}_{\Omega_k^{--}} + \frac{1}{2} (\Delta_{k+1}\bar{X})^* \mathbf{H}[u](\theta_{k+1}, \bar{X}_{t_k}) \Delta_{k+1}\bar{X} \mathbb{I}_{\Omega_k^{--}} \\ &\quad + \sum_{|\alpha|=3} \frac{1}{\alpha!} (\Delta_{k+1}\bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\Omega_k^{--}} \\ &\quad + \int_0^1 d\xi \sum_{|\alpha|=4} \frac{(1-\xi)^4}{\alpha!} (\Delta_{k+1}\bar{X})^\alpha \frac{\partial^4 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k} + \xi \Delta_{k+1}\bar{X}) \mathbb{I}_{\Omega_k^{--}} \\ &=: S_k^{--1} + S_k^{--2} + S_k^{--3} + S_k^{--4}.\end{aligned}$$

We now use that $\Omega_k^{++} \cup \Omega_k^{--} = \Omega - (\Omega_k^{+-} \cup \Omega_k^{-+})$. Notice that $\Omega_k^{+-} \cup \Omega_k^{-+}$ belongs to the σ -field generated by (W_t) up to time t_{k+1} . In view of the first line of (3.9) and the fact that $\mathbb{E}^{\mathcal{F}^{t_k}} \Delta W_{k+1} = 0$, we get

$$\begin{aligned}\mathbb{E}^{x_0}(S_k^{++1} + S_k^{--1}) &= \frac{h_n}{2} \mathbb{E}^{x_0} [\langle \partial a(\bar{X}_{t_k}), \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle] \\ &\quad - \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^\sharp\bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} \right]\end{aligned}$$

Proceeding similarly and conditioning $(\Delta_{k+1}^\sharp \bar{X})^2$ w.r.t. the past of (W_t) up to time t_k , we obtain

$$\begin{aligned} \mathbb{E}^{x_0}(S_k^{++2} + S_k^{--2}) &= \frac{1}{2} \mathbb{E}^{x_0} [\text{Tr}[\sigma \mathbf{H}[u] \sigma^*](\theta_{k+1}, \bar{X}_{t_k})] h_n \\ &\quad - \frac{1}{2} \mathbb{E}^{x_0} \left[(\Delta_{k+1}^\sharp \bar{X})^* \mathbf{H}[u](\theta_{k+1}, \bar{X}_{t_k}) \Delta_{k+1}^\sharp \bar{X} \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} \right], \end{aligned}$$

and, since $\mathbb{E}^{x_0}(\Delta_{k+1} W)^\alpha = 0$ whenever $|\alpha| = 3$,

$$\begin{aligned} \mathbb{E}^{x_0}(S_k^{+++3} + S_k^{---3}) &= \sum_{|\alpha|=3} \frac{1}{\alpha!} \mathbb{E}^{x_0} \left[(\Delta_{k+1}^\sharp \bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \right] \\ &\quad - \sum_{|\alpha|=3} \frac{1}{\alpha!} \mathbb{E}^{x_0} \left[(\Delta_{k+1}^\sharp \bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} \right]. \end{aligned}$$

We have, combining the results of Corollary 2.21 and Proposition 4.5,

$$\left| \sum_{|\alpha|=3} \frac{1}{\alpha!} \mathbb{E}^{x_0} \left[(\Delta_{k+1}^\sharp \bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \right] \right| \leq C h_n^{3/2}. \quad (4.17)$$

In addition, and for the same reasons, we have

$$\mathbb{E}^{x_0} |S_k^{++4} + S_k^{--4}| \leq C h_n^2.$$

To summarize the calculations of this subsection, we have obtained

$$\begin{aligned} \mathbb{E}^{x_0} S_k &= \\ &\mathbb{E}^{x_0} \mathcal{L}u(\theta_{k+1}, \bar{X}_{t_k}) h_n + \mathbb{E}^{x_0} \left[\left(S_k - \langle \Delta_{k+1}^\sharp \bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \right) \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} \right] \\ &- \mathbb{E}^{x_0} \left[\frac{1}{2} (\Delta_{k+1}^\sharp \bar{X})^* \mathbf{H}[u](\theta_{k+1}, \bar{X}_{t_k}) \Delta_{k+1}^\sharp \bar{X} \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} + \sum_{|\alpha|=3} \frac{1}{\alpha!} (\Delta_{k+1}^\sharp \bar{X})^\alpha \frac{\partial^3 u}{\partial x^\alpha}(\theta_{k+1}, \bar{X}_{t_k}) \mathbb{I}_{\Omega_k^{+-} \cup \Omega_k^{-+}} \right] \\ &+ O(h_n^{3/2}) \\ &=: \mathbb{E}^{x_0} \mathcal{L}u(\theta_{k+1}, \bar{X}_{t_k}) h_n + \mathbb{E}^{x_0} \mathcal{R}_k^{(1)} - \mathbb{E}^{x_0} \mathcal{R}_k^{(2)} + O(h_n^{3/2}). \end{aligned} \quad (4.18)$$

We now estimate the remaining terms $\mathbb{E}^{x_0} \mathcal{R}_k^{(1)}$ and $\mathbb{E}^{x_0} \mathcal{R}_k^{(2)}$.

4.5 Control of the term $\mathbb{E}^{x_0} \mathcal{R}_k^{(1)}$. Expansion around a well chosen point in Γ

On the event Ω_k^{+-} we have that $\bar{X}_{t_{k+1}}$ and \bar{X}_{t_k} are close to Γ . On this event, we also have that $\hat{X}_{t_{k+1}} \in D_-$ and $\bar{X}_{t_k} \in D_+$. Remember our definition of $(F^{\gamma+}(x), \pi_\Gamma^{\gamma+}(x))$ for $x \in D_-$.

4.5.1 Decomposition of $\mathbb{E}^{x_0} \mathcal{R}_k^{(1)}$

As the function u is continuous across the surface Γ at point $\pi_\Gamma^{\gamma^+}(x)$, we get

