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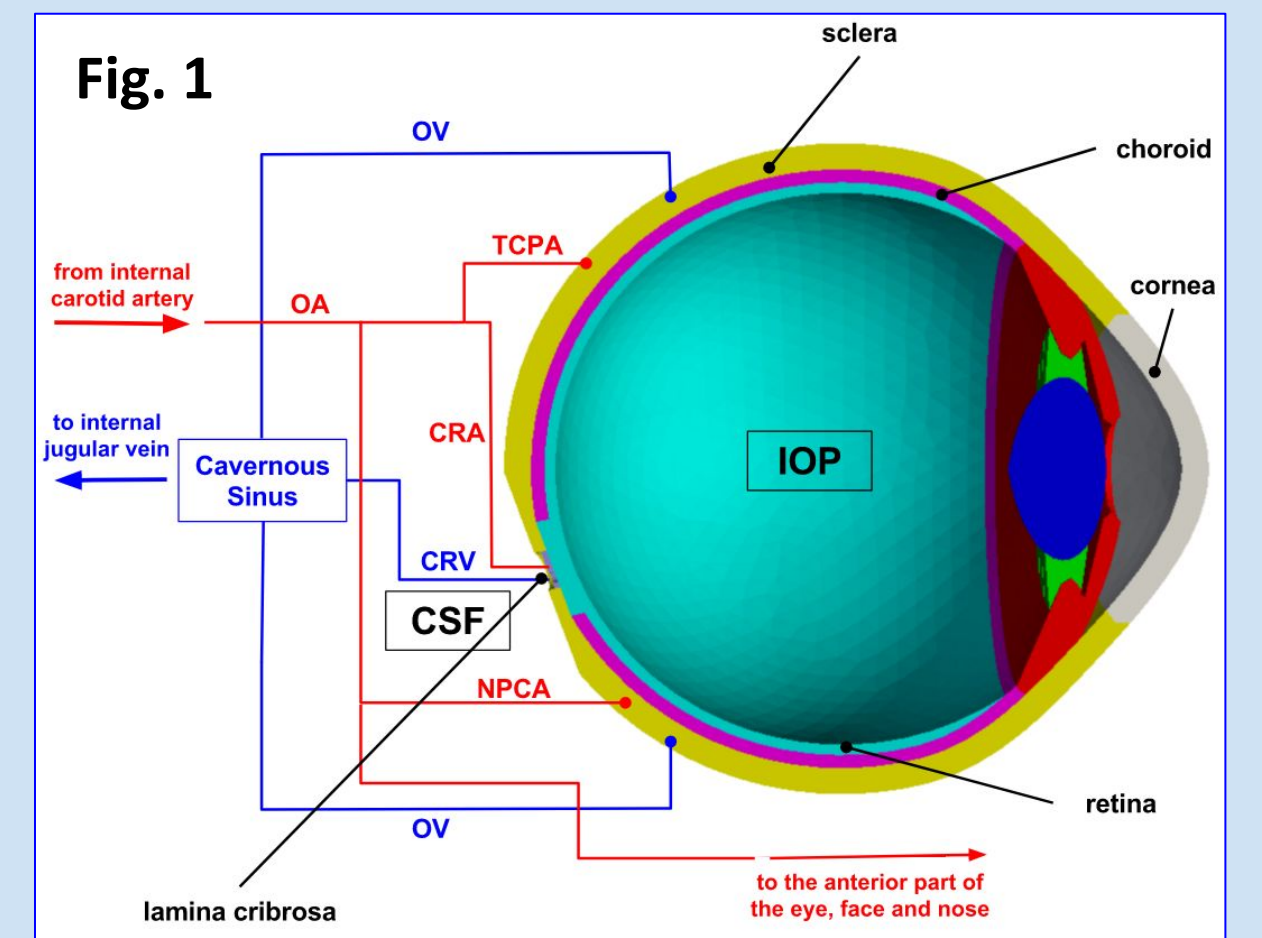
# Unconditionally stable operator splitting method for a multiscale application in ophthalmology

## 1. Motivation

In the context of ophthalmology, we are developing a multiscale **Ocular Mathematical Virtual Simulator (O-MVS<sup>[1]</sup>)**, which combines a three-dimensional poro-elastic system, representing the biomechanics and hemodynamics of the ocular posterior tissues, coupled with a lumped-parameter circuit that describes the blood circulation in the major vessels in the back of the eye.

