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RANK BASED APPROACH ON GRAPHS WITH STRUCTURED NEIGHBORHOOD

BENJAMIN BERGOUGNOUX AND MAMADOU MOUSTAPHA KANTÉ

ABSTRACT.

1. INTRODUCTION

Connectivity problems such as CONNECTED DOMINATING SET, FEEDBACK VERTEX SET or HAMILTONIAN CYCLE were for a long time a curiosity in FPT world as they admit trivial $k^{O(k)}$ time algorithm parameterized by tree-width, but no lower-bound were known. Indeed, for good reason, in [3], Bodlaender et al. proposed a general toolkit called *rank-based approach* to design deterministic $2^{O(k)} \cdot n^{O(1)}$ time algorithm parametrized by tree-width to solve these problems. The idea of this is to encode the set of solutions by a binary matrix and then show that this matrix admits a base of size $2^{O(k)}$ computable by a greedy algorithm, which *represents* the set of optimal solutions.

Nevertheless, despite the broad interest on tree-width, only sparse graphs can have bounded tree-width. But many NP-hard problems are tractable on dense graphs classes. Most of the time, this tractability can be explained by the ability of these graphs to be recursively decomposed along vertex partitions (A, B) where the adjacency between A and B is simple to describe, *i.e.*, they have a *structured neighborhood*. A lot of graph parameters have been defined to characterize this ability, the most remarkable ones are certainly clique-width [6], rank-width [12], and maximum induced matching width (called mim-width) [16].

Introduced by Courcelle and Olariu [6], the modeling power of clique-width is strictly stronger than tree-width. In other words, if a graph class has bounded tree-width, then it has bounded clique-width [6], but the converse is false as cliques have clique-width at most 2 and unbounded tree-width. While rank-width has the same modeling power as clique-width, mim-width has the strongest of all these complexity measures and is even bounded on interval graphs [1]. Despite their generality, a lot of NP-hard problems admit polynomial time algorithms parametrized by these parameters. But dealing with these parameters is known to be harder than manipulating tree-width.

Unlike tree-width, algorithm parameterized by clique-width, rank-width and mim-width for connectivity problems, were not investigated, except for some special cases as FEEDBACK VERTEX SET which is proved to admit a $2^{O(k)} \cdot n^{O(1)}$ time algorithm parameterized by clique-width in [2], a $2^{O(k^2)} \cdot n^{O(1)}$ time algorithm parameterized by rank-width [8], and a $n^{O(k)}$ time algorithm parameterized by mim-width [10].

One successful way to work with these different parameters is through the notion of *neighbor equivalence* introduced in [5]. This equivalence relation is defined among

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the subsets of a cut $A \subseteq V(G)$ and for an integer d , such that two subsets $X, Y \subseteq A$ are equivalent if every vertex not in A either has the same number of neighbors in X and Y , or at least d in each. The neighbor equivalence was the key in the design of efficient algorithm to solve some well-studied and well-known difficult problems such as DOMINATING SET [5, 9, 13] or FEEDBACK VERTEX SET [8, 10].

Our Contributions and approach. One of our main contribution is the modification of the rank-based approach to fit with the neighbor equivalence (presented in Section 3). The resulting framework simplifies and generalizes the original one from [3]. In particular, we obtain algorithmic results for clique-width, rank-width, mim-width and also \mathbb{Q} -rank-width, an useful variant of rank-width used in [13].

In Section 4, we apply our framework on connectivity problems with locally checkable properties, such as such as CONNECTING DOMINATION SET, CONNECTED VERTEX COVER or NODE WEIGHTED STEINER TREE. All these problems are some special case of the connected version of the (σ, ρ) - DOMINATING SET problem. This problem was introduced in [15] and studied in graphs of bounded clique-width and rank-width in [5, 13]. We recall its definition at the beginning of Section 4. We provide an algorithm for CONNECTED (σ, ρ) - DOMINATING SET whose running time depends only on the number of vertices of the graph and the number of equivalence classes of the neighbor equivalence. Consequently, we obtain efficient parameterized algorithms with parameters and running time given in the Table 1. Up to a constant in the exponent, these running times match those known for basic problems such as DOMINATING SET [5, 13]. These results highlight the importance of the neighbor equivalence for these parameters.

Clique-width	Rank-width	\mathbb{Q} -rank-width	Mim-width
$2^{O(k)} \cdot n^{O(1)}$	$2^{O(k^2)} \cdot n^{O(1)}$	$2^{O(k \cdot \log(k))} \cdot n^{O(1)}$	$n^{O(k)}$

TABLE 1. Running times of our algorithms for the different parameters.

In Section 5, we integrate the notion of acyclicity to our framework. As a result, we obtain efficient algorithms for FEEDBACK VERTEX SET and MAXIMUM INDUCED TREE whose running times are described in the Table 1. Contrary to Section 4, we were not able to obtain an algorithm that relies only on the neighbor equivalence. Indeed, some specific part of our algorithms use the structural properties of the different parameters. We leave open the question of an algorithm with a running time polynomial on the number of vertices and equivalence classes of the neighbor equivalence for these two problems.

2. PRELIMINARIES

The size of a set V is denoted by $|V|$ and its power set is denoted by 2^V . We write $A \setminus B$ for the set difference of A from B , and we write $A \uplus B$ for the disjoint union of A and B . We often write x to denote the singleton set $\{x\}$. We let $\min(\emptyset) := +\infty$ and $\max(\emptyset) := -\infty$. For two sets A and B , we let

$$A \otimes B := \begin{cases} \emptyset & \text{if } A = \emptyset \text{ or } B = \emptyset, \\ \{X \cup Y \mid X \in A \wedge Y \in B\} & \text{otherwise.} \end{cases}$$

Graphs. Our graph terminology is standard, and we refer to [7]. The vertex set of a graph G is denoted by $V(G)$ and its edge set by $E(G)$. For every vertex set $A \subseteq V(G)$, we denote by \bar{A} , the set $V(G) \setminus A$. An edge between two vertices x and y is denoted by xy (respectively yx). The subgraph of G induced by a subset X of its vertex set is denoted by $G[X]$, and we write $G \setminus X$ to denote the induced subgraph $G[V(G) \setminus X]$. The set of vertices that is adjacent to x is denoted by $N(x)$, and for $U \subseteq V(G)$, $N(U) := (\bigcup_{v \in U} N(v)) \setminus U$. For every $A \subseteq V(G)$, we denote by M_A the adjacency matrix of the cut (A, \bar{A}) , *i.e.*, the (A, \bar{A}) -matrix such that $M_A[x, y] = 1$ if $y \in N(x)$, 0 otherwise.

Let $X \subseteq V(G)$, we denote by $\text{cc}(X)$ the partition $\{V(C) \mid C \text{ is a connected component of } G[X]\}$ of X . Moreover, a consistent cuts of X is ordered bipartition (X_1, X_2) of X such that $N(X_1) \cap X_2 = \emptyset$. We denote by $\text{cuts}(X)$ the set of all consistent cuts of X .

Neighbor-equivalence. Let $A \subseteq V(G)$ and $d \in \mathbb{N}$. Two subsets X and Y of A are d -neighbor equivalent w.r.t. \bar{A} , denoted by $X \equiv_A^d Y$, if $\min(d, |X \cap N(x)|) = \min(d, |Y \cap N(x)|)$ for all $x \in \bar{A}$. It is not hard to check that \equiv_A^d is an equivalence relation. We denote the number of equivalence classes of \equiv_A^d by $\text{nec}_d(A)$.

It is known that $\text{nec}_d(A) = \text{nec}_d(\bar{A})$, see [11, Theorem 1.2.3]. The following fact follows directly from the definition of d -neighbor equivalent. We use it several times in our proof.

Fact 1. *Let $A, B \subseteq V(G)$ such that $A \subseteq B$, and let $d \in \mathbb{N}$. For all $X, Y \subseteq A$, if $X \equiv_A^d Y$ then $X \equiv_B^d Y$.*

For each $X \subseteq A$, let us denote by $\text{rep}_A^d(X)$ the lexicographically¹ smallest set $R \subseteq A$ such that $|R|$ is minimized and $R \equiv_A^d X$. Moreover, we denote by \mathcal{R}_A^d the set $\{\text{rep}_A^d(X) \mid X \subseteq A\}$. In order to compute \mathcal{R}_A^d , we use the following lemma.

Lemma 2.1 ([5]). *For every $A \subseteq V(G)$, one can compute in time $O(\text{nec}(A)^d \cdot \log(\text{nec}(A)^d) \cdot |V(G)|^2)$, the sets \mathcal{R}_A^d and a data structure that given a set $X \subseteq A$ compute $\text{rep}_A^d(X)$ in time $O(\log(\text{nec}(A)^d) \cdot |A| \cdot |V(G)|)$.*

Rooted Layout. A *rooted binary tree* is a binary tree with a distinguished vertex r , called the *root*. Since we manipulate at the same time graphs and trees representing them the vertices of trees will be called *nodes*.

A *rooted layout* of G is a pair (T, \mathcal{L}) of a rooted binary tree T and a bijective function \mathcal{L} between $V(G)$ and the leaves of T . For each node x of T , let L_x be the set of all the leaves l of T such that the path from the root of T to l contains x . We denote by V_x , the set of vertices that are in bijection with L_x .

All the structural parameters used in this section are defined similarly, the only difference is the set function used. Given, a set function $f : 2^{V(G)} \rightarrow \mathbb{N}$ and a rooted layout (T, \mathcal{L}) , the width of a node x of T is $f(V_x)$ and the width of (T, \mathcal{L}) , denoted by $f(T, \mathcal{L})$, is $\max\{f(x) \mid x \in V(T)\}$. Finally, the width of G is the minimum width over all rooted layouts of G .

Neighbor-width. For every $d \in \mathbb{N}$, the d -neighbor-width is the parameter obtained through the set function nec_d .

Clique-width / Module-width. We won't define *clique-width*, but its equivalent measure *module-width* [14] as done in [4] for FEEDBACK VERTEX SET. Module-width is defined through the set function $\text{mw} : 2^{V(G)} \rightarrow \mathbb{N}$ such that $\text{mw}(A)$ is the number of vertices with distinct neighborhood over the cut (A, \bar{A}) , *i.e.* the cardinal of $\{N(v) \cap \bar{A} \mid v \in A\}$. It is also the number of different rows in M_A . The following theorem shows the link between module-width and clique-width.

¹Given a arbitrary ordering of $V(G)$.

Theorem 2.2 ([14, Theorem 6.6]). *For every n -vertex graph G , $\text{mw}(G) \leq \text{cw}(G) \leq 2 \cdot \text{mw}(G)$, where $\text{cw}(G)$ denotes the clique-width of G . One can moreover translate, in time at most $O(n^2)$, a given decomposition into the other one with width at most the given bounds.*

Rank-width. Rank-width is defined through the set function $\text{rw} : 2^{V(G)} \rightarrow \mathbb{N}$ such that $\text{rw}(A)$ is the rank over $GF(2)$ of the matrix M_A .

\mathbb{Q} -rank-width. \mathbb{Q} -rank-width is associated to the set function $\mathbb{Q}\text{-rw} : 2^{V(G)} \rightarrow \mathbb{N}$ such that $\mathbb{Q}\text{-rw}(A)$ is the rank over \mathbb{Q} of the matrix M_A .