$$\begin{aligned}
& \mathbb{E}^{x_0} \left(\left(S_k - \langle \Delta_{k+1}^\# \bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) \rangle \right) \mathbb{I}_{\Omega_k^{+-}} \right) \\
&= \mathbb{E}^{x_0} \left[\left(\left(u(\theta_{k+1}, \bar{X}_{t_{k+1}}) - u(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \right) + \left(u(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) - u(\theta_{k+1}, \bar{X}_{t_k}) \right) \right) \mathbb{I}_{\Omega_k^{+-}} \right] \\
&\quad - \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^\# \bar{X}, \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \\
&\quad - \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^\# \bar{X}, \nabla_x u(\theta_{k+1}, \bar{X}_{t_k}) - \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \\
&= \left\{ \mathbb{E}^{x_0} \left[\langle \bar{X}_{t_{k+1}} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}}), \nabla_x u_-(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \right. \\
&\quad - \mathbb{E}^{x_0} \left[\langle \bar{X}_{t_k} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}}), \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \\
&\quad \left. - \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^\# \bar{X}, \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \right\}_{:=L_k^{+-1}} \\
&\quad + \left\{ \int_0^1 d\xi \sum_{|\alpha|=2} \frac{(1-\xi)^2}{\alpha!} \mathbb{E}^{x_0} \left[(\hat{X}_{t_{k+1}} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}}))^\alpha \right. \right. \\
&\quad \quad \left. \left. \times \frac{\partial^\alpha u}{\partial x^\alpha}(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) + \xi(\hat{X}_{t_{k+1}} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \right] \mathbb{I}_{\Omega_k^{+-}} \right] \\
&\quad - \int_0^1 d\xi \sum_{|\alpha|=2} \frac{(1-\xi)^2}{\alpha!} \mathbb{E}^{x_0} \left[(\bar{X}_{t_k} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}}))^\alpha \right. \\
&\quad \quad \left. \left. \times \frac{\partial^\alpha u}{\partial x^\alpha}(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) + \xi(\bar{X}_{t_k} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \right] \mathbb{I}_{\Omega_k^{+-}} \right] \left. \right\}_{:=L_k^{+-2}} \\
&\quad - \left\{ \mathbb{E}^{x_0} \left[\langle \Delta_{k+1}^\# \bar{X}, \nabla_x u_+(\theta_{k+1}, \bar{X}_{t_k}) - \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \mathbb{I}_{\Omega_k^{+-}} \right] \right\}_{:=L_k^{+-3}} \\
&= L_k^{+-1} + L_k^{+-2} + L_k^{+-3}.
\end{aligned}$$

4.5.2 Canceling the term L_k^{+-1} using the transmission condition

Observe that due to the fact that

$$\left(\hat{X}_{t_k} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}}) \right) + \left(\hat{X}_{t_{k+1}} - \hat{X}_{t_k} \right) = \hat{X}_{t_{k+1}} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}}).$$

we have that

$$\begin{aligned}
L_k^{+-1} &= \mathbb{E}^{x_0} \left[\left(\langle \bar{X}_{t_{k+1}} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}}), \nabla_x u_-(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \right. \right. \\
&\quad \left. \left. - \langle \hat{X}_{t_{k+1}} - \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}}), \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \right) \mathbb{I}_{\Omega_k^{+-}} \right] \\
&= \mathbb{E}^{x_0} \left[F^{\gamma^+}(\hat{X}_{t_{k+1}}) \left(\langle -\gamma_-(\pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})), \nabla_x u_-(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \right. \right. \\
&\quad \left. \left. - \langle \gamma_+(\pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})), \nabla_x u_+(\theta_{k+1}, \pi_\Gamma^{\gamma^+}(\hat{X}_{t_{k+1}})) \rangle \right) \mathbb{I}_{\Omega_k^{+-}} \right] \\
&= 0,
\end{aligned}$$

where we have used the vector problem solved by $(F^{\gamma^+}, \pi_\Gamma^{\gamma^+})$ and Equation (3.5) (i.e. the transmission condition (\star) and the definition of $\gamma_\pm(x)$).

4.5.3 The term L_k^{+-2}

We now turn to the term L_k^{+-2} .

The term L_k^{+-2} is the sum of two terms. These two terms are treated similarly, so we concentrate only on the first. Let α such that $|\alpha| = 2$. We have that

$$\mathbb{E}^{x_0} \left[\left| (\bar{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}))^{\alpha} \right| \mathbb{I}_{\Omega_k^{+-}} \right] \leq c_1 \mathbb{E}^{x_0} \left[|\bar{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})|^2 \mathbb{I}_{\Omega_k^{+-}} \right] \leq c_2 \mathbb{E}^{x_0} \left[|\Delta_{k+1}^{\#} X|^2 \mathbb{I}_{\Omega_k^{+-}} \right]$$

The same kind of treatment can be performed for the second term of L_k^{+-2} . Conditioning w.r.t \mathcal{F}_{t_k} and applying the Cauchy-Schwarz inequality in the conditionnal expectation, we find using the result of Lemma 4.3,

$$|L_k^{+-2}| \leq C \mathbb{E}^{x_0} \left[\mathbb{E}^{\mathcal{F}_{t_k}} \left[|\Delta_{k+1}^{\#} X|^4 \right]^{1/2} \mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{+-})^{1/2} \right] \leq C h_n \mathbb{E}^{x_0} \mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{+-})^{1/2}.$$

4.5.4 The term L_k^{+-3}

For the term L_k^{+-3} , we may perform a Taylor's expansion to the term

$$\nabla_x u_+(\theta_{k+1}, \bar{X}_{t_k}) - \nabla_x u_+(\theta_{k+1}, \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})).$$

Using Corollary 2.21 and the Cauchy-Schwarz inequality, we find that

$$|L_k^{+-3}| \leq C \mathbb{E}^{x_0} \left[|\Delta_{k+1}^{\#} \bar{X}| |\bar{X}_{t_k} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}})| \mathbb{I}_{\Omega_k^{+-}} \right] \leq C \mathbb{E}^{x_0} \left[|\Delta_{k+1}^{\#} X|^2 \mathbb{I}_{\Omega_k^{+-}} \right]. \quad (4.19)$$

Finally, as for the term L_k^{+-2} , we find that

$$|L_k^{+-3}| \leq C h_n \mathbb{E}^{x_0} \mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{+-})^{1/2}.$$

Using the same method for the other side Ω_k^{-+} , we find that

$$\mathbb{E}^{x_0} \mathcal{R}_k^{(1)} \leq C h_n \mathbb{E}^{x_0} \left(\mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{+-})^{1/2} + \mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{-+})^{1/2} \right).$$

4.6 Summing up

The term $\mathbb{E}^{x_0} \mathcal{R}_k^{(2)}$ can be estimated using the same techniques used in the previous section and we omit the details.

Using now the fact that $\partial_t u - \mathcal{L}u = 0$, we finally find that

$$\epsilon_T^{x_0} \leq C h_n \mathbb{E}^{x_0} \sum_{k=0}^{n-1} \left(\mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{+-})^{1/2} + \mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{-+})^{1/2} \right) + C \sqrt{h_n}. \quad (4.20)$$

Observe – using the result of Lemma 4.3 – that

$$\begin{aligned} \mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{+-})^{1/2} &= \mathbb{P}^{\mathcal{F}_{t_k}} \left(\bar{X}_{t_k} \in D_+, \hat{X}_{t_{k+1}} \in D_- \right)^{1/2} \\ &\leq \mathbb{P}^{\mathcal{F}_{t_k}} \left(\|\hat{X}_{t_{k+1}} - \bar{X}_{t_k}\| \geq d(\bar{X}_{t_k}, \Gamma) \right)^{1/2} \\ &\leq K(T) \exp \left(-\frac{1}{2} \frac{d^2(\bar{X}_{t_k}, \Gamma)}{h_n} \right) \end{aligned}$$

and the same kind of inequality holds true for $\mathbb{P}^{\mathcal{F}_{t_k}}(\Omega_k^{-+})^{1/2}$.

