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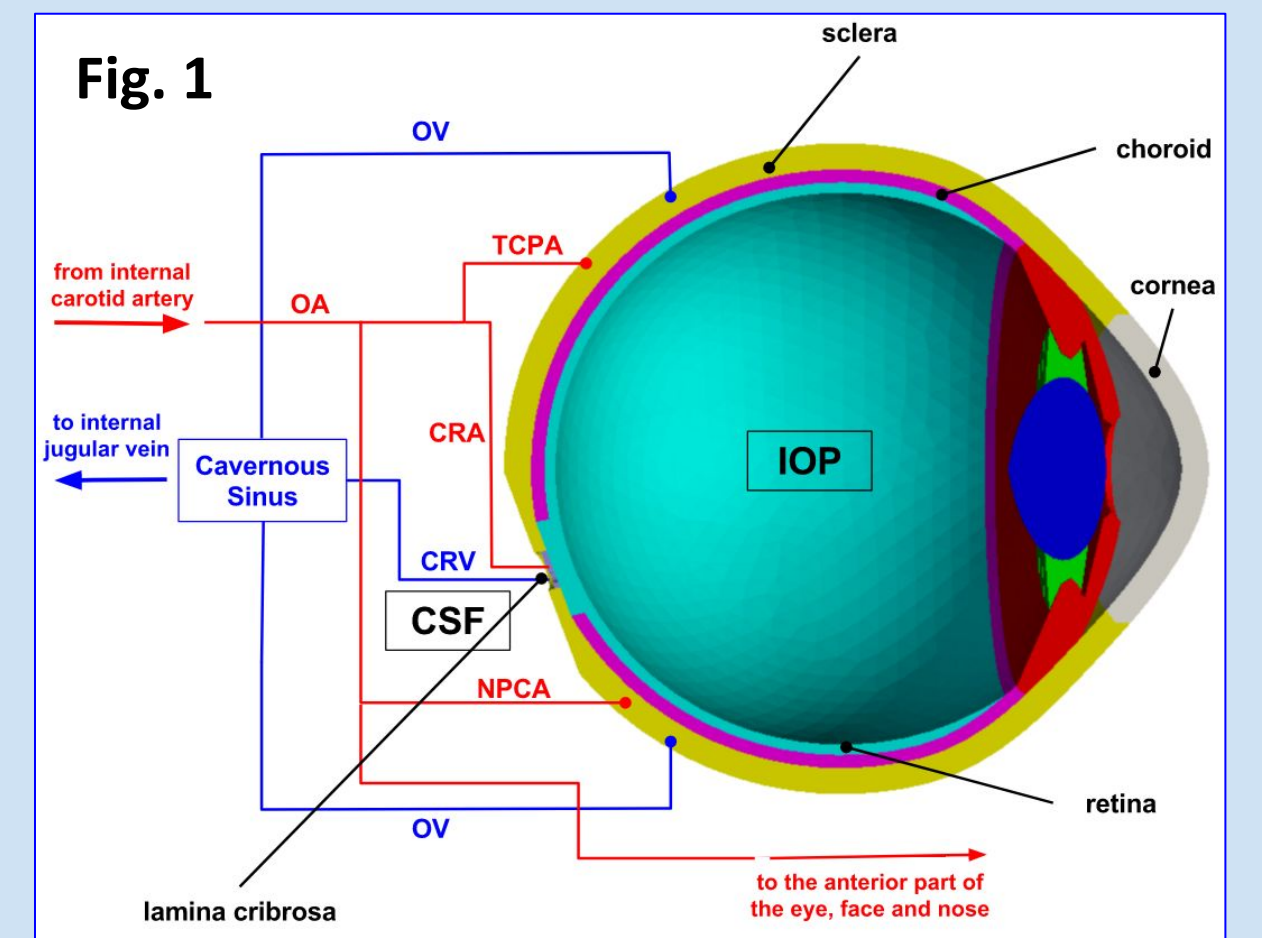
Unconditionally stable operator splitting method for a multiscale application in ophthalmology