Mim-width. Mim-width is defined through the set function $\text{mim} : 2^{V(G)} \rightarrow \mathbb{N}$ where $\text{mim}(A)$ is the size of the maximum induced matching of the cut (A, \bar{A}) , i.e., the maximum induced matching of the graph with vertex set $V(G)$ and edge set $\{uv \in E(G) \mid u \in A, v \in \bar{A}\}$. This is equivalent to the maximum $|X|$ over all $X \subseteq A$, such that for all $v \in X$, we have $(N(v) \cap \bar{A}) \setminus N(X \setminus \{v\}) \neq \emptyset$.

The following lemma shows how the neighbor-width is bounded by the others parameters.

Lemma 2.3. *For every $A \subseteq V(G)$ and $d \in \mathbb{N}$, the d -neighbor-width of A , i.e. $\text{nec}_d(A)$, is upper bounded by the following terms:*

- (1) $(d+1)^{\text{mw}(A)}$,
- (2) $2^{d \cdot \text{rw}(A)^2}$,
- (3) $(d \cdot \mathbb{Q}\text{-rw}(A) + 1)^{\mathbb{Q}\text{-rw}(A)}$,
- (4) $n^{d \cdot \text{mim}(A)}$.

Proof. The second and the third upper bounds have been proved respectively in [13, Theorem 4.2]. The fourth is proved in [1, Lemma 2]. It remains to prove that $\text{nec}_d(A) \leq (d+1)^{\text{mw}(A)}$. Let $Y, W \subseteq \bar{A}$. Observe that for every vertex $u, v \in A$ such that $u \equiv_A^1 v$, we have $N(u) \cap A = N(v) \cap A$. Thus, $Y \equiv_A^d W$ if and only if for all $v \in A$, we have $\max(|N(\text{rep}_A^1(v)) \cap Y|, d) = \max(|N(\text{rep}_A^1(v)) \cap W|, d)$. By definition, there is $\text{mw}(A)$ vertices with different neighborhood over the cut (A, \bar{A}) . Thus, the set $\{\text{rep}_A^1(v) \mid v \in A\}$ has size $\text{mw}(A)$. Since $\max(|N(v) \cap Y|, d) \in [0, d]$ for all $v \in A$, we deduce that $\text{nec}(\bar{A}) \leq (d+1)^{\text{mw}(A)}$. The inequalities follows from the fact that $\text{nec}(\bar{A}) = \text{nec}(A)$. \square

In the following, we fix G a n -vertex graph and (T, \mathcal{L}) a rooted layout of G . We fix also $w : V(G) \rightarrow \mathbb{Z}$ be a weight function over the vertices of G and $\text{opt} \in \{\min, \max\}$.

3. REPRESENTATIVE SETS

In the following, we define a notion of representativity similar to the one defined in [3] and adapted to the notion of neighbor-equivalence. Our notion of representativity is defined *w.r.t.* some node x of T and a representative $R' \in \mathcal{R}_{V_x}^1$.

Definition 1 ((x, R') -representativity). For every $\mathcal{A} \subseteq 2^{V(G)}$ and $Y \subseteq V(G)$, we define $\text{best}(\mathcal{A}, Y)$ as follow

$$\text{best}(\mathcal{A}, Y) := \text{opt}\{w(X) \mid X \in \mathcal{A} \wedge G[X \cup Y] \text{ is connected}\}.$$

Let $\mathcal{A}, \mathcal{B} \subseteq 2^{V_x}$. We say that \mathcal{B} (x, R') -represents \mathcal{A} if for every $Y \subseteq \bar{V}_x$ such that $Y \equiv_{V_x}^1 R'$, we have $\text{best}(\mathcal{A}, Y) = \text{best}(\mathcal{B}, Y)$.

Observe that the (x, R') -representativity is an equivalence relation. The set \mathcal{A} is meant to represent the set of partial solutions of V_x , we have computed. We expect to complete these partial solutions with the partial solution of \bar{V}_x equivalent to R'

for $\equiv_{\overline{V}_x}^1$. If \mathcal{B} (x, R')-represents \mathcal{A} , then we can safely substitute \mathcal{B} to \mathcal{A} because the quality of the output of the dynamic programming will remain the same. Indeed, the best solutions obtained by the union of a partial solution in \mathcal{A} and a set of \overline{V}_x will have the same weight as those obtained from \mathcal{B} . The goal is to substitute \mathcal{A} by a set \mathcal{B} as small as possible. To this end, we show that given a set $\mathcal{A} \subseteq 2^{\overline{V}_x}$, we can compute efficiently a small subset of \mathcal{A} that (x, R')-represents \mathcal{A} . Similarly to [3], the small set of representatives we want to compute is an optimal basis of some matrix. To compute this optimal basis, we use the following lemma. The constant ω denotes the matrix multiplication exponent, which is known to be strictly less than 2.3727 due to [17].

Lemma 3.1 ([3]). *Let M be a binary $n \times m$ -matrix with $m \leq n$ and let $w : \{1, \dots, n\} \rightarrow \mathbb{N}$ be a weight function on the rows of M . Then, one can find a basis of maximum (or minimum) weight of the row space of M in time $O(nm^{\omega-1})$.*

Theorem 3.2. *Let $R \in \mathcal{R}_{\overline{V}_x}^1$. Given $\mathcal{A} \subseteq 2^{\overline{V}_x}$ such that for each $X \in \mathcal{A}$, we have $X \equiv_{\overline{V}_x}^1 R$. There exists an algorithm reduce that outputs in time $O(|\mathcal{A}| \cdot \text{nec}_1(V_x)^{2 \cdot (\omega-1)} \cdot n^2)$, a subset $\mathcal{B} \subseteq \mathcal{A}$ such that \mathcal{B} (x, R')-represents \mathcal{A} and $|\mathcal{B}| \leq \text{nec}_1(V_x)^2$.*

Proof. Assume, w.l.o.g., that $\text{opt} = \max$, the proof for $\text{opt} = \min$ is symmetric.

Observe that for every $Y \equiv_{\overline{V}_x}^1 \emptyset$, we have $N(Y) \cap V_x = N(\emptyset) \cap V_x = \emptyset$. It follows that for every $Y \subseteq \overline{V}_x$ such that $Y \equiv_{\overline{V}_x}^1 \emptyset$ and $Y \neq \emptyset$, we have $\text{best}(\mathcal{A}, Y) = -\infty$. Moreover, by definition of best , we have $\text{best}(\mathcal{A}, \emptyset) = \{w(X) \in \mathcal{A} \mid G[X]$ is connected $\}$. Hence, if $R' = \emptyset$, then it is sufficient to return $\mathcal{B} = \{X\}$, where X is an element of \mathcal{A} of maximum weight that induces a connected graph.

Assume from now that $R' \neq \emptyset$. Let $X \in \mathcal{A}$. If there exists $C \in \text{cc}(X)$ such that $N(C) \cap R' = \emptyset$, then for all $Y \equiv_{\overline{V}_x}^1 R'$, we have $N(C) \cap Y = \emptyset$. Moreover, as $R' \neq \emptyset$, we have $Y \neq \emptyset$. Consequently, for every $Y \equiv_{\overline{V}_x}^1 R'$, the graph $G[X \cup Y]$ is not connected. We can conclude that $\mathcal{A} \setminus \{X\}$ (x, R') represents \mathcal{A} . Thus, we can safely remove from \mathcal{A} all such sets, this can be done in time $|\mathcal{A}| \cdot n^2$. From now, we assume that for all $X \in \mathcal{A}$ and for all $C \in \text{cc}(X)$, we have $N(C) \cap R' \neq \emptyset$.

Symmetrically, if for some $Y \subseteq \overline{V}_x$ there exists $C \in \text{cc}(Y)$ such that $N(C) \cap R = \emptyset$, then for every $X \in \mathcal{A}$, the graph $G[X \cup Y]$ is not connected. Let \mathcal{D} be the set of all subsets Y of \overline{V}_x such that $Y \equiv_{\overline{V}_x}^1 R'$ and, for all $C \in \text{cc}(Y)$, we have $N(C) \cap R \neq \emptyset$.

Observe that the sets in $2^{\overline{V}_x} \setminus \mathcal{D}$ do not matter in the (x, R')-representativity.

For every $Y \in \mathcal{D}$, we let v_Y be one fixed vertex of Y . In the following, we denote by \mathcal{F} the set $\{(R'_1, R'_2) \in \mathcal{R}_{\overline{V}_x}^1 \times \mathcal{R}_{\overline{V}_x}^1\}$. Let \mathcal{M} , \mathcal{C} , and $\overline{\mathcal{C}}$ be, respectively, an $(\mathcal{A}, \mathcal{D})$ -matrix, an $(\mathcal{A}, \mathcal{F})$, and an $(\mathcal{F}, \mathcal{D})$ -matrix such that

$$\begin{aligned} \mathcal{M}[X, Y] &:= \begin{cases} 1 & \text{if } G[X \cup Y] \text{ is connected,} \\ 0 & \text{otherwise.} \end{cases} \\ \mathcal{C}[X, (R'_1, R'_2)] &:= \begin{cases} 1 & \text{if } \exists (X_1, X_2) \in \text{cuts}(X) \text{ such that } N(X_1) \cap R'_2 = \emptyset \text{ and } N(X_2) \cap R'_1 = \emptyset, \\ 0 & \text{otherwise.} \end{cases} \\ \overline{\mathcal{C}}[(R'_1, R'_2), Y] &:= \begin{cases} 1 & \text{if } \exists (Y_1, Y_2) \in \text{cuts}(Y) \text{ such that } v_Y \in Y_1, Y_1 \equiv_{\overline{V}_x}^1 R'_1, \text{ and } Y_2 \equiv_{\overline{V}_x}^1 R'_2, \\ 0 & \text{otherwise.} \end{cases} \end{aligned}$$

Intuitively, \mathcal{M} contains all the information we need. Indeed, it is easy to see that an optimal basis of \mathcal{M} in $GF(2)$ is a (x, R')-representative set of \mathcal{A} . But \mathcal{M} is too big to be computable efficiently. In fact, we will prove that it is enough to compute an optimal basis of \mathcal{C} . As \mathcal{C} is small, i.e., $|\mathcal{A}| \cdot \text{nec}_1(V_x)^2$, the running time of the computation and the size of the basis is also small.

We start by proving that $\mathcal{M} =_2 \mathcal{C} \cdot \bar{\mathcal{C}}$, where $=_2$ denotes the equality in $GF(2)$. Let $X \in \mathcal{A}$ and $Y \in \mathcal{D}$. We want to prove the following equality

$$(\mathcal{C} \cdot \bar{\mathcal{C}})[X, Y] = \sum_{(R'_1, R'_2) \in \mathcal{F}} \mathcal{C}[X, (R'_1, R'_2)] \cdot \bar{\mathcal{C}}[(R'_1, R'_2), Y] = 2^{|\text{cc}(X \cup Y)|-1}.$$

We prove this equality with the two following claims.