Finally,

$$\epsilon_T^{x_0} \leq K(T) h_n \mathbb{E}^{x_0} \sum_{k=0}^{n-1} \exp \left(-\frac{1}{2} \frac{d^2(\bar{X}_{t_k}, \Gamma)}{h_n} \right) + C \sqrt{h_n},$$

and we conclude the proof of Theorem 4.1 using the result of Lemma 4.4 (note that if we sum up all the dependancies of our constants, we indeed have that K in (4.1) depends on d, λ, Λ, f and T).

5 Numerical experiments

Example 1. In this example $d = 2$ and the domain D is the open unit disc, i.e.,

$$D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 < 1\}.$$

Note that the boundary of D is the unit circle $\partial D = \{(x_1, x_2) \in \mathbb{R}^2 : x_1^2 + x_2^2 = 1\}$.

The subdomains D_+ and D_- are defined by

$$D_+ = \{(x_1, x_2) \in D \text{ with } x_2 > 0\} \quad \text{and} \quad D_- = \{(x_1, x_2) \in D \text{ with } x_2 < 0\},$$

so that the interface is $\Gamma = \{(x_1, 0) \in \mathbb{R}^2 : -1 \leq x_1 \leq 1\}$.

The diffusion matrix is defined by

$$a(x) = a_+(x)\mathbb{I}_{x \in D_+} + a_-(x)\mathbb{I}_{x \in \bar{D}_-},$$

with

$$a_{\pm}(x) = P_{\pm}^* D_{\pm}(x) P_{\pm}$$

where P_{\pm} are rotation (therefore orthogonal) matrices

$$P_{\pm} = \begin{pmatrix} \cos(\theta_{\pm}) & -\sin(\theta_{\pm}) \\ \sin(\theta_{\pm}) & \cos(\theta_{\pm}) \end{pmatrix}$$

(for $\theta_{\pm} \in [0, 2\pi)$), and $D_{\pm}(x)$ are diagonal matrix-valued functions

$$D_{\pm}(x) = \begin{pmatrix} \lambda_{\pm}^1 + \epsilon_{\pm} x_2 & 0 \\ 0 & \lambda_{\pm}^2 + \epsilon_{\pm} x_2 \end{pmatrix}$$

where $\lambda_{\pm}^1, \lambda_{\pm}^2 > 0$ and $\epsilon_{\pm} < \lambda_{\pm}^i$ for $i = 1, 2$. Note that this ensures that $a(x)$ satisfies the uniform ellipticity assumption **(E)**.

We take $\theta_+ = \frac{\pi}{4}$, $\theta_- = \frac{\pi}{3}$, $\lambda_+^1 = 1$, $\lambda_+^2 = 9$, $\lambda_-^1 = 2$, $\lambda_-^2 = 3$, $\epsilon_+ = 0.5$ and $\epsilon_- = 1.9$. This gives

$$a_+(x) = \frac{1}{2} \begin{pmatrix} 5 + 0.5x_2 & 4 \\ 4 & 5 + 0.5x_2 \end{pmatrix} \quad \text{and} \quad a_-(x) = \frac{1}{2} \begin{pmatrix} \frac{11}{4} + 1.9x_2 & \frac{\sqrt{3}}{4} \\ \frac{\sqrt{3}}{4} & \frac{9}{4} + 1.9x_2 \end{pmatrix}.$$

We have the Cholesky decompositions $2a_{\pm}(x) = \sigma_{\pm}\sigma_{\pm}^*(x)$, with

$$\sigma_+(x) = \begin{pmatrix} \sqrt{5 + 0.5x_2} & 0 \\ 4/\sqrt{5 + 0.5x_2} & \sqrt{5 + 0.5x_2 - 16/(5 + 0.5x_2)} \end{pmatrix}$$

and

$$\sigma_-(x) = \begin{pmatrix} \sqrt{\frac{11}{4} + 1.9x_2} & 0 \\ \frac{\sqrt{3}}{4}/\sqrt{\frac{11}{4} + 1.9x_2} & \sqrt{\frac{9}{4} + 1.9x_2 - 3/(44 + 30.4x_2)} \end{pmatrix},$$

so that $2a(x) = \sigma\sigma^*(x)$ with $\sigma(x) = \sigma_+(x)\mathbb{I}_{x \in D_+} + \sigma_-(x)\mathbb{I}_{x \in \bar{D}_-}$. Besides we have

$$\partial a(x) = \begin{pmatrix} 0 \\ 0.25 \end{pmatrix} \mathbb{I}_{x \in D_+} + \begin{pmatrix} 0 \\ 0.95 \end{pmatrix} \mathbb{I}_{x \in \bar{D}_-}.$$

Having this quantities in hand we can perform our scheme \bar{X} .

Note that when the scheme crosses the interface Γ , we compute the quantities $\pi_{\Gamma}^{\gamma_{\pm}}(\hat{X}_{t_{k+1}})$, $F^{\gamma_{\pm}}(\hat{X}_{t_{k+1}})$ in the following way (we will detail the procedure for $\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}})$ and $F^{\gamma_+}(\hat{X}_{t_{k+1}})$). Recall that we have

$$\hat{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}}) = F^{\gamma_+}(\hat{X}_{t_{k+1}})\gamma_+(\pi_{\Gamma}^{\gamma_+}(\hat{X}_{t_{k+1}})).$$

But here $\nu = (0, 1)^*$ so that for any $x \in \Gamma$

$$\gamma_+(x) = \frac{1}{2} \begin{pmatrix} 4 \\ 5 + 0.5x_2 \end{pmatrix}$$

and $(\pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}))_2 = 0$ so that $(\hat{X}_{t_{k+1}} - \pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}))_2 = (\hat{X}_{t_{k+1}})_2$. This yields

$$F^{\gamma+}(\hat{X}_{t_{k+1}}) = \frac{(\hat{X}_{t_{k+1}})_2}{2.5},$$

and then

$$\pi_{\Gamma}^{\gamma+}(\hat{X}_{t_{k+1}}) = \begin{pmatrix} (\hat{X}_{t_{k+1}})_1 - F^{\gamma+}(\hat{X}_{t_{k+1}}) \times 2 \\ 0 \end{pmatrix}.$$

We wish here to treat the elliptic transmission problem

$$(\mathcal{E}_{\Gamma}^0) \left\{ \begin{array}{ll} \mathcal{L}u(x) = 0 & \forall x \in D \\ \langle a_+ \nabla_x u_+(y) - a_- \nabla u_-(y), \nu(y) \rangle = 0 & \forall y \in \Gamma \\ u(y+) = u(y-) & \forall y \in \Gamma \\ u(x) = f(x) & \forall x \in \partial D \end{array} \right.$$

Consider then on one side our study of the convergence in the parabolic case, and on the other side the Feynman-Kac representation for elliptic PDEs available in the smooth case (see for instance Theorem 5.7.2 in [15]). One can hope that

$$\mathbb{E}^x[f(\bar{X}_{\bar{\tau}})] \xrightarrow{h_n \rightarrow 0} u(x),$$

where \bar{X} denotes our scheme and $\bar{\tau} = \inf\{t \geq 0 : \bar{X}_t \notin D\}$.

