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A META-ALGORITHM FOR SOLVING CONNECTIVITY AND ACYCLICITY CONSTRAINTS ON LOCALLY CHECKABLE PROPERTIES PARAMETERIZED BY GRAPH WIDTH MEASURES

BENJAMIN BERGOUGNOUX AND MAMADOU MOUSTAPHA KANTÉ

ABSTRACT. In this paper, we design a framework to obtain efficient algorithms for several problems with a global constraint (acyclicity or connectivity) such as CONNECTED DOMINATING SET, NODE WEIGHTED STEINER TREE, MAXIMUM INDUCED TREE, LONGEST INDUCED PATH, and FEEDBACK VERTEX SET. We design a meta-algorithm that solves all these problems and whose running time is upper bounded by $2^{O(k)} \cdot n^{O(1)}$, $2^{O(k \log(k))} \cdot n^{O(1)}$, $2^{O(k^2)} \cdot n^{O(1)}$ and $n^{O(k)}$ where k is respectively the clique-width, \mathbb{Q} -rank-width, rank-width and maximum induced matching width of a given decomposition. Our meta-algorithm simplifies and unifies the known algorithms for each of the parameters and its running time matches asymptotically also the running times of the best known algorithms for basic NP-hard problems such as VERTEX COVER and DOMINATING SET. Our framework is based on the d -neighbor equivalence defined in [Bui-Xuan, Telle and Vatshelle, TCS 2013]. The results we obtain highlight the importance of this equivalence relation on the algorithmic applications of width measures.

We also prove that our framework could be useful for W[1]-hard problems parameterized by clique-width such as MAX CUT and MAXIMUM MINIMAL CUT. For these latter problems, we obtain $n^{O(k)}$, $n^{O(k)}$ and $n^{2^{O(k)}}$ time algorithms where k is respectively the clique-width, the \mathbb{Q} -rank-width and the rank-width of the input graph.

1. INTRODUCTION

Tree-width is one of the most well-studied graph parameters in the graph algorithm community, due to its numerous structural and algorithmic properties. Nevertheless, despite the broad interest on tree-width, only sparse graphs can have bounded tree-width. But, many NP-hard problems are tractable on dense graph classes. For many graph classes, this tractability can be explained through other width measures. The most remarkable ones are certainly clique-width [11], rank-width [27], and maximum induced matching width (*a.k.a.* mim-width) [33].

We obtain most of these parameters through the well-known notion of *layout* (*a.k.a.* *branch-decomposition*) introduced in [31]. A layout of a graph G is a tree T whose leaves are in bijection with the vertices of G . Every edge e of the layout is associated with a vertex bipartition of G through the two connected components obtained by the removal of e . Given a symmetric function $f : 2^{V(G)} \rightarrow \mathbb{N}$, one can associate with each layout T a measure, called usually f -width, defined as the maximum $f(A)$ over all the vertex bipartitions (A, \bar{A}) of $V(G)$ associated with the edges of T . For instance, rank-width is defined from the function $f(A)$ which corresponds to the rank over $GF(2)$ of the adjacency matrix between the vertex sets A and \bar{A} ; if we take the rank over \mathbb{Q} , we obtain a useful variant of rank-width introduced in [28], called \mathbb{Q} -rank-width. For mim-width, $f(A)$ is the maximum size of an induced matching in the bipartite graph between A and \bar{A} .

These other width measures have a modeling power strictly stronger than the modeling power of tree-width. For example, if a graph class has bounded tree-width, then it has bounded clique-width [11], but the converse is false as cliques have clique-width at most 2 and unbounded tree-width. While (\mathbb{Q} -)rank-width has the same modeling power as clique-width, mim-width has the strongest one among all these width measures and is even bounded on interval graphs

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[1]. Despite their generality, a lot of NP-hard problems admit polynomial time algorithms when one of these width measures is fixed. But, dealing with these width measures is known to be harder than manipulating tree-width.

Concerning their computations, it is not known whether the clique-width (respectively mim-width) of a graph can be approximated within a constant factor in time $f(k) \cdot n^{O(1)}$ (resp. $n^{f(k)}$) for some function f . However, for (Q-)rank-width, there is a $2^{3k} \cdot n^4$ time algorithm that, given a graph G as input and $k \in \mathbb{N}$, either outputs a decomposition for G of (Q-)rank-width at most $3k + 1$ or confirms that the rank-width of G is more than k [28, 29].

Finding efficient algorithms parameterized by one of these width measures is by now standard for problems based on local constraints [10, 32]. In contrast, the task is quite complicated for problems involving a global constraint, *e.g.*, connectivity or acyclicity. For a long time, our knowledge on the parameterized complexity of this latter kind of problems, with parameters the common width measures, was quite limited even for tree-width. For a while, the FPT community used to think that for problems involving global constraints the naive $k^{O(k)} \cdot n^{O(1)}$ time algorithm, k being the tree-width of the input graph, could not be improved. But, quite surprisingly, in 2011, Cygan et al. introduced in [12] a technique called *Cut & Count* to design Monte Carlo $2^{O(k)} \cdot n^{O(1)}$ time algorithms for a wide range of problems with global constraints, including HAMILTONIAN CYCLE, FEEDBACK VERTEX SET, and CONNECTED DOMINATING SET. Later, Bodlaender et al. proposed in [6] a general toolkit, called *rank-based approach*, to design deterministic $2^{O(k)} \cdot n$ time algorithms to solve a wider range of problems.

In a recent paper [3], the authors adapted the rank-based approach of [6] to obtain $2^{O(k)} \cdot n$ time algorithms, k being the clique-width of a given decomposition, for many problems with a global constraint, *e.g.* CONNECTED DOMINATING SET and FEEDBACK VERTEX SET.

Unlike tree-width and clique-width, algorithms parameterized by rank-width and mim-width for problems with a global constraint, were not investigated, except for some special cases such as FEEDBACK VERTEX SET [18, 24] and LONGEST INDUCED PATH [23].

One successful way to design efficient algorithms with these width measures is through the notion of *d-neighbor equivalence*. This concept was introduced by Bui-Xuan, Telle and Vatshelle in [10]. Formally, given $A \subseteq V(G)$ and $d \in \mathbb{N}^+$, two sets $X, Y \subseteq A$ are *d-neighbor equivalent w.r.t. A* if, for all $v \in V(G) \setminus A$, we have $\min(d, |N(v) \cap X|) = \min(d, |N(v) \cap Y|)$, where $N(v)$ is the set of neighbors of v in G . Notice that X and Y are 1-neighbor equivalent *w.r.t. A* if and only if both have the same neighborhood in $V(G) \setminus A$.

The *d-neighbor equivalence* gives rise to a width measure, called in this paper *d-neighbor-width*. This width measure, based also on layouts, is defined from the function $\mathbf{s-nec}_d(A)$ which corresponds to the maximum number of equivalence classes of the *d-neighbor equivalence* over A and $V(G) \setminus A$. It is worth noticing that the boolean-width of a layout introduced in [8] corresponds to the binary logarithm of the 1-neighbor-width.

These notions were used by Bui-Xuan et al. in [10] to design efficient algorithms for the family of problems called (σ, ρ) -DOMINATING SET problems. This family of problems was introduced by Telle and Proskurowski in [32]. Given a pair (σ, ρ) of finite or co-finite subsets of \mathbb{N} and a graph G , a (σ, ρ) -*dominating set* of G is a subset D of $V(G)$ such that, for each vertex $x \in V(G)$, the number of neighbors of x in D is in σ if $x \in D$ and in ρ otherwise. A problem is a (σ, ρ) -DOMINATING SET problem if it consists in finding a minimum (or maximum) (σ, ρ) -*dominating set*. For instance, the DOMINATING SET problem asks for the computation of a minimum $(\mathbb{N}, \mathbb{N} \setminus \{0\})$ -dominating set. Many NP-hard problems based on local constraints belong to this family, see [10, Table 1].

Bui-Xuan et al. [10] designed a meta-algorithm that, given a rooted layout \mathcal{L} , solve any (σ, ρ) -DOMINATING SET problem in time $\mathbf{s-nec}_d(\mathcal{L})^{O(1)} \cdot n^{O(1)}$ where d is a constant depending on the considered problem. The known upper bounds on $\mathbf{s-nec}_d(\mathcal{L})$ (see Lemma 2.4) and the algorithm of [10] give efficient algorithms to solve any (σ, ρ) -DOMINATING SET problem, with parameters tree-width, clique-width, (Q-)rank-width and mim-width. The running times of these algorithms are given in Table 1.

TABLE 1. Upper bounds on $\mathfrak{s}\text{-nec}_d(\mathcal{L})^{O(1)} \cdot n^{O(1)}$ with \mathcal{L} a layout and d a constant.

Tree-width	Clique-width	Rank-width	Q-rank-width	Mim-width
$2^{O(k)} \cdot n^{O(1)}$	$2^{O(k)} \cdot n^{O(1)}$	$2^{O(k^2)} \cdot n^{O(1)}$	$2^{O(k \log(k))} \cdot n^{O(1)}$	$n^{O(k)}$

Our contributions. In this paper, we design a framework based on the 1-neighbor equivalence relation and some ideas from the rank-based approach of [6] to design efficient algorithms for many problems involving a connectivity constraint. This framework provides tools to reduce the size of the sets of partial solutions we compute at each step of a dynamic programming algorithm. We prove that many ad-hoc algorithms for these problems can be unified into a single meta-algorithm that is almost the same as the one from [10] computing a (σ, ρ) -dominating set.

We use our framework to design a meta-algorithm that, given a rooted layout \mathcal{L} , solves any connectivity variant (a solution must induce a connected graph) of a (σ, ρ) -DOMINATING SET problem. This includes some well-known problems such as CONNECTED DOMINATING SET, CONNECTED VERTEX COVER or NODE WEIGHTED STEINER TREE. The running time of our algorithm is polynomial in n and $\mathfrak{s}\text{-nec}_d(\mathcal{L})$, with d a constant that depends on σ and ρ . Consequently, each connectivity variant of a (σ, ρ) -DOMINATING SET problem admits an algorithm with the running times given in Table 1.

We introduce some new concepts to deal with acyclicity. We use these concepts to deal with the AC variants¹ (a solution must induce a tree) of (σ, ρ) -DOMINATING SET problems. Both MAXIMUM INDUCED TREE and LONGEST INDUCED PATH are AC variants of (σ, ρ) -DOMINATING SET problems. We prove that we can modify slightly the meta-algorithm for connectivity constraints so that it can solve these AC variants in the running times given in Table 1. To obtain these results, we rely heavily on the d -neighbor equivalence. However, we were not able to upper bound the running time of this algorithm by a polynomial in n and $\mathfrak{s}\text{-nec}_d(\mathcal{L})$ for some constant d .

We then reduce any acyclic variant (a solution must induce a forest) of a (σ, ρ) -DOMINATING SET problem to solving the AC-variant of the same problem. In particular, this shows that we can use the algorithm for MAXIMUM INDUCED TREE to solve the FEEDBACK VERTEX SET problem.

Up to a constant in the exponent, the running times of our meta-algorithms and their algorithmic consequences match those of the best known algorithms for basic problems such as VERTEX COVER and DOMINATING SET [10, 28]. Moreover, the $2^{O(k)} \cdot n^{O(1)}$ time algorithms we obtain for clique-width are optimal under the well-known Exponential Time Hypothesis (ETH) [21]. That is, unless ETH fails, there are no $2^{o(k)} \cdot n^{O(1)}$ time algorithms, k being the clique-width of a given decomposition, for the NP-hard problems considered in this paper. This follows from the facts that the clique-width of a graph is at most its number of vertices and that (under well-know Karp reduction [21, 22]) those problems do not admit a $2^{o(n)} \cdot n^{O(1)}$ time algorithms unless ETH fails.

Surprisingly, our results reveal that the d -neighbor equivalence relation can be used for problems which are not based on local constraints. This highlights the importance and the generalizing power of this concept on many width measures: for many problems and many width measures, one obtains the “best” algorithms by using the upper bounds on $\mathfrak{s}\text{-nec}_d(\mathcal{L})$ (see Lemma 3.3). We prove that the d -neighbor equivalence relation could be also useful for problems that are W[1]-hard parameterized by clique-width. For doing so, we prove that, given an n -vertex graph and a rooted layout \mathcal{L} , we can use the n -neighbor equivalence to solve MAX CUT in time $\mathfrak{s}\text{-nec}_n^{O(1)}(\mathcal{L}) \cdot n^{O(1)}$. This meta-algorithm gives the best known algorithms parameterized by clique-width, Q-rank-width and rank-width for MAX CUT. We proved also that we can combine

¹AC stands for “acyclic and connected”.

our algorithm for MAX CUT and our framework to solve MAXIMUM MINIMAL CUT a variant of MAX CUT with two connectivity constraints discussed in [14, 15]. Finally, we also show the wide applicability of our framework by explaining how to use it for solving locally checkable partitioning problems with multiple global constraints.

Our approach. To solve these problems, our algorithms do a bottom-up traversal of a given layout \mathcal{L} of the input graph G and at each step we compute a set of partial solutions. In our case, the steps of our algorithms are associated with the vertex bipartitions (A, \bar{A}) induced by the edges of a layout and the partial solutions are subsets of A .

Let us explain our approach for the variant of (σ, ρ) -DOMINATING SET with the connected and AC variants of the DOMINATING SET problem. At each step, our algorithms compute, for each pair (R, R') where R (resp. R') is a 1-neighbor equivalence class of A (resp. \bar{A}), a set of partial solutions $\mathcal{A}_{R,R'} \subseteq R$. The way we compute these sets guarantees that the partial solutions in $\mathcal{A}_{R,R'}$ will be completed with sets in R' . Consequently, we have information about how we will complete our partial solutions since every $Y \in R'$ has the same neighborhood in A .

To deal with the local constraint of these problems, namely the domination constraint, we use the ideas of Bui-Xuan et al. [10]. For each pair (R, R') , let us say that $X \subseteq A$ is *coherent* with (R, R') if: (1) $X \in R$ and (2) $X \cup Y$ dominates A in the graph G for every $Y \in R'$. To compute a minimum dominating set, Bui-Xuan et al. proved that it is enough to keep, for each pair (R, R') , a partial solution X of minimum weight that is coherent with (R, R') . Intuitively, if a partial solution X that is coherent with (R, R') could be completed into a dominating set of G , then it is the case for every partial solution coherent with (R, R') . This is due to the fact that any pair of sets in R (resp. R') dominate the same vertices in \bar{A} (resp. A).

To solve the connectivity variant, we compute, for each (R, R') , a set $\mathcal{A}_{R,R'}$ of partial solutions coherent with (R, R') . Informally, $\mathcal{A}_{R,R'}$ has to be as small as possible, but if a partial solution coherent with (R, R') leads to a minimum connected dominating set, then $\mathcal{A}_{R,R'}$ must contain a similar partial solution. To deal with this intuition, we introduce the relation of *R' -representativity* between sets of partial solutions. We say that \mathcal{A}^* *R' -represents* a set \mathcal{A} , if, for all sets $Y \in R'$, we have $\text{best}(\mathcal{A}, Y) = \text{best}(\mathcal{A}^*, Y)$ where $\text{best}(\mathcal{B}, Y)$ is the minimum weight of a set $X \in \mathcal{B}$ such that $G[X \cup Y]$ is connected.

The main tool of our framework is a function `reduce` that, given a set of partial solutions \mathcal{A} and a 1-neighbor equivalence class R' of \bar{A} , outputs a subset of \mathcal{A} that R' -represents \mathcal{A} and whose size is upper bounded by $\text{s-nec}_1(\mathcal{L})^2$. To design this function, we use ideas from the rank-based approach of [6]. That is, we define a small matrix \mathcal{C} with $|\mathcal{A}|$ rows and $\text{s-nec}_1(\mathcal{L})^2$ columns. Then, we show that a basis of maximum weight of the row space of \mathcal{C} corresponds to an R' -representative set of \mathcal{A} . Since \mathcal{C} has $\text{s-nec}_1(\mathcal{L})^2$ columns, the size of a basis of \mathcal{C} is smaller than $\text{s-nec}_1(\mathcal{L})^2$. By calling `reduce` after each computational step, we keep the sizes of the sets of partial solutions polynomial in $\text{s-nec}_1(\mathcal{L})$. Besides, the definition of R' -representativity guarantees that the set of partial solutions computed for the root of \mathcal{L} contains a minimum connected dominating set.

For the AC variant of dominating set, we need more information in order to deal with the acyclicity. We obtain this extra information by considering that R (resp. R') is a 2-neighbor equivalence class over A (resp. \bar{A}). This way, for all sets $X \subseteq A$, the set X^{+2} of vertices in X that have at least 2 neighbors in R' corresponds exactly to the set of vertices in X that have at least 2 neighbors in Y , for all $Y \in R'$. The vertices in X^{+2} play a major role in the acyclicity constraint because if we control them, then we control the number of edges between X and the sets $Y \in R'$. To obtain this control, we prove in particular that if there exists $Y \in R'$ such that $X \cup Y$ is a forest, then $|X^{+2}| \leq 2\text{mim}(A)$ where $\text{mim}(A)$ is the size of an induced matching in the bipartite graph between A and \bar{A} . Consequently, the size of X^{+2} must be small, otherwise the partial solution X can be discarded.

We need also a new notion of representativity. We say that \mathcal{A}^* R' -ac-represents a set \mathcal{A} , if, for all sets $Y \in R'$, we have $\text{best}^{\text{acy}}(\mathcal{A}, Y) = \text{best}^{\text{acy}}(\mathcal{A}^*, Y)$ where $\text{best}^{\text{acy}}(\mathcal{B}, Y)$ is the minimum weight of a set $X \in \mathcal{B}$ such that $G[X \cup Y]$ is a tree.

As for the R' -representativity, we provide a function that, given a set of partial solutions \mathcal{A} and a 2-neighbor equivalence class R' of \overline{A} , outputs a small subset \mathcal{A}^* of \mathcal{A} that R' -ac-represents \mathcal{A} . Unfortunately, we were not able to upper bound the size of \mathcal{A}^* by a polynomial in n and $\text{s-nec}_d(\mathcal{L})$ (for some constant d). Instead, we prove that, for clique-width, rank-width, \mathbb{Q} -rank-width and mim-width, the size of \mathcal{A}^* can be upper bounded by, respectively, $2^{O(k)} \cdot n$, $2^{O(k^2)} \cdot n$, $2^{O(k \log(k))} \cdot n$ and $n^{O(k)}$. The key to compute \mathcal{A}^* is to decompose \mathcal{A} into a small number of sets $\mathcal{A}_1 \dots, \mathcal{A}_\ell$, called R' -consistent, where the notion of R' -ac-representativity matches the notion of R' -representativity. More precisely, any R' -representative set of an R' -consistent set \mathcal{A} is also an R' -ac-representative set of \mathcal{A} .

To compute an R' -ac-representative set of \mathcal{A} it is then enough to compute an R' -representative set for each R' -consistent set in the decomposition of \mathcal{A} . The union of these R' -representative sets is an R' -ac-representative set of \mathcal{A} . Besides the notion of representativity, the algorithm for the AC variant of DOMINATING SET is very similar to the one for finding a minimum connected dominating set.

It is worth noticing that we can not use the same trick as in [6] to ensure the acyclicity, that is counting the number of edges induced by the partial solutions. Indeed, we would need to differentiate at least n^k partial solutions (for any parameter k considered in Table 1) in order to update this information. We give more explanation on this statement at the beginning of Section 6.

For MAX CUT, we use the n -neighbor equivalence with n the number of vertices of G . For every $X, W \subseteq A$, we prove that if X and W are n -neighbor equivalent over A , then, for every $Y \subseteq \overline{A}$, the number of edges between X and Y are the same as the number of edges between W and Y . It follows, that if the number of edges between X and $A \setminus X$ is greater than the number of edges between W and $A \setminus W$, then $X \cup Y$ is a better solution than $W \cup Y$ for every $Y \subseteq \overline{A}$. Consequently, it is sufficient to keep a partial solution for each n -neighbor equivalence class over A . This observation leads to an $\text{s-nec}_n(\mathcal{L})^{O(1)} \cdot n^{O(1)}$ time algorithm for MAX CUT.

Relation to previous works. Our framework can be used on tree-decompositions to obtain $2^{O(k)} \cdot n^{O(1)}$ time algorithms parameterized by tree-width for all the problems considered in that paper. Indeed, given a vertex separator S of size k , the number of d -neighbor equivalence classes over S (resp. $V(G) \setminus S$) is upper bounded by 2^k (resp. $(d+1)^k$). For this reason, we can consider our framework as a generalization of the rank-based approach of [6]. Our framework generalizes also the clique-width adaptation of the rank-based approach used in [3] to obtain $2^{O(k)} \cdot n$ time algorithms, k being the clique-width of a given decomposition, for CONNECTED (σ, ρ) -DOMINATING SET problem and FEEDBACK VERTEX SET. However, the constants in the running times of the algorithms in [3, 6] are better than those of our algorithms. For instance, the authors in [3] obtained a $15^k \cdot 2^{(\omega+1) \cdot k} \cdot k^{O(1)} \cdot n$ time algorithm for FEEDBACK VERTEX SET, while in this paper, we design a $54^k \cdot 2^{2(\omega+1) \cdot k} \cdot n^4$ time algorithm for this latter problem. Indeed, our approach is based on a more general parameter and is not optimized neither for tree-width nor clique-width. Our framework simplifies the algorithms in [3, 6] because contrary to [3, 6] we do not use weighted partitions to encode the partial solutions. Consequently, the definitions of the dynamic programming tables and the computational steps of our algorithms are simpler than those in [3, 6]. This is particularly true for FEEDBACK VERTEX SET where the use of weighted partitions to encode the partial solutions in [3] implies to take care of many technical details concerning the acyclicity.

The results we obtain simplify the $2^{O(k^2)} \cdot n^{O(1)}$ time algorithm parameterized by rank-width for FEEDBACK VERTEX SET from [18] and the $n^{O(k)}$ time algorithms parameterized by mim-width for FEEDBACK VERTEX SET and LONGEST INDUCED PATH from [23, 24]. We also notice that contrary to the algorithm for MAX CUT (and its variants) given in [14, 15, 16], there is no

need to assume that the graph is given with a clique-width expression as our algorithm can be parameterized by \mathbb{Q} -rank-width, which is always smaller than clique-width and for which also a fast FPT $(3k + 1)$ -approximation algorithm exists [28, 29].

Concerning mim-width, we provide unified polynomial-time algorithms for the considered problems for all well-known graph classes having bounded mim-width and for which a layout of bounded mim-width can be computed in polynomial time [1] (*e.g.*, interval graphs, circular arc graphs, permutation graphs, Dilworth- k graphs and k -polygon graphs for all fixed k). Notice that we also generalize one of the results from [26] proving that the CONNECTED VERTEX COVER problem is solvable in polynomial time for circular arc graphs.

It is worth noticing that the approach used in [12] called Cut & Count can also be generalized to the d -neighbor-width for any CONNECTED (σ, ρ) -DOMINATING SET problem with more or less the same arguments used in this paper (see the PhD thesis [2, Theorem 4.66]). However, it is not clear how to generalize the Cut & Count approach to solve the acyclic variants of the CONNECTED (σ, ρ) -DOMINATING SET problems, with the width measures considered in this paper.

Summary. We give some general definitions and notations in Section 2 and present the d -neighbor equivalence relation in Section 3. The framework based on the 1-neighbor equivalence relation is given in Section 4. The applications to connectivity and acyclicity constraints on (σ, ρ) -DOMINATING SET problems are given in Sections 5 and 6. The algorithms concerning MAX CUT and variants are given in Section 7. Finally, we conclude with some open questions and by giving some examples of problems which might be interesting to tackle with the help of the d -neighbor equivalence relation in Section 8.

2. PRELIMINARIES

Let V be a finite set. The size of a set V is denoted by $|V|$ and its power set is denoted by 2^V . A set function $f : 2^V \rightarrow \mathbb{N}$ is *symmetric* if, for all $S \subseteq V$, we have $f(S) = f(V \setminus S)$. We write $A \setminus B$ for the set difference of A from B . We denote by \mathbb{N} the set of non-negative integers and by \mathbb{N}^+ the set $\mathbb{N} \setminus \{0\}$. We let $\min(\emptyset) := +\infty$ and $\max(\emptyset) := -\infty$.

Graphs. Our graph terminology is standard and we refer to [13]. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. For every vertex set $X \subseteq V(G)$, when the underlying graph is clear from context, we denote by \overline{X} the set $V(G) \setminus X$. An edge between two vertices x and y is denoted by xy or yx . The set of vertices that is adjacent to x is denoted by $N_G(x)$. For a set $U \subseteq V(G)$, we define $N_G(U) := \bigcup_{x \in U} N_G(x)$. If the underlying graph is clear, then we may remove G from the subscript.

The subgraph of G induced by a subset X of its vertex set is denoted by $G[X]$. For two disjoint subsets X and Y of $V(G)$, we denote by $G[X, Y]$ the bipartite graph with vertex set $X \cup Y$ and edge set $\{xy \in E(G) : x \in X \text{ and } y \in Y\}$. Moreover, we write $E(X, Y)$ to denote the set $E(G[X, Y])$ and we denote by $M_{X, Y}$ the adjacency matrix between X and Y , *i.e.*, the (X, Y) -matrix such that $M_{X, Y}[x, y] = 1$ if $y \in N(x)$ and 0 otherwise.

For two subsets \mathcal{A} and \mathcal{B} of $2^{V(G)}$, we define the *merging* of \mathcal{A} and \mathcal{B} , denoted by $\mathcal{A} \otimes \mathcal{B}$, as

$$\mathcal{A} \otimes \mathcal{B} := \begin{cases} \emptyset & \text{if } \mathcal{A} = \emptyset \text{ or } \mathcal{B} = \emptyset, \\ \{X \cup Y : X \in \mathcal{A} \text{ and } Y \in \mathcal{B}\} & \text{otherwise.} \end{cases}$$

For a graph G , we denote by $\text{cc}(G)$ the partition $\{V(C) : C \text{ is a connected component of } G\}$. Let $X \subseteq V(G)$. A *consistent cut* of X is an ordered bipartition (X_1, X_2) of X such that $N(X_1) \cap X_2 = \emptyset$. We denote by $\text{ccut}(X)$ the set of all consistent cuts of X . In our proofs, we use the following facts.