Claim 3.2.1. *We have $\mathcal{C}[X, (R'_1, R'_2)] \cdot \bar{\mathcal{C}}[(R'_1, R'_2), Y] = 1$ if and only if there exists $(W_1, W_2) \in \text{cuts}(X \cup Y)$ such that $v_Y \in W_1$, $W_1 \cap Y \equiv_{V_x}^1 R'_1$, and $W_2 \cap Y \equiv_{V_x}^1 R'_2$.*

Proof. By definition, we have $\mathcal{C}[X, (R'_1, R'_2)] \cdot \bar{\mathcal{C}}[(R'_1, R'_2), Y] = 1$, if and only if

- (a) $\exists (Y_1, Y_2) \in \text{cuts}(Y)$ such that $(v_Y \in Y_1) \wedge (Y_1 \equiv_{V_x}^1 R'_1) \wedge (Y_2 \equiv_{V_x}^1 R'_2)$ and,
- (b) $\exists (X_1, X_2) \in \text{cuts}(X)$ such that $(N(X_1) \cap R'_2 = \emptyset) \wedge (N(X_2) \cap R'_1 = \emptyset)$.

Let $W_1, W_2 \subseteq X \cup Y$ and for each $i \in \{1, 2\}$, let $X_i := W_i \cap X$ and $Y_i := W_i \cap Y$. Observe that $(W_1, W_2) \in \text{cuts}(X \cup Y)$ if and only if (1) $(X_1, X_2) \in \text{cuts}(X)$, (2) $(Y_1, Y_2) \in \text{cuts}(Y)$, (3) $N(X_1) \cap Y_2 = \emptyset$ and $N(X_2) \cap Y_1 = \emptyset$.

By definition of $\equiv_{V_x}^1$, the condition (3) is equivalent to $N(X_1) \cap \text{rep}_{V_x}^1(Y_2) = \emptyset$ and $N(X_2) \cap \text{rep}_{V_x}^1(Y_1) = \emptyset$. From these observations, one easily proves the claim. \square

Claim 3.2.2. *Let $(W_1, W_2), (W'_1, W'_2) \in \text{cuts}(X \cup Y)$.*

We have $W_1 \equiv_{V_x}^1 W'_1$ and $W_2 \equiv_{V_x}^1 W'_2$ if and only if $(W_1, W_2) = (W'_1, W'_2)$.

Proof. We start by an observation of the connected component of $X \cup Y$. As $Y \in \mathcal{D}$, for all $C \in \text{cc}(Y)$, we have $N(C) \cap R \neq \emptyset$. Moreover, by assumption, that for all $C \in \text{cc}(X)$, we have $N(C) \cap R' \neq \emptyset$. Since $X \equiv_{V_x}^1 R$ and $Y \equiv_{V_x}^1 R'$, every connected component of $\text{cc}(X \cup Y)$ contains at least one vertex of X and one vertex of Y .

Assume toward a contradiction that $W_1 \equiv_{V_x}^1 W'_1$, $W_2 \equiv_{V_x}^1 W'_2$ and $W_1 \neq W'_1$. Since $W_1 \neq W'_1$, there exists $C \in \text{cc}(X \cup Y)$ such that $C \subseteq W_1$ and $C \subseteq W'_2$. As $C \cap X \neq \emptyset$, we have $N(C \cap X) \cap W_1 \neq \emptyset$ and $N(C \cap X) \cap W'_2 \neq \emptyset$. Since $W_2 \equiv_{V_x}^1 W'_2$, we have $N(C \cap X) \cap W_2 \neq \emptyset$. This contradicts the fact that $(W_1, W_2) \in \text{cuts}(X \cup Y)$ and proves the claim. \square

Together, these two claims prove that

$$(\mathcal{C} \cdot \bar{\mathcal{C}})[X, Y] = |\{(W_1, W_2) \in \text{cuts}(X \cup Y) \mid v_Y \in W_1\}|.$$

It follows that $(\mathcal{C} \cdot \bar{\mathcal{C}})[X, Y] = 2^{|\text{cc}(X \cup Y)|-1}$ since every connected component of $G[X \cup Y]$ can be in both sides of a consistent cuts at the exception of the connected component containing v_Y . Hence, $(\mathcal{C} \cdot \bar{\mathcal{C}})[X, Y]$ is odd if and only if $|\text{cc}(X \cup Y)| = 1$. We can conclude that $\mathcal{M} =_2 \mathcal{C} \cdot \bar{\mathcal{C}}$.

Let $\mathcal{B} \subseteq \mathcal{A}$ be a basis of maximum weight of the row space of \mathcal{C} over $GF(2)$. We claim that \mathcal{B} (x, R') -represents \mathcal{A} .

Let $Y \subseteq \bar{V}_x$ such that $Y \equiv_{V_x}^1 R'$. Observe that, by definition of \mathcal{D} , if $Y \notin \mathcal{D}$, then $\text{best}(\mathcal{A}, Y) = \text{best}(\mathcal{B}, Y) = -\infty$. Thus it is sufficient to prove that for every $Y \in \mathcal{D}$, we have $\text{best}(\mathcal{A}, Y) = \text{best}(\mathcal{B}, Y)$. Let $X \in \mathcal{A}$ and $Y \in \mathcal{D}$.

Since \mathcal{B} is a basis of \mathcal{C} , there exists $\mathcal{B}' \subseteq \mathcal{B}$ such that for each $(R'_1, R'_2) \in \mathcal{F}$, $\mathcal{C}[X, (R'_1, R'_2)] =_2 \sum_{W \in \mathcal{B}'} \mathcal{C}[W, (R'_1, R'_2)]$. Thus, we have the following equalities

$$\begin{aligned} \mathcal{M}[X, Y] &= \sum_{(R'_1, R'_2) \in \mathcal{F}} \mathcal{C}[X, (R'_1, R'_2)] \cdot \bar{\mathcal{C}}[(R'_1, R'_2), Y] \\ &= \sum_{(R'_1, R'_2) \in \mathcal{F}} \left(\sum_{W \in \mathcal{B}'} \mathcal{C}[W, (R'_1, R'_2)] \right) \cdot \bar{\mathcal{C}}[(R'_1, R'_2), Y] \\ &= \sum_{W \in \mathcal{B}'} \sum_{(R'_1, R'_2) \in \mathcal{F}} \mathcal{C}[W, (R'_1, R'_2)] \cdot \bar{\mathcal{C}}[(R'_1, R'_2), Y] \\ &= \sum_{W \in \mathcal{B}'} (\mathcal{C} \cdot \bar{\mathcal{C}})[W, Y] =_2 \sum_{W \in \mathcal{B}'} \mathcal{M}[W, Y]. \end{aligned}$$

If $\mathcal{M}[X, Y] = 1$, *i.e.*, $G[X, Y]$ is connected, then there is a odd number of set W in \mathcal{B}' such that $\mathcal{M}[W, Y] = 1$. Let $W \in \mathcal{B}'$ such that $\mathcal{M}[W, Y] = 1$ and $w(W)$ is maximum. Assume towards a contradiction that $w(W) < w(X)$. Observe that $(\mathcal{B} \setminus \{W\}) \cup \{X\}$ is also a basis of \mathcal{C} since the set of independent row sets of a matrix forms a matroid. Since $w(W) < w(X)$, $(\mathcal{B} \setminus \{W\}) \cup \{X\}$ is a basis of weight strictly greater than \mathcal{B} , yielding a contradiction. Thus $w(X) \leq w(W)$. Hence, for all $Y \in \mathcal{D}$ and all $X \in \mathcal{A}$, if $G[X \cup Y]$ is connected then there exists $W \in \mathcal{B}$ such that $G[W \cup Y]$ is connected and $w(X) \leq w(W)$. This is sufficient to prove that \mathcal{B} (x, R') -represents \mathcal{A} .

It remains to prove the running time. Observe that \mathcal{C} is easy to compute. Indeed, one easily checks that $\mathcal{C}[X, (R'_1, R'_2)] = 1$ if and only if for each $C \in \text{cc}(X)$, we have $N(C) \cap R'_1 = \emptyset$ or $N(C) \cap R'_2 = \emptyset$. Thus, each entry of \mathcal{C} is computable in time $O(n^2)$. Since \mathcal{C} has $|\mathcal{A}| \cdot |\mathcal{R}_{V_x}^1|^2 = |\mathcal{A}| \cdot \text{nec}_1(V_x)^2$ entries, we can compute \mathcal{C} in time $O(|\mathcal{A}| \cdot \text{nec}_1(V_x)^2 \cdot n^2)$. Furthermore, by Lemma 3.1, an optimal basis of \mathcal{C} can be computed in time $O(|\mathcal{A}| \cdot \text{nec}_1(V_x)^{2 \cdot (\omega-1)})$. We conclude that \mathcal{B} can be computed in time $O(|\mathcal{A}| \cdot \text{nec}_1(V_x)^{2 \cdot (\omega-1)} \cdot n^2)$. \square

Now to boots up a dynamic programming algorithm P on some rooted layout (T, \mathcal{L}) of G , we can use the function `reduce` to keep the size of the sets of partial solutions we manipulate to $\text{nec}_1(T, \mathcal{L})^2$. To prove the correctness of this new dynamic programming P' , we need to prove that set of solutions computed by P' $(V(G), \emptyset)$ -represents the set of solutions computed by P . For doing so, we need to prove that at each steps of the dynamic programming, we have representatives, *i.e.*, the operations we use preserve the representativity. The following fact states that we can use the union without restriction, it follows directly from Definition 1 of (x, R') -representativity.

Fact 2. *If \mathcal{B} and \mathcal{D} respectively (x, R') -represents \mathcal{A} and \mathcal{C} then $\mathcal{B} \cup \mathcal{D}$ (x, R') -represents $\mathcal{A} \cup \mathcal{C}$.*

The second operation we use in our dynamic programming is \otimes , to use it we need the following notion of compatibility. Suppose now, that x is an internal node of T with a and b as children. Let $R \in \mathcal{R}_{V_x}^d$ and let $d \in \mathbb{N}$ such that $d > 0$.

Definition 2 (d - (R, R') -compatibility). We say that $(A, A') \in \mathcal{R}_{V_a}^d \times \mathcal{R}_{V_a}^d$ and $(B, B') \in \mathcal{R}_{V_b}^d \times \mathcal{R}_{V_b}^d$ are d - (R, R') -compatible if $A \cup B \equiv_{V_x}^d R$, $A' \equiv_{V_a}^d B \cup R'$, and $B' \equiv_{V_b}^d A \cup R'$.

Lemma 3.3. *Let (A, A') and (B, B') d - (R, R') -compatible.*

Let $\mathcal{A}' \subseteq \mathcal{A} \subseteq 2^{V_a}$ such that for all $X \in \mathcal{A}$, $X \equiv_{V_a}^d A$ and let $\mathcal{B}' \subseteq \mathcal{B} \subseteq 2^{V_b}$ such that for all $W \in \mathcal{B}$, $W \equiv_{V_b}^d B$.

If \mathcal{A}' (a, A')-represents \mathcal{A} and \mathcal{B}' (b, B')-represents \mathcal{B} then

$$\otimes(\mathcal{A}', \mathcal{B}') \text{ (} x, R'\text{)-represents } \otimes(\mathcal{A}, \mathcal{B}).$$

Proof. Let $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^1 R'$. We start by proving the following facts

- (a) For every $W \in \mathcal{B}$, we have $W \cup Y \equiv_{\overline{V_a}}^1 A'$,
- (b) For every $X \in \mathcal{A}$, we have $X \cup Y \equiv_{\overline{V_b}}^1 B'$.