1. Motivation

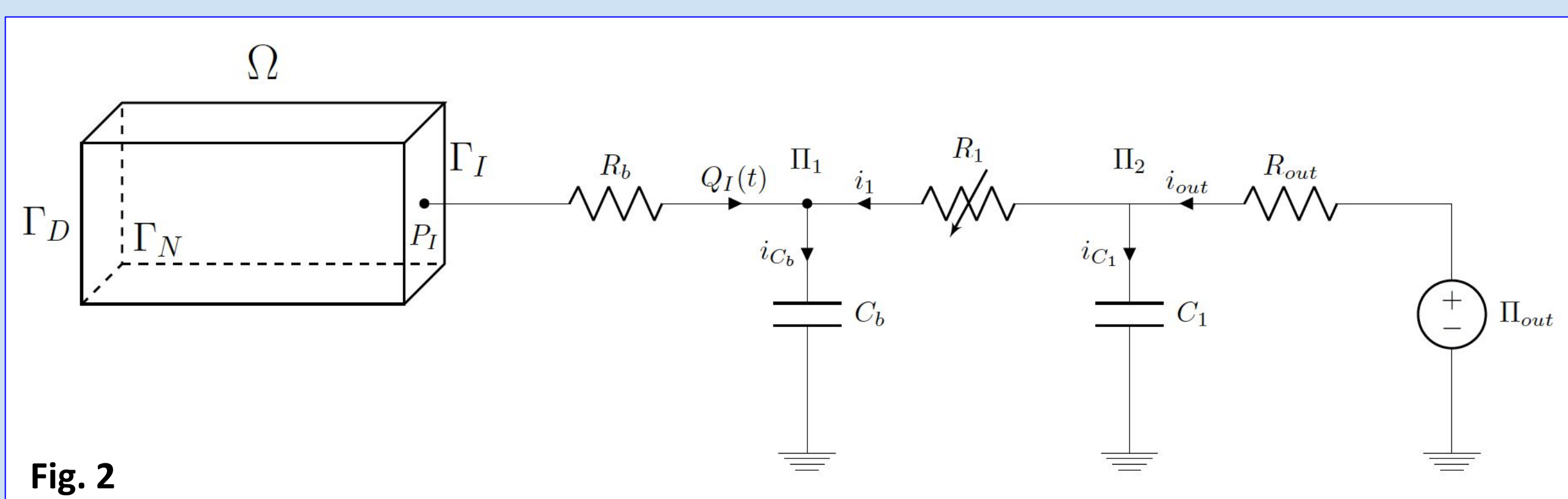
In the context of ophthalmology, we are developing a multiscale **Ocular Mathematical Virtual Simulator (O-MVS^[1])**, which combines a three-dimensional poro-elastic system, representing the biomechanics and hemodynamics of the ocular posterior tissues, coupled with a lumped-parameter circuit that describes the blood circulation in the major vessels in the back of the eye.

The **multiscale** approach provides:

- a rough description of the whole system + 3D zoom in a specific region of interest;
- a reliable method for computing correct boundary conditions at the artificial boundaries of a region of interest.



2. Porous media 3d-0d model



Darcy problem

$$\begin{cases} \frac{\partial p}{\partial t} + \nabla \cdot \underline{u} = f & \text{in } \Omega \times (0, T) \\ \underline{k} \nabla p + \underline{u} = \underline{0} & \text{in } \Omega \times (0, T) \\ \underline{u} \cdot \underline{n} = g_N & \text{on } \Gamma_N \times (0, T) \\ p = g_D & \text{on } \Gamma_D \times (0, T) \\ \int_{\Gamma_I} \underline{u} \cdot \underline{n} = Q_I & \text{on } \Gamma_I \times (0, T) \\ p|_{\Gamma_I} = P_I & \text{on } \Gamma_I \times (0, T) \\ Q_I = \frac{P_I - \Pi_1}{R_b} & \text{on } \Gamma_I \times (0, T) \end{cases}$$

Systemic circuit

$$\begin{cases} C \frac{d\Pi}{dt} + \underline{A} \Pi = \underline{s} + \underline{b} & \text{in } (0, T) \\ C = \begin{bmatrix} C_b & 0 \\ 0 & C_1 \end{bmatrix} & \underline{A} = \begin{bmatrix} \frac{1}{R_b} + \frac{1}{R_1} & -\frac{1}{R_1} \\ -\frac{1}{R_1} & \frac{1}{R_1} + \frac{1}{R_{out}} \end{bmatrix} \\ \underline{s} = \begin{bmatrix} 0 \\ \frac{\Pi_{out}}{R_{out}} \end{bmatrix} & \underline{b} = \begin{bmatrix} \frac{P_I}{R_b} \\ 0 \end{bmatrix} \end{cases}$$

NB: the use of Hybridizable Discontinuous Galerkin method for the Darcy problem naturally allows the spatial coupling with the circuit thanks to the Integral Boundary Condition handling.