Fact 2.1. *Let $X \subseteq V(G)$. For every $C \in \text{cc}(G[X])$ and every $(X_1, X_2) \in \text{ccut}(X)$, we have either $C \subseteq X_1$ or $C \subseteq X_2$.*

We deduce from the above fact that $|\text{ccut}(X)| = 2^{|\text{cc}(G[X])|}$.

Fact 2.2. *Let X and Y be two disjoint subsets of $V(G)$. We have $(W_1, W_2) \in \text{ccut}(X \cup Y)$ if and only if the following conditions are satisfied*

- (1) $(W_1 \cap X, W_2 \cap X) \in \text{ccut}(X)$,
- (2) $(W_1 \cap Y, W_2 \cap Y) \in \text{ccut}(Y)$ and
- (3) $N(W_1 \cap X) \cap (W_2 \cap Y) = \emptyset$ and $N(W_2 \cap X) \cap (W_1 \cap Y) = \emptyset$.

Rooted Layout. A *rooted binary tree* is a binary tree with a distinguished vertex called the *root*. Since we manipulate at the same time graphs and trees representing them, the vertices of trees will be called *nodes*.

A *rooted layout* of G is a pair $\mathcal{L} = (T, \delta)$ of a rooted binary tree T and a bijective function δ between $V(G)$ and the leaves of T . For each node x of T , let L_x be the set of all the leaves l of T such that the path from the root of T to l contains x . We denote by $V_x^\mathcal{L}$ the set of vertices that are in bijection with L_x , i.e., $V_x^\mathcal{L} := \{v \in V(G) : \delta(v) \in L_x\}$. When \mathcal{L} is clear from the context, we may remove \mathcal{L} from the superscript.

All the width measures dealt with in this paper are special cases of the following one, the difference being in each case the used set function. Given a set function $f : 2^{V(G)} \rightarrow \mathbb{N}$ and a rooted layout $\mathcal{L} = (T, \delta)$, the f -width of a node x of T is $f(V_x^\mathcal{L})$ and the f -width of (T, δ) , denoted by $f(T, \delta)$ (or $f(\mathcal{L})$), is $\max\{f(V_x^\mathcal{L}) : x \in V(T)\}$. Finally, the f -width of G is the minimum f -width over all rooted layouts of G .

Clique-width / Module-width. We won't define *clique-width*, but its equivalent measure *module-width* [30]. The module-width of a graph G is the **mw**-width where $\text{mw}(A)$ is the cardinality of $\{N(v) \cap \bar{A} : v \in A\}$ for all $A \subseteq V(G)$. One also observes that $\text{mw}(A)$ is the number of different rows in $M_{A, \bar{A}}$. The following theorem shows the link between module-width and clique-width.

Theorem 2.3 ([30, Theorem 6.6]). *For every n -vertex graph G , $\text{mw}(G) \leq \text{cw}(G) \leq 2\text{mw}(G)$, where $\text{cw}(G)$ denotes the clique-width of G . One can moreover translate, in time at most $O(n^2)$, a given decomposition into the other one with width at most the given bounds.*

(\mathbb{Q})-Rank-width. The rank-width and \mathbb{Q} -rank-width are, respectively, the **rw**-width and **rw $_{\mathbb{Q}}$** -width where $\text{rw}(A)$ (resp. $\text{rw}_{\mathbb{Q}}(A)$) is the rank over $GF(2)$ (resp. \mathbb{Q}) of the matrix $M_{A, \bar{A}}$ for all $A \subseteq V(G)$.

Mim-width. The **mim**-width of a graph G is the **mim**-width of G where $\text{mim}(A)$ is the size of a maximum induced matching of the graph $G[A, \bar{A}]$ for all $A \subseteq V(G)$.

It is worth noticing that module-width is the only parameter associated with a set function that is not symmetric.

The following lemma provides some upper bounds between **mim**-width and the other parameters that we use in Section 6. All of these upper bounds are proved in [33].

Lemma 2.4 ([33]). *Let G be a graph. For every $A \subseteq V(G)$, $\text{mim}(A)$ is upper bounded by $\text{mw}(A)$, $\text{rw}(A)$ and $\text{rw}_{\mathbb{Q}}(A)$.*

Proof. Let $A \subseteq V(G)$. It is clear that $\text{mim}(A)$ is upper bounded by the number of different rows in $M_{A, \bar{A}}$, so $\text{mim}(A) \leq \text{mw}(A)$. Let S be the vertex set of a maximum induced matching of the graph $G[A, \bar{A}]$. Observe that the restriction of the matrix $M_{A, \bar{A}}$ to rows and columns in S is the identity matrix. Hence, $\text{mim}(A)$ is upper bounded both by $\text{rw}(A)$ and $\text{rw}_{\mathbb{Q}}(A)$. \square

3. THE d -NEIGHBOR EQUIVALENCE

Let G be a graph. The following definition is from [10]. Let $A \subseteq V(G)$ and $d \in \mathbb{N}^+$. Two subsets X and Y of A are *d -neighbor equivalent over A* , denoted by $X \equiv_A^d Y$, if $\min(d, |X \cap N(u)|) = \min(d, |Y \cap N(u)|)$ for all $u \in \bar{A}$. It is not hard to check that \equiv_A^d is an equivalence relation. See Figure 1 for an example of 2-neighbor equivalent sets.

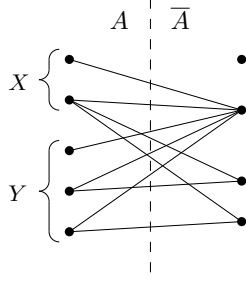


FIGURE 1. We have $X \equiv_A^2 Y$, but it is not the case that $X \equiv_A^3 Y$.

For all $d \in \mathbb{N}^+$, we let $\text{nec}_d : 2^{V(G)} \rightarrow \mathbb{N}$ where, for all $A \subseteq V(G)$, $\text{nec}_d(A)$ is the number of equivalence classes of \equiv_A^d . Notice that while nec_1 is a symmetric function [25, Theorem 1.2.3], nec_d is not necessarily symmetric for $d \geq 2$. For example, if a vertex x of G has c neighbors, then, for every $d \in \mathbb{N}^+$, we have $\text{nec}_d(\{x\}) = 2$ and $\text{nec}_d(\overline{\{x\}}) = 1 + \min(d, c)$. It is worth noticing that, for every $d \in \mathbb{N}^+$, $\text{nec}_d(A)$ and $\text{nec}_d(\overline{A})$ are at most $\text{nec}_1(A)^{d \log_2(\text{nec}_1(A))}$, for each $A \subseteq V(G)$ [10].

The following fact follows directly from the definition of the d -neighbor equivalence relation. We use it several times in our proofs.

Fact 3.1. *Let $A, B \subseteq V(G)$ such that $A \subseteq B$ and let $d \in \mathbb{N}^+$. For all $X, Y \subseteq A$, if $X \equiv_A^d Y$, then $X \equiv_B^d Y$.*

In order to manipulate the equivalence classes of \equiv_A^d , one needs to compute a representative for each equivalence class in polynomial time. This is achieved with the following notion of a representative. Let G be a graph with an arbitrary ordering of $V(G)$ and let $A \subseteq V(G)$. For each $X \subseteq A$, let us denote by $\text{rep}_A^d(X)$ the lexicographically smallest set $R \subseteq A$ among all $R \equiv_A^d X$ of minimum size. Moreover, we denote by \mathcal{R}_A^d the set $\{\text{rep}_A^d(X) : X \subseteq A\}$. It is worth noticing that the empty set always belongs to \mathcal{R}_A^d , for all $A \subseteq V(G)$ and $d \in \mathbb{N}^+$. Moreover, we have $\mathcal{R}_{V(G)}^d = \mathcal{R}_\emptyset^d = \{\emptyset\}$ for all $d \in \mathbb{N}^+$. In order to compute \mathcal{R}_A^d , we use the following lemma.

Lemma 3.2 ([10]). *Let G be an n -vertex graph. For every $A \subseteq V(G)$ and $d \in \mathbb{N}^+$, one can compute in time $O(\text{nec}_d(A) \cdot n^2 \cdot \log(\text{nec}_d(A)))$, the sets \mathcal{R}_A^d and a data structure that, given a set $X \subseteq A$, computes $\text{rep}_A^d(X)$ in time $O(|A| \cdot n \cdot \log(\text{nec}_d(A)))$.*

d -neighbor-width. For every graph G and $d \in \mathbb{N}^+$, the d -neighbor-width is the parameter obtained through the symmetric function $\text{s-nec}_d : 2^{V(G)} \rightarrow \mathbb{N}$ such that

$$\text{s-nec}_d(A) = \max(\text{nec}_d(A), \text{nec}_d(\overline{A})).$$

The following lemma shows how the d -neighbor-width is upper bounded by the other parameters, most of the upper bounds were already proved in [1, 28].

Lemma 3.3 ([1, 28, 33]). *Let G be a graph. For every $A \subseteq V(G)$ and $d \in \mathbb{N}^+$, we have the following upper bounds on $\text{nec}_d(A)$ and $\text{nec}_d(\overline{A})$:*

- | | |
|------------------------------------|-------------------------------------------------------------------------|
| (a) $(d+1)^{\text{mw}(A)}$, | (c) $(d \cdot \text{rw}_\mathbb{Q}(A) + 1)^{\text{rw}_\mathbb{Q}(A)}$, |
| (b) $2^{d \cdot \text{rw}(A)^2}$, | (d) $n^{d \cdot \text{mim}(A)}$. |

Proof. The first upper bound was proved in [33, Lemma 5.2.2]. The second upper bound was implicitly proved in [33] and is due to the fact that $\text{nec}_d(A) \leq \text{mw}(A)^{d \cdot \text{mim}(A)}$ [33, Lemma 5.2.3]. Since $\text{mim}(A) \leq \text{rw}(A)$ by Lemma 2.4 and $\text{mw}(A) \leq 2^{\text{rw}(A)}$ [29], we deduce that $\text{nec}_d(A) \leq 2^{d \cdot \text{rw}(A)^2}$. The third upper bound was proved in [28, Theorem 4.2]. The fourth was proved in [1, Lemma 2]. \square

Lemma 3.3 implies the following.

Corollary 3.4. *If there exists an algorithm that, given an n -vertex graph G and a rooted layout \mathcal{L} of G , solves a problem Π on G in time $\mathbf{s-nec}_c(\mathcal{L})^{O(1)} \cdot n^{O(1)}$ for some constant c , then Π is decidable on G within the following running times:*

- $2^{O(\mathbf{mw}(\mathcal{L}))} \cdot n^{O(1)}$,
- $\mathbf{rw}_{\mathbb{Q}}(G)^{O(\mathbf{rw}_{\mathbb{Q}}(G))} \cdot n^{O(1)}$,
- $2^{O(\mathbf{rw}(G)^2)} \cdot n^{O(1)}$,
- $n^{O(\mathbf{mim}(\mathcal{L}))}$.

Observe that the running times given in Corollary 3.4 for rank-width and \mathbb{Q} -width use the width of the input graph and not the width of the rooted layout. This follows from the fact that, given a graph G , we can compute a rooted layout of rank-width $3\mathbf{rw}(G) + 1$ (resp. \mathbb{Q} -rank width $3\mathbf{rw}_{\mathbb{Q}}(G) + 1$) in time $2^{O(\mathbf{rw}(G))} \cdot n^3$ (resp. $2^{\mathbf{rw}_{\mathbb{Q}}(G)} \cdot n^{O(1)}$) [28, 29].

To deal with the MAX CUT problem and the acyclic variants of (σ, ρ) -DOMINATING SET problems, we use the n -neighbor equivalence with n being the number of vertices of the input graph. For both problems, we use the following property of the n -neighbor equivalence.

Lemma 3.5. *Let G be a graph and $A \subseteq V(G)$. For all $X, W \subseteq A$ such that $X \equiv_A^n W$ and for every $Y \subseteq \bar{A}$, we have $|E(X, Y)| = |E(W, Y)|$.*

Proof. Let $X, W \subseteq A$ such that $X \equiv_A^n W$. Observe that, for every $v \in \bar{A}$, we have $\min(n, |N(v) \cap X|) = |N(v) \cap X|$. We deduce that, for every $v \in \bar{A}$, we have $|N(v) \cap X| = |N(v) \cap W|$. Thus, for every $Y \subseteq \bar{A}$, we have

$$|E(X, Y)| = \sum_{v \in Y} |N(v) \cap X| = \sum_{v \in Y} |N(v) \cap W| = |E(W, Y)|.$$

□

For the acyclic variants of (σ, ρ) -DOMINATING SET problems, we use the n -neighbor equivalence over vertex sets of small size. Given an integer $t \in \mathbb{N}$, a graph G and $A \subseteq V(G)$, we denote by $\mathbf{nec}_n^{\leq t}(A)$ the number of equivalence classes generated by the n -neighbor equivalence over the set $\{X \subseteq A : |X| \leq t\}$. To deduce the algorithmic consequences on rank-width and \mathbb{Q} -rank-width of our algorithm, we need the following upper bounds on $\mathbf{nec}_n^{\leq t}(A)$.

Lemma 3.6. *Let G be a graph. For every $A \subseteq V(G)$, $\mathbf{nec}_n^{\leq t}(A)$ is upper bounded by $(t+1)^{\mathbf{rw}_{\mathbb{Q}}(A)}$, $(t+1)^{\mathbf{mw}(A)}$ and $2^{t \cdot \mathbf{rw}(A)}$.*

Proof. We start by proving that $|\mathbf{nec}_n^{\leq t}(A)| \leq (t+1)^{\mathbf{rw}_{\mathbb{Q}}(A)}$. Observe that this latter inequality was implicitly proved in [28, Theorem 4.2] and that we use the same arguments here. For $X \subseteq A$, let $\sigma(X)$ be the vector corresponding to the sum over \mathbb{Q} of the row vectors of $M_{A, \bar{A}}$ corresponding to X . Observe that, for every $X, W \subseteq A$, we have $X \equiv_A^n W$ if and only if $\sigma(X) = \sigma(W)$. Hence, it is enough to prove that $|\{\sigma(X) : X \subseteq A \wedge |X| \leq t\}| \leq (t+1)^{\mathbf{rw}_{\mathbb{Q}}(A)}$.

Let C be a set of $\mathbf{rw}_{\mathbb{Q}}(A)$ linearly independent columns of $M_{A, \bar{A}}$. Since the rank over \mathbb{Q} of $M_{A, \bar{A}}$ is $\mathbf{rw}_{\mathbb{Q}}(A)$, every linear combination of row vectors of $M_{A, \bar{A}}$ is completely determined by its entries in C . For every $X \in \mathcal{A}$, the values in $\sigma(X)$ are between 0 and t . Hence, we conclude that $|\{\sigma(X) : X \subseteq A \wedge |X| \leq t\}| = \mathbf{nec}_n^{\leq t}(A) \leq (t+1)^{\mathbf{rw}_{\mathbb{Q}}(A)}$. Since we have $\mathbf{rw}_{\mathbb{Q}}(A) \leq \mathbf{mw}(A)$ [28, Theorem 3.6], we deduce that $\mathbf{nec}_n^{\leq t}(A) \leq (t+1)^{\mathbf{mw}(A)}$.

Concerning rank-width, since there are at most $2^{\mathbf{rw}(A)}$ different rows in the matrix $M_{A, \bar{A}}$, we conclude that $\mathbf{nec}_n^{\leq t}(A) \leq 2^{t \cdot \mathbf{rw}(A)}$. □

For the algorithms in Section 6, we use the n -neighbor equivalence over sets of size at most $2\mathbf{mim}(A)$, so we do not need any upper bounds on $\mathbf{nec}_n^{\leq t}(A)$ in function of \mathbf{mim} -width as it is trivially upper bounded by $n^{2\mathbf{mim}(A)}$. Moreover, for module-width, we will use another fact to get the desired running time.

It is worth noticing that Upper bounds (b) and (c) of Lemma 3.3 are implied by 3.6 and the fact, for every $d \in \mathbb{N}$ and $A \subseteq V(G)$, a representative of minimum size of a d -neighbor equivalence class over A has size at most $d \cdot \mathbf{rw}_{\mathbb{Q}}(A)$ and $d \cdot \mathbf{rw}(A)$ [10, 28].

For MAX CUT, we use the n -neighbor equivalence over sets of arbitrary size. In particular, we use the following property of this equivalence relation.

Fact 3.7. *Let $A \subseteq V(G)$. For every $X, W \subseteq A$ such that $X \equiv_A^n W$, we have $A \setminus X \equiv_A^n A \setminus W$.*

Proof. Let $X, W \subseteq A$ such that $X \equiv_A^n W$ and v be a vertex of \overline{A} . We have

$$|N(v) \cap A| = |N(v) \cap X| + |N(v) \cap (A \setminus X)|.$$

As $X \equiv_A^n W$, by Lemma 3.5, we have $|N(v) \cap X| = |N(v) \cap W|$. We deduce that

$$|N(v) \cap A| - |N(v) \cap W| = |N(v) \cap (A \setminus X)|.$$

Since $|N(v) \cap A| - |N(v) \cap W| = |N(v) \cap (A \setminus W)|$, we conclude that $|N(v) \cap (A \setminus X)| = |N(v) \cap (A \setminus W)|$. As this equality holds for every $v \in \overline{A}$, we can conclude that $A \setminus X \equiv_A^n A \setminus W$. \square

The following lemma provides some upper bound on $\text{nec}_n(A)$ and $\text{nec}_n(\overline{A})$.

Lemma 3.8. *Let G be an n -vertex graph. For every $A \subseteq V(G)$, we have the following upper bounds on $\text{nec}_n(A)$ and $\text{nec}_n(\overline{A})$:*

$$(a) \ n^{\text{mw}(A)}, \quad (b) \ n^{\text{rw}_{\mathbb{Q}}(A)}, \quad (c) \ n^{2^{\text{rw}(A)}}.$$

Proof. Let $A \subseteq V(G)$. If $A = V(G)$, then obviously we have $\text{nec}_n(A) = 1$. Otherwise, we have $\text{nec}_n(A) = \text{nec}_n^{\leq n-1}(A)$ and by Lemma 3.6, we conclude that $\text{nec}_n(A) \leq n^{\text{rw}_{\mathbb{Q}}(A)}$.

Since $\text{rw}_{\mathbb{Q}}(A) \leq \text{mw}(A)$ [28, Theorem 3.6], we deduce that $\text{nec}_n(A) \leq n^{\text{mw}(A)}$. Moreover, as any binary matrix M of rank k over $GF(2)$ has at most 2^k different rows, we have $\text{mw}(A) \leq 2^{\text{rw}(A)}$. Thus, we conclude that $\text{nec}_n(A) \leq n^{2^{\text{rw}(A)}}$. These upper bounds on $\text{nec}_n(A)$ holds also on $\text{nec}_n(\overline{A})$ because $\text{rw}_{\mathbb{Q}}$ and rw are symmetric functions. \square

Observe that the upper bounds of Lemma 3.8 are almost tight, that is, for every $k \in \mathbb{N}$, we prove in the appendix that there exists a graph G and $A \subseteq V(G)$ such that $\text{rw}(A) = k + 1$, $\text{mw}(A) = \text{rw}_{\mathbb{Q}}(A) = 2^k$ and $\text{nec}_n(A) \in (n/2^k - 1)^{2^k}$.

Lemma 3.8 has the following consequences.

Corollary 3.9. *If there exists an algorithm that, given a graph G and a rooted layout \mathcal{L} of G , solves a problem Π on G in time $s\text{-nec}_n(\mathcal{L})^{O(1)} \cdot n^{O(1)}$ for some constant c , then Π is decidable on G within the following running times:*

$$\bullet \ n^{O(\text{mw}(G))}, \quad \bullet \ n^{O(\text{rw}_{\mathbb{Q}}(G))}, \quad \bullet \ n^{2^{O(\text{rw}(G))}}.$$

It is worth noticing that the running time given by Corollary 3.9 for module-width depends on the width of the graph. This is due to the facts that, for every graph G , we have $\text{rw}_{\mathbb{Q}}(G) \leq \text{mw}(G)$ [28, Theorem 3.6] and we can compute a rooted layout of G of \mathbb{Q} -rank-width at most $3\text{rw}_{\mathbb{Q}}(G) + 1$ in time $2^{O(\text{rw}_{\mathbb{Q}}(G))} \cdot n^{O(1)}$ [28, Theorem 3.1].

4. REPRESENTATIVE SETS

In the following, we fix G an n -vertex graph, (T, δ) a rooted layout of G and $w : V(G) \rightarrow \mathbb{Q}$ a weight function over the vertices of G . We also assume that $V(G)$ is ordered.

In this section, we define a notion of representativity between sets of partial solutions *w.r.t.* the connectivity. Our notion of representativity is defined *w.r.t.* some node x of T and the 1-neighbor equivalence class of some set $R' \subseteq \overline{V}_x$. In our algorithms, R' will always belong to $\mathcal{R}_{\overline{V}_x}^d$ for some $d \in \mathbb{N}^+$. Our algorithms compute a set of partial solutions for each $R' \in \mathcal{R}_{\overline{V}_x}^d$. The partial solutions computed for R' will be completed with sets equivalent to R' *w.r.t.* $\equiv_{\overline{V}_x}^d$. Intuitively, the R' 's represent some expectation about how we will complete our sets of partial solutions. For the connectivity and the domination, $d = 1$ is enough but if we need more information for some reasons (for example the (σ, ρ) -domination or the acyclicity), we may take

$d > 1$. This is not a problem as the d -neighbor equivalence class of R' is included in the 1-neighbor equivalence class of R' . Hence, in this section, we fix a node x of T and a set $R' \subseteq \overline{V_x}$ to avoid to overload the statements by the sentence “let x be a node of T and $R' \subseteq \overline{V_x}$ ”. We let $\text{opt} \in \{\min, \max\}$; if we want to solve a maximization (or minimization) problem, we use $\text{opt} = \max$ (or $\text{opt} = \min$). We use it also, as here, in the next sections.

We recall that two subsets Y, W of $\overline{V_x}$ are 1-neighbor equivalent *w.r.t.* $\overline{V_x}$ if they have the same neighborhood in V_x , *i.e.*, $N(Y) \cap V_x = N(W) \cap V_x$.

Definition 4.1 ((x, R') -representativity). *For every $\mathcal{A} \subseteq 2^{V(G)}$ and $Y \subseteq V(G)$, we define*

$$\text{best}(\mathcal{A}, Y) := \text{opt}\{\mathbf{w}(X) : X \in \mathcal{A} \text{ and } G[X \cup Y] \text{ is connected}\}.$$

Let $\mathcal{A}, \mathcal{B} \subseteq 2^{V_x}$. We say that \mathcal{B} (x, R') -represents \mathcal{A} if, for every $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^1 R'$, we have $\text{best}(\mathcal{A}, Y) = \text{best}(\mathcal{B}, Y)$.

Notice that the (x, R') -representativity is an equivalence relation. The set \mathcal{A} is meant to represent a set of partial solutions of $G[V_x]$ which have been computed. We expect to complete these partial solutions with partial solutions of $G[\overline{V_x}]$ which are equivalent to R' *w.r.t.* $\equiv_{\overline{V_x}}^1$. If \mathcal{B} (x, R') -represents \mathcal{A} , then we can safely substitute \mathcal{A} by \mathcal{B} because the quality of the output of the dynamic programming algorithm will remain the same. Indeed, for every subset Y of $\overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^1 R'$, the optimum solutions obtained by the union of a partial solution in \mathcal{A} and Y will have the same weight as the optimum solution obtained from the union of a set in \mathcal{B} and Y .

The next theorem presents the main tool of our framework: a function `reduce` that, given a set of partial solutions \mathcal{A} , outputs a subset of \mathcal{A} that (x, R') -represents \mathcal{A} and whose size is upper bounded by $\mathfrak{s}\text{-nec}_1(\mathcal{L})^2$. To design this function, we use ideas from the rank-based approach of [6]. That is, we define a small matrix \mathcal{C} with $|\mathcal{A}|$ rows and $\mathfrak{s}\text{-nec}_1(V_x)^2$ columns. Then, we show that a basis of maximum weight of the row space of \mathcal{C} corresponds to an (x, R') -representative set of \mathcal{A} . Since \mathcal{C} has $\mathfrak{s}\text{-nec}_1(\mathcal{L})^2$ columns, the size of a basis of \mathcal{C} is smaller than $\mathfrak{s}\text{-nec}_1(\mathcal{L})^2$. To compute this basis, we use the following lemma. The constant ω denotes the matrix multiplication exponent, which is known to be strictly less than 2.3727 due to [34].

Lemma 4.2 ([6]). *Let M be a binary $n \times m$ -matrix with $m \leq n$ and let $\mathbf{w} : \{1, \dots, n\} \rightarrow \mathbb{Q}$ be a weight function on the rows of M . Then, one can find a basis of maximum (or minimum) weight of the row space of M in time $O(nm^{\omega-1})$.*

In order to compute a small (x, R') -representative set of a set $\mathcal{A} \subseteq 2^{V_x}$, the following theorem requires that the sets in \mathcal{A} are pairwise equivalent *w.r.t.* $\equiv_{\overline{V_x}}^1$. This is useful since in our algorithm we classify our sets of partial solutions with respect to this property. We need this to guarantee that the partial solutions computed for R' will be completed with sets equivalent to R' *w.r.t.* $\equiv_{\overline{V_x}}^d$. However, if one wants to compute a small (x, R') -representative set of a set \mathcal{A} that does not respect this property, then it is enough to compute an (x, R') -representative set for each 1-neighbor equivalence class of \mathcal{A} . The union of these (x, R') -representative sets is an (x, R') -representative set of \mathcal{A} .