The **multiscale** approach provides:

- a rough description of the whole system + 3D zoom in a specific region of interest;
- a reliable method for computing correct boundary conditions at the artificial boundaries of a region of interest.



## 2. Porous media 3d-0d model

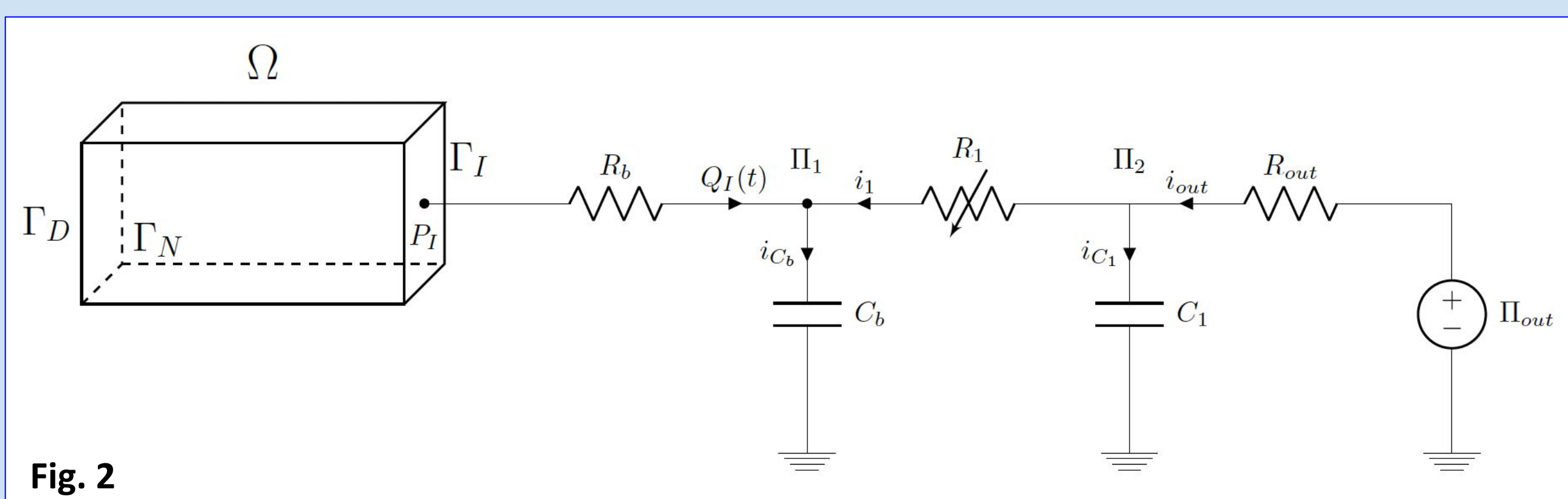


Fig. 2

### Darcy problem

$$\begin{cases} \frac{\partial p}{\partial t} + \nabla \cdot \underline{u} = f & \text{in } \Omega \times (0, T) \\ \underline{k} \nabla p + \underline{u} = \underline{0} & \text{in } \Omega \times (0, T) \\ \underline{u} \cdot \underline{n} = g_N & \text{on } \Gamma_N \times (0, T) \\ p = g_D & \text{on } \Gamma_D \times (0, T) \\ \int_{\Gamma_I} \underline{u} \cdot \underline{n} = Q_I & \text{on } \Gamma_I \times (0, T) \\ p|_{\Gamma_I} = P_I & \text{on } \Gamma_I \times (0, T) \\ Q_I = \frac{P_I - \Pi_1}{R_b} & \text{on } \Gamma_I \times (0, T) \end{cases}$$