3. Time discretization

Monolithic approach

- + The coupled problem is treated as a whole;
- more difficult to implement and sometimes gives rise to badly conditioned problems.

New operator splitting method

- + May allow to use existing solvers already developed for solving sub-problems separately;
- accuracy issues related to the choice of the time step.

Given p^n, Π^n for $n \geq 0$, solve

Step 1.

$$\begin{cases} \frac{\partial p}{\partial t} + \nabla \cdot \underline{u} = f & \text{in } \Omega \times (t^n, t^{n+1}) \\ \underline{k} \nabla p + \underline{u} = \underline{0} & \text{in } \Omega \times (t^n, t^{n+1}) \\ C \frac{d\Pi}{dt} = \underline{s}(\Pi, t) & \text{in } (t^n, t^{n+1}) \end{cases}$$

+ interface and boundary conditions

$$p(t^n) = p^n \quad \Pi(t^n) = \Pi^n$$

and then set

$$\begin{aligned} p^{n+\frac{1}{2}} &= p(t^{n+\frac{1}{2}}) \\ \Pi^{n+\frac{1}{2}} &= \Pi(t^{n+\frac{1}{2}}) \\ \underline{u}^{n+\frac{1}{2}} &= \underline{u}(t^{n+\frac{1}{2}}) \end{aligned}$$

Next step: comparison with other existing coupling methods.

Sub-steps are designed so that the **energy at the semi-discrete** time level mirrors the behavior of the **energy of the full coupled system** in the spirit of [3].

Step 2.

$$\begin{cases} \frac{\partial p}{\partial t} = 0 & \text{in } \Omega \times (t^n, t^{n+1}) \\ C \frac{d\Pi}{dt} + \underline{A} \Pi = \underline{b}(t) & \text{in } (t^n, t^{n+1}) \\ p(t^n) = p^{n+\frac{1}{2}} & \Pi(t^n) = \Pi^{n+\frac{1}{2}} \end{cases}$$

and then set

$$\begin{aligned} p^{n+1} &= p(t^{n+1}) \quad (= p(t^{n+\frac{1}{2}})) \\ \Pi^{n+1} &= \Pi(t^{n+1}) \\ \underline{u}^{n+1} &= \underline{u}(t^{n+1}) \quad (= \underline{u}(t^{n+\frac{1}{2}})) \end{aligned}$$

4. Main features

The proposed operator splitting method

- is based on a **semi-discretization in time** inspired by the energy estimate^[2];
- solves in a separate step potential **nonlinearities** in the 0D network;
- allows **flexibility** on the choice of the numerical method and discretization for each sub-step;
- does not require any iteration between the two sub-steps;
- is **unconditionally stable** with respect to the choice of the time step;
- has at most **first-order accuracy** in time with the current implementation

Theorem (stability)

Consider the coupled Darcy-0D problem and assume that:

- the circuit properties are such that the tensor \underline{A} is constant and \underline{B} is positive definite, where \underline{B} gathers all the contributions from resistive, capacitive and inductive elements in the lumped circuit;
- there are no forcing terms, namely $\mathcal{F}_{pde}(t) = \mathcal{F}_{ode}(t) = 0 \quad \forall t \geq 0$, where \mathcal{F}_{pde} represents the forcing on the system due to body forces and external pressures acting on the Darcy domain and \mathcal{F}_{ode} the forcing on the system due to generators of current and voltage within the lumped circuit, respectively;
- the Darcy-circuit connection is of resistive type.

Under these assumptions, the algorithm given by Steps 1 and 2 is unconditionally stable.

5. Results

The convergence tests were performed comparing numerical simulations [$t_{step} = 0.1, 0.05, 0.025, 0.01$] with an analytical solution.

- Fig. 3 shows the spatial distribution of the pressure with the information of the flux direction;
- Fig. 4 illustrates the convergence study for the 3d pressure and flux;
- Figs. 5, 6 and 7 display the comparison between different pressures in the circuit using different time steps and the analytical solution.

The simulation tests confirm the theoretical results and the main features of this operator splitting method.