Theorem 4.3. *Let $R \in \mathcal{R}_{V_x}^1$ and $R' \subseteq \overline{V_x}$. Then, there exists an algorithm `reduce` that, given $\mathcal{A} \subseteq 2^{V_x}$ such that $X \equiv_{\overline{V_x}}^1 R$ for all $X \in \mathcal{A}$, outputs in time $O(|\mathcal{A}| \cdot \text{nec}_1(V_x)^{2(\omega-1)} \cdot n^2)$ a subset $\mathcal{B} \subseteq \mathcal{A}$ such that \mathcal{B} (x, R') -represents \mathcal{A} and $|\mathcal{B}| \leq \text{nec}_1(V_x)^2$.*

Proof. We assume w.l.o.g. that $\text{opt} = \max$, the proof is symmetric for $\text{opt} = \min$. First, we suppose that $R' \equiv_{\overline{V_x}}^1 \emptyset$. Observe that, for every $Y \equiv_{\overline{V_x}}^1 \emptyset$, we have $N(Y) \cap V_x = N(\emptyset) \cap V_x = \emptyset$. It follows that, for every $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^1 \emptyset$ and $Y \neq \emptyset$, we have $\text{best}(\mathcal{A}, Y) = -\infty$. Moreover, by definition of `best`, we have $\text{best}(\mathcal{A}, \emptyset) = \max\{\mathbf{w}(X) : X \in \mathcal{A} \text{ and } G[X] \text{ is connected}\}$. Hence, if $R' \equiv_{\overline{V_x}}^1 \emptyset$, then it is sufficient to return $\mathcal{B} = \{X\}$, where X is an element of \mathcal{A} of maximum weight that induces a connected graph.

Assume from now that R' is not equivalent to \emptyset *w.r.t.* $\equiv_{\overline{V}_x}^1$. Let $X \in \mathcal{A}$. If there exists $C \in \text{cc}(G[X])$ such that $N(C) \cap R' = \emptyset$, then, for all $Y \equiv_{\overline{V}_x}^1 R'$, we have $N(C) \cap Y = \emptyset$. Moreover, as R' is not equivalent to \emptyset *w.r.t.* $\equiv_{\overline{V}_x}^1$, we have $Y \neq \emptyset$. Consequently, for every $Y \equiv_{\overline{V}_x}^1 R'$, the graph $G[X \cup Y]$ is not connected. We can conclude that $\mathcal{A} \setminus \{X\}$ (x, R') represents \mathcal{A} . Thus, we can safely remove from \mathcal{A} all such sets and this can be done in time $|\mathcal{A}| \cdot n^2$. From now on, we may assume that, for all $X \in \mathcal{A}$ and for all $C \in \text{cc}(G[X])$, we have $N(C) \cap R' \neq \emptyset$. It is worth noticing that if $R = \emptyset$ or more generally $N(R) \cap R' = \emptyset$, then by assumption, $\mathcal{A} = \emptyset$.

Indeed, if $N(R) \cap R' = \emptyset$, then, for every $X \in \mathcal{A}$, we have $N(X) \cap R' = N(R) \cap R' = \emptyset$ and in particular, for every $C \in \text{cc}(G[X])$, we have $N(C) \cap R' = \emptyset$ (and we have assumed that no such set exists in \mathcal{A}).

Symmetrically, if, for some $Y \subseteq \overline{V}_x$ there exists $C \in \text{cc}(G[Y])$ such that $N(C) \cap R = \emptyset$, then, for every $X \in \mathcal{A}$, the graph $G[X \cup Y]$ is not connected. Let \mathcal{D} be the set of all subsets Y of \overline{V}_x such that $Y \equiv_{\overline{V}_x}^1 R'$ and, for all $C \in \text{cc}(G[Y])$, we have $N(C) \cap R \neq \emptyset$. Notice that the sets in $2^{\overline{V}_x} \setminus \mathcal{D}$ do not matter for the (x, R') -representativity.

For every $Y \in \mathcal{D}$, we let v_Y be one fixed vertex of Y . In the following, we denote by \mathfrak{R} the set $\{(R'_1, R'_2) \in \mathcal{R}_{\overline{V}_x}^1 \times \mathcal{R}_{\overline{V}_x}^1\}$. Let \mathcal{M}, \mathcal{C} and $\overline{\mathcal{C}}$ be, respectively, an $(\mathcal{A}, \mathcal{D})$ -matrix, an $(\mathcal{A}, \mathfrak{R})$ -matrix and an $(\mathfrak{R}, \mathcal{D})$ -matrix such that

$$\begin{aligned} \mathcal{M}[X, Y] &:= \begin{cases} 1 & \text{if } G[X \cup Y] \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases} \\ \mathcal{C}[X, (R'_1, R'_2)] &:= \begin{cases} 1 & \text{if } \exists (X_1, X_2) \in \text{ccut}(X) \text{ such that } N(X_1) \cap R'_2 = \emptyset \text{ and } N(X_2) \cap R'_1 = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \\ \overline{\mathcal{C}}[(R'_1, R'_2), Y] &:= \begin{cases} 1 & \text{if } \exists (Y_1, Y_2) \in \text{ccut}(Y) \text{ such that } v_Y \in Y_1, Y_1 \equiv_{\overline{V}_x}^1 R'_1 \text{ and } Y_2 \equiv_{\overline{V}_x}^1 R'_2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Intuitively, \mathcal{M} contains all the information we need. Indeed, it is easy to see that a basis of maximum weight of the row space of \mathcal{M} in $GF(2)$ is an (x, R') -representative set of \mathcal{A} . But, \mathcal{M} is too big to be computable efficiently. Instead, we prove that a basis of maximum weight of the row space of \mathcal{C} is an (x, R') -representative set of \mathcal{A} . This follows from the fact that $(\mathcal{C} \cdot \overline{\mathcal{C}})[X, Y]$ equals the number of consistent cuts (W_1, W_2) in $\text{ccut}(X \cup Y)$ such that $v_Y \in W_1$. That is $(\mathcal{C} \cdot \overline{\mathcal{C}})[X, Y] = 2^{|\text{cc}(G[X \cup Y])| - 1}$. Consequently, $\mathcal{M} =_2 \mathcal{C} \cdot \overline{\mathcal{C}}$, where $=_2$ denotes the equality in $GF(2)$, *i.e.*, $(\mathcal{C} \cdot \overline{\mathcal{C}})[X, Y]$ is odd if and only if $G[X \cup Y]$ is connected. We deduce the running time of `reduce` and the size of `reduce`(\mathcal{A}) from the size of \mathcal{C} (*i.e.* $|\mathcal{A}| \cdot \text{nec}_1(V_x)^2$) and the fact that \mathcal{C} is easy to compute.

We start by proving that $\mathcal{M} =_2 \mathcal{C} \cdot \overline{\mathcal{C}}$. Let $X \in \mathcal{A}$ and $Y \in \mathcal{D}$. We want to prove the following equality

$$(\mathcal{C} \cdot \overline{\mathcal{C}})[X, Y] = \sum_{(R'_1, R'_2) \in \mathfrak{R}} \mathcal{C}[X, (R'_1, R'_2)] \cdot \overline{\mathcal{C}}[(R'_1, R'_2), Y] = 2^{|\text{cc}(G[X \cup Y])| - 1}.$$

We prove this equality with the following two claims.

Claim 4.4. *We have $\mathcal{C}[X, (R'_1, R'_2)] \cdot \overline{\mathcal{C}}[(R'_1, R'_2), Y] = 1$ if and only if there exists $(W_1, W_2) \in \text{ccut}(X \cup Y)$ such that $v_Y \in W_1$, $W_1 \cap Y \equiv_{\overline{V}_x}^1 R'_1$ and $W_2 \cap Y \equiv_{\overline{V}_x}^1 R'_2$.*

Proof. By definition, we have $\mathcal{C}[X, (R'_1, R'_2)] \cdot \overline{\mathcal{C}}[(R'_1, R'_2), Y] = 1$, if and only if

- (a) $\exists (Y_1, Y_2) \in \text{ccut}(Y)$ such that $v_Y \in Y_1$, $Y_1 \equiv_{\overline{V}_x}^1 R'_1$, $Y_2 \equiv_{\overline{V}_x}^1 R'_2$ and
- (b) $\exists (X_1, X_2) \in \text{ccut}(X)$ such that $N(X_1) \cap R'_2 = \emptyset$ and $N(X_2) \cap R'_1 = \emptyset$.

Let $(Y_1, Y_2) \in \text{ccut}(Y)$ and $(X_1, X_2) \in \text{ccut}(X)$ that satisfy, respectively, Properties (a) and (b). By definition of $\equiv_{\overline{V}_x}^1$, we have $N(X_1) \cap Y_2 = \emptyset$ because $N(X_1) \cap R'_2 = \emptyset$ and $Y_2 \equiv_{\overline{V}_x}^1 R'_2$. Symmetrically, we have $N(X_2) \cap Y_1 = \emptyset$. By Fact 2.2, we deduce that $(X_1 \cup Y_1, X_2 \cup Y_2) \in \text{ccut}(X \cup Y)$. This proves the claim. \square

Claim 4.5. *Let (W_1, W_2) and $(W'_1, W'_2) \in \text{ccut}(X \cup Y)$. We have $W_1 \cap Y \equiv_{\overline{V_x}}^1 W'_1 \cap Y$ and $W_2 \cap Y \equiv_{\overline{V_x}}^1 W'_2 \cap Y$ if and only if $W_1 = W'_1$ and $W_2 = W'_2$.*

Proof. We start by an observation about the connected components of $X \cup Y$. As $Y \in \mathcal{D}$, for all $C \in \text{cc}(G[Y])$, we have $N(C) \cap R \neq \emptyset$. Moreover, by assumption, for all $C \in \text{cc}(G[X])$, we have $N(C) \cap R' \neq \emptyset$. Since $X \equiv_{\overline{V_x}}^1 R$ and $Y \equiv_{\overline{V_x}}^1 R'$, every connected component of $G[X \cup Y]$ contains at least one vertex of X and one vertex of Y .

Suppose that $W_1 \cap Y \equiv_{\overline{V_x}}^1 W'_1 \cap Y$ and $W_2 \cap Y \equiv_{\overline{V_x}}^1 W'_2 \cap Y$. Assume towards a contradiction that $W_1 \neq W'_1$ and $W_2 \neq W'_2$. Since $W_1 \neq W'_1$, by Fact 2.1, we deduce that there exists $C \in \text{cc}(G[X \cup Y])$ such that either (1) $C \subseteq W_1$ and $C \subseteq W'_2$ or (2) $C \subseteq W'_1$ and $C \subseteq W_2$. We can assume w.l.o.g. that there exists $C \in \text{cc}(G[X \cup Y])$ such that $C \subseteq W_1$ and $C \subseteq W'_2$. From the above observation, C contains at least one vertex of X and one of Y and we have $N(C \cap X) \cap (W_1 \cap Y) \neq \emptyset$ and $N(C \cap X) \cap (W'_2 \cap Y) \neq \emptyset$. But, since $W_2 \cap Y \equiv_{\overline{V_x}}^1 W'_2 \cap Y$, we have $N(C \cap X) \cap (W_2 \cap Y) \neq \emptyset$. This implies in particular that $N(W_1) \cap W_2 \neq \emptyset$. It is a contradiction with the fact that $(W_1, W_2) \in \text{ccut}(X \cup Y)$.

The other direction being trivial we can conclude the claim. \square

Notice that Claim 4.5 implies that, for every $(R'_1, R'_2) \in \mathfrak{R}$, there exists at most one consistent cut $(W_1, W_2) \in \text{ccut}(X \cup Y)$ such that $v_Y \in W_1$, $W_1 \cap Y \equiv_{\overline{V_x}}^1 R'_1$ and $W_2 \cap Y \equiv_{\overline{V_x}}^1 R'_2$. We can thus conclude from these two claims that

$$(\mathcal{C} \cdot \overline{\mathcal{C}})[X, Y] = |\{(W_1, W_2) \in \text{ccut}(X \cup Y) : v_Y \in W_1\}|.$$

By Fact 2.1, we deduce that $(\mathcal{C} \cdot \overline{\mathcal{C}})[X, Y] = 2^{|\text{cc}(G[X \cup Y])| - 1}$ since every connected component of $G[X \cup Y]$ can be in both sides of a consistent cut at the exception of the connected component containing v_Y . Hence, $(\mathcal{C} \cdot \overline{\mathcal{C}})[X, Y]$ is odd if and only if $|\text{cc}(G[X \cup Y])| = 1$. We conclude that $\mathcal{M} =_2 \mathcal{C} \cdot \overline{\mathcal{C}}$.

Let $\mathcal{B} \subseteq \mathcal{A}$ be a basis of maximum weight of the row space of \mathcal{C} over $GF(2)$. We claim that \mathcal{B} (x, R') -represents \mathcal{A} .

Let $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^1 R'$. Observe that, by definition of \mathcal{D} , if $Y \notin \mathcal{D}$, then $\text{best}(\mathcal{A}, Y) = \text{best}(\mathcal{B}, Y) = -\infty$. Thus it is sufficient to prove that, for every $Y \in \mathcal{D}$, we have $\text{best}(\mathcal{A}, Y) = \text{best}(\mathcal{B}, Y)$.

Let $X \in \mathcal{A}$ and $Y \in \mathcal{D}$. Recall that we have proved that $\mathcal{M}[X, Y] =_2 (\mathcal{C} \cdot \overline{\mathcal{C}})[X, Y]$. Since \mathcal{B} is a basis of \mathcal{C} , there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that, for each $(R'_1, R'_2) \in \mathfrak{R}$, we have $\mathcal{C}[X, (R'_1, R'_2)] =_2 \sum_{W \in \mathcal{B}'} \mathcal{C}[W, (R'_1, R'_2)]$. Thus, we have the following equality

$$\begin{aligned} \mathcal{M}[X, Y] &= _2 \sum_{(R'_1, R'_2) \in \mathfrak{R}} \mathcal{C}[X, (R'_1, R'_2)] \cdot \overline{\mathcal{C}}[(R'_1, R'_2), Y] \\ &= _2 \sum_{(R'_1, R'_2) \in \mathfrak{R}} \left(\sum_{W \in \mathcal{B}'} \mathcal{C}[W, (R'_1, R'_2)] \right) \cdot \overline{\mathcal{C}}[(R'_1, R'_2), Y] \\ &= _2 \sum_{W \in \mathcal{B}'} \left(\sum_{(R'_1, R'_2) \in \mathfrak{R}} \mathcal{C}[W, (R'_1, R'_2)] \cdot \overline{\mathcal{C}}[(R'_1, R'_2), Y] \right) \\ &= _2 \sum_{W \in \mathcal{B}'} (\mathcal{C} \cdot \overline{\mathcal{C}})[W, Y] =_2 \sum_{W \in \mathcal{B}'} \mathcal{M}[W, Y]. \end{aligned}$$

If $\mathcal{M}[X, Y] = 1$ (i.e. $G[X \cup Y]$ is connected), then there is an odd number of sets W in \mathcal{B}' such that $\mathcal{M}[W, Y] = 1$ (i.e. $G[W \cup Y]$ is connected). Hence, there exists at least one $W \in \mathcal{B}'$ such that $G[W \cup Y]$ is connected. Let $W \in \mathcal{B}'$ such that $\mathcal{M}[W, Y] = 1$ and $w(W)$ is maximum. Assume towards a contradiction that $w(W) < w(X)$. Notice that $(\mathcal{B} \setminus \{W\}) \cup \{X\}$ is also a basis of \mathcal{C} since the set of independent row sets of a matrix forms a matroid. Since $w(W) < w(X)$, the weight of the basis $(\mathcal{B} \setminus \{W\}) \cup \{X\}$ is strictly greater than the weight of the basis \mathcal{B} , yielding a contradiction. Thus $w(X) \leq w(W)$. Hence, for all $Y \in \mathcal{D}$ and all $X \in \mathcal{A}$, if $G[X \cup Y]$ is

connected, then there exists $W \in \mathcal{B}$ such that $G[W \cup Y]$ is connected and $w(X) \leq w(W)$. This is sufficient to prove that \mathcal{B} (x, R') -represents \mathcal{A} . Since \mathcal{B} is a basis, the size of \mathcal{B} is at most the number of columns of \mathcal{C} , thus, $|\mathcal{B}| \leq \text{nec}_1(V_x)^2$.

It remains to prove the running time. We claim that \mathcal{C} is easy to compute.

By Fact 2.1, $\mathcal{C}[X, (R'_1, R'_2)] = 1$ if and only if, for each $C \in \text{cc}(G[X])$, we have either $N(C) \cap R'_1 = \emptyset$ or $N(C) \cap R'_2 = \emptyset$. Thus, each entry of \mathcal{C} is computable in time $O(n^2)$. Since \mathcal{C} has $|\mathcal{A}| \cdot |\mathcal{R}_{V_x}^1|^2 = |\mathcal{A}| \cdot \text{nec}_1(V_x)^2 \cdot n^2$ entries, we can compute \mathcal{C} in time $O(|\mathcal{A}| \cdot \text{nec}_1(V_x)^2 \cdot n^2)$. Furthermore, by Lemma 4.2, a basis of maximum weight of \mathcal{C} can be computed in time $O(|\mathcal{A}| \cdot \text{nec}_1(V_x)^{2(\omega-1)})$. We conclude that \mathcal{B} can be computed in time $O(|\mathcal{A}| \cdot \text{nec}_1(V_x)^{2(\omega-1)} \cdot n^2)$. \square

Now to boost up a dynamic programming algorithm P on some rooted layout (T, δ) of G , we can use the function `reduce` to keep the size of the sets of partial solutions bounded by $\text{s-nec}_1(T, \delta)^2$. We call P' the algorithm obtained from P by calling the function `reduce` at every step of computation. We can assume that the set of partial solutions \mathcal{A}_r computed by P and associated with the root r of (T, δ) contains an optimal solution (this will be the cases in our algorithms). To prove the correctness of P' , we need to prove that \mathcal{A}'_r (r, \emptyset) -represents \mathcal{A}_r where \mathcal{A}'_r is the set of partial solutions computed by P' and associated with r . For doing so, we need to prove that at each step of the algorithm the operations we use preserve the (x, R') -representativity. The following fact states that we can use the union without restriction, it follows directly from Definition 4.1 of (x, R') -representativity.

Fact 4.6. *If \mathcal{B} and \mathcal{D} (x, R') -represents, respectively, \mathcal{A} and \mathcal{C} , then $\mathcal{B} \cup \mathcal{D}$ (x, R') -represents $\mathcal{A} \cup \mathcal{C}$.*

The second operation we use in our dynamic programming algorithms is the merging operator \otimes . In order to safely use it, we need the following notion of compatibility that just tells which partial solutions from V_a and V_b can be joined to possibly form a partial solution in V_x . (It was already used in [10] without naming it.)

Definition 4.7 (d - (R, R') -compatibility). *Suppose that x is an internal node of T with a and b as children. Let $d \in \mathbb{N}^+$ and $R \in \mathcal{R}_{V_x}^d$. We say that $(A, A') \in \mathcal{R}_{V_a}^d \times \mathcal{R}_{V_a}^d$ and $(B, B') \in \mathcal{R}_{V_b}^d \times \mathcal{R}_{V_b}^d$ are d - (R, R') -compatible if we have:*

- $A \cup B \equiv_{V_x}^d R$,
- $A' \equiv_{V_a}^d B \cup R'$ and
- $B' \equiv_{V_b}^d A \cup R'$.

The d - (R, R') -compatibility just tells which partial solutions from V_a and V_b can be joined to possibly form a partial solution in V_x .

Lemma 4.8. *Suppose that x is an internal node of T with a and b as children. Let $d \in \mathbb{N}^+$ and $R \in \mathcal{R}_{V_x}^d$. Let $(A, A') \in \mathcal{R}_{V_a}^d \times \mathcal{R}_{V_a}^d$ and $(B, B') \in \mathcal{R}_{V_b}^d \times \mathcal{R}_{V_b}^d$ that are d - (R, R') -compatible. Let $\mathcal{A} \subseteq 2^{V_a}$ such that, for all $X \in \mathcal{A}$, we have $X \equiv_{V_a}^d A$ and let $\mathcal{B} \subseteq 2^{V_b}$ such that, for all $W \in \mathcal{B}$, we have $W \equiv_{V_b}^d B$. If $\mathcal{A}' \subseteq \mathcal{A}$ (a, A') -represents \mathcal{A} and $\mathcal{B}' \subseteq \mathcal{B}$ (b, B') -represents \mathcal{B} , then $\mathcal{A}' \otimes \mathcal{B}'$ (x, R') -represents $\mathcal{A} \otimes \mathcal{B}$.*

Proof. We assume w.l.o.g. that $\text{opt} = \max$, the proof is symmetric for $\text{opt} = \min$. Suppose that $\mathcal{A}' \subseteq \mathcal{A}$ (a, A') -represents \mathcal{A} and $\mathcal{B}' \subseteq \mathcal{B}$ (b, B') -represents \mathcal{B} . To prove the lemma, it is sufficient to prove that $\text{best}(\mathcal{A}' \otimes \mathcal{B}', Y) = \text{best}(\mathcal{A} \otimes \mathcal{B}, Y)$ for every $Y \equiv_{V_x}^1 R'$.

Let $Y \subseteq \overline{V_x}$ such that $Y \equiv_{V_x}^1 R'$. We start by proving the following facts

- (a) for every $W \in \mathcal{B}$, we have $W \cup Y \equiv_{V_a}^1 A'$,
- (b) for every $X \in \mathcal{A}$, we have $X \cup Y \equiv_{V_b}^1 B'$.

Let $W \in \mathcal{B}$. Owing to the d - (R, R') -compatibility, we have $B \cup R' \equiv_{V_a}^d A'$. Since $W \equiv_{V_b}^d B$ and $V_b \subseteq \overline{V_a}$, we deduce, by Fact 3.1, that $W \equiv_{V_a}^d B$ and thus $W \cup R' \equiv_{V_a}^d A'$. In particular, we have

$W \cup R' \equiv_{\overline{V}_a}^1 A'$. Similarly, we have from Fact 3.1 that $W \cup Y \equiv_{\overline{V}_a}^1 A'$ because $Y \equiv_{\overline{V}_x}^1 R'$ and $\overline{V}_x \subseteq \overline{V}_a$. This proves Fact (a). The proof for Fact (b) is symmetric.

Now observe that, by the definitions of best and of the merging operator \otimes , we have (even if $\mathcal{A} = \emptyset$ or $\mathcal{B} = \emptyset$)

$$\text{best}(\mathcal{A} \otimes \mathcal{B}, Y) = \max\{w(X) + w(W) : X \in \mathcal{A} \wedge W \in \mathcal{B} \wedge G[X \cup W \cup Y] \text{ is connected}\}.$$

Since $\text{best}(\mathcal{A}, W \cup Y) = \max\{w(X) : X \in \mathcal{A} \wedge G[X \cup W \cup Y] \text{ is connected}\}$, we deduce that

$$\text{best}(\mathcal{A} \otimes \mathcal{B}, Y) = \max\{\text{best}(\mathcal{A}, W \cup Y) + w(W) : W \in \mathcal{B}\}.$$

Since \mathcal{A}' (a, A')-represents \mathcal{A} , by Fact (a), we have

$$\begin{aligned} \text{best}(\mathcal{A} \otimes \mathcal{B}, Y) &= \max\{\text{best}(\mathcal{A}', W \cup Y) + w(W) : W \in \mathcal{B}\} \\ &= \text{best}(\mathcal{A}' \otimes \mathcal{B}, Y). \end{aligned}$$

Symmetrically, we deduce from Fact (b) that $\text{best}(\mathcal{A}' \otimes \mathcal{B}, Y) = \text{best}(\mathcal{A}' \otimes \mathcal{B}', Y)$. This stands for every $Y \subseteq \overline{V}_x$ such that $Y \equiv_{\overline{V}_x}^1 R'$. Thus, we conclude that $\mathcal{A}' \otimes \mathcal{B}'$ (x, R')-represents $\mathcal{A} \otimes \mathcal{B}$. \square

5. CONNECTED (CO)- (σ, ρ) -DOMINATING SETS

Let σ and ρ be two (non-empty) finite or co-finite subsets of \mathbb{N} . We say that a subset D of $V(G)$ (σ, ρ) -dominates a subset $U \subseteq V(G)$ if,

$$|N(u) \cap D| \in \begin{cases} \sigma & \text{if } u \in D, \\ \rho & \text{otherwise (if } u \in U \setminus D). \end{cases}$$

A subset D of $V(G)$ is a (σ, ρ) -dominating set (resp. co- (σ, ρ) -dominating set) if D (resp. $V(G) \setminus D$) (σ, ρ) -dominates $V(G)$.

The CONNECTED (σ, ρ) -DOMINATING SET problem asks, given a weighted graph G , a maximum or minimum (σ, ρ) -dominating set which induces a connected graph. Similarly, one can define CONNECTED CO- (σ, ρ) -DOMINATING SET. Examples of some CONNECTED (CO)- (σ, ρ) -DOMINATING SET problems are shown in Table 2.

Let $d(\mathbb{N}) := 0$ and for a finite or co-finite subset μ of \mathbb{N} , let

$$d(\mu) := 1 + \min(\max(\mu), \max(\mathbb{N} \setminus \mu)).$$

Let $d := \max\{1, d(\sigma), d(\rho)\}$. The definition of d is motivated by the following observation which is due to the fact that, for all $\mu \subseteq \mathbb{N}$, if $d(\mu) \in \mu$, then μ is co-finite and contains $\mathbb{N} \setminus \{1, \dots, d(\mu) - 1\}$.

Fact 5.1. *Let $A \subseteq V(G)$ and let (σ, ρ) be a pair of finite or co-finite subsets of \mathbb{N} . Let $d := \max(1, d(\sigma), d(\rho))$. For all $X \subseteq A$ and $Y \subseteq \overline{A}$, $X \cup Y$ (σ, ρ) -dominates A if and only if $\min(d, |N(v) \cap X| + |N(v) \cap Y|)$ belongs to σ (resp. ρ) if $v \in X$ (resp. $v \notin X$).*

As in [10], we use the d -neighbor equivalence relation to characterize the (σ, ρ) -domination of the partial solutions.

TABLE 2. Examples of (CO)- (σ, ρ) -DOMINATING SET problems. To solve these problems, we use the d -neighbor equivalence with d the maximum between 1, $d(\sigma)$ and $d(\rho)$. Column d shows the value of d for each problem.

σ	ρ	d	Version	Standard name
\mathbb{N}	\mathbb{N}^+	1	CONNECTED	CONNECTED DOMINATING SET
$\{q\}$	\mathbb{N}	$q+1$	CONNECTED	CONNECTED INDUCED q -REGULAR SUBGRAPH
\mathbb{N}	$\{1\}$	2	CONNECTED	CONNECTED PERFECT DOMINATING SET
$\{0\}$	\mathbb{N}	1	CONNECTED CO	CONNECTED VERTEX COVER

We will need the following lemma in our proof.