Let $W \in \mathcal{B}$. Owing to the d -(R, R')-compatibility, we have $B \cup R' \equiv_{\overline{V_a}}^d A'$. Since $W \equiv_{\overline{V_b}}^d B$ and $\overline{V_b} \subseteq \overline{V_a}$, by Fact 1, we deduce that $W \equiv_{\overline{V_a}}^d B$ and thus $W \cup R' \equiv_{\overline{V_a}}^d A'$. Similarly, as $Y \equiv_{\overline{V_x}}^1 R'$ and $\overline{V_x} \subseteq \overline{V_a}$, we conclude that $W \cup Y \equiv_{\overline{V_a}}^1 A'$. That prove Fact (a). The proof for Fact (b) is symmetric.

Now observe that, by the definitions of **best** and \otimes , we have

$$\text{best}(\otimes(\mathcal{A}, \mathcal{B}), Y) = \max\{\text{best}(\mathcal{A}, W \cup Y) + w(W) \mid W \in \mathcal{B}\}.$$

Since \mathcal{A}' (a, A')-represents \mathcal{A} and by Fact (a), we have

$$\begin{aligned} \text{best}(\otimes(\mathcal{A}, \mathcal{B}), Y) &= \max\{\text{best}(\mathcal{A}', W \cup Y) + w(W) \mid W \in \mathcal{B}\} \\ &= \text{best}(\otimes(\mathcal{A}', \mathcal{B}), Y). \end{aligned}$$

Symmetrically, we deduce from Fact (b) that $\text{best}(\otimes(\mathcal{A}', \mathcal{B}), Y) = \text{best}(\otimes(\mathcal{A}', \mathcal{B}'), Y)$. This stands for every $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^1 R'$. Thus, we conclude that $\otimes(\mathcal{A}', \mathcal{B}')$ (x, R')-represents $\otimes(\mathcal{A}, \mathcal{B})$. \square

4. CONNECTED (CO)-(σ, ρ)-DOMINATING SETS

Let σ and ρ be (non-empty) finite or co-finite subsets of \mathbb{N} . We say that a subset D of $V(G)$ (σ, ρ)-dominates a subset U if for every vertex $u \in U$, $|N(u) \cap D| \in \sigma$, if $u \in D$ and otherwise $|N(u) \cap D| \in \rho$. A subset D of $V(G)$ is a *co- (σ, ρ) -dominating set* if $V(G) \setminus D$ is a *(σ, ρ) -dominating set* of $V(G)$. Here is a formal definition of one of the problem we are interested in.

Connected (σ, ρ) -Dominated Set

Input: A graph G with a weight function $w : V(G) \rightarrow \mathbb{N}$ and $\text{opt} \in \{\max, \min\}$.

Output: A connected (σ, ρ) -dominating set of optimum weight, *i.e.*, minimum if $\text{opt} = \min$ and maximum if $\text{opt} = \max$.

Similarly, one can define **CONNECTED CO- (σ, ρ) -DOMINATING SET**.

Examples of some vertex subset properties expressible as connected (co)- (σ, ρ) -dominating set are shown on Table 2.

Let $d := \max\{d(\sigma), d(\rho)\}$, where $d(\mathbb{N}) := 0$, and for a non-empty set $\mu \subset \mathbb{N}$, $d(\mu) := 1 + \max(\mu)$ if μ is finite, otherwise, $d(\mu) := 1 + \max(\mathbb{N} \setminus \mu)$. As in [5], we use the *d-neighbor equivalence* to characterize the (σ, ρ) -domination of the partial solutions. We will need the following lemma in our proof.

σ	ρ	d	Version	Standard name
\mathbb{N}	\mathbb{N}^+	1	<i>Normal</i>	CONNECTED DOMINATING SET
$\{q\}$	\mathbb{N}	$q+1$	<i>Normal</i>	CONNECTED INDUCED q -REGULAR SUBGRAPH
\mathbb{N}	$\{1\}$	2	<i>Normal</i>	CONNECTED PERFECT DOMINATING SET
$\{0\}$	\mathbb{N}	2	<i>Co</i>	CONNECTED VERTEX COVER

TABLE 2. Examples of connected (co)- (σ, ρ) -dominating sets with $\mathbb{N}^+ = \{1, \dots, +\infty\}$ and $d = \max\{d(\sigma), d(\rho)\}$.

Lemma 4.1 ([5]). *Let $A \subseteq V(G)$.*

Let $X \subseteq A$ and $Y, Y' \subseteq (V(G) \setminus A)$ such that $Y \equiv_A^d Y'$. Then $(X \cup Y)$ (σ, ρ) -dominates A if and only if $(X \cup Y')$ (σ, ρ) -dominates A .

In this section, we present an algorithm solving **CONNECTED (σ, ρ) -DOMINATING SET** with G , opt and w as inputs, its running time is $O(\text{nec}_d(T, \mathcal{L})^4 \cdot \text{nec}_1(T, \mathcal{L})^{2 \cdot (\omega+1)} \cdot n^3)$. The same algorithm, with some little modifications, will be able to solve **CONNECTED CO- (σ, ρ) -DOMINATING SET**.

Definition 3. Let $x \in V(T)$, for all $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$.

We define $\mathcal{A}_x[R, R'] \subseteq V_x$ as follow

$$\mathcal{A}_x[R, R'] := \{X \subseteq V_x \mid X \equiv_{V_x}^d R \text{ and } X \cup R' (\sigma, \rho)\text{-dominates } V_x\}.$$

For each node x of $V(T)$, our algorithm will compute a table \mathcal{D}_x that satisfies the following invariant.

Invariant. For every $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$, the set $\mathcal{D}_x[R, R']$ is a subset of $\mathcal{A}_x[R, R']$ of size at most $\text{nec}_1(T, \mathcal{L})^2$ that (x, R') -represents $\mathcal{A}_x[R, R']$.

Observe that, by the definition of $\mathcal{A}_r[\emptyset, \emptyset]$ and Definition 1 of (x, R') -representativity, if G admits a connected (σ, ρ) -dominating set, then $\mathcal{D}_r[R, R']$ must contain an optimal connected (σ, ρ) -dominating set.

The first step of our algorithm is to compute for each $x \in V(T)$, the sets $\mathcal{R}_{V_x}^d$ and $\mathcal{R}_{V_x}^d$ and a data structure to compute each calls at the functions rep_x^d and $\text{rep}_{V_x}^d$ in times $O(\log(\text{nec}_d(T, \mathcal{L})) \cdot n^2)$. As T has $2n - 1$ nodes, by Lemma 2.1, computing these sets and data structures can be done in time $O(\text{nec}_d(T, \mathcal{L}) \cdot \log(\text{nec}_d(T, \mathcal{L})) \cdot n^3)$.

In order to compute the table of the internal nodes of T , we need the following lemma to compute the table of a node $x \in V(T)$ an internal node of T with a and b as children and let $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$. This lemma uses the notion of compatibility from Definition 2 to give the following equality between $\mathcal{A}_x[R, R']$ and the tables \mathcal{A}_a and \mathcal{A}_b .

Lemma 4.2. *For all $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$, we have*

$$\mathcal{A}_x[R, R'] := \bigcup_{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible}} \bigotimes (\mathcal{A}_a[A, A'], \mathcal{A}_b[B, B'])$$

Proof. The lemma is an implication of the two following claims.

Claim 4.2.1. *For every $X \in \mathcal{A}_x[R, R']$, there exists (A, A') and (B, B') d - (R, R') -compatible such that $X \cap V_a \in \mathcal{A}_a[A, A']$ and $X \cap V_b \in \mathcal{A}_b[B, B']$.*

Proof. Let $X \in \mathcal{A}_x[R, R']$, $X_a := X \cap V_a$ and $X_b := X \cap V_b$.

Let $A := \text{rep}_a^d(X_a)$ and $A' := \text{rep}_{V_a}^d(X_b \cup R')$. Symmetrically, we define $B := \text{rep}_b^d(X_b)$ and $B' := \text{rep}_{V_b}^d(X_a \cup R')$.

We claim that $X_a \in \mathcal{A}_a[A, A']$. As $X \in \mathcal{A}_x[R, R']$, we know, by Definition 2, that $X \cup R' = X_a \cup X_b \cup R'$ is a (σ, ρ) -dominating set of V_x . In particular, $X_a \cup (X_b \cup R')$ (σ, ρ) -dominates V_a . Since $A' \equiv_{V_a}^d X_b \cup R'$, by Lemma 4.1, we conclude that $X_a \cup A'$ (σ, ρ) -dominates V_a . As $A \equiv_{V_a}^d X_a$, we have $X_a \in \mathcal{A}_a[A, A']$. By symmetry, we have $B \in \mathcal{A}_b[B, B']$. It remains to prove that (A, A') and (B, B') are d - (R, R') -compatible.

- By construction, we have $X_a \cup X_b \equiv_{V_x}^d R$. As $A \equiv_{V_a}^d X_a$ and from Fact 1, we have $A \cup X_b \equiv_{V_x}^d R$ and since $B \equiv_{V_b}^d X_b$, we deduce that $A \cup B \equiv_{V_x}^d R$.
- By definition, we have $A' \equiv_{V_a}^d X_b \cup R'$. As $B \equiv_{V_b}^d X_b$ and by Fact 1, we have $A' \equiv_{V_a}^d B \cup R'$. By symmetry, we have $B' \equiv_{V_b}^d R' \cup A$.

Thus, (A, A') and (B, B') are d - (R, R') -compatible. \square

Claim 4.2.2. *For every $X_a \in \mathcal{A}_a[A, A']$ and $X_b \in \mathcal{A}_b[B, B']$ such that (A, A') and (B, B') are d - (R, R') -compatible, we have $X_a \cup X_b \in \mathcal{A}_x[R, R']$*

Proof. Since $X_a \equiv_{V_a}^d A$ and $X_b \equiv_{V_b}^d B$, by Fact 1, we deduce that $X_a \cup X_b \equiv_{V_x}^d A \cup B$. Thus, by the definition of d - (R, R') -compatibility, we have $X_a \cup X_b \equiv_{V_x}^d R$.

It remains to prove that $X_a \cup X_b \cup R'$ (σ, ρ) -dominates V_x . As before, one can check that Fact 1 implies that $X_b \cup R' \equiv_{V_a}^d B \cup R'$. From the Lemma 4.1, we conclude that $X_a \cup X_b \cup R'$ (σ, ρ) -dominates V_a . Symmetrically, we prove that $X_a \cup X_b \cup R'$ (σ, ρ) -dominates V_b . As $V_x = V_a \cup V_b$, we deduce that $X_a \cup X_b \cup R'$ (σ, ρ) -dominates V_x . Hence, we have $X_a \cup X_b \in \mathcal{A}_x[R, R']$. \square

\square

We are now ready to prove the main theorem of this section.

Theorem 4.3. *Given G a n -vertex graph and (T, \mathcal{L}) a rooted layout of G . There exists an algorithm that compute an optimal connected (σ, ρ) -dominating set in time $O(\text{nec}_d(T, \mathcal{L})^4 \cdot \text{nec}_1(T, \mathcal{L})^{2 \cdot (\omega+1)} \cdot n^3)$.*

Proof. The algorithm computes for every node t of T the table \mathcal{D}_x by a bottom-up dynamic programming, starting at the leaves of T .