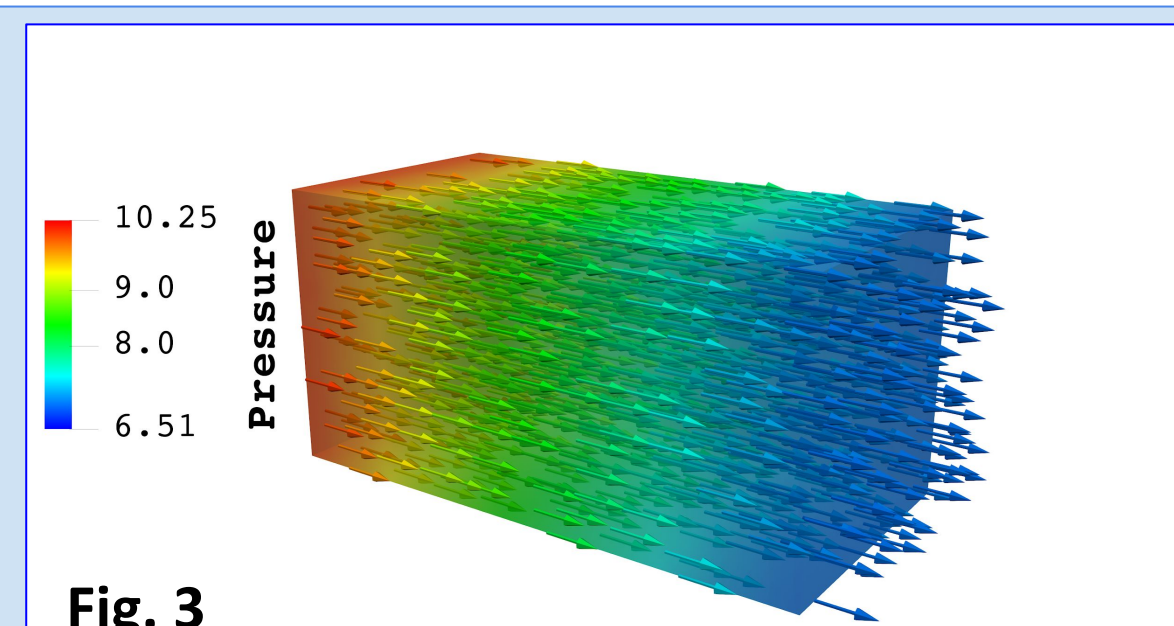


Fig. 3

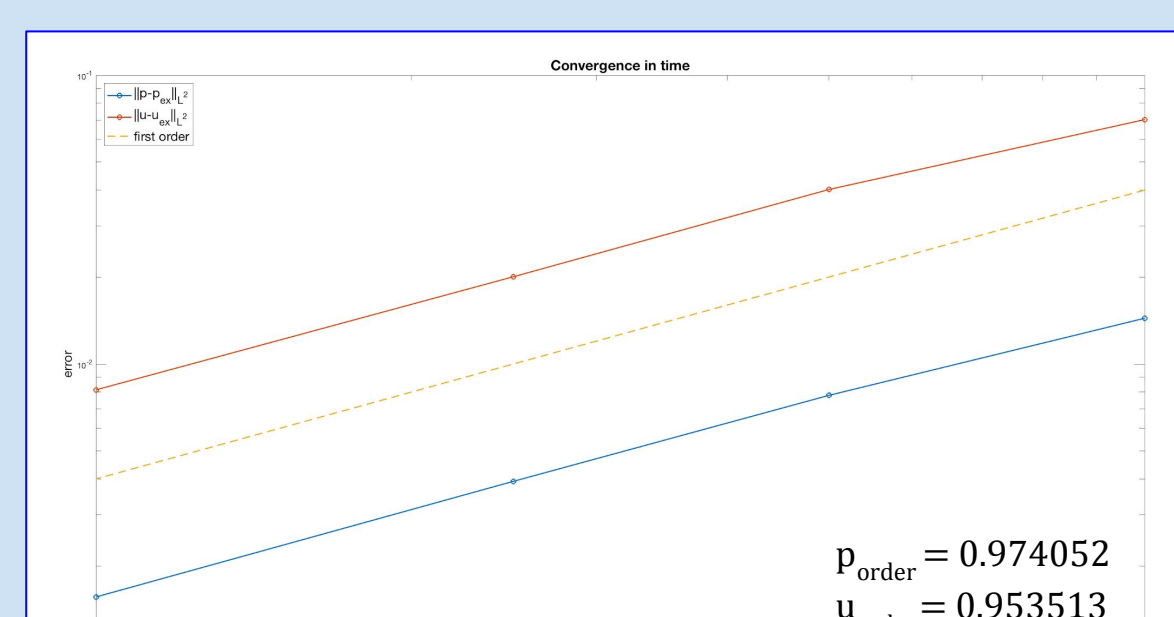