Lemma 5.2 ([10]). *Let $A \subseteq V(G)$. Let $X \subseteq A$ and $Y, Y' \subseteq \overline{A}$ such that $Y \equiv_{\overline{A}}^d Y'$. Then $(X \cup Y)$ (σ, ρ) -dominates A if and only if $(X \cup Y')$ (σ, ρ) -dominates A .*

In this section, we present an algorithm that computes a maximum (or minimum) connected (σ, ρ) -dominating set with a graph G and a layout (T, δ) of G as inputs. Its running time is $O(\mathbf{s-nec}_d(T, \delta)^{O(1)} \cdot n^3)$. The same algorithm, with some little modifications, will be able to find a minimum Steiner tree or a maximum (or minimum) connected co- (σ, ρ) -dominating set as well.

For each node x of T and for each pair $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d$, we will compute a set of partial solutions $\mathcal{D}_x[R, R']$ coherent with (R, R') that (x, R') -represents the set of all partial solutions coherent with (R, R') . We say that a set $X \subseteq V_x$ is coherent with (R, R') if $X \equiv_{V_x}^d R$ and $X \cup R'$ (σ, ρ) dominates V_x . Observe that by Lemma 5.2, we have $X \cup Y$ (σ, ρ) -dominates V_x , for all $Y \equiv_{\overline{V_x}}^d R'$ and for all $X \subseteq V_x$ coherent with (R, R') . We compute these sets by a bottom-up dynamic programming algorithm, starting at the leaves of T . The computational steps are trivial for the leaves. For the internal nodes of T , we simply use the notion of d - (R, R') -compatibility and the merging operator.

By calling the function `reduce` defined in Section 4, each set $\mathcal{D}_x[R, R']$ contains at most $\mathbf{s-nec}_1(T, \delta)^2$ partial solutions. If we want to compute a maximum (resp. minimum) connected (σ, ρ) -dominating set, we use the framework of Section 4 with `opt` = `max` (resp. `opt` = `min`). If G admits a connected (σ, ρ) -dominating set, then a maximum (or minimum) connected (σ, ρ) -dominating set can be found by looking at the entry $\mathcal{D}_r[\emptyset, \emptyset]$ with r the root of T .

We begin by defining the sets of partial solutions for which we will compute representative sets.

Definition 5.3. *Let $x \in V(T)$. For all pairs $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d$, we let $\mathcal{A}_x[R, R'] := \{X \subseteq V_x : X \equiv_{V_x}^d R \text{ and } X \cup R' \text{ } (\sigma, \rho)\text{-dominates } V_x\}$.*

For each node x of $V(T)$, our algorithm will compute a table \mathcal{D}_x that satisfies the following invariant.

Invariant. For every $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d$, the set $\mathcal{D}_x[R, R']$ is a subset of $\mathcal{A}_x[R, R']$ of size at most $\mathbf{s-nec}_1(T, \delta)^2$ that (x, R') -represents $\mathcal{A}_x[R, R']$.

Notice that, by the definition of $\mathcal{A}_r[\emptyset, \emptyset]$ (r being the root of T) and the definition of (x, R') -representativity, if G admits a connected (σ, ρ) -dominating set, then $\mathcal{D}_r[\emptyset, \emptyset]$ must contain a maximum (or minimum) connected (σ, ρ) -dominating set.

The following lemma provides an equality between the entries of the table \mathcal{A}_x and the entries of the tables \mathcal{A}_a and \mathcal{A}_b for each internal node $x \in V(T)$ with a and b as children. We use this lemma to prove, by induction, that the entry $\mathcal{D}_x[R, R']$ (x, R') -represents $\mathcal{A}_x[R, R']$ for every $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d$. Note that this lemma can be deduced from [10].

Lemma 5.4. *For all $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d$, we have*

$$\mathcal{A}_x[R, R'] = \bigcup_{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible}} \mathcal{A}_a[A, A'] \otimes \mathcal{A}_b[B, B'].$$

Proof. The lemma is implied by the two following claims.

Claim 5.5. *For all $X \in \mathcal{A}_x[R, R']$, there exist d - (R, R') -compatible pairs (A, A') and (B, B') such that $X \cap V_a \in \mathcal{A}_a[A, A']$ and $X \cap V_b \in \mathcal{A}_b[B, B']$.*

Proof. Let $X \in \mathcal{A}_x[R, R']$, $X_a := X \cap V_a$ and $X_b := X \cap V_b$. Let $A := \text{rep}_{V_a}^d(X_a)$ and $A' := \text{rep}_{\overline{V_a}}^d(X_b \cup R')$. Symmetrically, we define $B := \text{rep}_{V_b}^d(X_b)$ and $B' := \text{rep}_{\overline{V_b}}^d(X_a \cup R')$.

We claim that $X_a \in \mathcal{A}_a[A, A']$. As $X \in \mathcal{A}_x[R, R']$, we know, by Definition 5.3, that $X \cup R' = X_a \cup X_b \cup R'$ is a (σ, ρ) -dominating set of V_x . In particular, $X_a \cup (X_b \cup R')$ (σ, ρ) -dominates V_a . Since $A' \equiv_{\overline{V_a}}^d X_b \cup R'$, by Lemma 5.2, we conclude that $X_a \cup A'$ (σ, ρ) -dominates V_a . As $A \equiv_{V_a}^d X_a$, we have $X_a \in \mathcal{A}_a[A, A']$. By symmetry, we deduce $B \in \mathcal{A}_b[B, B']$.

It remains to prove that (A, A') and (B, B') are d - (R, R') -compatible.

- By construction, we have $X_a \cup X_b = X \equiv_{V_x}^d R$. As $A \equiv_{V_a}^d X_a$ and from Fact 3.1, we have $A \cup X_b \equiv_{V_x}^d R$. Since $B \equiv_{V_b}^d X_b$, we deduce that $A \cup B \equiv_{V_x}^d R$.
- By definition, we have $A' \equiv_{V_a}^d X_b \cup R'$. As $B \equiv_{V_b}^d X_b$ and by Fact 3.1, we have $A' \equiv_{V_a}^d B \cup R'$. Symmetrically, we deduce that $B' \equiv_{V_b}^d R' \cup A$.

Thus, (A, A') and (B, B') are d - (R, R') -compatible. \square

Claim 5.6. *For every $X_a \in \mathcal{A}_a[A, A']$ and $X_b \in \mathcal{A}_b[B, B']$ such that (A, A') and (B, B') are d - (R, R') -compatible, we have $X_a \cup X_b \in \mathcal{A}_x[R, R']$.*

Proof. Since $X_a \equiv_{V_a}^d A$ and $X_b \equiv_{V_b}^d B$, by Fact 3.1, we deduce that $X_a \cup X_b \equiv_{V_x}^d A \cup B$. Thus, by the definition of d - (R, R') -compatibility, we have $X_a \cup X_b \equiv_{V_x}^d R$.

It remains to prove that $X_a \cup X_b \cup R'$ (σ, ρ) -dominates V_x . As before, one can check that Fact 3.1 implies that $X_b \cup R' \equiv_{V_a}^d B \cup R'$. From Lemma 5.2, we conclude that $X_a \cup X_b \cup R'$ (σ, ρ) -dominates V_a . Symmetrically, we prove that $X_a \cup X_b \cup R'$ (σ, ρ) -dominates V_b . As $V_x = V_a \cup V_b$, we deduce that $X_a \cup X_b \cup R'$ (σ, ρ) -dominates V_x . Hence, we have $X_a \cup X_b \in \mathcal{A}_x[R, R']$. \square

\square

We are now ready to prove the main theorem of this section.

Theorem 5.7. *There exists an algorithm that, given an n -vertex graph G and a rooted layout (T, δ) of G , computes a maximum (or minimum) connected (σ, ρ) -dominating set in time $O(\text{s-nec}_d(T, \delta)^3 \cdot \text{s-nec}_1(T, \delta)^{2(\omega+1)} \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^3)$ with $d := \max\{1, d(\sigma), d(\rho)\}$.*

Proof. The algorithm is a usual bottom-up dynamic programming algorithm and computes for each node x of T the table \mathcal{D}_x .

The first step of our algorithm is to compute, for each $x \in V(T)$, the sets $\mathcal{R}_{V_x}^d$, $\mathcal{R}_{\overline{V_x}}^d$ and a data structure to compute $\text{rep}_{V_x}^d(X)$ and $\text{rep}_{\overline{V_x}}^d(Y)$, for any $X \subseteq V_x$ and any $Y \subseteq \overline{V_x}$, in time $O(\log(\text{s-nec}_d(T, \delta)) \cdot n^2)$. As T has $2n - 1$ nodes, by Lemma 3.2, we can compute these sets and data structures in time $O(\text{s-nec}_d(T, \delta) \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^3)$.

Let x be a leaf of T with $V_x = \{v\}$. Observe that, for all $(R, R') \in \mathcal{R}_d^{V_x} \times \mathcal{R}_d^{\overline{V_x}}$, we have $\mathcal{A}_x[R, R'] \subseteq 2^{V_x} = \{\emptyset, \{v\}\}$. Thus, our algorithm can directly compute $\mathcal{A}_x[R, R']$ and set $\mathcal{D}_x[R, R'] := \mathcal{A}_x[R, R']$. In this case, the invariant trivially holds.

Now let x be an internal node with a and b as children such that the invariant holds for a and b . For each $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d$, the algorithm computes $\mathcal{D}_x[R, R'] := \text{reduce}(\mathcal{B}_x[R, R'])$, where the set $\mathcal{B}_x[R, R']$ is defined as follows

$$\mathcal{B}_x[R, R'] := \bigcup_{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible}} \mathcal{D}_a[A, A'] \otimes \mathcal{D}_b[B, B'].$$

We claim that the invariant holds for x . Let $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d$.

We start by proving that the set $\mathcal{B}_x[R, R']$ is an (x, R') -representative set of $\mathcal{A}_x[R, R']$. By Lemma 4.8, for all d - (R, R') -compatible pairs (A, A') and (B, B') , we have

$$\mathcal{D}_a[A, A'] \otimes \mathcal{D}_b[B, B'] \text{ } (x, R')\text{-represents } \mathcal{A}_a[A, A'] \otimes \mathcal{A}_b[B, B'].$$

By Lemma 5.4 and by construction of $\mathcal{D}_x[R, R']$ and from Fact 4.6, we conclude that $\mathcal{B}_x[R, R']$ (x, R') -represents $\mathcal{A}_x[R, R']$.

From the invariant, we have $\mathcal{D}_a[A, A'] \subseteq \mathcal{A}_a[A, A']$ and $\mathcal{D}_b[B, B'] \subseteq \mathcal{A}_b[B, B']$, for all d - (R, R') -compatible pairs (A, A') and (B, B') . Thus, from Lemma 5.4, it is clear that by construction, we have $\mathcal{B}_x[R, R'] \subseteq \mathcal{A}_x[R, R']$. Hence, $\mathcal{B}_x[R, R']$ is a subset and an (x, R') -representative set of $\mathcal{A}_x[R, R']$.

Notice that, for each $X \in \mathcal{B}_x[R, R']$, we have $X \equiv_{V_x}^d R$. Thus, we can apply Theorem 4.3 and the function reduce on $\mathcal{B}_x[R, R']$. By Theorem 4.3, $\mathcal{D}_x[R, R']$ is a subset and an (x, R') -representative set of $\mathcal{B}_x[R, R']$. Thus $\mathcal{D}_x[R, R']$ is a subset of $\mathcal{A}_x[R, R']$. Notice that the (x, R') -representativity is an equivalence relation and in particular it is transitive. Consequently,

$\mathcal{D}_x[R, R']$ (x, R')-represents $\mathcal{A}_x[R, R']$. From Theorem 4.3, the size of $\mathcal{D}_x[R, R']$ is at most $\text{nec}_1(V_x)^2$ and that $\mathcal{D}_x[R, R'] \subseteq \mathcal{B}_x[R, R']$. As $\text{nec}_1(V_x) \leq \text{s-nec}_1(T, \delta)$ and $\mathcal{B}_x[R, R'] \subseteq \mathcal{A}_x[R, R']$, we conclude that the invariant holds for x .

By induction, the invariant holds for all nodes of T . The correctness of the algorithm follows from the fact that $\mathcal{D}_r[\emptyset, \emptyset]$ (r, \emptyset)-represents $\mathcal{A}_r[\emptyset, \emptyset]$.

Running Time. Let x be a node of T . Suppose first that x is a leaf of T . Then $|\mathcal{R}_{V_x}^d| \leq 2$ and $|\mathcal{R}_{V_x}^d| \leq d$. Thus, \mathcal{D}_x can be computed in time $O(d \cdot n)$.

Assume now that x is an internal node of T with a and b as children.

Notice that, by Definition 4.7, for every $(A, B, R') \in \mathcal{R}_{V_a}^d \times \mathcal{R}_{V_b}^d \times \mathcal{R}_{V_x}^d$, there exists only one tuple $(A', B', R) \in \mathcal{R}_{V_a}^d \times \mathcal{R}_{V_b}^d \times \mathcal{R}_{V_x}^d$ such that (A, A') and (B, B') are d -(R, R')-compatible. More precisely, you have to take $R = \text{rep}_{V_x}^d(A \cup B)$, $A' = \text{rep}_{V_a}^d(R' \cup B)$ and $B' = \text{rep}_{V_b}^d(R' \cup A)$. Thus, there are at most $\text{s-nec}_d(T, \delta)^3$ tuples (A, A', B, B', R, R') such that (A, A') and (B, B') are d -(R, R')-compatible. It follows that we can compute the intermediary table \mathcal{B}_x by doing the following.

- Initialize each entry of \mathcal{B}_x to \emptyset .
- For each $(A, B, R') \in \mathcal{R}_{V_a}^d \times \mathcal{R}_{V_b}^d \times \mathcal{R}_{V_x}^d$, compute $R' := \text{rep}_{V_x}^d(A \cup B)$, $A' = \text{rep}_{V_a}^d(R' \cup B)$ and $B' = \text{rep}_{V_b}^d(R' \cup A)$. Then, update $\mathcal{B}_x[R, R'] := \mathcal{B}_x[R, R'] \cup (\mathcal{D}_a[A, A'] \otimes \mathcal{D}_b[B, B'])$.

Each call to the functions $\text{rep}_{V_x}^d$, $\text{rep}_{V_a}^d$ and $\text{rep}_{V_b}^d$ takes $O(\log(\text{s-nec}_d(T, \delta)) \cdot n^2)$ time. We deduce that the running time to compute the entries of \mathcal{B}_x is

$$O \left(\text{s-nec}_d(T, \delta)^3 \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^2 + \sum_{(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d} |\mathcal{B}_x[R, R']| \cdot n^2 \right).$$

Observe that, for each $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$, by Theorem 4.3, the running time to compute $\text{reduce}(\mathcal{B}_x[R, R'])$ from $\mathcal{B}_x[R, R']$ is $O(|\mathcal{B}_x[R, R']| \cdot \text{s-nec}_1(T, \delta)^{2(\omega-1)} \cdot n^2)$. Thus, the total running time to compute the table \mathcal{D}_x from the table \mathcal{B}_x is

$$(1) \quad O \left(\sum_{(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d} |\mathcal{B}_x[R, R']| \cdot \log(\text{s-nec}_d(T, \delta)) \cdot \text{s-nec}_1(T, \delta)^{2(\omega-1)} \cdot n^2 \right).$$

For each (A, A') and (B, B') , the size of $\mathcal{D}_a[A, A'] \otimes \mathcal{D}_b[B, B']$ is at most $|\mathcal{D}_a[A, A']| \cdot |\mathcal{D}_b[B, B']| \leq \text{s-nec}_1(T, \delta)^4$. Since there are at most $\text{s-nec}_d(T, \delta)^3$ pairs that are d -(R, R')-compatible, we can conclude that

$$\sum_{(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d} |\mathcal{B}_x[R, R']| \leq \text{s-nec}_d(T, \delta)^3 \cdot \text{s-nec}_1(T, \delta)^4.$$

From Equation (1), we deduce that the entries of \mathcal{D}_x are computable in time

$$O(\text{s-nec}_d(T, \delta)^3 \cdot \text{s-nec}_1(T, \delta)^{2(\omega+1)} \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^2).$$

Since T has $2n - 1$ nodes, the running time of our algorithm is $O(\text{s-nec}_d(T, \delta)^3 \cdot \text{s-nec}_1(T, \delta)^{2(\omega+1)} \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^3)$. \square

As a corollary, we can solve in time $\text{s-nec}_1(T, \delta)^{(2\omega+5)} \cdot \log(\text{s-nec}_1(T, \delta)) \cdot n^3$ the NODE-WEIGHTED STEINER TREE problem that asks, given a subset of vertices $K \subseteq V(G)$ called *terminals*, a subset T of minimal weight such that $K \subseteq T \subseteq V(G)$ and $G[T]$ is connected.

Corollary 5.8. *There exists an algorithm that, given an n -vertex graph G , a subset $K \subseteq V(G)$ and a rooted layout (T, δ) of G , computes a minimum node-weighted Steiner tree for (G, K) in time $O(\text{s-nec}_1(T, \delta)^{(2\omega+5)} \cdot \log(\text{s-nec}_1(T, \delta)) \cdot n^3)$.*

Proof. Observe that a Steiner tree is a minimum connected (\mathbb{N}, \mathbb{N}) -dominating set of G that contains K . Thus, it is sufficient to change the definition of the table \mathcal{A}_x as follows. Let $x \in V(T)$. For all $(R, R') \in \mathcal{R}_{V_x}^1 \times \mathcal{R}_{V_x}^1$, we define $\mathcal{A}_x[R, R'] \subseteq V_x$ as follows

$$\mathcal{A}_x[R, R'] := \{X \subseteq V_x : X \equiv_{V_x}^d R, K \cap V_x \subseteq X \text{ and } X \cup R' \text{ } (\mathbb{N}, \mathbb{N})\text{-dominates } V_x\}.$$

Notice that this modification will just modify the way we compute the table \mathcal{D}_x when x is a leaf of T associated with a vertex in K . With this definition of \mathcal{A}_x and by Definition 4.1 of (x, R') -representativity, if G contains an optimal solution, then $\mathcal{D}_r[\emptyset, \emptyset]$ contains an optimal solution of G . The running time comes from the running time of Theorem 5.7 with $d = 1$. \square

Observe that Corollary 5.8 simplifies and generalizes the algorithm from [6] for the EDGE-WEIGHTED STEINER TREE problem. Indeed the incidence graph of a graph of tree-width k has tree-width at most $k + 1$ and one can reduce the computation of an edge-weighted Steiner tree on a graph to the computation of a node-weighted Steiner tree on its incidence graph [20].

With few modifications, we can easily deduce an algorithm to compute a maximum (or minimum) connected $\text{co-}(\sigma, \rho)$ -dominating set.

Corollary 5.9. *There exists an algorithm that, given an n -vertex graph G and a rooted layout (T, δ) of G , computes a maximum (or minimum) connected $\text{co-}(\sigma, \rho)$ -dominating set in time $O(\text{s-nec}_d(T, \delta)^3 \cdot \text{s-nec}_1(T, \delta)^{(2\omega+5)} \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^3)$ with $d := \max\{1, d(\sigma), d(\rho)\}$.*

Proof. To find a maximum (or minimum) $\text{co-}(\sigma, \rho)$ -dominating set, we need to modify the definition of the table \mathcal{A}_x , the invariant and the computational steps of the algorithm from Theorem 5.7. For each vertex $x \in V(T)$, we define the set of indices of our table \mathcal{D}_x as $\mathbb{I}_x := \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^1 \times \mathcal{R}_{V_x}^1$.

For all $(R, R', \bar{R}, \bar{R}') \in \mathbb{I}_x$, we define $\mathcal{A}_x[R, R', \bar{R}, \bar{R}'] \subseteq 2^{V_x}$ as the following set

$$\{X \subseteq V_x : X \equiv_{V_x}^1 \bar{R}, (V_x \setminus X) \equiv_{V_x}^d R \text{ and } (V_x \setminus X) \cup R' \text{ } (\sigma, \rho)\text{-dominates } V_x\}.$$

It is worth noticing that the definition of \mathcal{A}_x does not depend on \bar{R}' , it is just more convenient to write the proof this way in order to obtain an algorithm similar to the one from Theorem 5.7.

Similarly to Theorem 5.7, for each node x of $V(T)$, our algorithm will compute a table \mathcal{D}_x that satisfies the following invariant.

Invariant. For every $(R, R', \bar{R}, \bar{R}') \in \mathbb{I}_x$, the set $\mathcal{D}_x[R, R', \bar{R}, \bar{R}']$ is a subset of $\mathcal{A}_x[R, R', \bar{R}, \bar{R}']$ of size at most $\text{s-nec}_1(T, \delta)^2$ that (x, \bar{R}') -represents $\mathcal{A}_x[R, R', \bar{R}, \bar{R}']$.

Intuitively, we use \bar{R} and \bar{R}' to deal with the connectivity constraint of the $\text{co-}(\sigma, \rho)$ -dominating set and R and R' for the (σ, ρ) -domination.

The following claim adapts Lemma 5.4 to the $\text{co-}(\sigma, \rho)$ -dominating set case.

Claim 5.10. *Let x be an internal node of T with a and b as children. For all $(R, R', \bar{R}, \bar{R}') \in \mathbb{I}_x$, we have*

$$\mathcal{A}_x[R, R', \bar{R}, \bar{R}'] := \bigcup_{\substack{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible} \\ (\bar{A}, \bar{A}'), (\bar{B}, \bar{B}') \text{ } 1\text{-}(\bar{R}, \bar{R}')\text{-compatible}}} \mathcal{A}_a[A, A', \bar{A}, \bar{A}'] \otimes \mathcal{A}_b[B, B', \bar{B}, \bar{B}'].$$

\square

The proof of this claim follows from the proof of Lemma 5.4. With these modifications, it is straightforward to check that the algorithm of Theorem 5.7 can be adapted to compute a minimum or maximum connected $\text{co-}(\sigma, \rho)$ -dominating set of $V(G)$. With the same analysis as in Theorem 5.7, one easily deduces that the running time of this modified algorithm is $O(\text{s-nec}_d(T, \delta)^3 \cdot \text{s-nec}_1(T, \delta)^{(2\omega+5)} \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^3)$. \square

6. ACYCLIC VARIANTS OF (CONNECTED) (σ, ρ) -DOMINATING SET

We call $\text{AC-}(\sigma, \rho)\text{-DOMINATING SET}$ (resp. $\text{ACYCLIC }(\sigma, \rho)\text{-DOMINATING SET}$) the family of problems which consists in finding a subset $X \subseteq V(G)$ of maximum (or minimum) weight such that X is a (σ, ρ) -dominating set of G and $G[X]$ is a tree (resp. a forest). Some examples of famous problems which belong to these family of problems are presented in Table 3.

TABLE 3. Examples of $\text{AC-}(\sigma, \rho)\text{-DOMINATING SET}$ problems and $\text{ACYCLIC }(\sigma, \rho)\text{-DOMINATING SET}$ problems. To solve these problems, we use the d -neighbor equivalence with $d := \max\{2, d(\sigma), d(\rho)\}$. Column d shows the value of d for each problem.

σ	ρ	d	Version	Standard name
\mathbb{N}	\mathbb{N}	2	AC	MAXIMUM INDUCED TREE
\mathbb{N}	\mathbb{N}	2	ACYCLIC	MAXIMUM INDUCED FOREST
$\{1, 2\}$	\mathbb{N}	3	AC	LONGEST INDUCED PATH
$\{1, 2\}$	\mathbb{N}	3	ACYCLIC	MAXIMUM INDUCED LINEAR FOREST

In this section, we present an algorithm that solves any $\text{AC-}(\sigma, \rho)\text{-DOMINATING SET}$ problem. Unfortunately, we were not able to obtain an algorithm whose running time is polynomial in n and the d -neighbor-width of the given layout (for some constant d). But, for the other parameters, by using their respective properties, we get the running time presented in Table 4 which are roughly the same as those in the previous section. Moreover, we show, via a polynomial reduction, that we can use our algorithm for $\text{AC-}(\sigma, \rho)\text{-DOMINATING SET}$ problems (with some modifications) to solve any $\text{ACYCLIC }(\sigma, \rho)\text{-DOMINATING SET}$ problem.

TABLE 4. Upper bounds on the running times of our algorithm for $\text{AC-}(\sigma, \rho)\text{-DOMINATING SET}$ problems with $\mathcal{L} = (T, \delta)$ and $d := \max\{2, d(\sigma), d(\rho)\}$.

Parameter	Running time
Module-width	$O((d+1)^{3\text{mw}(\mathcal{L})} \cdot 2^{(2\omega+3)\text{mw}(\mathcal{L})} \cdot \text{mw}(\mathcal{L}) \cdot n^4)$
Rank-width	$O(2^{(2\omega+3d+4)\text{rw}(\mathcal{L})^2} \cdot \text{rw}(\mathcal{L}) \cdot n^4)$
\mathbb{Q} -rank-width	$O((d \cdot \text{rw}_{\mathbb{Q}}(\mathcal{L}))^{(2\omega+5)\text{rw}_{\mathbb{Q}}(\mathcal{L})} \cdot \text{rw}_{\mathbb{Q}}(\mathcal{L}) \cdot n^4)$
Mim-width	$O(n^{(2\omega+3d+4)\text{mim}(\mathcal{L})+4} \cdot \text{mim}(\mathcal{L}))$

Let us first explain why we cannot use the same trick as in [6] on the algorithms of Section 5 to ensure the acyclicity, that is classifying the partial solutions X – associated with a node $x \in V(T)$ – with respect to $|X|$ and $|E(G[X])|$. Indeed, for two sets $X, W \subseteq V_x$ with $|X| = |W|$ and $|E(G[X])| = |E(G[W])|$, we have $|E(G[X \cup Y])| = |E(G[W \cup Y])|$, for all $Y \subseteq \overline{V}_x$, if and only if $X \equiv_{V_x}^n W$. Hence, the trick used in [6] would imply to classify the partial solutions with respect to their n -neighbor equivalence class. But, the upper bounds we have on $\text{nec}_n(V_x)$ with respect to module-width, (\mathbb{Q} -)rank-width would lead to an XP algorithm. In fact, for every $k \in \mathbb{N}$ and every $n \geq 2k$, one can construct an n -vertex bipartite graph $H_k[A, \overline{A}]$ where $\text{mw}(A) = k$ and $\text{nec}_n(A) = (n/\text{mw}(A))^{\text{mw}(A)}$ (see Figure 3 in Appendix A). Since both $\text{rw}(A)$ and $\text{rw}_{\mathbb{Q}}(A)$ are upper-bounded by $\text{mw}(A)$, we deduce that using the trick of [6] would give, for each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}\}$, an $n^{\Omega(f(T, \delta))}$ time algorithm.