Let x a leaf of T with $V_x = \{v\}$. Observe that $\mathcal{A}_x[R, R'] \subseteq 2^{V_x} = \{\emptyset, \{v\}\}$. Thus, our algorithm can directly compute $\mathcal{A}_x[R, R']$ and set $\mathcal{D}_x[R, R'] := \mathcal{A}_x[R, R']$. In this case, the invariant trivially holds.

Now let x an internal node with a and b as children such that the invariant holds for a and b . For each $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$, the algorithm computes $\mathcal{D}_x[R, R'] := \text{reduce}(\mathcal{B}_x[R, R'])$, where the set $\mathcal{B}_x[R, R']$ is defined as follow

$$\mathcal{B}_x[R, R'] := \bigcup_{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible}} \bigotimes (\mathcal{D}_a[A, A'], \mathcal{D}_b[B, B']).$$

We start by proving that the set $\mathcal{B}_x[R, R']$ is a subset and a (x, R') -representative of $\mathcal{A}_x[R, R']$. By Lemma 3.3, for each (A, A') and (B, B') d - (R, R') -compatible, we have

$$\bigotimes (\mathcal{D}_a[A, A'], \mathcal{D}_b[B, B']) \text{ } (x, R')\text{-represents } \bigotimes (\mathcal{A}_a[A, A'], \mathcal{A}_b[B, B']).$$

From Fact 2, we conclude that $\mathcal{B}_x[R, R']$ (x, R') -represents $\mathcal{A}_x[R, R']$. From the invariant, we have $\mathcal{D}_a[A, A'] \subseteq \mathcal{A}_a[A, A']$ and $\mathcal{D}_b[B, B'] \subseteq \mathcal{A}_b[B, B']$, for every indexes (A, A') and (B, B') . Thus, from Lemma 4.2, it is clear that by construction, we have $\mathcal{B}_x[R, R'] \subseteq \mathcal{A}_x[R, R']$. It follows that for each $X \in \mathcal{B}_x[R, R']$, we have $X \equiv_{V_x}^d R$ and thus, we can use the function `reduce` on $\mathcal{B}_x[R, R']$.

Now, we are ready to prove the invariant for x . Let $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$. By the Theorem 3.2, $\mathcal{D}_x[R, R']$ is a subset and a (x, R') -representative of $\mathcal{B}_x[R, R']$. Thus $\mathcal{D}_x[R, R']$ is a subset of $\mathcal{A}_x[R, R']$. Observe that the (x, R') -representativity is an equivalence relation, in particular, it is transitive. Consequently, $\mathcal{D}_x[R, R']$ (x, R') -represents $\mathcal{A}_x[R, R']$. From Theorem 3.2, the size of $\mathcal{D}_x[R, R']$ is at most $\text{nec}_1(V_x)^2$. Since $\text{nec}_1(V_x) \leq \text{nec}_1(T, \mathcal{L})$, we conclude that the invariant holds for x .

By induction, the invariant holds for every nodes of T . The correctness of the algorithm follows from the fact that $\mathcal{D}_r[\emptyset, \emptyset]$ (r, \emptyset) -represents $\mathcal{A}_r[\emptyset, \emptyset]$.

Running Time. Let x a node of T . If x is a leaf of T then $|\mathcal{R}_{V_x}^d| \leq 2$ and $|\mathcal{R}_{V_x}^d| \leq d$. Thus, \mathcal{D}_x can be computed in $O(d \cdot n)$. Otherwise, x is an internal node of T with a and b as children. Observe that for each $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d$, the running times to compute $\mathcal{B}_x[R, R']$ and `reduce`($\mathcal{B}_x[R, R']$) are respectively $O(|\mathcal{B}_x[R, R']| \cdot n^2)$ and

$O(|\mathcal{B}_x[R, R']| \cdot \text{nec}_1(T, \mathcal{L})^{2 \cdot (\omega-1)} \cdot n^2)$. Thus, the total running time to compute the table \mathcal{D}_x is

$$O\left(\sum_{(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d} |\mathcal{B}_x[R, R']| \cdot \text{nec}_1(T, \mathcal{L})^{2 \cdot (\omega-1)} \cdot n^2\right).$$

Observe that, by Definition 2, when you fix A, A', B and R' , there exists only one R and one B' such that (A, A') and (B, B') are d - (R, R') -compatible. Thus, there is at most $\text{nec}_d(T, \mathcal{L})^4$ indexes $(A, A'), (B, B')$ and (R, R') such that (A, A') and (B, B') are d - (R, R') -compatible. Moreover, for each (A, A') and (B, B') , the size of $\otimes(\mathcal{D}_a[A, A'], \mathcal{D}_b[B, B'])$ is at most $|\mathcal{D}_a[A, A']| \cdot |\mathcal{D}_b[B, B']| \leq \text{nec}_1(T, \mathcal{L})^4$. Hence, \mathcal{D}_x is computable in $O(\text{nec}_d(T, \mathcal{L})^4 \cdot \text{nec}_1(T, \mathcal{L})^{2 \cdot (\omega+1)} \cdot n^2)$ time. Since T has $2 \cdot n - 1$ nodes, the running time of our algorithm is $O(\text{nec}_d(T, \mathcal{L})^4 \cdot \text{nec}_1(T, \mathcal{L})^{2 \cdot (\omega+1)} \cdot n^3)$. \square

With few modifications, we can easily deduce an algorithm to compute a optimal out-connected (σ, ρ) -dominating set and one to compute a minimum node-weighted steiner tree.

Corollary 4.4. *Given G a n -vertex graph and (T, \mathcal{L}) a rooted layout of G . There exists an algorithm that compute an optimal connected co- (σ, ρ) -dominating set in time $\text{nec}_d(T, \mathcal{L})^4 \cdot \text{nec}(T, \mathcal{L})^{2 \cdot (\omega+3)} \cdot n^3$.*

Proof. We run the same algorithm with a different table. For every vertex $x \in V(T)$, we define the set of indexes of \mathcal{A}_x as $\mathbb{I}_x := \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^d \times \mathcal{R}_{V_x}^1 \times \mathcal{R}_{V_x}^1$.

For all $(R, R', \bar{R}, \bar{R}') \in \mathbb{I}_x$, we define $\mathcal{A}_x[R, R', \bar{R}, \bar{R}'] \subseteq V_x$ as follow

$$\mathcal{A}_x[R, R', \bar{R}, \bar{R}'] := \{X \subseteq V_x \mid X \equiv_{V_x}^1 \bar{R}, (V_x \setminus X) \equiv_{V_x}^d R \text{ and } (V_x \setminus X) \cup R' \text{ } (\sigma, \rho)\text{-dominate } V_x\}.$$

Intuitively, we use \bar{R} and \bar{R}' for the connectivity of the co- (σ, ρ) -dominating set and R and R' for the (σ, ρ) -domination. As for Theorem 4.3, for each node x of $V(T)$, our algorithm will compute a table \mathcal{D}_x that satisfies the following invariant.

Invariant. For every $(R, R', \bar{R}, \bar{R}') \in \mathbb{I}_x$, the set $\mathcal{D}_x[R, R', \bar{R}, \bar{R}']$ is a subset of $\mathcal{A}_x[R, R', \bar{R}, \bar{R}']$ of size at most $\text{nec}_1(T, \mathcal{L})^2$ that (x, \bar{R}') -represents $\mathcal{A}_x[R, R', \bar{R}, \bar{R}']$.

The following claim adapts Lemma 4.2 to the case co- (σ, ρ) -dominating set.

Claim 4.4.1. *Let x an internal node of T with a and b as children.*

For all $(R, R', \bar{R}, \bar{R}') \in \mathbb{I}_x$, we have

$$\mathcal{A}_x[R, R', \bar{R}, \bar{R}'] := \bigcup_{\substack{(A, A'), (B, B') \text{ } d\text{-}(R, R')\text{-compatible} \\ (\bar{A}, \bar{A}'), (\bar{B}, \bar{B}') \text{ } 1\text{-}(\bar{R}, \bar{R}')\text{-compatible}}} \otimes(\mathcal{A}_a[A, A', \bar{A}, \bar{A}'], \mathcal{A}_b[B, B', \bar{A}, \bar{A}'])$$

The proof of this claim follows from the proof of Lemma 4.2. With these modifications, it is straightforward to check that the algorithm of Theorem 4.3 compute an optimal connected co- (σ, ρ) -dominating set of $V(G)$. As for all internal node x of T with a and b as children, one easily check that the running time of this modified algorithm is $\text{nec}_d(T, \mathcal{L})^4 \cdot \text{nec}(T, \mathcal{L})^{2 \cdot (\omega+3)} \cdot n^3$. \square

The problem STEINER TREE ask; given a subset of vertices $K \subseteq V(G)$ called *terminals*; a subset T of minimal weight such that $K \subseteq T \subseteq V(G)$ and $G[T]$ is connected.

Corollary 4.5. *Given G a n -vertex graph, a subset $K \subseteq V(G)$, and (T, \mathcal{L}) a rooted layout of G . There exists an algorithm that compute an minimum node-weighted steiner tree for (G, K) in time $\text{nec}_1(T, \mathcal{L})^{2 \cdot (\omega+3)} \cdot n^3$.*

Proof. Here, we just have to change the definition of the table. Let $x \in V(T)$, for all $(R, R') \in \mathcal{R}_{V_x}^1 \times \mathcal{R}_{V_x}^1$. We define $\mathcal{A}_x[R, R'] \subseteq V_x$ as follow

$$\mathcal{A}_x[R, R'] := \{X \subseteq V_x \mid X \equiv_{V_x}^d R \text{ and } K \cap V_x \subseteq X\}.$$

With this definition and by Definition 1 of (x, R') -representativity, it is straightforward to check that if $\mathcal{D}_r[\emptyset, \emptyset]$ (r, \emptyset) -represents $\mathcal{A}_r[R, R']$, then $\mathcal{D}_r[\emptyset, \emptyset]$ contain an minimum steiner tree of G . The running time comes from the running time of Theorem 4.3 with $d = 1$. \square

5. MAXIMUM INDUCED TREE

In this section, we present an algorithm for the problem MAXIMUM INDUCED TREE, which consists in finding a subset T of $V(G)$ of maximum weight such that $G[T]$ is a tree. The same algorithm can be used to compute a *minimum feedback vertex set*, i.e., a subset $S \subseteq V(G)$ of minimum weight such that $G \setminus S$ is a forest. Unfortunately, we were not able to obtain an algorithm whose running time is polynomial in n and the neighbor-width. But for the other parameters, by using their respective properties, we get the running time presented in Table 3 which are roughly the same as those in the previous section.