We thus compute a Monte Carlo approximation of $\mathbb{E}^x[f(\bar{X}_{\bar{\tau}})]$ on one side (with $N = 10^6$ paths). Note that in this Monte Carlo procedure we have used a boundary shifting method, on order to reduce the bias introduced by the approximation of the exit time $\tau = \inf\{t \geq 0 : X_t \notin D\}$ by $\bar{\tau}$ (see [14] Subsection 5.4.3, and the references therein). On the other side we use the software FREEFEM to compute an approximation of $u(x)$ by a finite element method, using around 1.5×10^6 triangles and 7×10^5 vertices (finite elements basis consists of polynomial functions of order 1).

We take the function f to be

$$f(x) = \sin(3x_1) + \cos(4x_2).$$

Table 1 shows the results (we have also performed an Euler scheme without any correction at the interface Γ ; this is to highlight the fact that, even with a small time step, it is impossible to approach the right value with such a scheme).

Point x	Finite Element ($7 \cdot 10^5$ vertices)	Not Transformed Euler Scheme ($h_n = 10^{-n}$, $n = 5, 6$)	Transformed Euler Scheme ($h_n = 10^{-n}$, $n = 2, 4, 5, 6$)
$x = (0, 0.5)^*$	-0.1207	-	-0.136356
		-	-0.121001
		-0.105539	-0.121299
		-0.100914	-0.120821
$x = (0.9, 0.05)^*$	0.92527	-	0.824901
		-	0.924759
		0.815699	0.925370
		0.816116	0.925389
$x = (-0.3, -0.5)^*$	-0.745461	-	-0.737754
		-	-0.746226
		-0.766957	-0.745676
		-0.767783	-0.745829

Table 1: Approximated values of the solution $u(x)$ of (\mathcal{E}_T^0) at points $x = (0, 0.1)^*$, $(0.9, 0.2)^*$, $(-0.3, -0.5)^*$ computed with a finite element method ($7 \cdot 10^5$ vertices), a not-transformed Euler scheme and our transformed Euler scheme (with $N = 10^6$ Monte Carlo sample, and different values of h_n).

6 Appendix

Lemma 6.1. *Let $\Gamma = \mathbb{R}^{d-1} \times \{0\}$ ($d > 1$). We have*

$$\forall v \in H^{\frac{1}{2}}(\Gamma), \quad \forall 1 \leq j \leq d-1, \quad D_j v \in H^{-\frac{1}{2}}(\Gamma) \text{ with } \|D_j v\|_{H^{-1/2}(\Gamma)} \leq C \|v\|_{H^{1/2}(\Gamma)},$$

where C is a universal constant.

Proof. STEP1. We prove that for any $v \in H^s(\mathbb{R}^n)$, $s \in \mathbb{R}$, $n \in \mathbb{N}^*$, we have $D_j v \in H^{s-1}(\mathbb{R}^n)$, for any $1 \leq j \leq n$, with $\|D_j v\|_{H^{s-1}(\mathbb{R}^n)} \leq C \|v\|_{H^s(\mathbb{R}^n)}$.

We denote $\mathcal{F} : v \mapsto \hat{v}$, $v \in \mathcal{S}(\mathbb{R}^n)$.

Let $\varphi \in \mathcal{S}(\mathbb{R}^n)$, $1 \leq j \leq n$, we have $\mathcal{F}\{(-i2\pi x_j)\varphi(x)\}(\xi) = D_j \hat{\varphi}(\xi)$ (see [29] p72). From this and (1.2) it is an exercise to check that we get

$$\forall v \in \mathcal{S}'(\mathbb{R}^n), \quad \widehat{D_j v}(\xi) = \mathcal{F}\{D_j v\}(\xi) = (i2\pi \xi_j) \hat{v}(\xi).$$

Then we get, for any $v \in H^s(\mathbb{R}^n)$,

$$\int_{\mathbb{R}^n} |(1 + |\xi|^2)^{\frac{s-1}{2}} \widehat{D_j v}(\xi)|^2 d\xi \leq 4\pi^2 \int_{\mathbb{R}^n} (1 + |\xi|^2)^{s-1} |\xi|^2 |\hat{v}(\xi)|^2 d\xi \leq 4\pi^2 \int_{\mathbb{R}^n} (1 + |\xi|^2)^s |\hat{v}(\xi)|^2 d\xi < \infty.$$

Step 1 is proved.

STEP2. It suffices to notice that $\|v\|_{H^{1/2}(\Gamma)} = \|v_\zeta\|_{H^{1/2}(\mathbb{R}^{d-1})}$, with $v_\zeta(x') = v(x', 0)$, $x' \in \mathbb{R}^{d-1}$, that $(D_j v)_\zeta = D_j v_\zeta$, and to use Step 1 with $s = \frac{1}{2}$ and $\|D_j v\|_{H^{-1/2}(\Gamma)} = \|(D_j v)_\zeta\|_{H^{-1/2}(\mathbb{R}^{d-1})}$ (cf [29] p98). \square

Proof of Lemma 2.3. To show the result for instance on D_+ , it suffices to notice that for any $\varphi \in C_c^\infty(D_+; \mathbb{R})$ we have

$$\begin{aligned} \langle (D_i v)_+, \varphi \rangle_{H^{-1}(D_+), H_0^1(D_+)} &= \langle D_i v, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = - \int_{\mathbb{R}^d} v D_i \varphi = - \int_{D_+} v_+ D_i \varphi \\ &= \langle D_i v_+, \varphi \rangle_{H^{-1}(D_+), H_0^1(D_+)}. \end{aligned}$$

One may of course proceed in the same fashion on D_- . \square

In order to prove Lemma 2.4 we first need a lemma.

Lemma 6.2. Let $u \in L^2(\mathbb{R}^d)$ with $u_{\pm} \in H^1(\mathbb{R}_{\pm}^d)$ and $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. We have

$$-\int_{\mathbb{R}_{\pm}^d} u_{\pm} D_j \varphi = \int_{\mathbb{R}_{\pm}^d} (D_j u_{\pm}) \varphi, \quad \forall 1 \leq j \leq d-1, \quad (6.1)$$

and

$$-\int_{\mathbb{R}_{\pm}^d} u_{\pm} D_d \varphi = \pm \int_{\Gamma} \gamma(u_{\pm}) \gamma(\varphi) d\sigma + \int_{\mathbb{R}_{\pm}^d} (D_d u_{\pm}) \varphi \quad (6.2)$$

(we recall that here $\Gamma = \mathbb{R}^{d-1} \times \{0\}$).