In the following, we introduce some new concepts that extends the framework designed in Section 4 in order to manage acyclicity. All along, we give intuitions on these concepts through

a concrete example: MAXIMUM INDUCED TREE. Finally, we present the algorithms for the AC- (σ, ρ) -DOMINATING SET problems and the algorithms for ACYCLIC (σ, ρ) -DOMINATING SET problems.

We start by defining a new notion of representativity to deal with the acyclicity constraint. This new notion of representativity is defined *w.r.t.* the 2-neighbor equivalence class of a set $R' \subseteq \overline{V_x}$. We consider 2-neighbor equivalence classes instead of 1-neighbor equivalence classes in order to manage the acyclicity (see the following explanations). Similarly to Section 4, every concept introduced in this section is defined with respect to a node x of T and a set $R' \subseteq \overline{V_x}$. To simplify this section, we fix a node x of T and $R' \subseteq \overline{V_x}$. In our algorithm, R' will always belong to $\mathcal{R}_{\overline{V_x}}^d$ for some $d \in \mathbb{N}^+$ with $d \geq 2$. For MAXIMUM INDUCED TREE $d = 2$ is enough and in general, we use $d := \max\{2, d(\sigma), d(\rho)\}$.

The following definition extends Definition 4.1 of Section 4 to deal with the acyclicity. We let $\text{opt} \in \{\min, \max\}$; if we want to solve a maximization (or minimization) problem, we use $\text{opt} = \max$ (or $\text{opt} = \min$).

Definition 6.1 ((x, R') -acy-representativity). *For every $\mathcal{A} \subseteq 2^{V(G)}$ and $Y \subseteq V(G)$, we define*

$$\text{best}(\mathcal{A}, Y)^{\text{acy}} := \text{opt}\{\mathbf{w}(X) : X \in \mathcal{A} \text{ and } G[X \cup Y] \text{ is a tree}\}.$$

Let $\mathcal{A}, \mathcal{B} \subseteq 2^{V_x}$. We say that \mathcal{B} (x, R') -acy-represents \mathcal{A} if, for every $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^2 R'$, we have $\text{best}^{\text{acy}}(\mathcal{A}, Y) = \text{best}^{\text{acy}}(\mathcal{B}, Y)$.

As for the (x, R') -representativity, we need to prove that the operations we use in our algorithm preserve the (x, R') -acy-representativity. The following fact follows from Definition 6.1 of (x, R') -acy-representativity.

Fact 6.2. *If \mathcal{B} and \mathcal{D} (x, R') -acy-represents, respectively, \mathcal{A} and \mathcal{C} , then $\mathcal{B} \cup \mathcal{D}$ (x, R') -acy-represents $\mathcal{A} \cup \mathcal{C}$.*

The following lemma is an adaptation of Lemma 4.8 to the notion of (x, R') -acy-representativity. The proof is almost the same as the one of Lemma 4.8. We refer to Definition 4.7 for the notion of d - (R, R') -compatibility.

Lemma 6.3. *Let $d \in \mathbb{N}^+$ such that $d \geq 2$. Suppose that x is an internal node of T with a and b as children. Let $R \in \mathcal{R}_{\overline{V_x}}^d$. Let $(A, A') \in \mathcal{R}_{\overline{V_a}}^d \times \mathcal{R}_{\overline{V_a}}^d$ and $(B, B') \in \mathcal{R}_{\overline{V_b}}^d \times \mathcal{R}_{\overline{V_b}}^d$ that are d - (R, R') -compatible. Let $\mathcal{A} \subseteq 2^{V_a}$ such that, for all $X \in \mathcal{A}$, we have $X \equiv_{\overline{V_a}}^d A$ and let $\mathcal{B} \subseteq 2^{V_b}$ such that, for all $W \in \mathcal{B}$, we have $W \equiv_{\overline{V_b}}^d B$.*

If $\mathcal{A}' \subseteq \mathcal{A}$ (a, A') -acy-represents \mathcal{A} and $\mathcal{B}' \subseteq \mathcal{B}$ (b, B') -acy-represents \mathcal{B} , then

$$\mathcal{A}' \otimes \mathcal{B}' \text{ } (x, R')^{\text{acy}}\text{-represents } \mathcal{A} \otimes \mathcal{B}.$$

Proof. Suppose that $\mathcal{A}' \subseteq \mathcal{A}$ (a, A') -acy-represents \mathcal{A} and $\mathcal{B}' \subseteq \mathcal{B}$ (b, B') -acy-represents \mathcal{B} .

In order to prove this lemma, it is sufficient to prove that, for each $Y \equiv_{\overline{V_x}}^2 R'$, we have $\text{best}^{\text{acy}}(\mathcal{A}' \otimes \mathcal{B}', Y) = \text{best}^{\text{acy}}(\mathcal{A} \otimes \mathcal{B}, Y)$.

Let $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^2 R'$. We claim the following facts

- (a) for every $W \in \mathcal{B}$, we have $W \cup Y \equiv_{\overline{V_a}}^2 A'$,
- (b) for every $X \in \mathcal{A}$, we have $X \cup Y \equiv_{\overline{V_b}}^2 B'$.

Let $W \in \mathcal{B}$. By Fact 3.1, we have that $W \equiv_{\overline{V_a}}^d B$ because $V_b \subseteq \overline{V_a}$ and $W \equiv_{\overline{V_b}}^d B$. Since $d \geq 2$, we have $W \equiv_{\overline{V_a}}^2 B$. By Fact 3.1, we deduce also that $Y \equiv_{\overline{V_a}}^2 R'$. Since (A, A') and (B, B') are d - (R, R') -compatible, we have $A' \equiv_{\overline{V_a}}^d R' \cup B$. In particular, we have $A' \equiv_{\overline{V_a}}^2 R' \cup B$ because $d \geq 2$. We can conclude that $W \cup Y \equiv_{\overline{V_a}}^2 A'$ for every $W \in \mathcal{B}$. The proof for Fact (b) is symmetric.

Now observe that, by the definitions of best^{acy} and of the merging operator \otimes , we have

$$\text{best}^{\text{acy}}(\mathcal{A} \otimes \mathcal{B}, Y) = \text{opt}\{\mathbf{w}(X) + \mathbf{w}(W) : X \in \mathcal{A} \wedge W \in \mathcal{B} \wedge G[X \cup W \cup Y] \text{ is a tree}\}.$$

Since $\text{best}^{\text{acy}}(\mathcal{A}, W \cup Y) = \text{opt}\{\mathbf{w}(X) : X \in \mathcal{A} \wedge G[X \cup W \cup Y] \text{ is a tree}\}$, we deduce that

$$\text{best}^{\text{acy}}(\mathcal{A} \otimes \mathcal{B}, Y) = \text{opt}\{\text{best}^{\text{acy}}(\mathcal{A}, W \cup Y) + \mathbf{w}(W) : W \in \mathcal{B}\}.$$

Since \mathcal{A}' (a, A')-represents \mathcal{A} and by Fact (a), we have

$$\begin{aligned} \text{best}^{\text{acy}}(\mathcal{A} \otimes \mathcal{B}, Y) &= \text{opt}\{\text{best}^{\text{acy}}(\mathcal{A}', W \cup Y) + \mathbf{w}(W) : W \in \mathcal{B}\} \\ &= \text{best}^{\text{acy}}(\mathcal{A}' \otimes \mathcal{B}, Y). \end{aligned}$$

Symmetrically, we deduce from Fact (b) that $\text{best}^{\text{acy}}(\mathcal{A}' \otimes \mathcal{B}, Y) = \text{best}^{\text{acy}}(\mathcal{A}' \otimes \mathcal{B}', Y)$. This stands for every $Y \subseteq \overline{V}_x$ such that $Y \equiv_{\overline{V}_x}^2 R'$. Thus, we conclude that $\mathcal{A}' \otimes \mathcal{B}'$ (x, R')-acy-represents $\mathcal{A} \otimes \mathcal{B}$. \square

In order to compute a maximum induced tree, we design an algorithm similar to those of Section 5. That is, for each $(R, R') \in \mathcal{R}_{V_x}^2 \times \mathcal{R}_{\overline{V}_x}^2$, our algorithm will compute a set $\mathcal{D}_x[R, R'] \subseteq 2^{V_x}$ that is an (x, R') -acy-representative set of small size of the set $\mathcal{A}_x[R] := \{X \subseteq V_x \text{ such that } X \equiv_{V_x}^2 R\}$. This is sufficient to compute a maximum induced tree of G since we have $\mathcal{A}_r[\emptyset] = 2^{V(G)}$ with r the root of T . Thus, by Definition 6.1, any (r, \emptyset) -acy-representative set of $\mathcal{A}_r[\emptyset]$ contains a maximum induced tree.

The key to compute the tables of our algorithm is a function that, given $\mathcal{A} \subseteq 2^{V_x}$, computes a small subset of \mathcal{A} that (x, R') -represents \mathcal{A} . This function starts by removing from \mathcal{A} some sets that will never give a tree with a set $Y \equiv_{\overline{V}_x}^2 R'$. For doing so, we characterize the sets $X \in \mathcal{A}$ such that $G[X \cup Y]$ is a tree for some $Y \equiv_{\overline{V}_x}^2 R'$. We call these sets R' -important. The following gives a formal definition of these important and unimportant partial solutions.

Definition 6.4 (R' -important). *We say that $X \subseteq V_x$ is R' -important if there exists $Y \subseteq \overline{V}_x$ such that $Y \equiv_{\overline{V}_x}^2 R'$ and $G[X \cup Y]$ is a tree, otherwise, we say that X is R' -unimportant.*

By definition, any set obtained from a set \mathcal{A} by removing R' -unimportant sets is an (x, R') -acy-representative set of \mathcal{A} . The following lemma gives some necessary conditions on R' -important sets. It follows that any set which does not respect one of these conditions can safely be removed from \mathcal{A} . These conditions are the key to obtain the running times of Table 4.

At this point, we need to introduce the following notations. For every $X \subseteq V_x$, we define $X^0 := \{v \in X : N(v) \cap R' = \emptyset\}$, $X^1 := \{v \in X : |N(v) \cap R'| = 1\}$ and $X^{2+} := \{v \in X : |N(v) \cap R'| \geq 2\}$. From the definition of the 2-neighbor equivalence, we have the following property which is the key to manage acyclicity.

Observation 6.5. *For every $X \subseteq V_x$ and $Y \subseteq \overline{V}_x$ such that $Y \equiv_{\overline{V}_x}^2 R'$, the vertices in X^0 have no neighbor in Y , those in X^1 have exactly one neighbor in Y and those in X^{2+} have at least 2 neighbors in Y .*

Lemma 6.6. *If $X \subseteq V_x$ is R' -important, then $G[X]$ is a forest and the following properties are satisfied:*

- (1) *for every pair of distinct vertices a and b in X^{2+} , we have $N(a) \cap \overline{V}_x \neq N(b) \cap \overline{V}_x$,*
- (2) *$|X^{2+}|$ is upper bounded by $2\text{mim}(V_x)$, $2\text{rw}(V_x)$, $2\text{rw}_{\mathbb{Q}}(V_x)$ and $2\log_2(\text{nec}_1(V_x))$.*

Proof. Obviously, any R' -important set must induce a forest. Let $X \subseteq V_x$ be an R' -important set. Since X is R' -important, there exists $Y \subseteq \overline{V}_x$ such that $Y \equiv_{\overline{V}_x}^2 R'$ and $G[X \cup Y]$ is a tree.

Assume towards a contradiction that there exist two distinct vertices a and b in X^{2+} such that $N(a) \cap \overline{V}_x = N(b) \cap \overline{V}_x$. Since a and b belong to X^{2+} and $Y \equiv_{\overline{V}_x}^2 R'$, both a and b have at least two neighbors in Y . Thus, a and b have at least two common neighbors in Y . We conclude that $G[X \cup Y]$ admits a cycle of length four, yielding a contradiction. We conclude that Property (1) holds for every R' -important set.

Now, we prove that Property (2) holds for X . Observe that, by Lemma 2.4, $\text{mim}(V_x)$ is upper bounded by $\text{rw}(V_x)$, $\text{rw}_{\mathbb{Q}}(V_x)$ and $\log_2(\text{nec}_1(V_x))$. Thus, in order to prove Property (2), it is sufficient to prove that $|X^{2+}| \leq 2\text{mim}(V_x)$.

We claim that $|X^{2+}| \leq 2k$ where k is the size of a maximum induced matching of $F := G[X^{2+}, Y]$. Since F is an induced subgraph of $G[V_x, \overline{V_x}]$, we have $k \leq \text{mim}(V_x)$ and this is enough to prove Property (2). Notice that F is a forest because F is a subgraph of $G[X \cup Y]$, which is a tree.

In the following, we prove that F admits a *good bipartition*, that is a bipartition $\{X_1, X_2\}$ of $X^{2+} \cap V(F)$ such that, for each $i \in \{1, 2\}$ and, for each $v \in X_i$, there exists $y_v \in Y \cap V(F)$ such that $N_F(y_v) \cap X_i = \{v\}$. Observe that this is enough to prove Property (2) since if F admits a good bipartition $\{X_1, X_2\}$, then $|X_1| \leq k$ and $|X_2| \leq k$. Indeed, if F admits a good bipartition $\{X_1, X_2\}$, then, for each $i \in \{1, 2\}$, the set of edges $M_i = \{vy_v : v \in X_i\}$ is an induced matching of F . In order to prove that F admits a good bipartition it is sufficient to prove that each connected component of F admits a good bipartition.

Let $C \in \text{cc}(F)$ and $u \in C \cap X^{2+}$. As F is a forest, $F[C]$ is a tree. Observe that the distance in F between each vertex $v \in C \cap X^{2+}$ and u is even because $F := G[X^{2+}, Y]$. Let C_1 (resp. C_2) be the set of all vertices $v \in C \cap X^{2+}$ such that there exists an odd (resp. even) integer $\ell \in \mathbb{N}$ so that the distance between v and u in F is $2i$. We claim that $\{C_1, C_2\}$ is a good bipartition of $F[C]$.

Let $i \in \{1, 2\}$, $v \in C_i$ and $\ell \in \mathbb{N}$ such that the distance between v and u in F is 2ℓ . Let P be the set of vertices in $V(F) \setminus \{v\}$ that share a common neighbor with v in F . We want to prove that there exists $y \in Y$ such that $N_F(y) \cap C_i = \{v\}$. For doing so, it is sufficient to prove that $N_F(v) \setminus N_F(C_i \setminus \{v\}) = N_F(v) \setminus N_F(P \cap C_i) \neq \emptyset$. Observe that, for every $v' \in P$, the distance between v' and u in F is either $2\ell - 2$, 2ℓ or $2\ell + 2$ because $F[C]$ is a tree and the distance between v and u is 2ℓ . By construction of $\{C_1, C_2\}$, every vertex at distance $2\ell - 2$ and $2\ell + 2$ from u must belong to C_{3-i} . Thus, every vertex in $P \cap C_i$ is at distance 2ℓ from u . If $\ell = 0$, then we are done because $v = u$ and $P \cap C_i = \emptyset$. Assume that $\ell \neq 0$. As $F[C]$ is a tree, v has only one neighbor w at distance $2\ell - 1$ from u in F . Because $F[C]$ is a tree, we deduce that $N_F(v) \cap N_F(P \cap C_i) = \{w\}$. Since $v \in X^{2+}$, v has at least two neighbors in $F = G[X^{2+}, Y]$ (because $Y \cong_{\overline{V_x}}^2 R'$), we conclude that $N_F(v) \setminus N_F(P \cap C_i) \neq \emptyset$. Hence, we deduce that $\{C_1, C_2\}$ is a good bipartition of $F[C]$.

We deduce that every connected component of F admits a good bipartition and thus F admits a good bipartition. Thus, $|X^{2+}| \leq 2\text{mim}(V_x)$. \square

These vertices in X^{2+} play a major role in the acyclicity and the computation of representatives in the following sense. By removing from \mathcal{A} the sets that do not respect the two above properties, we are able to decompose \mathcal{A} into a small number of sets $\mathcal{A}_1, \dots, \mathcal{A}_t$ such that an (x, R') -representative set of \mathcal{A}_i is an $(x, R')^{\text{acy}}$ -representative set of \mathcal{A}_i for each $i \in \{1, \dots, t\}$. We find an $(x, R')^{\text{acy}}$ -representative set of \mathcal{A} , by computing an (x, R') -representative set \mathcal{B}_i for each \mathcal{A}_i with the function `reduce`. This is sufficient because $\mathcal{B}_1 \cup \dots \cup \mathcal{B}_t$ is an $(x, R')^{\text{acy}}$ -representative set of \mathcal{A} thanks to Fact 6.2.

The following definition characterizes the sets $\mathcal{A} \subseteq 2^{V_x}$ for which an (x, R') -representative set is also an $(x, R')^{\text{acy}}$ -representative set.

Definition 6.7. *We say that $\mathcal{A} \subseteq 2^{V_x}$ is R' -consistent if, for each $Y \subseteq \overline{V_x}$ such that $Y \cong_{\overline{V_x}}^2 R'$, if there exists $W \in \mathcal{A}$ such that $G[W \cup Y]$ is a tree, then, for each $X \in \mathcal{A}$, either $G[X \cup Y]$ is a tree or $G[X \cup Y]$ is not connected.*

The following lemma proves that an (x, R') -representative set of an R' -consistent set is also an $(x, R')^{\text{acy}}$ -representative set of this later.

Lemma 6.8. *Let $\mathcal{A} \subseteq 2^{V_x}$. For all $\mathcal{D} \subseteq \mathcal{A}$, if \mathcal{A} is R' -consistent and \mathcal{D} (x, R') -represents \mathcal{A} , then \mathcal{D} $(x, R')^{\text{acy}}$ -represents \mathcal{A} .*

Proof. We assume that $\text{opt} = \max$, the proof for $\text{opt} = \min$ is similar. Let $Y \cong_{\overline{V_x}}^2 R'$. If $\text{best}^{\text{acy}}(\mathcal{A}, Y) = -\infty$, then we also have $\text{best}^{\text{acy}}(\mathcal{D}, Y) = -\infty$ because $\mathcal{D} \subseteq \mathcal{A}$.

Assume now that $\text{best}^{\text{acy}}(\mathcal{A}, Y) \neq -\infty$. Thus, there exists $W \in \mathcal{A}$ such that $G[W \cup Y]$ is a tree. Since \mathcal{A} is R' -consistent, for all $X \in \mathcal{A}$, the graph $G[X \cup Y]$ is either a tree or it is not

connected. Thus, by Definition 4.1 of best , we have $\text{best}^{\text{acy}}(\mathcal{A}, Y) = \text{best}(\mathcal{A}, Y)$. As $\mathcal{D} \subseteq \mathcal{A}$, we have also $\text{best}^{\text{acy}}(\mathcal{D}, Y) = \text{best}(\mathcal{D}, Y)$. We conclude by observing that if \mathcal{D} (x, R') -represents \mathcal{A} , then $\text{best}^{\text{acy}}(\mathcal{D}, Y) = \text{best}^{\text{acy}}(\mathcal{A}, Y)$. \square

The next lemma proves that, for each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$, we can decompose a set $\mathcal{A} \subseteq 2^{V_x}$ into a small collection $\{\mathcal{A}_1, \dots, \mathcal{A}_t\}$ of pairwise disjoint subsets of \mathcal{A} such that each \mathcal{A}_i is R' -consistent. Even though some parts of the proof are specific to each parameter, the ideas are roughly the same. First, we remove the sets X in \mathcal{A} that do not induce a forest. If $f = \text{mw}$, we remove the sets in \mathcal{A} that do not respect Condition (1) of Lemma 6.6, otherwise, we remove the sets that do not respect the upper bound associated with f from Condition (2) of Lemma 6.6. These sets can be safely removed as, by Lemma 6.6, they are R' -unimportant. After removing these sets, we obtain the decomposition of \mathcal{A} by taking the equivalence classes of some equivalence relation that is roughly the n -neighbor equivalence relation. Owing to the set of R' -unimportant sets we have removed from \mathcal{A} , we prove that the number of equivalence classes of this latter equivalence relation respects the upper bound associated with f that is described in Table 5.

Lemma 6.9. *Let $\mathcal{A} \subseteq 2^{V_x}$. For each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$, there exist pairwise disjoint subsets $\mathcal{A}_1, \dots, \mathcal{A}_t$ of \mathcal{A} computable in time $O(|\mathcal{A}| \cdot \mathcal{N}_f(T, \delta) \cdot n^2)$ such that*

- $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_t$ (x, R') -represents \mathcal{A} ,
- \mathcal{A}_i is R' -consistent for each $i \in \{1, \dots, t\}$ and
- $t \leq \mathcal{N}_f(T, \delta)$,

where $\mathcal{N}_f(T, \delta)$ is the term defined in Table 5.

TABLE 5. Upper bounds $\mathcal{N}_f(T, \delta)$ on the size of the decomposition of Lemma 6.9 for each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$.

f	mw	rw $_{\mathbb{Q}}$	rw	mim
$\mathcal{N}_f(T, \delta)$	$2^{\text{mw}(T, \delta)} \cdot 2n$	$(2\text{rw}_{\mathbb{Q}}(T, \delta) + 1)^{\text{rw}_{\mathbb{Q}}(T, \delta)} \cdot 2n$	$2^{2\text{rw}(T, \delta)^2} \cdot 2n$	$2n^{2\text{mim}(T, \delta)+1}$

Proof. We define the equivalence relation \sim on 2^{V_x} such that $X \sim W$ if we have $X^{2+} \equiv_{V_x}^n W^{2+}$ and $|E(G[X])| - |X \setminus X^1| = |E(G[W])| - |W \setminus W^1|$.

The following claim proves that an equivalence class of \sim is an R' -consistent set. Intuitively, we use the property of the n -neighbor equivalence to prove that, for every $X, W \subseteq V_x$ such that $X \sim W$ and for all $Y \equiv_{V_x}^2 R'$, we have $|E(G[X \cup Y])| = |X \cup Y| - 1$ if and only if $|E(G[W \cup Y])| = |W \cup Y| - 1$. Consequently, if $X \sim W$ and both sets induce with Y a connected graph, then both sets induce with Y a tree (because a graph F is a tree if and only if F is connected and $|E(F)| = |V(F)| - 1$). We conclude from these observation the following claim.

Claim 6.10. *Let $\mathcal{B} \subseteq \mathcal{A}$. If, for all $X, W \in \mathcal{B}$, we have $X \sim W$, then \mathcal{B} is R' -consistent.*

Proof. Suppose that $X \sim W$ for all $X, W \in \mathcal{B}$. In order to prove that \mathcal{B} is R' -consistent, it is enough to prove that, for each $X, W \in \mathcal{B}$ and $Y \equiv_{V_x}^2 R'$, if $G[X \cup Y]$ is a tree and $G[W \cup Y]$ is connected, then $G[W \cup Y]$ is a tree.

Let $Y \equiv_{V_x}^2 R'$ and $X, W \in \mathcal{B}$. Assume that $G[X \cup Y]$ is a tree and that $G[W \cup Y]$ is connected. We want to prove that $G[W \cup Y]$ is a tree.

Since $G[X \cup Y]$ is a tree, we have $|E(G[X \cup Y])| = |X \cup Y| - 1$. Since the vertices in X^0 have no neighbors in Y , we can decompose $|E(G[X \cup Y])| = |X \cup Y| - 1$ to obtain the following equality

$$(2) \quad |E(G[Y])| + |E(X^{2+}, Y)| + |E(X^1, Y)| + |E(G[X])| = |X \cup Y| - 1.$$

Since every vertex in X^1 has exactly one neighbor in Y (because $Y \equiv_{\overline{V}_x}^2 R'$), we have $|E(X^1, Y)| = |X^1|$. Thus, Equation (1) is equivalent to

$$(3) \quad |E(G[Y])| + |E(X^{2+}, Y)| + |E(G[X])| = |X \setminus X^1| + |Y| - 1.$$

Since $X \sim W$, we have $|E(G[X])| - |X \setminus X^1| = |E(G[W])| - |W \setminus W^1|$. Moreover, owing to $X^{2+} \equiv_{\overline{V}_x}^n W^{2+}$ and Lemma 3.5, we have $|E(G(X^{2+}, Y))| = |E(G(W^{2+}, Y))|$. We conclude that Equation (2) is equivalent to

$$(4) \quad |E(G[Y])| + |E(W^{2+}, Y)| + |E(G[W])| = |W \setminus W^1| + |Y| - 1.$$

With the same arguments to prove that (3) is equivalent to $|E(G[X \cup Y])| = |X \cup Y| - 1$, we can show that (3) is equivalent to $|E(G(W \cup Y))| = |W \cup Y| - 1$. By assumption, $G[W \cup Y]$ is connected and thus we conclude that $G[W \cup Y]$ is a tree. \square

We are now ready to decompose \mathcal{A} . We start by removing from \mathcal{A} all the sets that do not induce a forest. Trivially, this can be done in time $O(|\mathcal{A}| \cdot n)$. Moreover, these sets are R' -unimportant and thus we keep an (x, R') ^{acy}-representative set of \mathcal{A} . Before explaining how we proceed separately for each parameter, we need the following observation which follows from the removal of all the sets in \mathcal{A} that do not induce a forest.

Observation 6.11. *For all $X \in \mathcal{A}$, we have $|E(G[X])| - |X \setminus X^1| \in \{-n, \dots, n\}$.*

Concerning module-width. We remove all the sets X in \mathcal{A} that do not respect Condition (1) of Lemma 6.6. By Lemma 6.6, these sets are R' -unimportant and thus we keep an (x, R') ^{acy}-representative set of \mathcal{A} . After removing these sets, for each $X \in \mathcal{A}$, every pair (a, b) of distinct vertices in X^{2+} have a different neighborhood in \overline{V}_x . By definition of module-width, we have $\text{mw}(V_x) = |\{N(v) \cap \overline{V}_x : v \in V_x\}|$. Moreover, observe that, for every $X, W \in \mathcal{A}$, if $\{N(v) \cap \overline{V}_x : v \in X^{2+}\} = \{N(v) \cap \overline{V}_x : v \in W^{2+}\}$, then $X^{2+} \equiv_{\overline{V}_x}^n W^{2+}$.