### Systemic circuit

$$\begin{cases} C \frac{d\Pi}{dt} + \underline{A} \Pi = \underline{s} + \underline{b} & \text{in } (0, T) \\ C = \begin{bmatrix} C_b & 0 \\ 0 & C_1 \end{bmatrix} & \underline{A} = \begin{bmatrix} \frac{1}{R_b} + \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_{out}} \end{bmatrix} \\ \underline{s} = \begin{bmatrix} 0 \\ \frac{\Pi_{out}}{R_{out}} \end{bmatrix} & \underline{b} = \begin{bmatrix} \frac{P_I}{R_b} \\ 0 \end{bmatrix} \end{cases}$$

**NB:** the use of Hybridizable Discontinuous Galerkin method for the Darcy problem naturally allows the spatial coupling with the circuit thanks to the Integral Boundary Condition handling.

## 3. Time discretization

### Monolithic approach

- + The coupled problem is treated as a whole;
- more difficult to implement and sometimes gives rise to badly conditioned problems.

### New operator splitting method

- + May allow to use existing solvers already developed for solving sub-problems separately;
- accuracy issues related to the choice of the time step.

Given  $p^n, \Pi^n$  for  $n \geq 0$ , solve

Step 1.

$$\begin{cases} \frac{\partial p}{\partial t} + \nabla \cdot \underline{u} = f & \text{in } \Omega \times (t^n, t^{n+1}) \\ \underline{k} \nabla p + \underline{u} = \underline{0} & \text{in } \Omega \times (t^n, t^{n+1}) \\ C \frac{d\Pi}{dt} = \underline{s}(\Pi, t) & \text{in } (t^n, t^{n+1}) \end{cases}$$

+ interface and boundary conditions

$$p(t^n) = p^n \quad \Pi(t^n) = \Pi^n$$

and then set

$$\begin{aligned} p^{n+\frac{1}{2}} &= p(t^{n+\frac{1}{2}}) \\ \Pi^{n+\frac{1}{2}} &= \Pi(t^{n+\frac{1}{2}}) \\ \underline{u}^{n+\frac{1}{2}} &= \underline{u}(t^{n+\frac{1}{2}}) \end{aligned}$$

**Next step:** comparison with other existing coupling methods.

Sub-steps are designed so that the **energy at the semi-discrete** time level mirrors the behavior of the **energy of the full coupled system** in the spirit of [3].

Step 2.

$$\begin{cases} \frac{\partial p}{\partial t} = 0 & \text{in } \Omega \times (t^n, t^{n+1}) \\ C \frac{d\Pi}{dt} + \underline{A} \Pi = \underline{b}(t) & \text{in } (t^n, t^{n+1}) \\ p(t^n) = p^{n+\frac{1}{2}} & \Pi(t^n) = \Pi^{n+\frac{1}{2}} \end{cases}$$

and then set

$$\begin{aligned} p^{n+1} &= p(t^{n+1}) \quad (= p(t^{n+\frac{1}{2}})) \\ \Pi^{n+1} &= \Pi(t^{n+1}) \\ \underline{u}^{n+1} &= \underline{u}(t^{n+1}) \quad (= \underline{u}(t^{n+\frac{1}{2}})) \end{aligned}$$

## 4. Main features

The proposed operator splitting method

- is based on a **semi-discretization in time** inspired by the energy estimate<sup>[2]</sup>;
- solves in a separate step potential **nonlinearities** in the 0D network;
- allows **flexibility** on the choice of the numerical method and discretization for each sub-step;
- does not require any iteration between the two sub-steps;
- is **unconditionally stable** with respect to the choice of the time step;
- has at most **first-order accuracy** in time with the current implementation

### Theorem (stability)

Consider the coupled Darcy-0D problem and assume that:

- the circuit properties are such that the tensor  $\underline{A}$  is constant and  $\underline{B}$  is positive definite, where  $\underline{B}$  gathers all the contributions from resistive, capacitive and inductive elements in the lumped circuit;
- there are no forcing terms, namely  $\mathcal{F}_{pde}(t) = \mathcal{F}_{ode}(t) = 0 \forall t \geq 0$ , where  $\mathcal{F}_{pde}$  represents the forcing on the system due to body forces and external pressures acting on the Darcy domain and  $\mathcal{F}_{ode}$  the forcing on the system due to generators of current and voltage within the lumped circuit, respectively;
- the Darcy-circuit connection is of resistive type.

Under these assumptions, the algorithm given by Steps 1 and 2 is unconditionally stable.

## 5. Results

The convergence tests were performed comparing numerical simulations [ $t_{\text{step}} = 0.1, 0.05, 0.025, 0.01$ ] with an analytical solution.

- Fig. 3 shows the spatial distribution of the pressure with the information of the flux direction;
- Fig. 4 illustrates the convergence study for the 3d pressure and flux;
- Figs. 5, 6 and 7 display the comparison between different pressures in the circuit using different time steps and the analytical solution.

**The simulation tests confirm the theoretical results and the main features of this operator splitting method.**

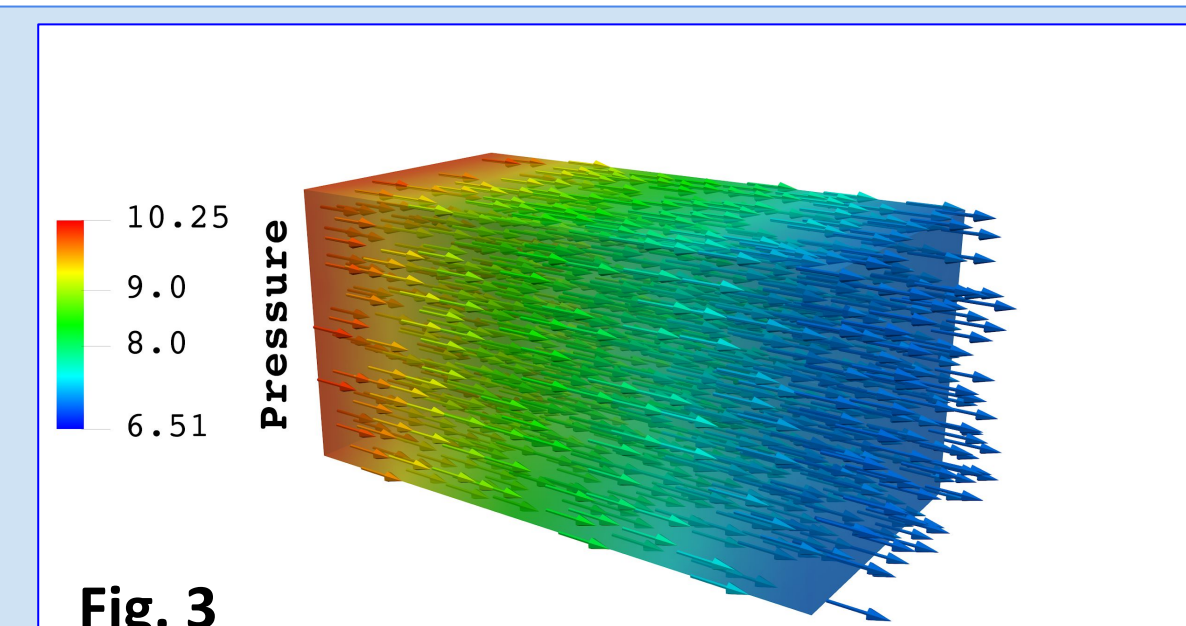


Fig. 3

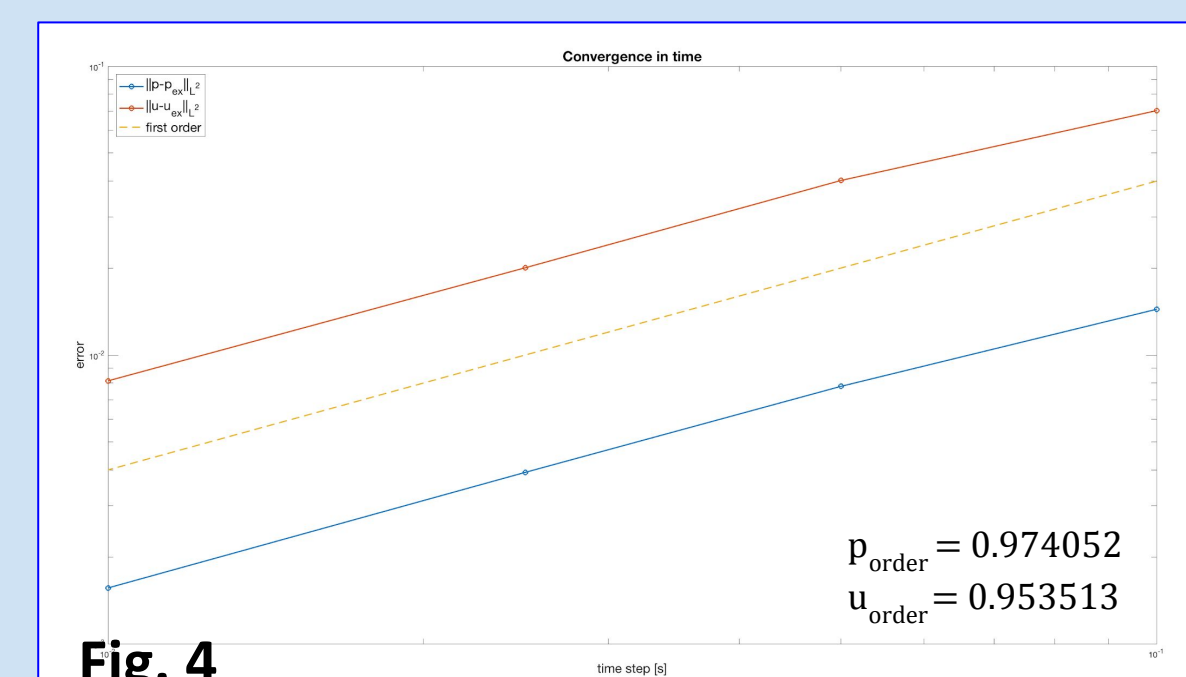


Fig. 4

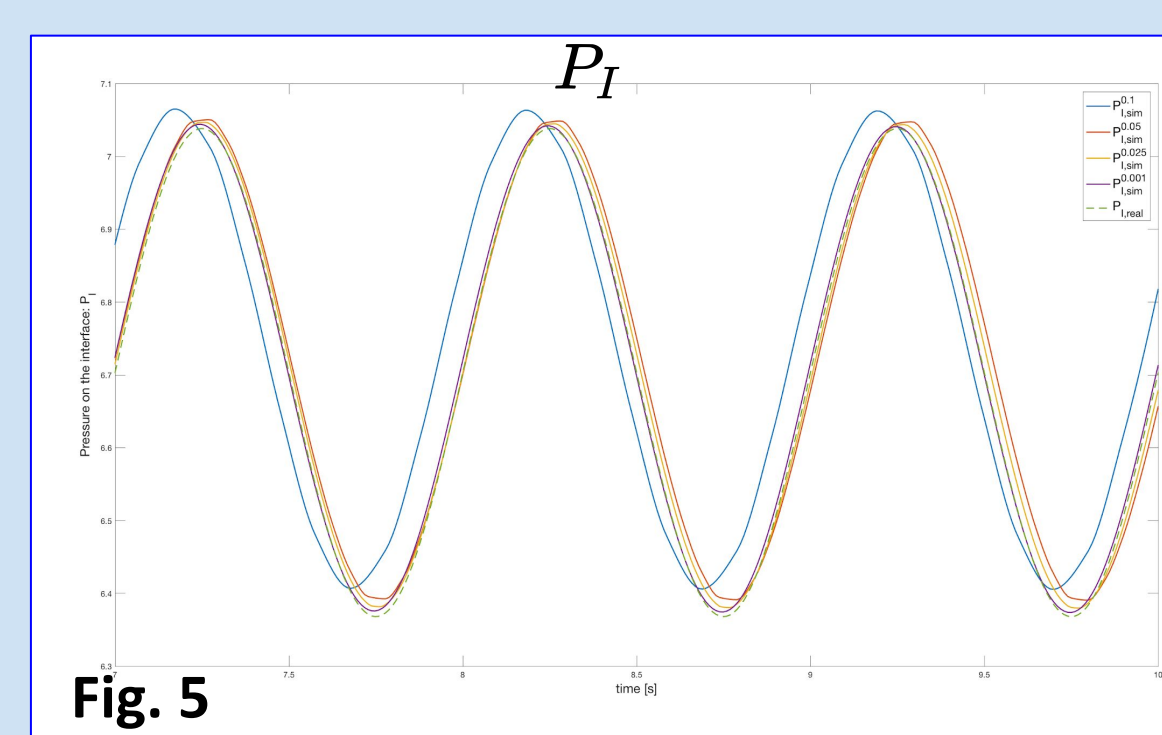


Fig. 5

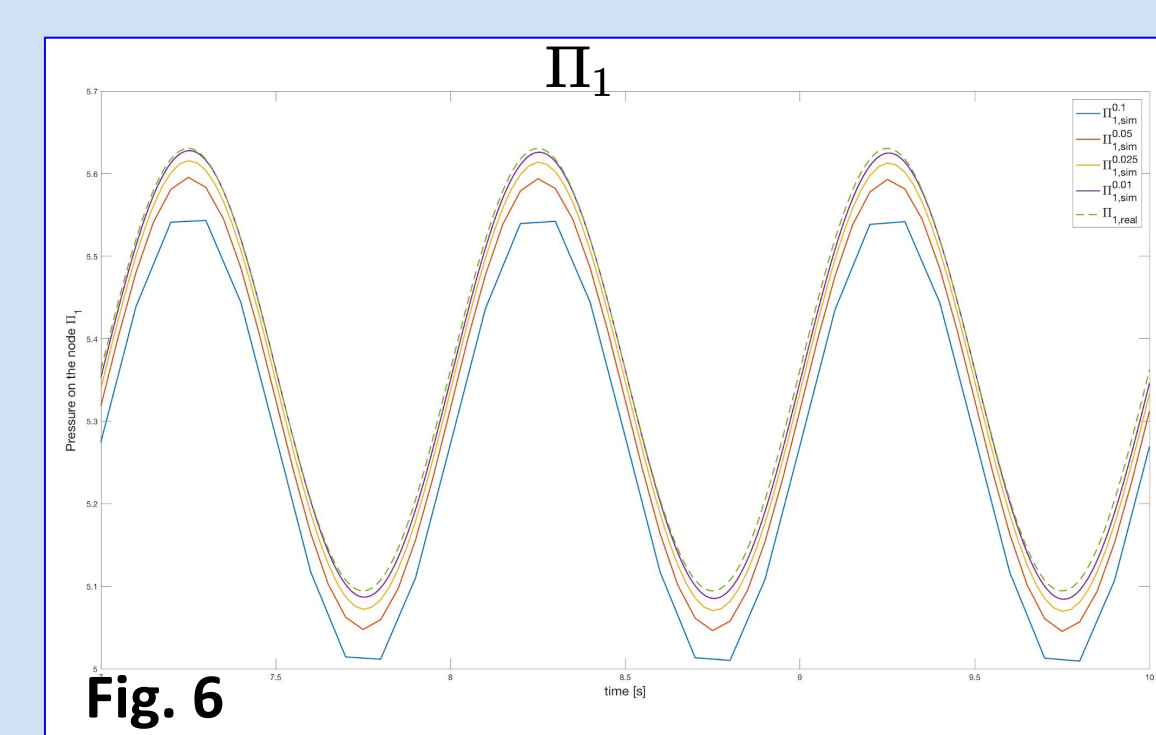


Fig. 6

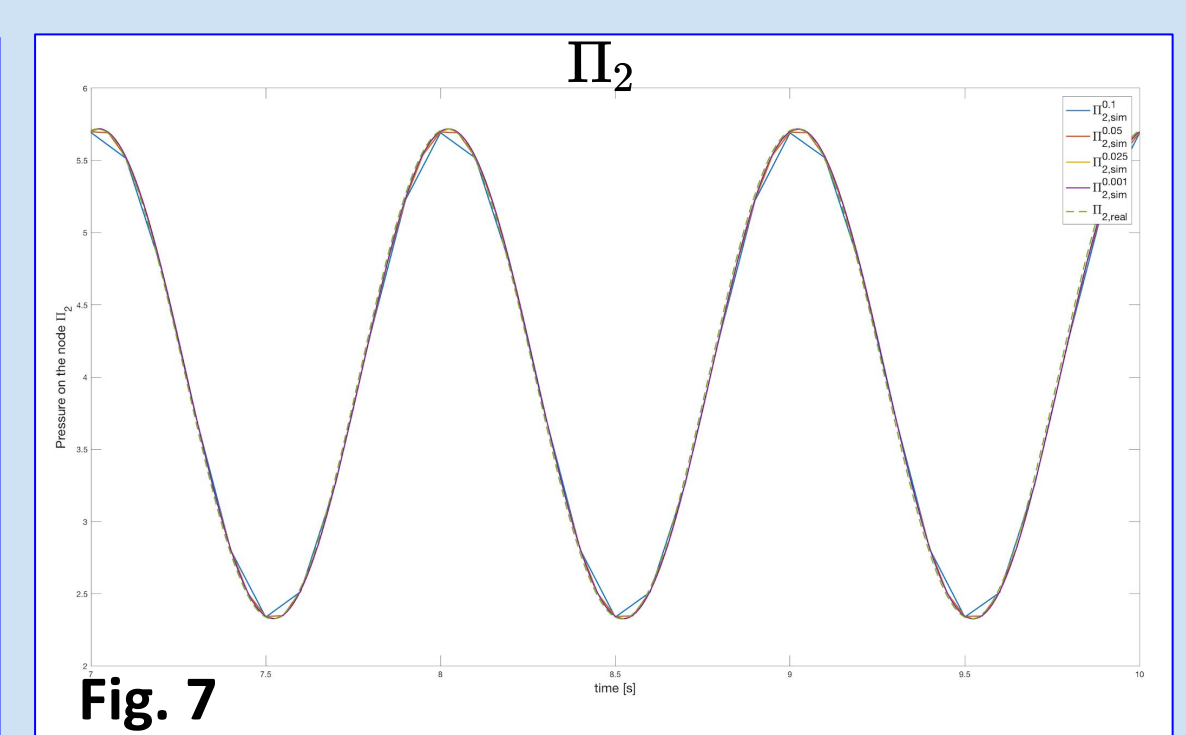
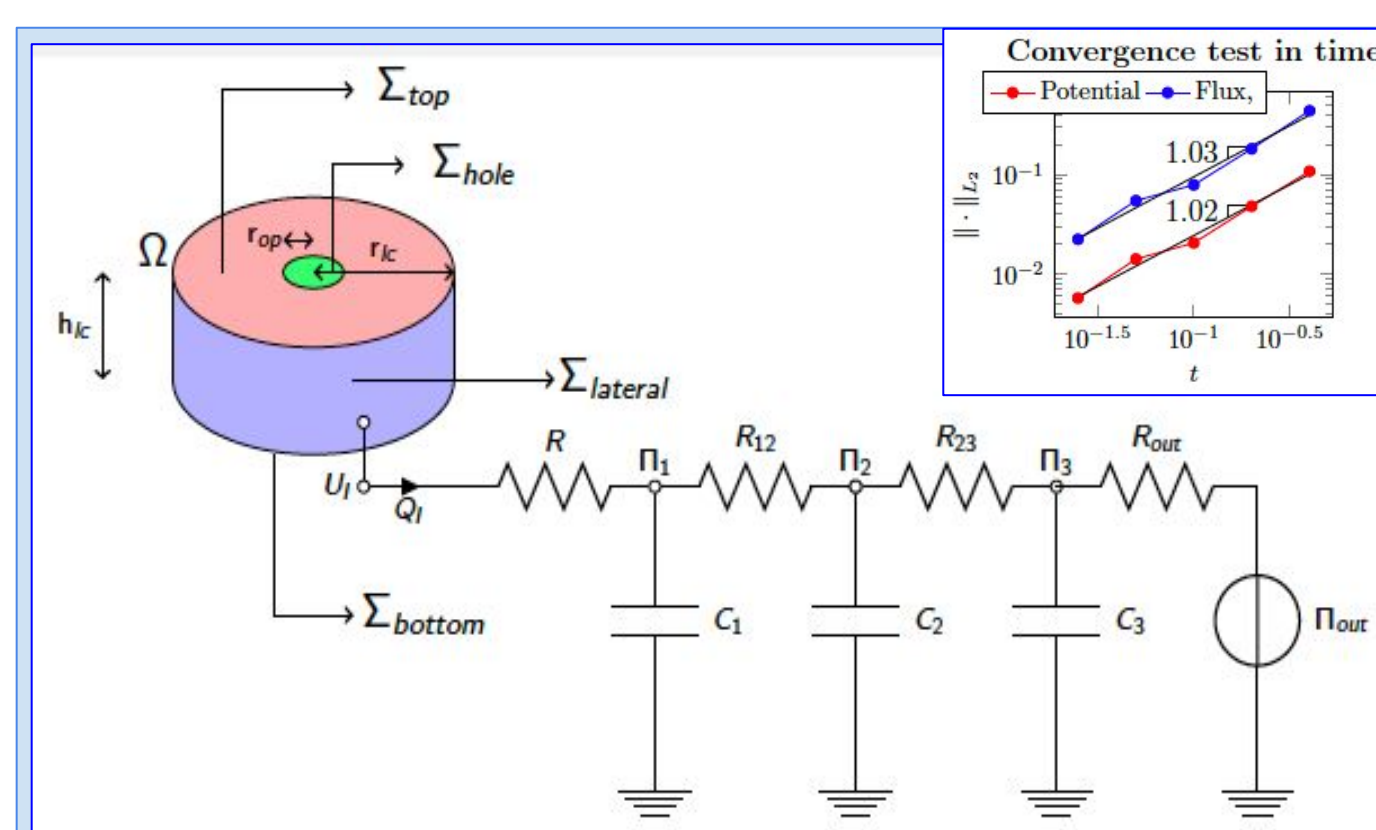


Fig. 7

## 6. Perspectives

We study the **perfusion of the lamina cribrosa**, a collagen structure located in the optic nerve head that plays a crucial role in several diseases such as glaucoma.

These pilot investigations support the idea that **hemodynamic models should have a systemic view on the circulation**. Specifically, our results<sup>[1]</sup> suggest that a localized systemic variation can have a significant effect on the whole system.



## 7. Conclusions

We have formulated and implemented a numerical algorithm combining **HDG discretization with an unconditionally stable operator splitting method** for multiscale problems in order to simulate the physiological behavior of blood flow in the ocular posterior segment.

More generally, the algorithm might be utilized for other problems arising in **engineering and life sciences**, where a **robust and efficient coupling** between a 3d poro-elastic model and a 0d fluid flow circuit described by a system of ODEs is required.

## References

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