Proof. We establish the formulae on $D_+ = \mathbb{R}_+^d$, as the case \mathbb{R}_-^d can be treated in a similar manner. Let (u_n) a sequence in $C_c^\infty(\mathbb{R}^d; \mathbb{R})$, s.t. $\|u_+ - u_n|_{D_+}\|_{H^1(\mathbb{R}_+^d)} \rightarrow 0$, $n \rightarrow \infty$, where $\|\cdot\|_{H^1(\mathbb{R}_+^d)}$ denotes the usual Sobolev norm. Let $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Let $1 \leq j \leq d-1$. Using integration by parts with respect to the j -th variable for smooth functions vanishing at infinity in the e_j -direction, we have

$$-\int_{\mathbb{R}_+^d} u_+ D_j \varphi = \lim_{n \rightarrow \infty} \left(-\int_{\mathbb{R}_+^d} u_n D_j \varphi \right) = \lim_{n \rightarrow \infty} \left(\int_{\mathbb{R}_+^d} (D_j u_n) \varphi \right) = \int_{\mathbb{R}_+^d} (D_j u_+) \varphi,$$

which proves (6.1). Further, we have

$$\begin{aligned} -\int_{\mathbb{R}_+^d} u_{\pm} D_d \varphi &= \lim_{n \rightarrow \infty} \left(-\int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} u_n D_d \varphi \right) \\ &= \lim_{n \rightarrow \infty} \left(-\int_{\mathbb{R}^{d-1}} \left(\lim_{x_d \rightarrow \infty} (u_n \varphi)(\cdot, x_d) \right. \right. \\ &\quad \left. \left. - \lim_{x_d \rightarrow 0, x_d > 0} (u_n \varphi)(\cdot, x_d) \right) dx_1 \dots dx_{d-1} + \int_{\mathbb{R}_+^d} (D_d u_n) \varphi \right) \\ &= \lim_{n \rightarrow \infty} \left(\int_{\Gamma} \gamma(u_n) \gamma(\varphi) d\sigma + \int_{\mathbb{R}_+^d} (D_d u_n) \varphi \right) \\ &= \int_{\Gamma} \gamma(u_+) \gamma(\varphi) d\sigma + \int_{\mathbb{R}_+^d} (D_d u_+) \varphi, \end{aligned}$$

where we have used the continuity of the trace operator at the last line. Equation (6.2) is proved. \square

Proof of Lemma 2.4. We first treat the case $D_{\pm} = \mathbb{R}_{\pm}^d$. Let $u \in L^2(\mathbb{R}^d)$ with $u_{\pm} \in H^1(\mathbb{R}_{\pm}^d)$. We will prove that for any $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ and any $1 \leq j \leq d$, we have

$$\langle D_j u, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = \langle D_j u_+, \varphi \rangle_{L^2(\mathbb{R}_+^d)} + \langle D_j u_-, \varphi \rangle_{L^2(\mathbb{R}_-^d)} + \delta_{jd} \int_{\Gamma} [u]_{\Gamma} \gamma(\varphi) d\sigma \quad (6.3)$$

(δ_{jd} stands for the Kronecker symbol). This is sufficient to provide the result.

Let then $\varphi \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$. Let $1 \leq j \leq d-1$. We have

$$\langle D_j u, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = -\int_{\mathbb{R}^d} u D_j \varphi = -\int_{\mathbb{R}_+^d} u_+ D_j \varphi - \int_{\mathbb{R}_-^d} u_- D_j \varphi.$$

Using then $u_{\pm} \in H^1(\mathbb{R}_{\pm}^d)$ and the fact that φ vanishes at infinity in the e_j -direction, we get, using an integration by parts formula (see Lemma 6.2),

$$\langle D_j u, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = \int_{\mathbb{R}_+^d} (D_j u_+) \varphi + \int_{\mathbb{R}_-^d} (D_j u_-) \varphi.$$

Further, we have in the same manner

$$\langle D_d u, \varphi \rangle_{H^{-1}(\mathbb{R}^d), H_0^1(\mathbb{R}^d)} = -\int_{\mathbb{R}^d} u D_d \varphi = -\int_{\mathbb{R}_+^d} u_+ D_d \varphi - \int_{\mathbb{R}_-^d} u_- D_d \varphi,$$

but here integration by parts will provide different results, as φ_{\pm} vanish at $\mathbb{R}^{d-1} \times \{\pm\infty\}$ but not at $\mathbb{R}^{d-1} \times \{0\}$. Indeed we have from Lemma 6.2,

$$-\int_{\mathbb{R}^{d-1} \times \mathbb{R}_+} u_+ D_d \varphi = \int_{\Gamma} \gamma(u_+) \gamma(\varphi) d\sigma + \int_{\mathbb{R}_+^d} (D_d u_+) \varphi \quad (6.4)$$

In the same manner we have

$$-\int_{\mathbb{R}_-^d} u_- D_d \varphi = -\int_{\Gamma} \gamma(u_-) \gamma(\varphi) d\sigma + \int_{\mathbb{R}_-^d} (D_d u_-) \varphi. \quad (6.5)$$

Summing (6.4) and (6.5) we get (6.3). If Γ is bounded and smooth one may cover it by a finite number of balls, use change of coordinates, cutoff functions and several times the result for $D_{\pm} = \mathbb{R}_{\pm}^d$. We will then get the desired result. \square

Proof of Proposition 2.16. We prove the result for $r = 0$. Again we fix a cutoff function $\chi \in C^\infty(D_-; \mathbb{R})$, $0 \leq \chi \leq 1$, with $\chi \equiv 1$ on D'_- and $\text{supp}(\chi) \subset D_-$. Note that again the derivatives of χ are bounded by some constant depending on $\text{dist}(D'_-, \partial \text{supp}(\chi))$, thus on d'_- .

Proceeding as at the beginning of STEP 1 - SUBSTEP a) of the proof of Proposition 2.17 one may show that $\chi u_- - A_-(\chi u_-) = \bar{f}_-$ on D_- with

$$\bar{f}_- = (\chi f)_- + \sum_{i,j=1}^d (a_-)_{ij} D_i \chi D_j u_- + D_i((a_-)_{ij} (D_j \chi) u_-) \in L^2(D_-).$$

Thus using (2.5) one gets

$$\langle \chi u_-, v \rangle_{L^2(D_-)} + \mathcal{E}_-(\chi u_-, v) = \langle \bar{f}_-, v \rangle_{L^2(D_-)}, \quad \forall v \in H_0^1(D_-).$$

In the computations below we drop for a moment the subscript $-$ on the functions. For any $1 \leq l \leq d$, one may now take $v = -\Delta_{l,-h}(\Delta_{l,h}(\chi u))$ in the above equation, with h sufficiently small, namely $|h| < \text{dist}(\text{supp}(\chi), \Gamma)$. Thus in the same manner than for (2.15) one gets

$$\begin{aligned} -\int_{D_-} \bar{f} \Delta_{l,-h}(\Delta_{l,h}(\chi u)) &= \int_{D_-} |\Delta_{l,h}(\chi u)|^2 + \sum_{i,j=1}^d \int_{D_-} a_{ij} D_j(\Delta_{l,h}(\chi u)) D_i(\Delta_{l,h}(\chi u)) \\ &\quad + \sum_{i,j=1}^d \int_{D_-} (\Delta_{l,h} a_{ij}) D_i(\chi u) (\cdot + h e_l) \Delta_{l,h}(D_i(\chi u)) \end{aligned}$$