We deduce that the number of n -neighbor equivalence classes over the set $\{X^{2+} : X \in \mathcal{A}\}$ is at most $2^{\text{mw}(V_x)}$. Thus, the number of equivalence classes of \sim over \mathcal{A} is at most $2^{\text{mw}(V_x)} \cdot 2n \leq \mathcal{N}_{\text{mw}}(T, \delta)$. The factor $2n$ comes from Observation 6.11 and appears also in all subsequent upper-bounds.

Concerning mim-width. We remove from \mathcal{A} all the sets X such that $|X^{2+}| > 2\text{mim}(V_x)$. By Lemma 6.6, these sets are R' -unimportant and thus we keep an (x, R') ^{acy}-representative set of \mathcal{A} . Observe that this can be done in time $O(n^{\text{mim}(V_x)+1} + |\mathcal{A}| \cdot n^2)$ because $\text{mim}(V_x)$ can be computed in time $O(n^{\text{mim}(V_x)} + 1)$. Since $|X^{2+}| \leq 2\text{mim}(V_x)$ for every $X \in \mathcal{A}$, we deduce that the number of equivalence classes of \sim over \mathcal{A} is at most $2n^{2\text{mim}(V_x)+1} \leq \mathcal{N}_{\text{mim}}(T, \delta)$.

Concerning rank-width. We remove from \mathcal{A} all the sets X such that $|X^{2+}| > 2\text{rw}(V_x)$ because they are R' -unimportant by Lemma 6.6. By Lemma 3.6, we know that $\text{nec}_n^{\leq 2\text{rw}(V_x)}(V_x)$ is upper bounded by $2^{2\text{rw}(V_x)^2}$. We can therefore conclude that the number of equivalence classes of \sim over \mathcal{A} is at most $2^{2\text{rw}(V_x)^2} \cdot 2n \leq \mathcal{N}_{\text{rw}}(T, \delta)$.

Concerning \mathbb{Q} -rank-width. We remove all the sets $X \in \mathcal{A}$ such that $|X^{2+}| > 2\text{rw}_{\mathbb{Q}}(V_x)$. By Lemma 6.6, we keep an (x, R') ^{acy}-representative set of \mathcal{A} . By Lemma 3.6, we know that $\text{nec}_n^{\leq 2\text{rw}_{\mathbb{Q}}(V_x)}(V_x)$ is upper bounded by $(2\text{rw}_{\mathbb{Q}}(V_x) + 1)^{\text{rw}_{\mathbb{Q}}(V_x)}$. We conclude that the number of equivalence classes of \sim over \mathcal{A} is at most $(2\text{rw}_{\mathbb{Q}}(V_x) + 1)^{\text{rw}_{\mathbb{Q}}(V_x)} \cdot 2n \leq \mathcal{N}_{\text{rw}_{\mathbb{Q}}}(T, \delta)$.

It remains to prove the running time. Observe that, for module-width and (\mathbb{Q})-rank-width, the removal of R' -unimportant sets can be done in time $O(|\mathcal{A}| \cdot n^2)$. Indeed, $\text{mw}(V_x)$, $\text{rw}(V_x)$ and $\text{rw}_{\mathbb{Q}}(V_x)$ can be computed in time $O(n^2)$. For 1-neighbor-width, we can assume that the size of $\text{nec}_1(V_x)$ is given because the first step of our algorithm for AC- (σ, ρ) -DOMINATING SET problems is to compute $\mathcal{R}_{V_x}^d$ for some $d \in \mathbb{N}^+$ and one can easily compute $\text{nec}_1(V_x)$ while computing $\mathcal{R}_{V_x}^d$. Notice that we can decide whether $X \sim W$ in time $O(n^2)$. Therefore, for

each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$, we can therefore compute the equivalence classes of \mathcal{A} in time $O(|\mathcal{A}| \cdot \mathcal{N}_f(T, \delta) \cdot n^2)$. \square

We are now ready to give an analog of Theorem 4.3 for the (x, R') ^{acy}-representativity.

Theorem 6.12. *Let $R \in \mathcal{R}_{V_x}^2$. For each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$, there exists an algorithm $\text{reduce}_f^{\text{acy}}$ that, given a set $\mathcal{A} \subseteq 2^{V_x}$ such that $X \equiv_{V_x}^2 R$ for every $X \in \mathcal{A}$, outputs in time $O(|\mathcal{A}| \cdot (\text{nec}_1(V_x)^{2(\omega-1)} + \mathcal{N}_f(T, \delta)) \cdot n^2)$, a subset $\mathcal{B} \subseteq \mathcal{A}$ such that $\mathcal{B} (x, R')$ ^{acy}-represents \mathcal{A} and $|\mathcal{B}| \leq \mathcal{N}_f(T, \delta) \cdot \text{nec}_1(V_x)^2$.*

Proof. Let $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$. By Lemma 6.9, we can compute in time $O(|\mathcal{A}| \cdot \mathcal{N}_f(T, \delta) \cdot n^2)$ a collection $\{\mathcal{A}_1, \dots, \mathcal{A}_t\}$ of pairwise disjoint subsets of \mathcal{A} such that

- $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_t (x, R')$ ^{acy}-represents \mathcal{A} ,
- \mathcal{A}_i is R' -consistent for each $i \in \{1, \dots, t\}$,
- $t \leq \mathcal{N}_f(T, \delta)$.

For each $X \in \mathcal{A}$, we have $X \equiv_{V_x}^1 R$ because $X \equiv_{V_x}^2 R$. Since $\mathcal{A}_1, \dots, \mathcal{A}_t \subseteq \mathcal{A}$, we can apply Theorem 4.3 to compute, for each $i \in \{1, \dots, t\}$, the set $\mathcal{B}_i := \text{reduce}(\mathcal{A}_i)$. By Theorem 4.3, for each $i \in \{1, \dots, t\}$, the set \mathcal{B}_i is a subset and an (x, R') -representative set of \mathcal{A}_i whose size is bounded by $\text{nec}_1(V_x)^2$. Moreover, as \mathcal{A}_i is R' -consistent, we have $\mathcal{B}_i (x, R')$ ^{acy}-represents \mathcal{A}_i by Lemma 6.8.

Let $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_t$. Since $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_t (x, R')$ ^{acy}-represents \mathcal{A} , we deduce from Fact 6.2 that $\mathcal{B} (x, R')$ ^{acy}-represents \mathcal{A} . Furthermore, we have $|\mathcal{B}| \leq \mathcal{N}_f(T, \delta) \cdot \text{nec}_1(V_x)^2$ owing to $t \leq \mathcal{N}_f(T, \delta)$ and $|\mathcal{B}_i| \leq \text{nec}_1(V_x)^2$ for all $i \in \{1, \dots, t\}$.

It remains to prove the running time. By Theorem 4.3, we can compute $\mathcal{B}_1, \dots, \mathcal{B}_t$ in time $O(|\mathcal{A}_1 \cup \dots \cup \mathcal{A}_t| \cdot \text{nec}_1(V_x)^{2(\omega-1)} \cdot n^2)$. Since the sets $\mathcal{A}_1, \dots, \mathcal{A}_t$ are subsets of \mathcal{A} and pairwise disjoint, we have $|\mathcal{A}_1 \cup \dots \cup \mathcal{A}_t| \leq |\mathcal{A}|$. That proves the running time and concludes the theorem. \square

We are now ready to present an algorithm that solves any AC- (σ, ρ) -DOMINATING SET problem. This algorithm follows the same ideas as the algorithm from Theorem 5.7, except that we use $\text{reduce}_f^{\text{acy}}$ instead of reduce .

Theorem 6.13. *For each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$, there exists an algorithm that, given an n -vertex graph G and a rooted layout (T, δ) of G , solves any AC- (σ, ρ) -DOMINATING SET problem, in time*

$$O(\text{s-nec}_d(T, \delta)^3 \cdot \text{s-nec}_1(T, \delta)^{2(\omega+1)} \cdot \mathcal{N}_f(T, \delta)^2 \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^3),$$

with $d := \max\{2, d(\sigma), d(\rho)\}$.

Proof. Let $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$. If we want to compute a solution of maximum (resp. minimum) weight, then we use the framework of Section 4 with $\text{opt} = \max$ (resp. $\text{opt} = \min$).

The first step of our algorithm is to compute, for each $x \in V(T)$, the sets $\mathcal{R}_{V_x}^d$, $\overline{\mathcal{R}}_{V_x}^d$ and a data structure to compute $\text{rep}_{V_x}^d(X)$ and $\text{rep}_{\overline{V_x}}^d(Y)$, for any $X \subseteq V_x$ and any $Y \subseteq \overline{V_x}$, in time $O(\log(\text{s-nec}_d(T, \delta)) \cdot n^2)$. As T has $2n - 1$ nodes, by Lemma 3.2, we can compute these sets and data structures in time $O(\text{s-nec}_d(T, \delta) \cdot \log(\text{s-nec}_d(T, \delta)) \cdot n^3)$.

For each node $x \in T$ and, for each $(R, R') \in \mathcal{R}_{V_x}^d \times \overline{\mathcal{R}}_{V_x}^d$, we define $\mathcal{A}_x[R, R'] \subseteq 2^{V_x}$ as follows

$$\mathcal{A}_x[R, R'] := \{X \subseteq V_x : X \equiv_{V_x}^d R \text{ and } X \cup R' (\sigma, \rho)\text{-dominates } V_x\}.$$

We deduce the following claim from the proof of Claim 5.4.

Claim 6.14. *For every internal node $x \in V(T)$ with a and b as children and $(R, R') \in \mathcal{R}_{V_x}^d \times \overline{\mathcal{R}}_{V_x}^d$, we have*

$$\mathcal{A}_x[R, R'] = \bigcup_{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible}} \mathcal{A}_a[A, A'] \otimes \mathcal{A}_b[B, B'].$$

\square

For each node x of $V(T)$, our algorithm will compute a table \mathcal{D}_x that satisfies the following invariant.

Invariant. For every $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$, the set $\mathcal{D}_x[R, R']$ is a subset of $\mathcal{A}_x[R, R']$ of size at most $\mathcal{N}_f(T, \delta) \cdot \text{nec}_1(V_x)^2$ that (x, R') -acyclic-represents $\mathcal{A}_x[R, R']$.

Notice that by Definition of (x, R') -acyclic-representativity, if the invariant holds for r , then $\mathcal{D}_r[\emptyset, \emptyset]$ contains a set X of maximum (or minimum) weight such that X is a (σ, ρ) -dominating set of G and $G[X]$ is a tree.

The algorithm is a usual bottom-up dynamic programming algorithm and computes for each node x of T the table \mathcal{D}_x .

Let x be a leaf of T with $V_x = \{v\}$. Observe that $\mathcal{A}_x[R, R'] \subseteq 2^{V_x} = \{\emptyset, \{v\}\}$. Thus, our algorithm can directly compute $\mathcal{A}_x[R, R']$ and set $\mathcal{D}_x[R, R'] := \mathcal{A}_x[R, R']$. In this case, the invariant trivially holds.

Now, take x an internal node of T with a and b as children such that the invariant holds for a and b . For each $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$, the algorithm computes $\mathcal{D}_x[R, R'] := \text{reduce}_f^{\text{acy}}(\mathcal{B}_x[R, R'])$, where the set $\mathcal{B}_x[R, R']$ is defined as follows

$$\mathcal{B}_x[R, R'] := \bigcup_{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible}} \mathcal{D}_a[A, A'] \otimes \mathcal{D}_b[B, B'].$$

Similarly to the proof of Theorem 5.7, we deduce from Fact 6.2, Lemma 6.3, Claim 6.14 and Theorem 6.12, that $\mathcal{D}_x[R, R']$ is a subset and an (x, R') -acyclic-representative set of $\mathcal{A}_x[R, R']$. By Theorem 6.12, we have $|\mathcal{D}_x[R, R']| \leq \mathcal{N}_f(T, \delta) \cdot \text{s-nec}_1(T, \delta)^2$.

Consequently, the invariant holds for x and by induction, it holds for all the nodes of T . The correctness of the algorithm follows.

Running Time. The running time of our algorithm is almost the same as the running time given in Theorem 6.13. The only difference is the factor $\mathcal{N}_f(T, \delta)^2$ which is due to the following fact: by the invariant condition, for each (A, A') and (B, B') , the size of $\mathcal{D}_a[A, A'] \otimes \mathcal{D}_b[B, B']$ is at most $\mathcal{N}_f(T, \delta)^2 \cdot \text{s-nec}_1(T, \delta)^4$. \square

By constructing for any graph G a graph G^* such that the width measure of G^* is linear in the width measure of G and any optimum acyclic (σ, ρ) -dominating set of G corresponds to an optimum AC- (σ, ρ) -dominating set of G^* and vice-versa, we obtain the following which allows for instance to compute a feedback vertex set in time $n^{O(c)}$, c the mim-width.

Theorem 6.15. *For each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$, there exists an algorithm that, given an n -vertex graph G and a rooted layout (T, δ) of G , solves any ACYCLIC (σ, ρ) -DOMINATING SET problem in time*

$$O(\text{s-nec}_d(T, \delta)^3 \cdot \text{s-nec}_1(T, \delta)^{2(\omega+1)} \cdot \mathcal{N}_f(T, \delta)^{O(1)} \cdot n^3),$$

with $d := \max\{2, d(\sigma), d(\rho)\}$.

Proof. Let $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$. Suppose that we want to compute a maximum acyclic (σ, ρ) -dominating set. The proof for computing a minimum acyclic (σ, ρ) -dominating set is symmetric.

The first step of this proof is to construct a $2n + 1$ -vertex graph G^* from G and a layout (T^*, δ^*) of G^* from (T, δ) in time $O(n^2)$ such that (T^*, δ^*) respects the following inequalities:

- (1) for every $d \in \mathbb{N}^+$, $\text{s-nec}_d(T^*, \delta^*) \leq (d + 1) \cdot \text{s-nec}_d(T, \delta)$,
- (2) for every $f \in \{\text{mim}, \text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}\}$, $f(T^*, \delta^*) \leq f(T, \delta) + 1$.

The second step of this proof consists in showing how the algorithm of Theorem 6.13 can be modified to find a maximum acyclic (σ, ρ) -dominating set of G by running this modified algorithm on G^* and (T^*, δ^*) .

We construct G^* as follows. Let β be a bijection from $V(G)$ to a set V^+ disjoint from $V(G)$. The vertex set of G^* is $V(G) \cup V^+ \cup \{v_0\}$ with v_0 a vertex distinct from those in $V(G) \cup V^+$. We extend the weight function w of G to G^* such that the vertices of $V(G)$ have the same weight

as in G and the weight of the vertices in $V^+ \cup \{v_0\}$ is 0. Finally, the edge set of G^* is defined as follows

$$E(G^*) := E(G) \cup \{\{v, \beta(v)\}, \{v_0, \beta(v)\} : v \in V(G)\}.$$

We now construct $\mathcal{L} = (T^*, \delta^*)$ from $\mathcal{L} := (T, \delta)$. We obtain T^* and δ^* by doing the following transformations on T and δ :

- For each leaf ℓ of T with $\delta(\ell) = \{v\}$, we transform ℓ into an internal node by adding two new nodes a_ℓ and b_ℓ as its children such that $\delta^*(a_\ell) = v$ and $\delta^*(b_\ell) = \beta(v)$.
- The root of T^* is a new node r whose children are the root of T and a new node a_r with $\delta^*(a_r) = v_0$.

In order to simplify the proof, we use the following notations.

For each node $x \in V(T^*)$, we let $\overline{V_x^{\mathcal{L}}} := V(G^*) \setminus V_x^{\mathcal{L}}$ and, for each node $x \in V(T)$, we let $\overline{V_x^{\mathcal{L}}} := V(G) \setminus V_x^{\mathcal{L}}$.

Now, we prove that (T^*, δ^*) respects Inequalities (1) and (2). Let x be a node of T^* . Observe that if $x \in V(T^*) \setminus V(T)$, then the set $V_x^{\mathcal{L}}$ either contains one vertex or equals $V(G^*)$. Hence, in this case, the inequalities hold because $\text{ne}_d(V_x^{\mathcal{L}}) \leq d$ for each $d \in \mathbb{N}^+$ and $f(V_x^{\mathcal{L}}) \leq 1$ for each $f \in \{\text{mim}, \text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}\}$.

Now, assume that x is also a node of T . Hence, by construction, we have

$$\begin{aligned} V_x^{\mathcal{L}} &= V_x^{\mathcal{L}} \cup \{\beta(v) : v \in V_x^{\mathcal{L}}\}. \\ \overline{V_x^{\mathcal{L}}} &= \overline{V_x^{\mathcal{L}}} \cup \{\beta(v) : v \in \overline{V_x^{\mathcal{L}}}\} \cup \{v_0\} \end{aligned}$$

Now, we prove Inequality (1). Let $d \in \mathbb{N}^+$. By construction of G^* and \mathcal{L} , for each vertex $v \in V_x^{\mathcal{L}}$, we have $\beta(v) \in V_x^{\mathcal{L}}$ and

$$(5) \quad N_{G^*}(v) \cap \overline{V_x^{\mathcal{L}}} = N_G(v) \cap \overline{V_x^{\mathcal{L}}},$$

$$(6) \quad N_{G^*}(\beta(v)) \cap \overline{V_x^{\mathcal{L}}} = \{v_0\}.$$

We deduce that, for every $X, Y \subseteq V_x^{\mathcal{L}}$, we have $X \equiv_{V_x^{\mathcal{L}}}^d Y$ if and only

- $X \cap V(G) \equiv_{V_x^{\mathcal{L}}}^d Y \cap V(G)$ and
- $\min(d, |N(v_0) \cap X|) = \min(d, |N(v_0) \cap Y|)$.

Similarly, we deduce that, for every $X, Y \subseteq \overline{V_x^{\mathcal{L}}}$, we have $X \equiv_{\overline{V_x^{\mathcal{L}}}}^d Y$ if and only if

- $X \cap V(G) \equiv_{\overline{V_x^{\mathcal{L}}}}^d Y \cap V(G)$ and
- $X \cap \{v_0\} = Y \cap \{v_0\}$.

Thus, we can conclude that $\text{s-ne}_d(V_x^{\mathcal{L}}) \leq (d+1) \cdot \text{s-ne}_d(V_x^{\mathcal{L}})$. Consequently, Inequality (1) holds.

We deduce Inequality (2) from Figure 2 describing the adjacency matrix between $V_x^{\mathcal{L}}$ and $\overline{V_x^{\mathcal{L}}}$.

$$V_x^{\mathcal{L}} \left\{ \begin{array}{l} V^+ \cap V_x^{\mathcal{L}} \\ V_x^{\mathcal{L}} \end{array} \right. \left(\begin{array}{c|c|c} \overline{V_x^{\mathcal{L}}} & v_0 & V^+ \cap \overline{V_x^{\mathcal{L}}} \\ \hline 0 & \begin{array}{c} \vdots \\ 1 \\ \vdots \end{array} & 0 \\ \hline M_{V_x^{\mathcal{L}}, \overline{V_x^{\mathcal{L}}}} & & 0 \end{array} \right)$$

FIGURE 2. The adjacency matrix between $V_x^{\mathcal{L}}$ and $\overline{V_x^{\mathcal{L}}}$.

Now, we explain how we modify the algorithm of Theorem 6.13 in order to find a maximum acyclic (σ, ρ) -dominating set of G by calling this algorithm on G^* . For doing so, we modify the definition of the table \mathcal{A}_x , the invariant and the computational steps of the algorithm of Theorem 6.13. The purpose of these modifications is to restrict the (σ, ρ) -domination to the vertices of $V(G)$. For doing so, we consider the set of nodes $S := V(T) \cup \{r, a_r\}$. Observe that, for every node x in S , there are no edges in $G[V_x^{\mathcal{L}}, \overline{V_x^{\mathcal{L}}}]$ between a vertex in $V(G)$ and a vertex in $V(G^*) \setminus V(G)$. This is not true for the nodes of $V(T^*) \setminus S$. For this reason, our algorithm ignores the nodes in $V(T^*) \setminus S$ and computes a table only for the nodes in S .

For every $x \in S$ and every $(R, R') \in \mathcal{R}_{V_x^{\mathcal{L}}}^d \times \mathcal{R}_{\overline{V_x^{\mathcal{L}}}}^d$ we define $\mathcal{A}_x[R, R'] \subseteq 2^{V_x^{\mathcal{L}}}$ as follows

$$\mathcal{A}_x[R, R'] := \{X \subseteq V_x^{\mathcal{L}} : X \equiv_{V_x^{\mathcal{L}}}^d R \text{ and } (X \cup R') \cap V(G) \text{ } (\sigma, \rho)\text{-dominates } V_x^{\mathcal{L}} \cap V(G)\}.$$

We claim that if G admits an acyclic (σ, ρ) -dominating set D , then there exists $D' \in \mathcal{A}_r[\emptyset, \emptyset]$ such that $D' \cap V(G) = D$ and $G^*[D']$ is a tree. Let D be an acyclic (σ, ρ) -dominating set of G with $\text{cc}(G[D]) = \{C_1, \dots, C_t\}$. For each $i \in \{1, \dots, t\}$, let v_i be a vertex in C_i . One easily checks that $G^*[D \cup \{\beta(v_i) : 1 \leq i \leq t\} \cup v_0]$ is a tree. Moreover, by definition of $\mathcal{A}_r[\emptyset, \emptyset]$, for every $X \in \mathcal{A}_r[\emptyset, \emptyset]$, if $G[X]$ is a tree, then $X \cap V(G)$ is an acyclic (σ, ρ) -dominating set of G . Hence, if G admits an acyclic (σ, ρ) -dominating set, any $(r, \emptyset)^{\text{acy}}$ -representative set of $\mathcal{A}_r[\emptyset, \emptyset]$ contains a set X such that $X \cap V(G)$ is a maximum acyclic (σ, ρ) -dominating set of G .

For every node $x \in S$, we compute a table \mathcal{D}_x satisfying the following invariant.

Invariant. For each node $x \in S$ and each $(R, R') \in \mathcal{R}_{V_x^{\mathcal{L}}}^d \times \mathcal{R}_{\overline{V_x^{\mathcal{L}}}}^d$, the set $\mathcal{D}_x[R, R']$ is a subset of $\mathcal{A}_x[R, R']$ of size at most $\mathcal{N}_f(T^*, \delta^*) \cdot \text{nec}_1(V_x^{\mathcal{L}})^2$ that $(x, R')^{\text{acy}}$ -represents $\mathcal{A}_x[R, R']$.

Before we explain how to compute the table \mathcal{D}_x , for each $x \in S$, we need the following fact and claim. We deduce the following fact from Lemma 5.2 and the fact that, for every node x in S , there are no edges in $G[V_x^{\mathcal{L}}, \overline{V_x^{\mathcal{L}}}]$ between a vertex in $V(G)$ and a vertex in $V(G^*) \setminus V(G)$.

Fact 6.16. *Let $x \in S$. Let $X \subseteq V_x^{\mathcal{L}}$ and $Y, R' \subseteq \overline{V_x^{\mathcal{L}}}$ such that $Y \equiv_{\overline{V_x^{\mathcal{L}}}}^d R'$. Then $(X \cup R') \cap V(G)$ (σ, ρ) -dominates $V_x^{\mathcal{L}} \cap V(G)$ if and only if $(X \cup Y) \cap V(G)$ (σ, ρ) -dominates $V_x^{\mathcal{L}} \cap V(G)$. \square*

We deduce the following claim from Fact 6.16 and Lemma 5.4.

Claim 6.17. *Let $x \in S \setminus \{a_r\}$ such that x is not a leaf in T . Let a and b be the children of x in T^* . For every $(R, R') \in \mathcal{R}_{V_x^{\mathcal{L}}}^d \times \mathcal{R}_{\overline{V_x^{\mathcal{L}}}}^d$, we have*

$$\mathcal{A}_x[R, R'] = \bigcup_{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible}} \mathcal{A}_a[A, A'] \otimes \mathcal{A}_b[B, B'].$$

\square

The algorithm starts by computing the table \mathcal{D}_x for each node $x \in S$ such that $x = a_r$ or x is a leaf of T . Since $|V_x^{\mathcal{L}}| \leq 2$, our algorithm directly computes $\mathcal{A}_x[R, R']$ and set $\mathcal{D}_x[R, R'] := \mathcal{A}_x[R, R']$ for every $(R, R') \in \mathcal{R}_{V_x^{\mathcal{L}}}^d \times \mathcal{R}_{\overline{V_x^{\mathcal{L}}}}^d$.

For the other nodes our algorithm computes the table \mathcal{D}_x exactly as the algorithm of Theorem 6.13.

The correctness of this algorithm follows from Theorem 6.13 and Claim 6.17. By Theorem 6.13, the running time of this algorithm is

$$O(\text{s-nec}_2(\mathcal{L})^3 \cdot \text{s-nec}_1(\mathcal{L})^{2(\omega+1)} \cdot \mathcal{N}_f(\mathcal{L})^2 \cdot n^3).$$

We deduce the running time in function of \mathcal{L} from Inequalities (1) and (2).

\square

7. MAX CUT

Prior to this work, the d -neighbor-equivalence relation was used only for problems with a locally checkable property like DOMINATING SET [10, 19, 28]. We prove in this paper that the d -neighbor-equivalence relation can also be useful for problems with a connectivity constraint and an acyclicity constraint. Is this notion also useful for other kinds of problems? Can we use it for the problems which are unlikely to admit FPT algorithms parameterized by clique-width, \mathbb{Q} -rank-width and rank-width? This is the case for well-known problems such as HAMILTONIAN CYCLE, EDGE DOMINATING SET and MAX CUT. The complexity of these problems parameterized by clique-width is well-known. Indeed, for each of these problems, we have an ad-hoc $n^{O(k)}$ time algorithm with k the clique-width of a given k -expression [4, 16]. On the other hand, little is known concerning rank-width and \mathbb{Q} -rank-width. For mim-width, we know that HAMILTONIAN CYCLE is para-NP-hard parameterized by the mim-width of a given rooted layout [24]. As these problems are W[1]-hard parameterized by clique-width, we cannot expect to rely only on the d -neighbor equivalence relation for d a constant.