Fig. 4

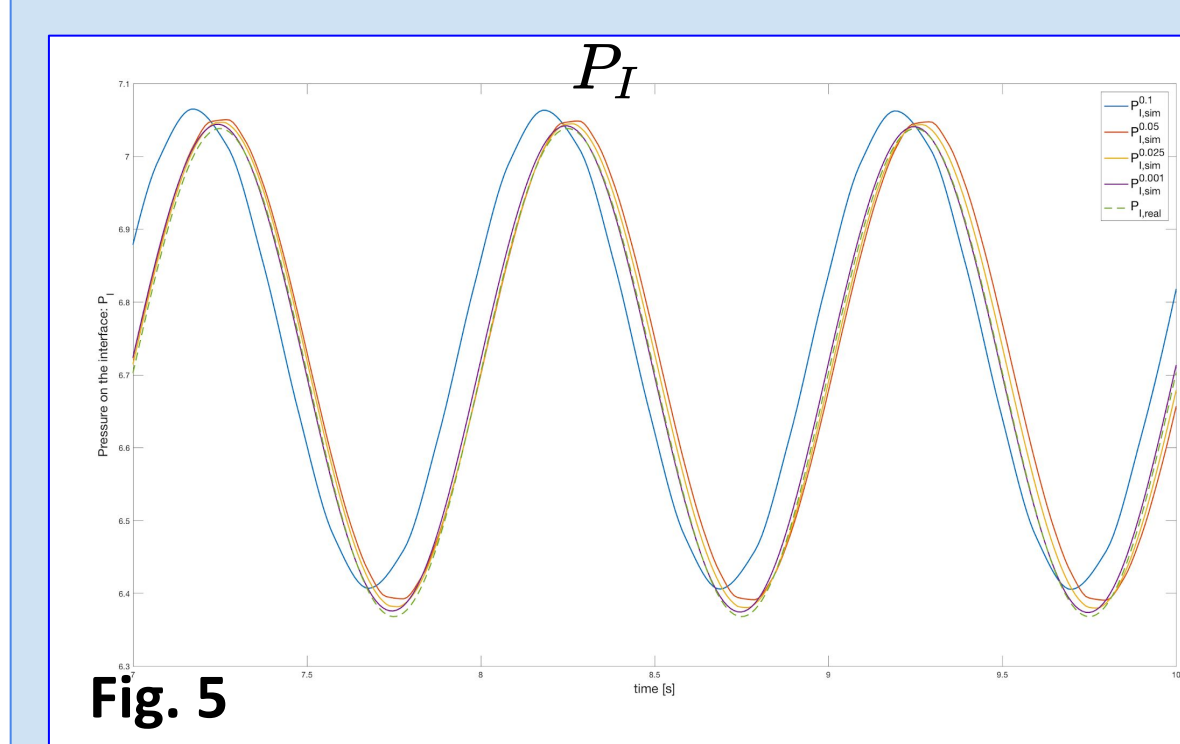


Fig. 5