(note that this time there is no boundary term). So that combining again **(E)**, Young's inequality and the fact that $|\Delta_{l,h} a_{ij}| \leq C$ (for some constant C depending on $\max_{1 \leq i,j \leq d} \max_{1 \leq k \leq d} \sup_{x \in D_-} |\partial_{x_k} a_{ij}(x)|$ but not on h) we get

$$\begin{aligned} c \|\Delta_{l,h}(\chi u)\|_{H^1(D_-)}^2 &\leq \frac{1}{2\varepsilon} \|\bar{f}\|_{L^2(D_-)}^2 + \frac{\varepsilon}{2} \|\Delta_{l,h}(\chi u)\|_{H^1(D_-)}^2 \\ &\quad + \frac{Cd^2}{2\delta} \|\nabla(\chi u)\|_{L^2(D_-)}^2 + \frac{Cd^2\delta}{2} \|\nabla(\Delta_{l,h}(\chi u))\|_{L^2(D_-)}^2 \end{aligned}$$

for any $\varepsilon, \delta > 0$ (the constant c depends on λ). Adjusting now ε and δ we get constants $c', C' > 0$, depending d, λ, C , but not on h s.t.

$$c' \|\Delta_{l,h}(\chi u)\|_{H^1(D_-)}^2 \leq C' (\|\bar{f}\|_{L^2(D_-)}^2 + \|\nabla(\chi u)\|_{L^2(D_-)}^2)$$

and thus, considering $M = C' (\|\bar{f}\|_{L^2(D_-)}^2 + \|\nabla(\chi u)\|_{L^2(D_-)}^2) / c'$ we have

$$\forall 1 \leq i \leq d, \quad \|D_i(\Delta_{l,h}(\chi u))\|_{L^2(D_-)}^2 = \|\Delta_{l,h}(D_i(\chi u))\|_{L^2(D_-)}^2 \leq M.$$

Obviously $\|\bar{f}_-\|_{L^2(D_-)}^2 \leq C'' (\|u_-\|_{H^1(D_-)}^2 + \|f_-\|_{L^2(D_-)}^2)$ with C'' depending on d'_- .

Thus (Lemma 2.15-ii) again) we have proved that $D_l D_i(\chi u) \in L^2(D_-)$ for any $1 \leq l \leq d, 1 \leq i \leq d$ with

$$\|D_l D_i(\chi u)\|_{L^2(D_-)}^2 \leq C_- (\|u_-\|_{H^1(D_-)}^2 + \|f_-\|_{L^2(D_-)}^2),$$

with a constant C_- depending on the announced quantities. Remembering that $\chi \equiv 1$ on D'_- this shows the result for u_- and $r = 0$. An induction argument in the spirit of STEP 2 of the proof of Proposition 2.17 takes care of $r > 0$. \square

Before proving Lemma 6.4 we need the following lemma, which allows to weaken the assumptions on the coefficients in Proposition 2.6, under the condition that the function under investigation is smooth.

Lemma 6.3. *Let G_2 an open bounded subset of \mathbb{R}^d with $G_2 \cap \Gamma \neq \emptyset$. Let $\chi \in C_c^\infty(G_2; \mathbb{R})$. Let some coefficients a_{ij}^u satisfy $(a_{\pm}^u)_{ij} \in H^1(D_{\pm}^2)$ (where $D_{\pm}^2 = G_2 \cap D_{\pm}$; note that $\chi_{\pm} \in H^2(D_{\pm}^2)$). Then, using Einstein's convention for summation over repeated indexes,*

$$\int_{D_+} (a_+^u)_{ij} D_j \chi_+ D_i v = - \int_{D_+} (D_i((a_+^u)_{ij} D_j \chi_+)) v - \left(\nu_i \gamma[(a_+^u)_{ij} D_j \chi_+], \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_+)$$

and

$$\int_{D_-} (a_-^u)_{ij} D_j \chi_- D_i v = - \int_{D_-} (D_i((a_-^u)_{ij} D_j \chi_-)) v + \left(\nu_i \gamma[(a_-^u)_{ij} D_j \chi_-], \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_-).$$

Proof. We prove the result on D_+^2 , w.l.g. For each $1 \leq i, j \leq d$ one may construct a sequence $((a_+^{u,n})_{ij})_n$ in $C_c^\infty(\mathbb{R}^d; \mathbb{R})$ such that $\|(a_+^{u,n})_{ij} D_j^2 - (a_+^u)_{ij} D_j^2\|_{H^1(D_+^2)} \rightarrow 0$ as $n \rightarrow \infty$. Then the $(a_+^{u,n})_{ij}$'s are obviously Lipschitz in D_+^2 and from Proposition 2.6 we have

$$\int_{D_+} (a_+^{u,n})_{ij} D_j \chi_+ D_i v = - \int_{D_+} (D_i((a_+^{u,n})_{ij} D_j \chi_+)) v - \left(\nu_i \gamma[(a_+^{u,n})_{ij} D_j \chi_+], \gamma(v) \right)_{\Gamma}, \quad \forall v \in H^1(D_+).$$

We will now pass to the limit in the above equality when $n \rightarrow \infty$. We drop the subscript $+$ on functions in order to lighten notations. To start with, as the $D_j \chi$ are bounded, it is obvious that the $(a_+^{u,n})_{ij} D_j \chi$ converge to $(a^u)_{ij} D_j \chi$ in $H^1(D_+^2)$. So that it is immediate that $\int_{D_+} (a_+^{u,n})_{ij} D_j \chi_+ D_i v$ converges to $\int_{D_+} (a_+^u)_{ij} D_j \chi_+ D_i v$. Further, using the continuity of the trace operator $\gamma : H^1(D_+) \rightarrow H^{1/2}(\Gamma)$ it is clear that $\gamma[(a_+^{u,n})_{ij} D_j \chi]$ converges to $\gamma[(a^u)_{ij} D_j \chi]$ in $H^{1/2}(\Gamma)$. Therefore by Cauchy-Schwarz, $(\nu_i \gamma[(a_+^{u,n})_{ij} D_j \chi_+], \gamma(v))_{\Gamma}$ converges to $(\nu_i \gamma[(a_+^u)_{ij} D_j \chi_+], \gamma(v))_{\Gamma}$. To treat the third and last term it suffices to develop $(D_i((a_+^{u,n})_{ij} D_j \chi))$ into

$$D_i((a_+^{u,n})_{ij}) D_j \chi + (a_+^{u,n})_{ij} D_i D_j \chi.$$

Using again the boundedness of the $D_i \chi$'s and $D_i D_j \chi$'s, this is easily seen to be converging in $L^2(D_+^2)$ to $D_i((a^u)_{ij}) D_j \chi + (a^u)_{ij} D_i D_j \chi = D_i((a^u)_{ij} D_j \chi)$ (note that the $(a^u)_{ij} D_j \chi$ are in $H^1(D_+^2)$). The proof is completed. \square