In this section, we prove that, given an n -vertex graph and a rooted layout \mathcal{L} , we can use the n -neighbor equivalence to solve the MAX CUT problem in time $\mathfrak{s}\text{-nec}_n(\mathcal{L})^{O(1)} \cdot n^{O(1)}$. The MAX CUT problem asks, given a graph G , for the maximum $w \in \mathbb{N}$ such that there exists a subset $X \subseteq V(G)$ with $w = |E(X, \overline{X})|$.

We also prove that we can solve a connectivity variant of MAX CUT called MAXIMUM MINIMAL CUT (a.k.a. LARGEST BOND) in time $\mathfrak{s}\text{-nec}_n(\mathcal{L})^{O(1)} \cdot \mathfrak{s}\text{-nec}_1^{O(1)} \cdot n^{O(1)}$. A *minimal cut* of a graph G is a subset of vertices X such that $G[X]$ and $G[\overline{X}]$ are connected. The problem MAXIMUM MINIMAL CUT asks, given a graph G , for the computation of a minimal cut $X \subseteq V(G)$ such that $|E(X, \overline{X})|$ is maximum. The parameterized complexity of this problem was studied recently in [14, 15] and in both papers, the authors proved that MAXIMUM MINIMAL CUT is solvable in time $n^{O(\text{cw})}$ with cw the clique-width of a given decomposition.

It is worth mentioning that to deal with the two connectivity constraints of MAXIMUM MINIMAL CUT, we modify the framework developed in Section 4 and in particular the notion of representative set and the way we compute one. For doing so, we use a non-trivial trick which could be easily generalized to deal with an arbitrary number of connectivity constraints.

By Corollary 3.9, our results imply that MAX CUT and MAXIMUM MINIMAL CUT are solvable in time $n^{O(\text{mw}(G))}$, $n^{O(\text{rw}_{\mathbb{Q}}(G))}$ and $n^{2^{O(\text{rw}(G))}}$.

Let us give some explanations. Let G be an n -vertex graph G and (T, δ) a rooted layout of G . Suppose that we want to solve MAX CUT. Let x be a node of T and $X, W \subseteq V_x$ such that $X \equiv_{V_x}^n W$. From Lemma 7.1, we can show that if the number of edges between X and $V_x \setminus X$ is bigger than the number of edges between W and $V_x \setminus W$, then X is a better partial solution than W . That is, for every $Y \subseteq \overline{V_x}$, the set $X \cup Y$ is a better solution than $W \cup Y$. This follows from the fact that the number of edges of $E(V_x, \overline{V_x})$ between $X \cup Y$ and $\overline{X \cup Y}$ is the same as the number of edges of $E(V_x, \overline{V_x})$ between $W \cup Y$ and $\overline{W \cup Y}$.

It follows that it is enough to compute, for each node x and each $R \in \mathcal{R}_{V_x}^n$, the maximum $k \in \mathbb{N}$ such that $k = |E(X, V_x \setminus X)|$ for some $X \equiv_{V_x}^n R$. The integer k computed for the root of T and $R = \emptyset$ corresponds to the solution of MAX CUT.

Before, we present this algorithm, we need the following lemma that we use to compute the tables associated with an internal node.

Lemma 7.1. *Let a and b be the children of some internal node of T . Let $A \in \mathcal{R}_{V_a}^n$ and $B \in \mathcal{R}_{V_b}^n$. For every $X \subseteq V_a$ and $W \subseteq V_b$ such that $A \equiv_{V_a}^n X$ and $B \equiv_{V_b}^n W$, the number of edges between $X \cup W$ and $V_x \setminus (X \cup W)$ equals*

$$|E(X, V_a \setminus X)| + |E(W, V_b \setminus W)| + |E(A, V_b \setminus B)| + |E(B, V_a \setminus A)|.$$

Proof. Let $X \subseteq V_a$ and $W \subseteq V_b$ such that $A \equiv_{V_a}^n X$ and $B \equiv_{V_b}^n W$. First, observe that the number of edges between $X \cup W$ and $V_x \setminus (X \cup W)$ equals

$$|E(X, V_a \setminus X)| + |E(W, V_b \setminus W)| + |E(X, V_b \setminus W)| + |E(W, V_a \setminus X)|.$$

So, we only need to prove that $|E(X, V_b \setminus W)| = |E(A, V_b \setminus B)|$ and $|E(W, V_a \setminus X)| = |E(B, V_a \setminus A)|$. As it is symmetrical, it is sufficient to prove the first equality.

Since $B \equiv_{V_b}^n W$, by Fact 3.7, we have $V_b \setminus B \equiv_{V_b}^n V_b \setminus W$. Thus, by Lemma 3.5, we have $|E(X, V_b \setminus W)| = |E(X, V_b \setminus B)|$. Moreover, as $A \equiv_{V_a}^n X$ by applying Lemma 3.5 again, we deduce that $|E(X, V_b \setminus W)| = |E(X, V_b \setminus B)| = |E(A, V_b \setminus B)|$. \square

Theorem 7.2. *There exists an algorithm that, given an n -vertex graph G and a rooted layout (T, δ) , solves MAX CUT in time $O(\text{s-nec}_n(T, \delta)^2 \cdot \log(\text{s-nec}_n(T, \delta)) \cdot n^3)$.*

Proof. The first step of our algorithm is to compute, for each $x \in V(T)$, the sets $\mathcal{R}_{V_x}^n$ and a data structure to compute $\text{rep}_{V_x}^n(X)$, for each $X \subseteq V_x$, in time $O(\log(\text{s-nec}_n(T, \delta)) \cdot n^2)$. As T has $2n - 1$ nodes, by Lemma 3.2, we can compute these sets and data structures in time $O(\text{s-nec}_n(T, \delta) \cdot \log(\text{s-nec}_n(T, \delta)) \cdot n^3)$.

For every node $x \in V(T)$ and every $R \in \mathcal{R}_{V_x}^n$, we define $\mathcal{T}_x[R]$ as follows

$$\mathcal{T}_x[R] := \max\{|E(X, V_x \setminus X)| : X \subseteq V_x \text{ and } X \equiv_{V_x}^n R\}.$$

Observe that, for r the root of T , the entry of $\mathcal{T}_r[\emptyset]$ is the size of a maximum cut of G .

The algorithm is a usual bottom-up dynamic programming algorithm and computes for each node x of T the table \mathcal{T}_x . For the leaves x of T , we simply set $\mathcal{T}_x[R] := 0$ for every $R \in \mathcal{R}_{V_x}^n$. This is correct because the graph $G[V_x]$ contains only one vertex.

Let x be an internal node of T with a and b as children. To compute the tables for the internal nodes we need the following claim.

Claim 7.3. *For every $R \in \mathcal{R}_{V_x}^n$, we have $\mathcal{T}_x[R]$ equals*

$$\max\{\mathcal{T}_a[A] + \mathcal{T}_b[B] + |E(A, V_b \setminus B)| - |E(B, V_a \setminus A)| : (A, B) \in \mathcal{R}_{V_a}^n \times \mathcal{R}_{V_b}^n \text{ and } A \cup B \equiv_{V_x}^n R\}.$$

Claim 7.3 is implied by the two following facts.

Fact 7.4. *Let $R \in \mathcal{R}_{V_x}^n$. For every $(A, B) \in \mathcal{R}_{V_a}^n \times \mathcal{R}_{V_b}^n$ such that $A \cup B \equiv_{V_x}^n R$, we have*

$$\mathcal{T}_x[R] \geq \mathcal{T}_a[A] + \mathcal{T}_b[B] + |E(A, V_b \setminus B)| + |E(B, V_a \setminus A)|.$$

Proof. Let $(A, B) \in \mathcal{R}_{V_a}^n \times \mathcal{R}_{V_b}^n$ such that $A \cup B \equiv_{V_x}^n R$. By definition of $\mathcal{T}_a[A]$, there exists $X_a \subseteq V_a$ such that $X \equiv_{V_a}^n A$ and $\mathcal{T}_a[A] = |E(X_a, V_a \setminus X_a)|$. Symmetrically, there exists $X_b \subseteq V_b$ such that $X_b \equiv_{V_b}^n B$ and $\mathcal{T}_b[B] = |E(X_b, V_b \setminus X_b)|$.

From Lemma 7.1, the number of edges between $X_a \cup X_b$ and $V_x \setminus (X_a \cup X_b)$ equals $\mathcal{T}_a[A] + \mathcal{T}_b[B] + |E(A, V_b \setminus B)| + |E(B, V_a \setminus A)|$. By Fact 3.1, we deduce that $X_a \cup X_b \equiv_{V_x}^n A \cup B$. Thus, $X_a \cup X_b \equiv_{V_x}^n R$ and by definition, $\mathcal{T}_x[R]$ is bigger than the number of edges between $X_a \cup X_b$ and $V_x \setminus (X_a \cup X_b)$. This proves the fact. \square

Fact 7.5. *For every $R \in \mathcal{R}_{V_x}^n$, there exists $(A, B) \in \mathcal{R}_{V_a}^n \times \mathcal{R}_{V_b}^n$ such that $A \cup B \equiv_{V_x}^n R$ and*

$$\mathcal{T}_x[R] = \mathcal{T}_a[A] + \mathcal{T}_b[B] + |E(A, V_b \setminus B)| + |E(B, V_a \setminus A)|.$$

Proof. By definition of $\mathcal{T}_x[R]$, there exists a set $X \subseteq V_x$ such that $X \equiv_{V_x}^n R$ and $\mathcal{T}_x[R] = |E(X, V_x \setminus X)|$. For every $i \in \{a, b\}$, let $X_i = X \cap V_i$.

Let $A := \text{rep}_{V_a}^n(X_a)$ and let $B := \text{rep}_{V_b}^n(X_b)$. By definition, we have $X \equiv_{V_x}^n A \cup B$ and by Claim 7.4, we have

$$(7) \quad \mathcal{T}_x[R] \geq \mathcal{T}_a[A] + \mathcal{T}_b[B] + |E(A, V_b \setminus B)| + |E(B, V_a \setminus A)|.$$

Moreover, as $\mathcal{T}_x[R] = |E(X, V_x \setminus X)|$ and from Lemma 7.1, we deduce that

$$(8) \quad \mathcal{T}_x[R] = |E(X_a, V_a \setminus X_a)| + |E(X_b, V_b \setminus X_b)| + |E(A, V_b \setminus B)| + |E(B, V_a \setminus A)|.$$

By definition of $\mathcal{T}_a[A]$, we know that $\mathcal{T}_a[A] \geq |E(X_a, V_a \setminus X_a)|$. Symmetrically, we have $\mathcal{T}_b[B] \geq |E(X_b, V_b \setminus X_b)|$. Hence, we conclude from Inequality (7) and Equation (8) that $\mathcal{T}_x[R] = \mathcal{T}_a[A] + \mathcal{T}_b[B] + |E(V_a, V_b)| - |E(A, B)|$. \square

We deduce that we can compute the entries of \mathcal{T}_x by doing the following:

- For every $R \in \mathcal{R}_{V_x}^n$, initialize some temporary variable w_R to 0.
- For every $(A, B) \in \mathcal{R}_{V_a}^n \times \mathcal{R}_{V_b}^n$, compute $R = \text{rep}_{V_x}^n(A \cup B)$ and update w_R as follows

$$w_R := \max\{w_R, \mathcal{T}_a[A] + \mathcal{T}_b[B] + |E(A, V_b \setminus B)| + |E(B, V_a \setminus A)|\}.$$

From Claim 7.3, at the end of this subroutine, we have correctly updated $\mathcal{T}_x[R]$ which is precisely equal to w_R . Recall that each call to the functions $\text{rep}_{V_x}^n$ takes $\log(\text{s-nec}_n(T, \delta)) \cdot n^2$ time. Thus, the running time to compute the entries of \mathcal{T}_x is $O(\text{s-nec}_n(T, \delta)^2 \cdot \log(\text{s-nec}_n(T, \delta)) \cdot n^2)$. The total running time of our algorithm follows from the fact that T has $2n - 1$ nodes. \square

Theorem 7.6. *There exists an algorithm that, given an n -vertex graph G and a rooted layout (T, δ) , solves MAXIMUM MINIMAL CUT in time $O(\text{s-nec}_n(T, \delta)^2 \cdot \text{s-nec}_1(T, \delta)^{4(\omega+1.5)} \cdot n^3)$.*

Proof. As we have two connectivity constraints, *i.e.*, we ask both the solution and its complement to induce connected graphs, we have to modify our notion of representative set and how we compute one. Let $x \in V(T)$. For $\mathcal{A} \subseteq 2^{V_x}$ and $Y \subseteq \overline{V_x}$, we define

$$\text{best}^*(\mathcal{A}, Y) := \max\{|E(X, V_x \setminus X)| : X \in \mathcal{A} \text{ and } X \cup Y \text{ is a minimal cut}\}.$$

For $R_Y, R_{\overline{Y}} \in \mathcal{R}_{V_x}^1$ and $\mathcal{A}, \mathcal{B} \subseteq 2^{V_x}$, we say that \mathcal{B} $(x, R_Y, R_{\overline{Y}})$ -represents \mathcal{A} , if for every $Y \subseteq \overline{V_x}$ such that $R_Y \equiv \frac{1}{V_x} Y$ and $R_{\overline{Y}} \equiv \frac{1}{V_x} \overline{V_x} \setminus Y$, we have

$$\text{best}^*(\mathcal{B}, Y) = \text{best}^*(\mathcal{A}, Y).$$

For all $x \in V(T)$, we define $\mathbb{I}_x := \mathcal{R}_{V_x}^n \times \mathcal{R}_{V_x}^1 \times \mathcal{R}_{V_x}^1$. Moreover, for each $R \in \mathcal{R}_{V_x}^n$, we define $\mathcal{A}_x[R] := \{X \subseteq V_x : X \equiv_{V_x}^n R\}$. For every node $x \in V(T)$, our algorithm will compute a table \mathcal{T}_x satisfying the following invariant.

Invariant. For every $(R, R_Y, R_{\overline{Y}}) \in \mathbb{I}_x$, the entry $\mathcal{T}_x[R, R_Y, R_{\overline{Y}}]$ is a subset of $\mathcal{A}_x[R]$ of size at most $\text{s-nec}_1(T, \delta)^4$ that $(x, R_Y, R_{\overline{Y}})$ -represents $\mathcal{A}_x[R]$.

The following claim proves that we can compute a small representative set for this new notion of representativity.

Claim 7.7. *Let $x \in V(T)$ and $(R, R_Y, R_{\overline{Y}}) \in \mathbb{I}_x$. There exists an algorithm reduce^* that, given $\mathcal{A} \subseteq 2^{V_x}$ such that, for all $X \in \mathcal{A}$, we have $X \equiv_{V_x}^n R$, outputs in time $O(|\mathcal{A}| \cdot \text{s-nec}_1(V_x)^{4(\omega-1)} \cdot n^2)$ a subset $\mathcal{B} \subseteq \mathcal{A}$ such that \mathcal{B} $(x, R_Y, R_{\overline{Y}})$ -represents \mathcal{A} and $|\mathcal{B}| \leq \text{s-nec}_1(V_x)^4$.*

Proof. Let \mathcal{D} be the set that contains all $Y \subseteq \overline{V_x}$ such that $Y \equiv \frac{1}{V_x} R_Y$ and $\overline{V_x} \setminus Y \equiv \frac{1}{V_x} R_{\overline{Y}}$.

First assume that $R_Y = \emptyset$ or $R_{\overline{Y}} = \emptyset$. We distinguish the following cases:

- If $R_Y = \emptyset$ and $R_{\overline{Y}} \equiv \frac{1}{V_x} \overline{V_x}$, then return $\{X\}$ with $X \in \mathcal{A}$ such that X is a minimal cut and $|E(X, \overline{V_x})|$ is maximum. As explained in the proof of Theorem 4.3, for every $X \subseteq V_x$ and $Y \in \mathcal{D}$ such that $Y \neq \emptyset$, the graph $G[X \cup Y]$ is not connected because $N(X) \cap Y = \emptyset$. The correctness of this case follows from the fact that $\text{best}(\mathcal{A}, \emptyset)$ is the weight of a set $X \in \mathcal{A}$ which is a minimal cut.
- If $R_{\overline{Y}} = \emptyset$ and $R_Y \equiv \frac{1}{V_x} \overline{V_x}$, then return $\{X\}$ with $X \in \mathcal{A}$ such that $X \cup \overline{V_x}$ is a minimal cut and $|E(V_x \setminus X, \overline{V_x} \setminus X)|$ is maximum. This case is symmetrical to the previous one.
- Otherwise, we return \emptyset . This is correct because in this case $\mathcal{D} = \emptyset$. Indeed, for every $W \subseteq \overline{V_x}$, if $W \equiv \frac{1}{V_x} \emptyset$, then $\overline{V_x} \setminus W \equiv \frac{1}{V_x} \overline{V_x}$.

Assume from now that $R_Y \neq \emptyset$ and $R_{\overline{Y}} \neq \emptyset$. We start by removing from \mathcal{A} the sets X that satisfy at least one of the following properties:

- there exists $C \in \text{cc}(G[X])$ such that $N(C) \cap R_Y = \emptyset$, or

- there exists $C \in \text{cc}(G[V_x \setminus X])$ such that $N(C) \cap R_{\overline{Y}} = \emptyset$.

As explained in the proof of Theorem 4.3, we can safely remove all such sets as they never form a minimal cut with a set $Y \in \mathcal{D}$. Moreover, we remove from \mathcal{D} the sets Y that satisfy at least one of the following properties:

- there exists $C \in \text{cc}(G[Y])$ such that $N(C) \cap R = \emptyset$, or
- there exists $C \in \text{cc}(G[\overline{V_x} \setminus Y])$ with $N(C) \cap (V_x \setminus R) = \emptyset$.

One can check that, for all sets Y removed from \mathcal{D} and all $X \in \mathcal{A}$, we have $\text{best}^*(\mathcal{A}, Y) = -\infty$. In particular, if a set Y satisfies the second property, then for every $X \in \mathcal{A}$, the graph $G[\overline{X \cup Y}]$ is not connected. Indeed, by Fact 3.7, we have $V_x \setminus X \equiv_{V_x}^n V_x \setminus R$ for every $X \in \mathcal{A}$, so the second property implies that a connected component of $\overline{V_x} \setminus Y$ has no neighbor in $V_x \setminus X$.

For every $Y \in \mathcal{D}$, we let v_Y be one fixed vertex of Y . For every $R_1, R_2 \in \mathcal{R}_{V_x}^1$, $X \subseteq V_x$ and $Y \subseteq \overline{V_x}$, we define the following predicates:

$$\mathcal{C}(X, (R_1, R_2)) := \begin{cases} 1 & \text{if } \exists (X_1, X_2) \in \text{ccut}(X) \text{ such that } N(X_1) \cap R_2 = \emptyset \text{ and } N(X_2) \cap R_1 = \emptyset, \\ 0 & \text{otherwise.} \end{cases}$$

$$\overline{\mathcal{C}}((R_1, R_2), Y) := \begin{cases} 1 & \text{if } \exists (Y_1, Y_2) \in \text{ccut}(Y) \text{ such that } v_Y \in Y_1, Y_1 \equiv_{V_x}^1 R_1, \text{ and } Y_2 \equiv_{V_x}^1 R_2, \\ 0 & \text{otherwise.} \end{cases}$$

These predicates are similar to the eponymous matrices in the proof of Theorem 4.3. In the following, we denote by \mathfrak{R} the set of all quadruples of elements in $\mathcal{R}_{V_x}^1$. Let \mathcal{M}^* , \mathcal{C}^* , and $\overline{\mathcal{C}}^*$ be, respectively, an $(\mathcal{A}, \mathcal{D})$ -matrix, an $(\mathcal{A}, \mathfrak{R})$ -matrix, and an $(\mathfrak{R}, \mathcal{D})$ -matrix such that

$$\mathcal{M}^*[X, Y] := \begin{cases} 1 & \text{if } X \cup Y \text{ is a minimal cut,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\mathcal{C}^*[X, (R_Y^1, R_Y^2, R_{\overline{Y}}^1, R_{\overline{Y}}^2)] := \begin{cases} 1 & \text{if } \mathcal{C}(X, (R_Y^1, R_Y^2)) \text{ and } \mathcal{C}(V_x \setminus X, (R_{\overline{Y}}^1, R_{\overline{Y}}^2)) \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

$$\overline{\mathcal{C}}^*[(R_Y^1, R_Y^2, R_{\overline{Y}}^1, R_{\overline{Y}}^2), Y] := \begin{cases} 1 & \text{if } \overline{\mathcal{C}}((R_Y^1, R_Y^2), Y) \text{ and } \overline{\mathcal{C}}((R_{\overline{Y}}^1, R_{\overline{Y}}^2), \overline{V_x} \setminus Y) \text{ holds,} \\ 0 & \text{otherwise.} \end{cases}$$

With the same arguments used in the proof of Theorems 4.3 and 7.2, one can easily prove the following:

- for every $X \in \mathcal{A}$ and $Y \in \mathcal{D}$, we have

$$(\mathcal{C}^* \cdot \overline{\mathcal{C}}^*)[X, Y] = 2^{|\text{cc}(G[X \cup Y])|-1} \cdot 2^{|\text{cc}(G[\overline{X \cup Y}])|-1}.$$

Consequently, $(\mathcal{C}^* \cdot \overline{\mathcal{C}}^*)[X, Y]$ is odd if and only if $X \cup Y$ is a minimal cut. Hence, $\mathcal{M}^* =_2 \mathcal{C}^* \cdot \overline{\mathcal{C}}^*$, where $=_2$ denotes the equality in $GF(2)$.

- The matrix \mathcal{C}^* can be computed in time $O(|\mathcal{A}| \cdot \text{s-nec}_1(V_x)^4 \cdot n^2)$.
- Let $\mathcal{B} \subseteq \mathcal{A}$ be a basis of maximum weight of the row space of \mathcal{C}^* w.r.t. to the weight function $w(X) := |E(X, V_x \setminus X)|$. Then, one can prove with the same arguments as in Theorem 4.3 that \mathcal{B} $(x, R_Y, R_{\overline{Y}})$ -represents \mathcal{A} and such a set can be computed in time $O(|\mathcal{A}| \cdot \text{s-nec}_1(V_x)^{4(\omega-1)} \cdot n^2)$.

□

As for the other algorithms of this paper, the computation of $\mathcal{T}_x[R, R_Y, R_{\overline{Y}}]$ is trivial for the leaves x of T . Take x an internal node of T with a and b as children such that the invariant holds for a and b . For $(R, R_Y, R_{\overline{Y}}) \in \mathbb{I}_x$, $(A, A_Y, A_{\overline{Y}}) \in \mathbb{I}_a$ and $(B, B_Y, B_{\overline{Y}}) \in \mathbb{I}_b$, we say that $(A, A_Y, A_{\overline{Y}})$ and $(B, B_Y, B_{\overline{Y}})$ are $(R, R_Y, R_{\overline{Y}})$ -compatible if the following conditions are satisfied:

- $A \cup B \equiv_{V_x}^n R$,
- $A \cup R_Y \equiv_{V_b}^1 B_Y$ and $B \cup R_Y \equiv_{V_a}^1 A_Y$,
- $(V_a \setminus A) \cup R_{\overline{Y}} \equiv_{V_b}^1 B_{\overline{Y}}$ and $(V_b \setminus B) \cup R_{\overline{Y}} \equiv_{V_a}^1 A_{\overline{Y}}$.

For each $(R, R_Y, R_{\overline{Y}}) \in \mathbb{I}_x$, the algorithm computes $\mathcal{T}_x[R, R_Y, R_{\overline{Y}}] := \text{reduce}^*(\mathcal{B}_x[R, R_Y, R_{\overline{Y}}])$ where $\mathcal{B}_x[R, R_Y, R_{\overline{Y}}]$ is the following set

$$\mathcal{B}_x[R, R_Y, R_{\overline{Y}}] := \bigcup_{(A, A_Y, A_{\overline{Y}}) \text{ and } (B, B_Y, B_{\overline{Y}}) \text{ are } (R, R_Y, R_{\overline{Y}})\text{-compatible}} \mathcal{T}_a[A, A_Y, A_{\overline{Y}}] \otimes \mathcal{T}_b[B, B_Y, B_{\overline{Y}}].$$

Observe that, $\mathcal{B}[R, R_Y, R_{\overline{Y}}]$ is a valid input for reduce^* because, for all $X \in \mathcal{B}[R, R_Y, R_{\overline{Y}}]$, we have $X \equiv_{V_x}^n R$. Thus, the conditions of Claim 7.7 on the input of reduce^* are satisfied.

Finally, with the same arguments used to prove Theorem 5.7, one can easily prove that the invariant holds for x and the table \mathcal{T}_x can be computed in time

$$O(\text{s-nec}_n(V_x)^2 \cdot \text{s-nec}_1(V_x)^{4(\omega+1.5)} \cdot n^2).$$

The total running time follows from the fact that T has $2n - 1$ nodes. \square

8. CONCLUSION

This paper highlights the importance of the d -neighbor-equivalence relation in the design of algorithms parameterized by clique-width, (\mathbb{Q}) -rank-width and mim-width. We prove that, surprisingly, this notion is helpful for problems with global constraints and also MAX CUT: a W[1]-hard problem parameterized by clique-width. Can we use it for other W[1]-hard problems parameterized by clique-width such as HAMILTONIAN CYCLE, EDGE DOMINATING SET and GRAPH COLORING?

Recently, the d -neighbor-equivalence relation was used by the first author and collaborators in order to give a meta-algorithm for solving the SUBSET FEEDBACK VERTEX SET problem and the NODE MULTIWAY CUT problem, both problems in time $2^{O(\text{rw}_{\mathbb{Q}}(G)^2 \cdot \log(\text{rw}_{\mathbb{Q}}(G)))} \cdot n^4$, $2^{O(\text{rw}(G)^3)} \cdot n^4$ and $n^{O(\text{mim}(\mathcal{L})^2)}$ with G the input graph and \mathcal{L} a given decomposition [5]. It would be interesting to see whether the d -neighbor-equivalence relation can be helpful to deal with other kinds of constraints such as 2-connectivity and other generalizations of FEEDBACK VERTEX SET such as the ones studied in [7].