Parameter	Running time	\mathcal{N}
Neighbor-width	$O(\text{nec}_2(T, \mathcal{L})^2 \cdot \text{nec}_1(T, \mathcal{L})^{2(\omega+2)} \cdot \mathcal{N}^2 \cdot n^3)$	$\text{nec}_1(T, \mathcal{L})^{2 \cdot \log_2(\text{nec}(V_x))+2} \cdot 2n$
Mim-width	$O(n^{2 \cdot (\omega+6) \cdot \text{mim}(V_x)+4})$	$2 \cdot n^{2 \cdot \text{mim}(V_x)+1}$
Clique-width	$O(2^{2 \cdot (\omega+5) \cdot \text{cw}(V_x)} \cdot n^4)$	$2^{\text{cw}(V_x)} \cdot 2n$
Rank-width	$O(2^{2 \cdot (\omega+6) \cdot \text{rw}(V_x)^2} \cdot n^4)$	$2^{2 \cdot \text{rw}(V_x)^2} \cdot 2n$
\mathbb{Q} -rank-width	$O(2^{2 \cdot (\omega+6) \cdot \mathbb{Q}\text{-rw}(V_x) \cdot \log_2(2 \cdot \mathbb{Q}\text{-rw}(V_x)+1)} \cdot n)$	$2^{\mathbb{Q}\text{-rw}(V_x) \cdot \log_2(2 \cdot \mathbb{Q}\text{-rw}(V_x)+1)} \cdot 2n$

TABLE 3. Upper bounds on the running time of our algorithm for MAXIMUM INDUCED TREE and \mathcal{N} the number of equivalence classes of the relation \sim defined in Lemma 5.3

As we deal with a maximization problem, we use the notions of Section 3 with opt set to \max . Let x a node of T and $(R, R') \in \mathcal{R}_{V_x}^1 \times \mathcal{R}_{V_x}^2$. The following definition extend the Definition 1 of Section 3 to the acyclicity.

Definition 4 ((x, R') ^{acy}-representativity). For every $\mathcal{A} \subseteq 2^{V(G)}$ and $Y \subseteq V(G)$, we define $\text{best}(\mathcal{A}, Y)^{\text{acy}}$ as follow

$$\text{best}^{\text{acy}}(\mathcal{A}, Y) := \text{opt}\{\text{w}(X) \mid X \in \mathcal{A} \wedge G[X \cup Y] \text{ is a tree}\}.$$

Let $\mathcal{A}, \mathcal{B} \subseteq 2^{V_x}$. We say that \mathcal{B} (x, R') ^{acy}-represents \mathcal{A} if for every $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^2 R'$, we have $\text{best}^{\text{acy}}(\mathcal{A}, Y) = \text{best}^{\text{acy}}(\mathcal{B}, Y)$.

The main ingredient of our algorithm is an adaptation of Theorem 3.2 to the notion of (x, R') ^{acy}-representativity. The following concept is the key of this adaptation.

Definition 5. We say that $\mathcal{A} \subseteq 2^{V_x}$ is R' -consistent, if for all $Y \subseteq \overline{V_x}$ such that $Y \equiv_{\overline{V_x}}^2 R'$ and there exists $F \in \mathcal{A}$ such that $G[F \cup Y]$ is a tree, we have, for all $X \in \mathcal{A}$, either $G[X \cup Y]$ is a tree or $G[X \cup Y]$ is not connected.

The motivation behind this notion comes from the following lemma.

Lemma 5.1. *Let $\mathcal{D} \subseteq \mathcal{A} \subseteq 2^{V_x}$. If \mathcal{A} is R' -consistent and \mathcal{D} (x, R') -represents \mathcal{A} , then \mathcal{D} (x, R') ^{acy}-represents \mathcal{A} .*

Proof. Let $Y \equiv_{\overline{V}_x}^2 R'$. Observe that if for all $X \in \mathcal{A}$, the graph $G[X \cup Y]$ either is not connected or contains a cycle, then both $\text{best}(\mathcal{A}, Y)$ and $\text{best}^{\text{acy}}(\mathcal{A}, Y)$ equals $+\infty$. Assume that there exists $F \in \mathcal{A}$ such that $G[F \cup Y]$ is a tree. Since \mathcal{A} is R' -consistent, for every $X \in \mathcal{A}$, we have $G[X \cup \mathcal{A}]$ is either a tree or is not connected. Thus, by Definition 1 of best , we have $\text{best}^{\text{acy}}(\mathcal{A}, Y) = \text{best}(\mathcal{A}, Y)$. As $\mathcal{D} \subseteq \mathcal{A}$, we have also $\text{best}^{\text{acy}}(\mathcal{D}, Y) = \text{best}(\mathcal{D}, Y)$. We conclude by observing that $\text{best}(\mathcal{D}, Y) = \text{best}(\mathcal{D}, Y)$ because \mathcal{D} (x, R')-represents \mathcal{A} . \square

The key to compute a small $(x, R')^{\text{acy}}$ -representative of a set \mathcal{A} is to decompose \mathcal{A} into a small number of R' -consistent sets without losing crucial informations, *i.e.*, the forest of \mathcal{A} that could potentially be completed into a tree. The following gives a formal definition of these crucial informations.

Definition 6 (R' -important). We say that $X \subseteq V_x$ is R' -important if there exists $Y \equiv_{\overline{V}_x}^2 R'$ such that $G[X \cup Y]$ is a tree, otherwise, we say that X is R' -unimportant.

Trivially, any set obtained from a set \mathcal{A} by removing unimportant sets is a $(x, R')^{\text{acy}}$ -representative of \mathcal{A} . The following lemma gives some properties on important sets, any set that does not respect these properties can safely be removed. Moreover, these properties are the key to obtain the running times of Table 3. At this point, we need to introduce the following notations. For every $X \subseteq V_x$, we define $X^0 := \{v \in X \mid N(v) \cap R' = \emptyset\}$, $X^1 := \{v \in X \mid |N(v) \cap R'| = 1\}$, and $X^{2+} := \{v \in X \mid |N(v) \cap R'| \geq 2\}$. Observe that for every $Y \equiv_{\overline{V}_x}^2 R'$, the vertices in X^0 have no neighbor in Y , those in X^1 have exactly one neighbor in Y and those in X^{2+} have at least 2 neighbors in Y .

Lemma 5.2. *If $R' \neq \emptyset$ and $X \subseteq V_x$ is R' -important, then $G[X]$ is a forest and the following properties are satisfied :*

- (1) *For all $C \in \text{cc}(X)$, we have $C \setminus X^0 \neq \emptyset$,*
- (2) *For all $a, b \in X$ such that $a \neq b$ and $a \equiv_{\overline{V}_x}^1 b$, either $a, b \in X^0$ or we have $a, b \in X^1$ and a and b are not connected in $G[X]$.*
- (3) *We have $|X^{2+}| \leq 2 \cdot \text{mim}(V_x)$.*

Proof. Obviously, an important set must induced a forest. We prove the property (1) and (2) by making the following observations. For every $X \subseteq V_x$ and $Y \subseteq \overline{V}_x$ such that $Y \equiv_{\overline{V}_x}^2 R'$, we have:

- If X has a connected component C such that $N(C) \cap R' = \emptyset$, then $G[X \cup Y]$ is not connected. As $Y \equiv_{\overline{V}_x}^2 R'$, we have also $N(C) \cap Y = \emptyset$. Since $R' \neq \emptyset$ and thus $Y \neq \emptyset$, the graph $G[X \cup Y]$ is not connected.
- If there exists two different vertices a, b in X^{2+} such that $a \equiv_{\overline{V}_x}^1 b$, then $G[X \cup Y]$ contains a cycle. Indeed, both a and b have at least two neighbors in Y because $Y \equiv_{\overline{V}_x}^2 R'$ and $a, b \in X^{2+}$. Since $N(a) \cap \overline{V}_x = N(b) \cap \overline{V}_x$, we deduce that $G[X \cup Y]$ admits a cycle of length four.
- If there exists two different vertices a, b in X^1 such that a and b are connected in $G[X]$ and $a \equiv_x^1 b$, then $G[X \cup Y]$ contains a cycle. Indeed, $Y \equiv_{\overline{V}_x}^2 R'$ and $a, b \in X^1$, both a and b have neighbor in Y . Since $a \equiv_x^1 b$, this neighbor is the same for a and b . Because they are connected in $G[X]$, we conclude that $G[X \cup Y]$ contains a cycle.

To prove that X satisfied the property (3), we consider a important set X . Since X is important, there exists $Y \equiv_{\overline{V}_x}^2 R'$ such that $G[X \cup Y]$ is a tree. Since the edges between X and Y induce a forest, the edges between X^{2+} and Y also. We call F the forest induced by the edges between X^{2+} and Y . We want to prove that the size of X^{2+} is at most $2 \cdot \text{mim}(F)$, where $\text{mim}(F)$ is the size of a maximum induced

matching of F . Since F is an induced subgraph of the bipartite graph between V_x and \overline{V}_x , this is enough to prove the property (3).

Observe that F has no leaf contained in X^{2+} , thus we can conclude with the following claim.

Claim 5.2.1. *Let F a forest that are the disjoint union of two independent sets A and B . If A has no isolated vertex and every connected component of F has at most one leaves in A , then the size of A is at most $2 \cdot \text{mim}(F)$, i.e., the size of a maximum induced matching of F .*

Proof. In order to prove the claim, we prove by induction that F admits a *good* bipartition (A_1, A_2) of A , i.e., a bipartition such that for every $i \in \{1, 2\}$, every vertex $v \in A_i$ has at least one neighbor y such that $N(y) \cap A_i = v$. If $|A| = 1$ then it is trivial. Assume that $|A| \geq 2$. Take v a vertex of A such that v is a leaf if F has a leaf in A . Let y be neighbor of v .

Observe that every connected component of $F \setminus \{v, y\}$ has at most one leaf in A . Obviously, it is true for the connected components of $F \setminus \{v, y\}$ that are not connected to y in F . Let C the connected components of $F \setminus \{v, y\}$ connected to y in F . As C is connected to v in F , they are no leaf of F in $A \cap C$, otherwise, v would be a leaf and F would contain a connected component with two leaves. Since F is a forest, y has exactly one neighbor in C . Thus, $C \cap A$ contains at most one leaf in $F \setminus \{v, y\}$.

Obviously, $A \setminus v$ has no isolated vertex in $F \setminus \{v, y\}$. By induction hypothesis, $F \setminus \{v, y\}$ admits a good bipartition (A_1, A_2) of $A \setminus v$. We can suppose *w.l.o.g.*, that every neighbor of y in F (at the exception of v) is in A_2 . Indeed, for every connected component C of $F \setminus \{v, y\}$, if (C_1, C_2) is a good bipartition of $A \cap C$ then (C_2, C_1) is also a good bipartition. We conclude by observing that $(A_1 \cup v, A_2)$ is a good bipartition of A *w.r.t.* F . \square

\square

The next lemma shows how to decompose a set \mathcal{A} into a small number of R' -consistent sets whose union $(x, R')^{\text{acy}}$ -represents \mathcal{A} . Even if some parts of the proof is specific to each parameters, the ideas are roughly the same. Intuitively, to obtain this decomposition, we start by removing from \mathcal{A} the sets that does not respect the properties of Lemma 5.2. After that, we partition \mathcal{A} with the help of an equivalence relation and we prove that the number of equivalence classes respect the upper bound described in Table 3 by using the properties of Lemma 5.2.