Lemma 6.4. *With the notations within the proof of Proposition 2.17, STEP1-a), the assumption $[\mathcal{B}_\nu u]_{\Gamma} \in H^{\frac{1}{2}}(\Gamma_2)$ implies that $[\mathcal{B}_\nu(\chi u)]_{\Gamma} \in H^{\frac{1}{2}}(\Gamma_2)$.*

Proof. We aim at proving that

$$[\mathcal{B}_\nu(\chi u)]_{\Gamma} = \chi [\mathcal{B}_\nu u]_{\Gamma} + \sum_{j=1}^d \left\{ \gamma[(D_j \chi)_+ (a_+)_{dj} u_+] - \gamma[(D_j \chi)_- (a_-)_{dj} u_-] \right\} \quad \text{in } H^{-1/2}(\Gamma). \quad (6.6)$$

As $0 \leq \chi \leq 1$ with bounded derivatives, the $(a_{\pm})_{ij}$'s are of class $C^1(\overline{D_{\pm}^2}; \mathbb{R})$, and we have $[\mathcal{B}_\nu u]_{\Gamma} \in H^{\frac{1}{2}}(\Gamma_2)$ and $u_{\pm} \in H^1(D_{\pm})$, we will get the desired result. In fact (6.6) will follow simply from

$$\mathcal{B}_\nu^{\pm} u = \chi \mathcal{B}_\nu^{\pm} u + \sum_{j=1}^d \gamma[(D_j \chi)_{\pm} (a_{\pm})_{dj} u_{\pm}] \quad \text{in } H^{-1/2}(\Gamma) \quad (6.7)$$

(note that in (6.7) for example $\chi \mathcal{B}_\nu^+ u$ is the element of $H^{-1/2}(\Gamma)$ defined by $\langle \chi \mathcal{B}_\nu^+ u, \phi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)} = \langle \mathcal{B}_\nu^+ u, \gamma(\chi) \phi \rangle_{H^{-1/2}(\Gamma), H^{1/2}(\Gamma)}$ for any $\phi \in H^{1/2}(\Gamma)$).

Thus we aim at proving (6.7) on $D_+ = \mathbb{R}_+^d$ (D_- is treated in the same manner). Let $v \in C_c^\infty(\mathbb{R}^d; \mathbb{R})$ and consider its restriction v_+ on D_+ . From now on we drop the subscript $+$ on functions and conormal derivatives. Using Proposition 2.9 we have

$$(\mathcal{B}_\nu(\chi u), \gamma(v))_{\Gamma} = -\mathcal{E}_+(\chi u, v) - \int_{D_+} A_+(\chi u) v$$

(note that we know that $A_+(\chi u) \in L^2(D_+)$ thanks to (2.13), $\bar{f} \in L^2(D_+)$ and $u \in H^1(\mathbb{R}^d)$). But, using Einstein's convention for summation over repeated indexes, $D_j(\chi u) = (D_j\chi)u + \chi D_j u$ and $\chi D_i v = D_i(\chi v) - (D_i\chi)v$, we get

$$(\mathcal{B}_\nu(\chi u), \gamma(v))_\Gamma = -\mathcal{E}_+(u, \chi v) - \int_{D_+} a_{ij} u D_j \chi D_i v + \int_{D_+} a_{ij} D_j u D_i \chi v - \int_{D_+} A_+(\chi u) v.$$

Using again Proposition 2.9 one gets

$$\begin{aligned} (\mathcal{B}_\nu(\chi u), \gamma(v))_\Gamma &= \\ (\mathcal{B}_\nu u, \gamma(\chi)\gamma(v))_\Gamma &+ \int_{D_+} (A_+ u)\chi v - \int_{D_+} a_{ij} u D_j \chi D_i v + \int_{D_+} a_{ij} D_j u D_i \chi v - \int_{D_+} A_+(\chi u) v. \end{aligned}$$

Then using surjection and density arguments we will get (6.7) if we prove that

$$\int_{D_+} (A_+ u)\chi v - \int_{D_+} a_{ij} u D_j \chi D_i v + \int_{D_+} a_{ij} D_j u D_i \chi v - \int_{D_+} A_+(\chi u) v = \left(\gamma[(D_j\chi)a_{dj}u], \gamma(v) \right)_\Gamma. \quad (6.8)$$

But using now $\nu = (0, \dots, 0, 1)^*$, $\chi \in C_c^\infty(G_2; \mathbb{R})$, $a_{ij}u \in H^1(D_+^2)$, and Lemma 6.3 one gets

$$- \int_{D_+} a_{ij} u D_j \chi D_i v - \int_{D_+} (A_+^u \chi) v = \left(\gamma[(D_j\chi)a_{dj}u], \gamma(v) \right)_\Gamma$$

where we have denoted $A_+^u \chi = D_i(a_{ij}u D_j\chi) \in L^2(D_+)$. We now claim that, as an element of $\tilde{H}^{-1}(D_+)$, the form $H^1(D_+) \ni \varphi \mapsto \int_{D_+} (A_+^u \chi) \varphi$ coincides on D_+ with

$$H^1(D_+) \ni \varphi \mapsto - \int_{D_+} (A_+ u)\chi \varphi + \int_{D_+} A_+(\chi u) \varphi - \int_{D_+} a_{ij} D_j u D_i \chi \varphi,$$

which will provide (6.8). Let then $\varphi \in C_c^\infty(D_+; \mathbb{R})$ (see Remark 2.8). We have

$$\begin{aligned} \int_{D_+} (A_+^u \chi) \varphi &= - \int_{D_+} a_{ij} u D_j \chi D_i \varphi \\ &= - \int_{D_+} a_{ij} D_j(\chi u) D_i \varphi + \int_{D_+} a_{ij} \chi D_j u D_i \varphi \\ &= - \int_{D_+} a_{ij} D_j(\chi u) D_i \varphi + \int_{D_+} a_{ij} D_j u D_i(\chi \varphi) - \int_{D_+} a_{ij} D_j u (D_i \chi) \varphi \\ &= \int_{D_+} A_+(\chi u) \varphi - \int_{D_+} (A_+ u)\chi \varphi - \int_{D_+} a_{ij} D_j u (D_i \chi) \varphi, \end{aligned}$$

where we have used successively $u D_j \chi = D_j(\chi u) - \chi D_j(u)$ and $\chi D_i \varphi = D_i(\chi \varphi) - (D_i \chi)\varphi$. The proof is completed. \square

Lemma 6.5. *In the context of the proof of Proposition 2.17, STEP2-a), let $u \in L^2(\mathbb{R}^d)$ with $u_\pm \in H^2(D_\pm^1)$. Let $1 \leq i_0 \leq d-1$. Then*

$$\mathcal{B}_\nu^\pm(D_{i_0} u) = D_{i_0} \mathcal{B}_\nu^\pm(u) - \sum_{j=1}^d \gamma[D_{i_0}((a_\pm)_{dj}) D_j u_\pm] \text{ in } H^{-1/2}(\Gamma_1').$$

Proof. Let $1 \leq i_0 \leq d-1$. We denote $D = D_{i_0}$ and prove the result on $D_+ = \mathbb{R}_+^d$. Let $v \in C_c^\infty(G_1'; \mathbb{R})$ and consider its restriction v_+ on D_+^1 . From now on we drop the subscript $+$ on functions and conormal derivatives and use Einstein's convention for summation over repeated indexes, when possible.