Concerning mim-width, it is known that HAMILTONIAN CYCLE is NP-complete on graphs of mim-width 1, even when a rooted layout is provided [24]. To the best of our knowledge, we do not know whether MAX CUT is decidable in time $n^{f(\text{mim}(\mathcal{L}))}$ on an n -vertex graph given with a rooted layout \mathcal{L} for some function f . In fact, we do not even know if MAX CUT is decidable in polynomial time on interval graphs (which are known to have mim-width 1 [1]).

As explained in the introduction, the $2^{O(\text{mw}(\mathcal{L}))} \cdot n^{O(1)}$ time algorithms we obtain for clique-width are asymptotically optimal under the Exponential Time Hypothesis (ETH). This is also the case of our algorithms for MAX CUT parameterized by clique-width and \mathbb{Q} -rank-width. Indeed, Fomin et al. [16] proved that there is no $n^{o(k)} \cdot f(k)$ time algorithm, k being the clique-width of a given decomposition, for MAX CUT unless ETH fails. Since the clique-width of a graph is always bigger than its \mathbb{Q} -rank-width [28], this lower-bound holds also for \mathbb{Q} -rank-width.

However, for the other algorithmic results obtained in this paper, it is not known whether they are optimal under ETH. It would be particularly interesting to have tight upper bounds for rank-width since we know how to compute efficiently this parameter. To the best of our knowledge, there is no algorithm parameterized by rank-width that is known to be optimal under ETH. Even for “basic” problems such as VERTEX COVER or DOMINATING SET, the best algorithms [10] run in time $2^{O(k^2)} \cdot n^{O(1)}$, k being the rank-width of the graph. On the other hand, the best lower bounds state that, unless ETH fails, there are no $2^{o(k)} \cdot n^{O(1)}$ time algorithms parameterized by rank-width for VERTEX COVER (or DOMINATING SET) and no $n^{o(k)} \cdot f(k)$ time algorithm for MAX CUT [16].

Finally, Fomin et al. [17] have shown that we can use fast computation of representative sets in matroids to obtain deterministic $2^{O(\text{tw}(G))} \cdot n^{O(1)}$ time algorithms parameterized by tree-width for many connectivity problems. Is this approach also generalizable with d -neighbor-width? Can it be of any help for obtaining $2^{o(\text{rw}(G)^2)} \cdot n^{O(1)}$ time algorithms for problems like VERTEX COVER or DOMINATING SET?

Let's finish the conclusion by explaining how the methods used here can be adapted to solve any locally checkable partitioning problem with connectivity and/or acyclicity constraints. Given a $(q \times q)$ -matrix M with entries being finite or co-finite subsets of \mathbb{N} , an M -partition of a graph G is a partition $\{V_1, \dots, V_q\}$ of $V(G)$ such that for each $1 \leq i \leq q$ and each $v \in V_i$ we have $|N(v) \cap V_j| \in M[i, j]$, for each $1 \leq j \leq q$.

The M -PARTITION problem asks, for a given graph G , whether G admits an M -partition. NP-hard problems fitting into this framework include some homomorphism problems [10, Table 2] and the question of deciding whether an input graph has a partition into q (σ, ρ) -dominating sets, which is in most cases NP-complete for small values of q [10, Table 1]. It is proved in [10] that, the M -PARTITION problem can be solved in time $O(\text{s-nec}_d(T, \delta)^{3q} \cdot q \cdot n^4)$, with $d := \max_{1 \leq i, j \leq q} (d(M[i, j]))$, for any graph G given with a rooted layout (T, δ) . The algorithm is of the same flavor as for computing a (σ, ρ) -dominating set, except that, for each $x \in V(T)$, the table Tab_x associated with x is indexed by $(\mathcal{R}_{V_x}^d)^q \times (\mathcal{R}_{\bar{V}_x}^d)^q$ where $Tab_x[(R_1, \dots, R_q), (R'_1, \dots, R'_q)]$ is set to true if there is a partition (S_1, \dots, S_q) of V_x such that for each $1 \leq i, j \leq q$, $S_i \equiv_{V_x}^d R_i$ and $S_i \cup R'_i$ $M[i, j]$ -dominates S_j .

Let (A, C) - M -PARTITION be the problem that asks, given a graph G and two subsets A, C of $2^{\{1, \dots, q\}}$, whether G admits an M -partition $\{V_1, \dots, V_q\}$ such that, for every $\{i_1, \dots, i_t\} \in A$, the graph $G[V_{i_1} \cup \dots \cup V_{i_t}]$ is a forest and, for every $\{i_1, \dots, i_t\} \in C$, the graph $G[V_{i_1} \cup \dots \cup V_{i_t}]$ is connected. It is a routine verification to check that our framework can be extended to deal with (A, C) - M -PARTITION problems:

- The trick used in our algorithm for MAXIMUM MINIMAL CUT to deal with two connectivity constraints can be generalized to solve any (\emptyset, C) - M -PARTITION.
- We can generalize the concepts used in Section 6 and in particular the decomposition into R' -consistent sets to solve any (A, C) - M -PARTITION with $A \subseteq C$.
- We can use the idea of the reduction for the ACYCLIC (σ, ρ) -DOMINATING SET problems to solve any (A, C) - M -PARTITION problem with the algorithm solving the special case when $A \subseteq C$.

We can therefore state the following without proof.

Theorem 8.1. *For a $(q \times q)$ -matrix M with entries being finite or co-finite subsets of \mathbb{N} , let $d(M) := \max_{1 \leq i, j \leq q} (d(M[i, j]))$.*

- (1) *There exists an algorithm that, given an n -vertex graph G with a rooted layout (T, δ) of G , solves any (\emptyset, C) - M -PARTITION problem in time $O(\text{s-nec}_{d(M)}(T, \delta)^{3q} \cdot \text{s-nec}_1(T, \delta)^{2(\omega+1) \cdot \ell} \cdot q \cdot n^{O(1)})$ with $\ell = |C|$.*
- (2) *For each $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{mim}\}$, there exists an algorithm that, given an n -vertex graph G and a rooted layout (T, δ) of G , solves any (A, C) - M -PARTITION problem, in time $O(\text{s-nec}_{d(M)}(T, \delta)^{3q} \cdot \text{s-nec}_1(T, \delta)^{O(\ell)} \cdot \mathcal{N}_f(T, \delta)^{O(\ell)} \cdot n^{O(1)})$ with $\ell = |A| + |C|$.*

REFERENCES

- [1] Rémy Belmonte and Martin Vatshelle. Graph classes with structured neighborhoods and algorithmic applications. *Theoret. Comput. Sci.*, 511:54–65, 2013. URL: <http://dx.doi.org/10.1016/j.tcs.2013.01.011>, doi:10.1016/j.tcs.2013.01.011.
- [2] Benjamin Bergougnoux. *Matrix Decomposition and Algorithmic Application to (Hyper)Graphs*. PhD thesis, Université Clermont Auvergne, 2019. URL: <https://folk.uib.no/bbe089/pdf/thesis.pdf>.
- [3] Benjamin Bergougnoux and Mamadou Moustapha Kanté. Fast exact algorithms for some connectivity problems parameterized by clique-width. *Theor. Comput. Sci.*, 782:30–53, 2019. doi:10.1016/j.tcs.2019.02.030.
- [4] Benjamin Bergougnoux, Mamadou Moustapha Kanté, and O-joung Kwon. An optimal XP algorithm for hamiltonian cycle on graphs of bounded clique-width. In *Algorithms and Data Structures - 15th International Symposium, WADS 2017, St. John's, NL, Canada, July 31 - August 2, 2017, Proceedings*, pages 121–132, 2017. doi:10.1007/978-3-319-62127-2_11.
- [5] Benjamin Bergougnoux, Charis Papadopoulos, and Jan Arne Telle. Node multiway cut and subset feedback vertex set on graphs of bounded mim-width. *CoRR*, abs/1910.00887, 2019. URL: <http://arxiv.org/abs/1910.00887>, arXiv:1910.00887.

- [6] Hans L. Bodlaender, Marek Cygan, Stefan Kratsch, and Jesper Nederlof. Deterministic single exponential time algorithms for connectivity problems parameterized by treewidth. *Inform. and Comput.*, 243:86–111, 2015. URL: <http://dx.doi.org/10.1016/j.ic.2014.12.008>, doi:10.1016/j.ic.2014.12.008.
- [7] Édouard Bonnet, Nick Brettell, O-joung Kwon, and Dániel Marx. Generalized feedback vertex set problems on bounded-treewidth graphs: Chordality is the key to single-exponential parameterized algorithms. *Algorithmica*, 81(10):3890–3935, 2019. doi:10.1007/s00453-019-00579-4.
- [8] Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. Boolean-width of graphs. In Jianer Chen and Fedor V. Fomin, editors, *IWPEC*, volume 5917 of *Lecture Notes in Computer Science*, pages 61–74. Springer, 2009.
- [9] Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. H-join decomposable graphs and algorithms with runtime single exponential in rankwidth. *Discrete Applied Mathematics*, 158(7):809–819, 2010. doi:10.1016/j.dam.2009.09.009.
- [10] Binh-Minh Bui-Xuan, Jan Arne Telle, and Martin Vatshelle. Fast dynamic programming for locally checkable vertex subset and vertex partitioning problems. *Theoret. Comput. Sci.*, 511:66–76, 2013. URL: <http://dx.doi.org/10.1016/j.tcs.2013.01.009>, doi:10.1016/j.tcs.2013.01.009.
- [11] Bruno Courcelle and Stephan Olariu. Upper bounds to the clique width of graphs. *Discrete Applied Mathematics*, 101(1-3):77–114, 2000.
- [12] Marek Cygan, Jesper Nederlof, Marcin Pilipczuk, Michał Pilipczuk, Johan M. M. van Rooij, and Jakub Onufry Wojtaszczyk. Solving connectivity problems parameterized by treewidth in single exponential time (extended abstract). In *2011 IEEE 52nd Annual Symposium on Foundations of Computer Science—FOCS 2011*, pages 150–159. IEEE Computer Soc., Los Alamitos, CA, 2011. URL: <http://dx.doi.org/10.1109/FOCS.2011.23>, doi:10.1109/FOCS.2011.23.
- [13] Reinhard Diestel. *Graph Theory*. Number 173 in Graduate Texts in Mathematics. Springer, third edition, 2005.
- [14] Gabriel Lagoa Duarte, Daniel Lokshantov, Lehlton L. C. Pedrosa, Rafael Crivellari Saliba Schouery, and Uéverton dos Santos Souza. Computing the largest bond of a graph. *Corr*, 2019.
- [15] Hiroshi Eto, Tesshu Hanaka, Yasuaki Kobayashi, and Yusuke Kobayashi. Parameterized algorithms for maximum cut with connectivity constraints. *CoRR*, abs/1908.03389, 2019. URL: <http://arxiv.org/abs/1908.03389>, arXiv:1908.03389.
- [16] Fedor V. Fomin, Petr A. Golovach, Daniel Lokshantov, and Saket Saurabh. Almost optimal lower bounds for problems parameterized by clique-width. *SIAM J. Comput.*, 43(5):1541–1563, 2014. URL: <http://dx.doi.org/10.1137/130910932>, doi:10.1137/130910932.
- [17] Fedor V. Fomin, Daniel Lokshantov, Fahad Panolan, and Saket Saurabh. Efficient computation of representative families with applications in parameterized and exact algorithms. *J. ACM*, 63(4):29:1–29:60, 2016. URL: <http://doi.acm.org/10.1145/2886094>, doi:10.1145/2886094.
- [18] Robert Ganian and Petr Hliněný. On parse trees and Myhill-Nerode-type tools for handling graphs of bounded rank-width. *Discrete Appl. Math.*, 158(7):851–867, 2010. URL: <http://dx.doi.org/10.1016/j.dam.2009.10.018>, doi:10.1016/j.dam.2009.10.018.
- [19] Petr A. Golovach, Pinar Heggernes, Mamadou Moustapha Kanté, Dieter Kratsch, Sigve Hortemo Sæther, and Yngve Villanger. Output-polynomial enumeration on graphs of bounded (local) linear mim-width. *Algorithmica*, 80(2):714–741, 2018. doi:10.1007/s00453-017-0289-1.
- [20] Sudipto Guha and Samir Khuller. Improved methods for approximating node weighted Steiner trees and connected dominating sets. *Information and Computation*, 150(1):57 – 74, 1999. URL: <http://www.sciencedirect.com/science/article/pii/S0890540198927547>, doi:https://doi.org/10.1006/inco.1998.2754.
- [21] Russell Impagliazzo and Ramamohan Paturi. On the complexity of k-sat. *J. Comput. Syst. Sci.*, 62(2):367–375, 2001. doi:10.1006/jcss.2000.1727.
- [22] Russell Impagliazzo, Ramamohan Paturi, and Francis Zane. Which problems have strongly exponential complexity? In *39th Annual Symposium on Foundations of Computer Science, FOCS '98, November 8-11, 1998, Palo Alto, California, USA*, pages 653–663, 1998. doi:10.1109/SFCS.1998.743516.
- [23] Lars Jaffke, O-joung Kwon, and Jan Arne Telle. Polynomial-time algorithms for the longest induced path and induced disjoint paths problems on graphs of bounded mim-width. In *12th International Symposium on Parameterized and Exact Computation, IPEC 2017, September 6-8, 2017, Vienna, Austria*, pages 21:1–21:13, 2017. doi:10.4230/LIPIcs.IPEC.2017.21.
- [24] Lars Jaffke, O-joung Kwon, and Jan Arne Telle. A unified polynomial-time algorithm for feedback vertex set on graphs of bounded mim-width. In *35th Symposium on Theoretical Aspects of Computer Science, STACS 2018, February 28 to March 3, 2018, Caen, France*, pages 42:1–42:14, 2018. doi:10.4230/LIPIcs.STACS.2018.42.
- [25] Ki Hang Kim. *Boolean matrix theory and applications*, volume 70. Dekker, 1982.
- [26] Pedro Montealegre and Ioan Todinca. On distance-d independent set and other problems in graphs with "few" minimal separators. In *Graph-Theoretic Concepts in Computer Science - 42nd International Workshop, WG*

2016, Istanbul, Turkey, June 22-24, 2016, Revised Selected Papers, pages 183–194, 2016. doi:10.1007/978-3-662-53536-3_16.

- [27] Sang-Il Oum. *Graphs of Bounded Rank Width*. PhD thesis, Princeton University, 2005.
- [28] Sang-il Oum, Sigve Hortemo Sæther, and Martin Vatshelle. Faster algorithms for vertex partitioning problems parameterized by clique-width. *Theoret. Comput. Sci.*, 535:16–24, 2014. URL: <http://dx.doi.org/10.1016/j.tcs.2014.03.024>, doi:10.1016/j.tcs.2014.03.024.
- [29] Sang-il Oum and Paul Seymour. Approximating clique-width and branch-width. *J. Combin. Theory Ser. B*, 96(4):514–528, 2006. URL: <http://dx.doi.org/10.1016/j.jctb.2005.10.006>, doi:10.1016/j.jctb.2005.10.006.
- [30] Michaël Rao. *Décompositions de Graphes et Algorithmes Efficaces*. PhD thesis, Université Paul Verlaine, Metz, 2006.
- [31] Neil Robertson and Paul D. Seymour. Graph minors. X. obstructions to tree-decomposition. *Journal of Combinatorial Theory, Series B*, 52(2):153–190, 1991. URL: [http://dx.doi.org/10.1016/0095-8956\(91\)90061-N](http://dx.doi.org/10.1016/0095-8956(91)90061-N), doi:10.1016/0095-8956(91)90061-N.
- [32] Jan Arne Telle and Andrzej Proskurowski. Algorithms for vertex partitioning problems on partial k -trees. *SIAM J. Discrete Math.*, 10(4):529–550, 1997. URL: <http://dx.doi.org/10.1137/S0895480194275825>, doi:10.1137/S0895480194275825.
- [33] Martin Vatshelle. *New width parameters of graphs*. PhD thesis, University of Bergen, Bergen, Norway, 2012.
- [34] Virginia Vassilevska Williams. Multiplying matrices faster than coppersmith-winograd. In *Proceedings of the 44th Symposium on Theory of Computing Conference, STOC 2012, New York, NY, USA, May 19 - 22, 2012*, pages 887–898, 2012. URL: <http://doi.acm.org/10.1145/2213977.2214056>, doi:10.1145/2213977.2214056.

APPENDIX A. LOWER BOUNDS ON THE n -NEIGHBOR EQUIVALENCE

The following theorem proves that the upper bounds of Lemma 3.8 on $\text{nec}_n(A)$ are essentially tight. In this theorem, we use a graph closely related to the one used in [10, Theorem 4] to prove that, for any $k \in \mathbb{N}$, there exists a graph G and $A \subseteq V(G)$ such that $\text{rw}(A) = k$ and $\text{nec}_1(A) \in 2^{\Theta(k^2)}$.

Theorem A.1. *For every $k \in \mathbb{N}$, there exist an n -vertex graph G_k and $A \subseteq V(G)$ such that $\text{rw}(A) = k + 1$, $\text{rw}_{\mathbb{Q}}(A) = \text{mw}(A) = 2^k$ and $\text{nec}_n(A) = (n/2^k - 1)2^k$.*

Proof. In the following, we will manipulate multiple graphs that have some vertices in common. To avoid any confusion, for every $f \in \{\text{mw}, \text{rw}, \text{rw}_{\mathbb{Q}}, \text{nec}_n\}$, graph G and $A \subseteq V(G)$, we will denote the value of $f(A)$ in G by $f^G(A)$. For every $k \in \mathbb{N}$, we define the sets $A_k := \{a_S : S \subseteq [k]\}$, $B_k := \{b_S : S \subseteq [k]\}$ and the graph $H_k = (A_k \cup B_k, \{\{a_S, b_T\} : |S \cap T| \text{ is even}\})$. For instance, Figure 3 shows the graph H_2 .

For a matrix M , let us denote by $\text{rw}(M)$ and $\text{rw}_{\mathbb{Q}}(M)$ the rank of M over $GF(2)$ and \mathbb{Q} respectively. For every $k \in \mathbb{N}$, we denote by M_k the adjacency matrix M_{A_k, B_k} . Notice that $\text{rw}^{H_k}(A_k) = \text{rw}(M_k)$ and $\text{rw}_{\mathbb{Q}}^{H_k}(A_k) = \text{rw}_{\mathbb{Q}}(M_k)$. We also consider the matrix \overline{M}_k where for every $S, T \subseteq [k]$, we have $\overline{M}[a_S, b_T] = 1$ if $|S \cap T|$ is odd and 0 otherwise. Figure 3 shows the matrices M_1 and \overline{M}_1 .

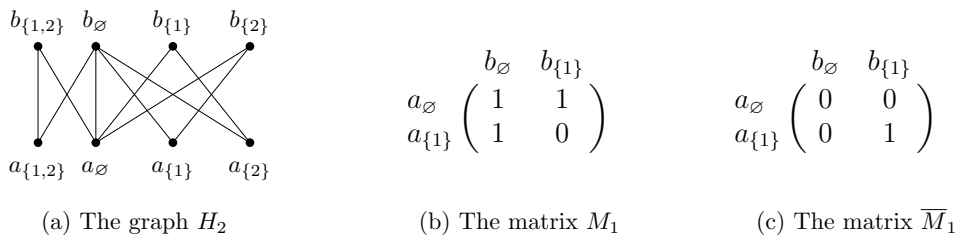


FIGURE 3. Graph H_k with $k = 2$ and the matrix M_2 .

We want to prove by recurrence on k that $\text{rw}(M_k) = k + 1$ and $\text{rw}_{\mathbb{Q}}(M_k) = 2^k$ for every $k \in \mathbb{N}$. This is true for $k = 0$ as we have $\text{rw}(M_0) = \text{rw}_{\mathbb{Q}}(M_0) = 1$.

Let $<$ be the total order on the subsets of \mathbb{N}^+ where $S < T$ if $\max(S \Delta T) \in T$. For example, we have $\emptyset < \{1\} < \{2\} < \{1, 2\} < \{3\} < \{1, 3\} < \{2, 3\} < \{1, 2, 3\}$. For every $k \in \mathbb{N}$, we

assume that the i th row (resp. column) of M_k and \overline{M}_k are associated, respectively, with a_S (resp. b_S) where S is the i th smallest subset of $[k]$ *w.r.t.* $<$. This way, for every $k \in \mathbb{N}$, the first $|A_{k+1}|/2 = 2^k$ rows (resp. columns) of M_{k+1} and \overline{M}_{k+1} are associated with the vertices a_S (resp. b_S) that belongs to A_k (resp. B_k), *i.e.*, we have $S \subseteq [k]$. On the other hand, the last 2^k rows (resp. columns) of M_k and \overline{M}_k are associated with the vertices a_S (resp. b_S) with $S = T \cup \{k+1\}$ with $T \subseteq [k]$. By definition, we deduce that, for every $k \in \mathbb{N}$, we have

$$(9) \quad M_{k+1} = \left(\begin{array}{c|c} M_k & M_k \\ \hline M_k & \overline{M}_k \end{array} \right) \text{ and } \overline{M}_{k+1} = \left(\begin{array}{c|c} \overline{M}_k & \overline{M}_k \\ \hline \overline{M}_k & M_k \end{array} \right).$$

Since row and column additions do not change the rank of a matrix and from the definition of M_k and \overline{M}_k , we deduce that, for every rank function $f \in \{\text{rw}, \text{rw}_{\mathbb{Q}}\}$, we have

$$f(M_{k+1}) = f \left(\begin{array}{c|c} M_k & M_k \\ \hline 0 & \overline{M}_k - M_k \end{array} \right) = f \left(\begin{array}{c|c} M_k & 0 \\ \hline 0 & \overline{M}_k - M_k \end{array} \right) = f(M_k) + f(\overline{M}_k - M_k).$$

In $GF(2)$, the matrix $\overline{M}_k - M_k$ is an all-1's matrix, we deduce that $\text{rw}(M_{k+1}) = \text{rw}(M_k) + 1$ for every $k \in \mathbb{N}$. As $\text{rw}(M_0) = 1$, by recurrence, this implies that $\text{rw}(M_k) = \text{rw}^{H_k}(A_k) = k + 1$.

Now, let us prove that $\text{rw}_{\mathbb{Q}}(M_k) = 2^k$. From the Equation (9), for every $k \in \mathbb{N}$, we deduce that

$$\overline{M}_{k+1} - M_{k+1} = \left(\begin{array}{c|c} \overline{M}_k - M_k & \overline{M}_k - M_k \\ \hline \overline{M}_k - M_k & M_k - \overline{M}_k \end{array} \right).$$

Hence, the rank over \mathbb{Q} of the matrix $\overline{M}_{k+1} - M_{k+1}$ equals

$$\text{rw}_{\mathbb{Q}} \left(\begin{array}{c|c} \overline{M}_k - M_k & \overline{M}_k - M_k \\ \hline 0 & 2(M_k - \overline{M}_k) \end{array} \right) = \text{rw}_{\mathbb{Q}} \left(\begin{array}{c|c} \overline{M}_k - M_k & 0 \\ \hline 0 & \overline{M}_k - M_k \end{array} \right) = 2\text{rw}_{\mathbb{Q}}(\overline{M}_k - M_k).$$

As $\text{rw}_{\mathbb{Q}}(M_0) = \text{rw}_{\mathbb{Q}}(\overline{M}_0 - M_0) = 1$, by recurrence, we deduce that $\text{rw}_{\mathbb{Q}}(\overline{M}_{k+1} - M_{k+1}) = 2^k$ and thus $\text{rw}_{\mathbb{Q}}(M_{k+1}) = 2^k$. We conclude that $\text{rw}_{\mathbb{Q}}^{H_k}(A_k) = 2^k$. As $\text{rw}_{\mathbb{Q}}^{H_k}(A_k) \leq \text{mw}^{H_k}(A_k) \leq |A_k| = 2^k$, we deduce that $\text{mw}^{H_k}(A_k) = 2^k$.

Let $k, t \in \mathbb{N}$ and G_k be the graph obtained from H_k by cloning t times each vertex in A_k . That is $V(G_k) = V(H_k) \cup \{a_S^2, \dots, a_S^t : S \subseteq [k]\}$ and for every $i \in \{2, \dots, t\}$ and $S \subseteq [k]$, we have $N(a_S^i) = N(a_S)$.

Let $A_k^* := A_k \cup \{a_S^2, \dots, a_S^t : S \subseteq [k]\}$. Obviously, we have $\text{rw}^{H_k}(A_k^*) = k + 1$, $\text{rw}_{\mathbb{Q}}^{H_k}(A_k^*) = \text{mw}^{H_k}(A_k^*) = 2^k$ because the adjacency matrix between A_k^* and B_k can be obtained from M_k by adding $t - 1$ copies of each row.

Let n be the number of vertices of G . We claim that $\text{nec}_n^{H_k}(A_k^*) = (n/2^k - 1)^{2^k}$. For every $X \subseteq A_k^*$, we define the column vector $v_X = (x_{\emptyset}, x_{\{1\}}, x_{\{2\}}, x_{\{1,2\}}, \dots, x_{[k]})$ such that, for every $S \subseteq [k]$, we have $x_S = |X \cap \{a_S, a_S^2, a_S^3, \dots, a_S^t\}|$. Observe that $M_k \cdot v_X = (y_{\emptyset}, y_{\{1\}}, \dots, y_{[k]})$ where, for every $S \subseteq [k]$, we have $y_S = |N(b_S) \cap X|$. Thus, for every $X, W \subseteq A_{H_k}^*$, we have $X \equiv_{A_{H_k}^*}^n W$ if and only if $M_k \cdot v_X = M_k \cdot v_W$. As $\text{rw}_{\mathbb{Q}}(M_k) = 2^k$, by the Rank-nullity theorem, we deduce that the linear application associated with M_k (over \mathbb{Q}) is a bijection. We conclude that, for every $X, W \subseteq A_{H_k}^*$, we have $X \equiv_{A_{H_k}^*}^n W$ if and only if $v_X = v_W$. Thus, $\text{nec}_n^{H_k}(A_k^*) = t^{2^k}$. By construction, we have $n = 2^k(t + 1)$. Consequently, $t = n/2^k - 1$ and $\text{nec}_n^{H_k}(A_k^*) = (n/2^k - 1)^{2^k}$. \square

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