Lemma 5.3. *Let $\mathcal{A} \subseteq 2^{V_x}$. There exists \mathcal{N} disjoint sets $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{N}} \subseteq \mathcal{A}$ such that $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_{\mathcal{N}}$ $(x, R')^{\text{acy}}$ -represents \mathcal{A} and for each $1 \leq i \leq \mathcal{N}$, we have \mathcal{A}_i is R' -consistent. Moreover, we can compute $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{N}}$ in time $O(|\mathcal{A}| \cdot \mathcal{N} \cdot n^2)$ and \mathcal{N} respects the upper bounds of described in Table 3.*

Proof. If $R' = \emptyset$ then for the same reasons as in the proof of Theorem 3.2, it is sufficient to return $\mathcal{B} = \{F\}$ where F is a tree of \mathcal{A} of maximum weight.

Assume now that $R' \neq \emptyset$. We begin by defining the equivalence relation and by proving that two equivalent sets form a R' -consistent set. We recall that M_{V_x} is the $V_x \times \overline{V}_x$ adjacency matrix of the cut (V_x, \overline{V}_x) over \mathbb{Q} . For a subset $X \subseteq V_x$, let $\sigma(X)$ be the sum of the row vectors of M_{V_x} corresponding to X . We define the equivalence relation \sim such that $F_1 \sim F_2$ if we have $\sigma(F_1^{2+}) = \sigma(F_2^{2+})$ and $|E(G[F_1])| - |F_1 \setminus F_1^1| = |E(G[F_2])| - |F_2 \setminus F_2^1|$.

Claim 5.3.1. *If $F_1 \sim F_2$, then $\{F_1, F_2\}$ is R' -consistent.*

Proof. In this proof, for $X \subseteq V_x$ and $Y \subseteq \overline{V}_x$, we denote by $E(X, Y) := \{uv \mid u \in X \wedge v \in Y\}$, i.e., the edges between X and Y .

Let $Y \equiv_{\overline{V}_x}^2 R'$ and $F_1, F_2 \in \mathcal{A}$ such that $F_1 \sim F_2$.

By Definition 5 of consistency, it is enough to prove that if $G[F_1 \cup Y]$ is a tree and $G[F_2 \cup Y]$ is connected, then $G[F_2 \cup Y]$ is a tree.

Observe that $G[F_1 \cup Y]$ is tree if and only if $G[F_1 \cup Y]$ is connected and $|E(G[F_1 \cup Y])| = |F_1 \cup Y| - 1$. By decomposing the different terms, we have $|E(G[F_1 \cup Y])| = |F_1 \cup Y| - 1$ if and only if

$$(1) \quad |E(G[Y])| + |E(F_1^{2+}, Y)| + |E(F_1^1, Y)| + |E(G[F_1])| = |F_1 \setminus F_1^1| + |F_1^1| + |Y| - 1$$

Since every vertex in F_1^1 has exactly one neighbor in R' and $Y \equiv_{\overline{V}_x}^1 R'$, we have $|E(F_1^1, Y)| = |F_1^1|$. Thus, the equation (1) is equivalent to

$$(2) \quad |E(F_1^{2+}, Y)| + |E(G[F_1])| - |F_1 \setminus F_1^1| = |Y| - |E(G[Y])| - 1$$

Since $F_1 \sim F_2$, we have $|E(G[F_1])| - |F_1 \setminus F_1^1| = |E(G[F_2])| - |F_2 \setminus F_2^1|$. Moreover, owing to $\sigma(F_1^{2+}) = \sigma(F_2^{2+})$, we deduce that $|E(F_1^{2+}, Y)| = |E(F_2^{2+}, Y)|$. We conclude that the equation (2) is equivalent to

$$(3) \quad |E(F_2^{2+}, Y)| + |E(G[F_2])| - |F_2 \setminus F_2^1| = |Y| - |E(G[Y])| - 1$$

With the same arguments to prove that (1) is equivalent to (2), we can show that (3) is equivalent to $|E(F_2 \cup Y)| = |F_2 \cup Y| - 1$. By assumption, $G[F_2 \cup Y]$ is connected and thus we conclude that $G[F_2 \cup Y]$ is a tree. \square

We are now ready to decompose \mathcal{A} . Before going into the details for each parameters. We start by removing from \mathcal{A} all the sets that does not induced a forest or that does not respect the properties (1) and (2) of Lemma 5.2. Trivially, this can be done in time $O(|\mathcal{A}| \cdot n^2)$. Since, these sets are unimportant, we keep a (x, R') -acy-representatives of \mathcal{A} . Observe that we can do the same with the sets that does not respect the property (3) or a weaker property. In the following, we explain how to end up with the upper bounds of Table 3 for each width separately.

For all the width, we use implicitly the following observation which follows from the removal of all the sets in \mathcal{A} that does not induce a forest.

Observation 5.3.2. *For all $F \in \mathcal{A}$, we have $-n \leq |E(G[F])| - |F \setminus F^1| \leq n$.*

Concerning the clique-width. The property (2) implies that for all $F \in \mathcal{A}$, the neighborhood in \overline{V}_x of the vertices in F^{2+} are pairwise distinct. By definition of clique-width, there is at most $\text{cw}(V_x)$ different rows in M_{V_x} . As two vertices with the same neighborhood in \overline{V}_x have the same row in M_{V_x} , we conclude that there is at most $2^{\text{cw}(V_x)}$ possible values for $\sigma(X^{2+})$. Thus, we have $\mathcal{N} \leq 2^{\text{cw}(V_x)} \cdot 2n$.

Concerning the mim-width. We remove from \mathcal{A} all the sets that does not respect the property (3). Let $F \in \mathcal{A}$. As $|F^{2+}| \leq 2 \cdot \text{mim}(V_x)$, there is at most $n^{2 \cdot \text{mim}(V_x)}$ possible values for F^{2+} . Hence, \mathcal{N} is upper bounded by $2 \cdot n^{2 \cdot \text{mim}(V_x) + 1}$.

Concerning the neighbor-width. Observe that $\text{mim}(V_x) \leq \log(\text{nec}_1(V_x))$. Indeed, an induced matching of size k generates 2^k different neighborhood, *i.e.*, $2^{\text{mim}(V_x)} \leq \text{nec}_1(V_x)$. Thus, we can remove all the set $F \in \mathcal{A}$ such that $|F^{2+}| > 2 \cdot \log_2(\text{nec}_1(V_x))$. Since there is at most $\text{nec}_1(V_x)$ different rows in M , we deduce that $\sigma(F^{2+})$ can take at most $\text{nec}_1(V_x)^{2 \cdot \log_2(\text{nec}_1(V_x)) + 1}$ values. Hence, we have $\mathcal{N} \leq \text{nec}_1(V_x)^{2 \cdot \log_2(\text{nec}_1(V_x))} \cdot 2n$.

Concerning the rank-width. Observe that $\text{mim}(V_x) \leq \text{rw}(V_x)$ because the matrix associated to an induced matching is the identity matrix. Thus, M has rank at least $\text{mim}(V_x)$. Consequently, we can remove from \mathcal{A} all the set F such that $|F^{2+}| > 2 \cdot \text{rw}(V_x)$. We know that there is at most $2^{\text{rw}(V_x)}$ different rows in M . We conclude that $\mathcal{N} \leq 2^{2 \cdot \text{rw}(V_x)^2} \cdot 2n$.

Concerning the \mathbb{Q} -rank-width. Similarly to the rank-width, we have $\text{mim}(V_x) \leq \mathbb{Q}\text{-rw}(V_x)$. Thus, we can remove all the set $F \in \mathcal{A}$ such that $|F^{2^+}| > 2 \cdot \mathbb{Q}\text{-rw}(V_x)$. Let $F \in \mathcal{A}$. As proved in [13], there is at most $2^{\mathbb{Q}\text{-rw}(V_x) \cdot \log(2 \cdot \mathbb{Q}\text{-rw}(V_x) + 1)}$ possible value for $\sigma(F^{2^+})$. Indeed, observe that the values associated to $\mathbb{Q}\text{-rw}(V_x)$ columns in $\sigma(F^{2^+})$ entirely determined $\sigma(F^{2^+})$. Since $|F^{2^+}| \leq 2 \cdot \mathbb{Q}\text{-rw}(V_x)$, the values in $\sigma(F^{2^+})$ are between zero and $2 \cdot \mathbb{Q}\text{-rw}(V_x)$. It follows that there is at most $(2 \cdot \mathbb{Q}\text{-rw}(V_x) + 1)^{\mathbb{Q}\text{-rw}(V_x)} = 2^{\mathbb{Q}\text{-rw}(V_x) \cdot \log(2 \cdot \mathbb{Q}\text{-rw}(V_x) + 1)}$ possibilities for $\sigma(F^{2^+})$. We conclude that $\mathcal{N} \leq 2^{\mathbb{Q}\text{-rw}(V_x) \cdot \log(2 \cdot \mathbb{Q}\text{-rw}(V_x) + 1)} \cdot 2n$.

It remains to prove the running time. Observe that we can decide $F_1 \sim F_2$ in time $O(n^2)$. Hence, we can compute the equivalence classes of \mathcal{A} in times $O(|\mathcal{A}| \cdot \mathcal{N} \cdot n^2)$. \square

We are now ready to give an adaptation of Theorem 3.2 to the notion of (x, R') -acy-representativity.

Theorem 5.4. *Given $\mathcal{A} \subseteq 2^{V_x}$ such that for each $F \in \mathcal{A}$, we have $F \equiv_x^1 R$. There exists an algorithm $\text{reduce}^{\text{acy}}$ that outputs in time $O((\text{nec}(V_x)^{2 \cdot (\omega-1)} + \mathcal{N}) \cdot |\mathcal{A}| \cdot n^2)$, a subset $\mathcal{B} \subseteq \mathcal{A}$ such that \mathcal{B} (x, R') -acy-represents \mathcal{A} and $|\mathcal{B}| \leq \mathcal{N} \cdot \text{nec}(V_x)^2$.*

Proof. By Lemma 5.3, we can compute in time $O(|\mathcal{A}| \cdot \mathcal{N} \cdot n^2)$ a collection of disjoint sets $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{N}} \subseteq \mathcal{A}$ such that $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_{\mathcal{N}}$ (x, R') -acy-represents \mathcal{A} and for each $1 \in [\mathcal{N}]$, the set \mathcal{A}_i is R' -consistent.

Since $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{N}} \subseteq \mathcal{A}$, for each $i \in [\mathcal{N}]$ and each set $F \in \mathcal{A}_i$, we have $F \equiv_x^1 R$. Thus, we can apply Theorem 3.2 to compute for each $i \in [\mathcal{N}]$ the set $\text{reduce}(\mathcal{A}_i) := \mathcal{B}_i$. For each $i \in [\mathcal{N}]$, we have $\mathcal{B}_i \subseteq \mathcal{A}_i$ such that \mathcal{B}_i (x, R') -represents \mathcal{A}_i and $|\mathcal{B}_i| \leq \text{nec}_1(V_x)^2$. As \mathcal{A}_i is R' -consistent, by Lemma 5.1, we have \mathcal{B}_i (x, R') -acy-represents \mathcal{A}_i .

Let $\mathcal{B} := \mathcal{B}_1 \cup \dots \cup \mathcal{B}_{\mathcal{N}}$. Since $\mathcal{A}_1 \cup \dots \cup \mathcal{A}_{\mathcal{N}}$ (x, R') -acy-represents \mathcal{A} , we conclude that \mathcal{B} (x, R') -acy-represents \mathcal{A} . Obviously, we have $|\mathcal{B}| \leq \mathcal{N} \cdot \text{nec}_1(V_x)^2$.