Note that since $u \in H^2(D_+^1)$ and the a_{ij} 's are in $C^2(\overline{D_+^2}; \mathbb{R})$, we have, using Proposition 2.6,

$$(\mathcal{B}_\nu u, \gamma(Dv))_\Gamma = -\mathcal{E}_+(u, Dv) - \int_{D_+} A_+ u Dv.$$

Now observe that $v \in H^2(D_+)$ so that $\gamma(Dv) = D\gamma(v)$. And that $(D\mathcal{B}_\nu(u), \gamma(v))_\Gamma = -(\mathcal{B}_\nu(u), D\gamma(v))_\Gamma$, since $\gamma(v) \in C_c^\infty(\Gamma'_1; \mathbb{R})$. So that

$$-(D\mathcal{B}_\nu u, \gamma(v))_\Gamma = -\mathcal{E}_+(u, Dv) - \int_{D_+} A_+ u Dv. \quad (6.9)$$

Let us consider the term $\mathcal{E}_+(u, Dv)$ in (6). Let us define $A_{k\ell}^j = a_{j\ell} \delta_{ki_0}$ where δ_{ki_0} is the Kronecker symbol. We have,

$$\mathcal{E}_+(u, Dv) = \int_{D_+} a_{ji} D_i u D_j (D_{i_0} v) = \int_{D_+} a_{ji} D_i u D_{i_0} (D_j v) = \int_{D_+} A_{k\ell}^j D_\ell u D_k (D_j v).$$

Applying the first Green identity of Proposition of 2.6 with the j dependent matrix $(A_{k\ell}^j)_{1 \leq k, \ell \leq d}$ instead of $(a_{ij})_{1 \leq i, j \leq d}$, we see that for any fixed $j \in \{1, \dots, d\}$

$$\sum_{1 \leq k, \ell \leq d} \int_{D_+} A_{k\ell}^j D_\ell u D_k (D_j v) = - \left\langle \sum_{1 \leq k, \ell \leq d} D_k (A_{k\ell}^j D_\ell u), D_j v \right\rangle_{L^2(D_+)} - \left(\sum_{\ell=1}^d \gamma (A_{d\ell}^j D_\ell u), \gamma(D_j v) \right)_\Gamma.$$

But, we see that from the definition of $A_{k\ell}^j$, the surface integral $\left(\sum_{\ell=1}^d \gamma (A_{d\ell}^j D_\ell u), \gamma(D_j v) \right)_\Gamma$ is null.

Further,

$$\begin{aligned} & - \left\langle \sum_{1 \leq k, \ell \leq d} D_k (A_{k\ell}^j D_\ell u), D_j v \right\rangle_{L^2(D_+)} \\ &= - \sum_{1 \leq k, \ell \leq d} \int_{D_+} D_k (A_{k\ell}^j) D_\ell u D_j v - \sum_{1 \leq k, \ell \leq d} \int_{D_+} A_{k\ell}^j D_\ell (D_k u) D_j v. \end{aligned} \quad (6.10)$$

And from the definition of $A_{k\ell}^j$,

$$- \left\langle \sum_{1 \leq k, \ell \leq d} D_k (A_{k\ell}^j D_\ell u), D_j v \right\rangle_{L^2(D_+)} = - \sum_{\ell=1}^d \int_{D_+} D(a_{j\ell}) D_\ell u D_j v - \sum_{\ell=1}^d \int_{D_+} a_{j\ell} D_\ell (Du) D_j v.$$

Finally, summing over $1 \leq j \leq d$, we get

$$\mathcal{E}_+(u, Dv) = -\mathcal{E}_+(Du, v) - \int_{D_+} D(a_{j\ell}) D_\ell u D_j v.$$

Using this and Proposition 2.9 (note that Du is in $H^1(D_+^{1'})$) in (6) we get

$$(\mathcal{B}_\nu(Du), \gamma(v))_\Gamma = (D\mathcal{B}_\nu(u), \gamma(v))_\Gamma - \int_{D_+} A_+ Du v - \int_{D_+} A_+ u Dv + \int_{D_+} D(a_{j\ell}) D_\ell u D_j v.$$

So that, similarly to the proof of Lemma 6.4, we will be done if we prove that

$$- \int_{D_+} A_+ Du v - \int_{D_+} A_+ u Dv + \int_{D_+} D(a_{j\ell}) D_\ell u D_j v = -(\gamma[D(a_{dj})D_j u], \gamma(v))_\Gamma. \quad (6.11)$$

But, using $\nu = (0, \dots, 0, 1)^*$ and again Proposition 2.6 (this time with the matrix $D(a_{ij})_{1 \leq i, j \leq d}$ instead of $(a_{ij})_{1 \leq i, j \leq d}$), we see that

$$\int_{D_+} D(a_{j\ell}) D_\ell u D_j v + \int_{D_+} D_j (D(a_{j\ell}) D_\ell u) v = -(\gamma(D(a_{dj})D_j u), \gamma(v))_\Gamma$$

(note that $D_j(D(a_{j\ell})D_\ell u) \in L^2(D_+^{1'})$, thanks to the smoothness of the $a_{j\ell}$'s and of u). So that we will get (6.11) if we see that, as an element of $\tilde{H}^{-1}(D_+^{1'})$, the form $H^1(D_+^{1'}) \ni \varphi \mapsto \int_{D_+^{1'}} D_j(D(a_{j\ell})D_\ell u) \varphi$

coincides on $D_+^{1'}$ with $H^1(D_+^{1'}) \ni \varphi \mapsto -\int_{D_+^{1'}} A_+(Du) \varphi - \int_{D_+^{1'}} A_+ u D\varphi$. Let then $\varphi \in C_c^\infty(D_+^{1'}; \mathbb{R})$. We have

$$\begin{aligned} -\int_{D_+^{1'}} A_+ u D\varphi &= \int_{D_+^{1'}} a_{j\ell} D\ell u D_j D\varphi = \int_{D_+^{1'}} a_{j\ell} D\ell u D D_j \varphi = -\int_{D_+^{1'}} D(a_{j\ell} D\ell u) D_j \varphi \\ &= -\int_{D_+^{1'}} D(a_{j\ell}) D\ell u D_j \varphi - \int_{D_+^{1'}} a_{j\ell} D(D\ell u) D_j \varphi \\ &= -\int_{D_+^{1'}} D(a_{j\ell}) D\ell u D_j \varphi - \int_{D_+^{1'}} a_{j\ell} D_\ell(Du) D_j \varphi \\ &= \int_{D_+^{1'}} D_j(D(a_{j\ell}) D\ell u) \varphi + \int_{D_+^{1'}} A_+(Du) \varphi, \end{aligned}$$

where we have used the fact that $D(a_{j\ell} D\ell u) = D(a_{j\ell}) D\ell u + a_{j\ell} D(D\ell u)$ in $L^2(D_+^{1'})$. The proof is completed. \square

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