It remains to prove the running time. By Theorem 3.2, we can compute $\mathcal{B}_1, \dots, \mathcal{B}_{\mathcal{N}}$ in time $O(|\mathcal{A}_1 \cup \dots \cup \mathcal{A}_{\mathcal{N}}| \cdot \text{nec}(V_x)^{2 \cdot (\omega-1)} \cdot n^2)$. Since the sets $\mathcal{A}_1, \dots, \mathcal{A}_{\mathcal{N}}$ are pairwise disjoint and subsets of \mathcal{A} , we deduce that $|\mathcal{A}_1 \cup \dots \cup \mathcal{A}_{\mathcal{N}}| \leq |\mathcal{A}|$. That proves the running time and conclude this proof. \square

We are now ready to present our algorithm to solve MAXIMUM INDUCED TREE. The algorithm follows the same ideas as the algorithms of the previous section, we just use $\text{reduce}^{\text{acy}}$ instead of reduce .

Theorem 5.5. *Given G a n -vertex graph and (T, \mathcal{L}) a rooted layout of G .*

There exists an algorithm that compute a maximum induced tree of G in time $O(\text{nec}_2(T, \mathcal{L})^2 \cdot \text{nec}_1(T, \mathcal{L})^{2(\omega+2)} \cdot \mathcal{N}^2 \cdot n^3)$.

Proof. For each node $x \in T$ and for all $(R, R') \in \mathcal{R}_{V_x}^1 \times \mathcal{R}_{V_x}^2$, we define $\mathcal{A}_x[R, R'] \subseteq V_x$ as follow

$$\mathcal{A}_x[R, R'] := \{X \subseteq V_x \mid X \equiv_x^1 R\}.$$

For each node x of $V(T)$, our algorithm will compute a table \mathcal{D}_x that satisfies the following invariant.

Invariant. For every $(R, R') \in \mathcal{R}_{V_x}^1 \times \mathcal{R}_{V_x}^2$, the set $\mathcal{D}_x[R, R']$ is a subset of $\mathcal{A}_x[R, R']$ of size at most $\mathcal{N} \cdot \text{nec}_1(V_x)^2$ that (x, R') -acy-represents $\mathcal{A}_x[R, R']$.

Observe that by Definition of (x, R') -acy-representativity, $\mathcal{D}_x[\emptyset, \emptyset]$ contains a maximum induced tree of G .

The algorithm computes for every node t of T the table \mathcal{D}_x by a bottom-up dynamic programming, starting at the leaves of T .

Let x a leaf of T with $V_x = \{v\}$. Observe that $\mathcal{A}_x[R, R'] \subseteq 2^{V_x} = \{\emptyset, \{v\}\}$. Thus, our algorithm can directly compute $\mathcal{A}_x[R, R']$ and set $\mathcal{D}_x[R, R'] := \mathcal{A}_x[R, R']$. In this case, the invariant trivially holds.

Now, take x an internal node of T with a and b as children such that the invariant holds for a and b . For each $(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d$, the algorithm computes $\mathcal{D}_x[R, R'] := \text{reduce}(\mathcal{B}_x[R, R'])$, where the set $\mathcal{B}_x[R, R']$ is defined as follow

$$\mathcal{B}_x[R, R'] := \bigcup_{(A, A'), (B, B') \text{ 2-}(R, R')\text{-compatible}} \bigotimes (\mathcal{D}_a[A, A'], \mathcal{D}_b[B, B']).$$

One easily checks that Lemma 3.3 and Fact 2 holds also for the notion of $(x, R')^{\text{acy}}$ -representativity. Thus, with the same argumentation as in the proof of Theorem 4.3, it follows that $\mathcal{D}_x[R, R']$ is a subset and a $(x, R')^{\text{acy}}$ -representative of $\mathcal{A}_x[R, R']$. By Theorem 5.4, we have $|\mathcal{D}_x[R, R']| \leq \mathcal{N} \cdot \text{nec}_1(T, \mathcal{L})^2$.

Consequently, the invariant holds for x and by induction, it holds for all vertex of T . The correctness of the algorithm follows.

Running Time. As for the proof of Theorem 4.3, the bottleneck is the call at the function $\text{reduce}^{\text{acy}}$. For each node $x \in V(T)$, the total running time to compute the table \mathcal{D}_x is

$$O\left(\mathcal{N} + \sum_{(R, R') \in \mathcal{R}_{V_x}^d \times \mathcal{R}_{\overline{V_x}}^d} |\mathcal{B}_x[R, R']| \cdot \text{nec}_1(T, \mathcal{L})^{2 \cdot (\omega-1)} \cdot n^2\right).$$

By Definition 2, when you fix A, A', B and R' , there exists only one R and one B' such that (A, A') and (B, B') are 2- (R, R') -compatible. Thus, there is at most $\text{nec}_2(T, \mathcal{L})^2 \cdot \text{nec}_1(T, \mathcal{L})^2$ indexes $(A, A'), (B, B')$ and (R, R') such that (A, A') and (B, B') are 2- (R, R') -compatible. Moreover, by the invariant, for each (A, A') and (B, B') , the size of $\bigotimes (\mathcal{D}_a[A, A'], \mathcal{D}_b[B, B'])$ is at most $|\mathcal{D}_a[A, A']| \cdot |\mathcal{D}_b[B, B']| \leq \mathcal{N}^2 \cdot \text{nec}_1(T, \mathcal{L})^4$. Hence, the running time to compute \mathcal{D}_x is

$$O(\text{nec}_2(T, \mathcal{L})^2 \cdot \text{nec}_1(T, \mathcal{L})^{2(\omega+2)} \cdot \mathcal{N}^2 \cdot n^2).$$

The total running time of our algorithm follows from the fact that T has $2n - 1$ nodes. \square

As corollary, we can solve the problem FEEDBACK VERTEX SET, which consist in finding the minimum set of vertices whose removal makes G acyclic.

Corollary 5.6. *Given G a n -vertex graph and (T, \mathcal{L}) a rooted layout of G .*

There exists an algorithm that compute a minimum feedback vertex set of G in time $O((2 \cdot \text{nec}_2(T, \mathcal{L}))^2 \cdot (2 \cdot \text{nec}_1(T, \mathcal{L}))^{2(\omega+2)} \cdot \mathcal{N}^2 \cdot n^3)$.

Proof. It is well-know that the complementary of a minimum feedback vertex set is a maximum induced forest. Thus, it is enough to compute a maximum induced forest of G .

The first step of this proof is to construct a graph G' from G in time $O(n^2)$ such that G has a maximum induced forest of weight W iff G' has a maximum induced tree of weight W . The second step is to construct from (T, \mathcal{L}) a layout (T', \mathcal{L}') of G' in time $O(n^2)$ such that $\text{nec}(T', \mathcal{L}') \leq 2 \cdot \text{nec}(T, \mathcal{L})$, $\text{cw}(T', \mathcal{L}') \leq 2 \cdot \text{cw}(T, \mathcal{L})$ and for every $k \in \{\text{mim}, \text{rw}, \text{Q-rw}\}$, we have $k(T', \mathcal{L}') \leq k(T, \mathcal{L})$.

This is sufficient since we can run our algorithm to find a MAXIMUM INDUCED TREE of G' and thus a maximum induced forest of G . The running time follows from Theorem 5.5 and by Lemma 5.3, by using the inequalities concerning (T, \mathcal{L}) and (T', \mathcal{L}') .

We construct G' as follow. Let $V(G) = \{v_1, \dots, v_n\}$. The vertex set of G' is $V(G) \cup \{v_0\} \cup \{v'_1, \dots, v'_n\}$. In G' , the vertices of $V(G)$ have the same weight as in

G and the weight of the vertices v_0, v'_1, \dots, v'_n is null. Finally, the edge set of G' is defined as follow

$$E(G') := E(G) \cup \{v_i v'_i, v_0 v'_i \mid 1 \leq i \leq n\}.$$

We claim that G has an induced forest of weight W if and only if G' has an induced tree of weight W . Let F an induced forest of G of weight W . Let $\text{cc}(F) = \{C_1, \dots, C_k\}$. For each $j \in [k]$, let $v_{i_j} \in C_j$. Let $T = F \cup \{v'_{i_1}, \dots, v'_{i_k}\} \cup v_0$. By construction, one easily checks that T is an induced tree of G' of weight W .

Now, let T an induced tree of G' of weight W . Obviously $T \cap V(G)$ induced a forest and since the weight of the vertices in $V(G') \setminus V(G)$ is null, the weight of $T \cap V(G)$ is W .

We construct (T', \mathcal{L}') directly from (T, \mathcal{L}) by doing the following. We transform each leaf x of T with $V_x = \{v_i\}$ into an internal node by connecting two nodes to x associated though \mathcal{L}' to the vertices v_i and v'_i . Finally, we add a node r' connected to the root of T and a leaf associated by \mathcal{L}' to v_0 .

We now prove that $\text{nec}(T', \mathcal{L}') \leq 2 \cdot \text{nec}(T, \mathcal{L})$. Let x a node of T' . If x is not also a node of T then, by construction, x is either a leaf, a node adjacent to the leaves of T or the root of T' . In this case, the neighbor-width of x are either 1 or 2.

Now, assume that x is also a node of T . Let $V_x \subseteq V(G)$ the vertex set associated to x in (T, \mathcal{L}) . By construction, x is associated in (T', \mathcal{L}') to the set $V'_x := V_x \cup \{v' \in V(G') \mid v \in V_x\}$. Let $\mathcal{S} := \{N(X) \cap \overline{V_x} \mid X \subseteq V_x\}$ and $\mathcal{S}' := \{N_{G'}(X) \cap \overline{V'_x} \mid X \subseteq V'_x\}$. By definition of the neighbor-width, we have $\text{nec}(V'_x) = |\mathcal{S}'|$ and $\text{nec}(V_x) = |\mathcal{S}|$. One easily checks that $\mathcal{S}' = \mathcal{S} \cup \{Y \cup v_0 \mid Y \in \mathcal{S}\}$. Thus, we have $|\mathcal{S}'| \leq 2 \cdot |\mathcal{S}|$ and then $\text{nec}(T', \mathcal{L}') \leq 2 \cdot \text{nec}(T, \mathcal{L})$.

We deduce the other inequalities from the Figure 1 describing the adjacency matrix between V'_x and $V(G') \setminus V'_x$.

	$V(G) \setminus V_x$	v_0	$V' \setminus V'_x$
$V' \cap V'_x$	0	$\begin{matrix} \vdots \\ 1 \\ \vdots \end{matrix}$	0
V_x	M	0	

FIGURE 1. The adjacency matrix between V'_x and $V(G') \setminus V'_x$. Where $V' = \{v'_1, \dots, v'_n\}$ and M is the adjacency matrix between V_x and $V(G) \setminus V_x$.

□

6. CONCLUDING REMARKS

In this paper, we provide a framework based on the rank-based approach and the neighbor equivalence to obtain efficient algorithms for graph with structured neighborhood and connectivity problems with locally checkable properties such as CONNECTED DOMINATING SET or NODE WEIGHTED STEINER TREE. The algorithmic consequences we get for clique-width, rank-width and mim-width confirm the importance of the neighbor equivalence for these parameters.

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