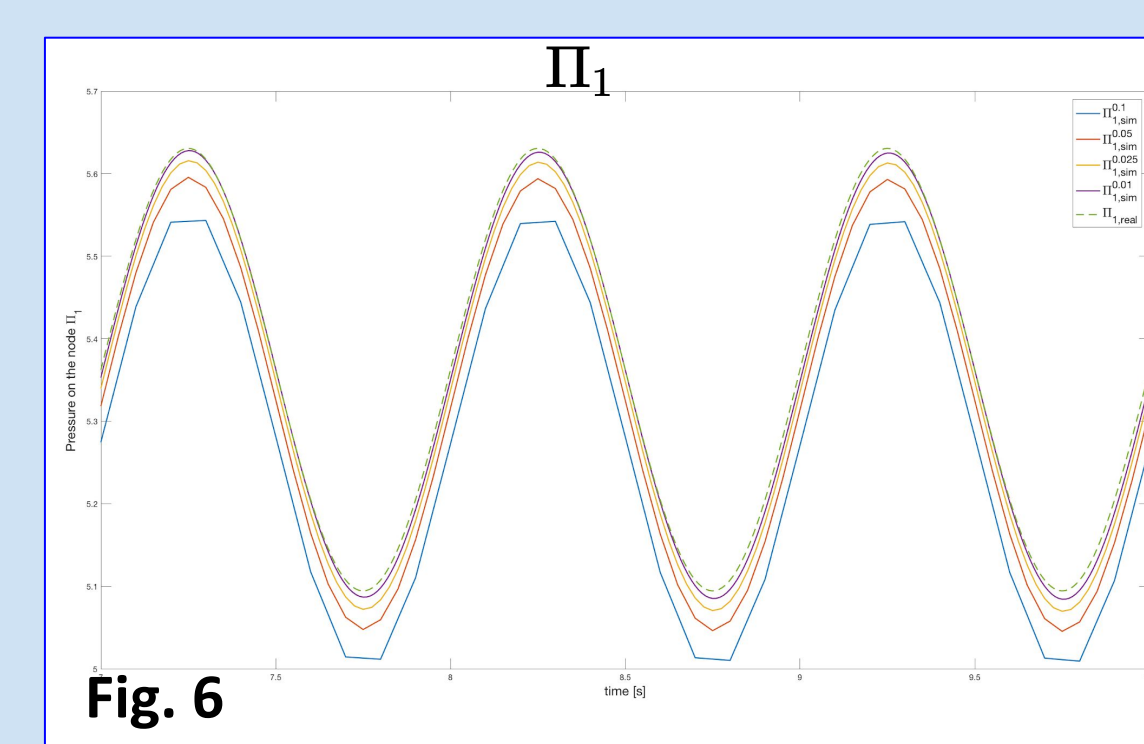


Fig. 6

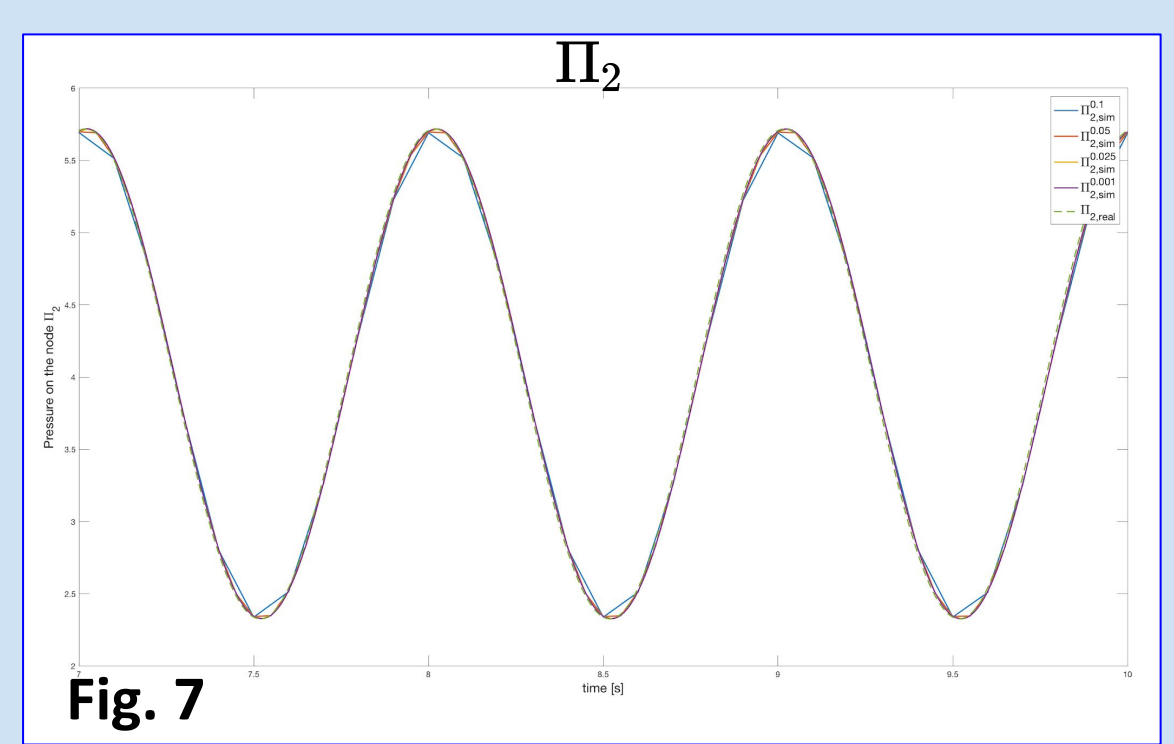
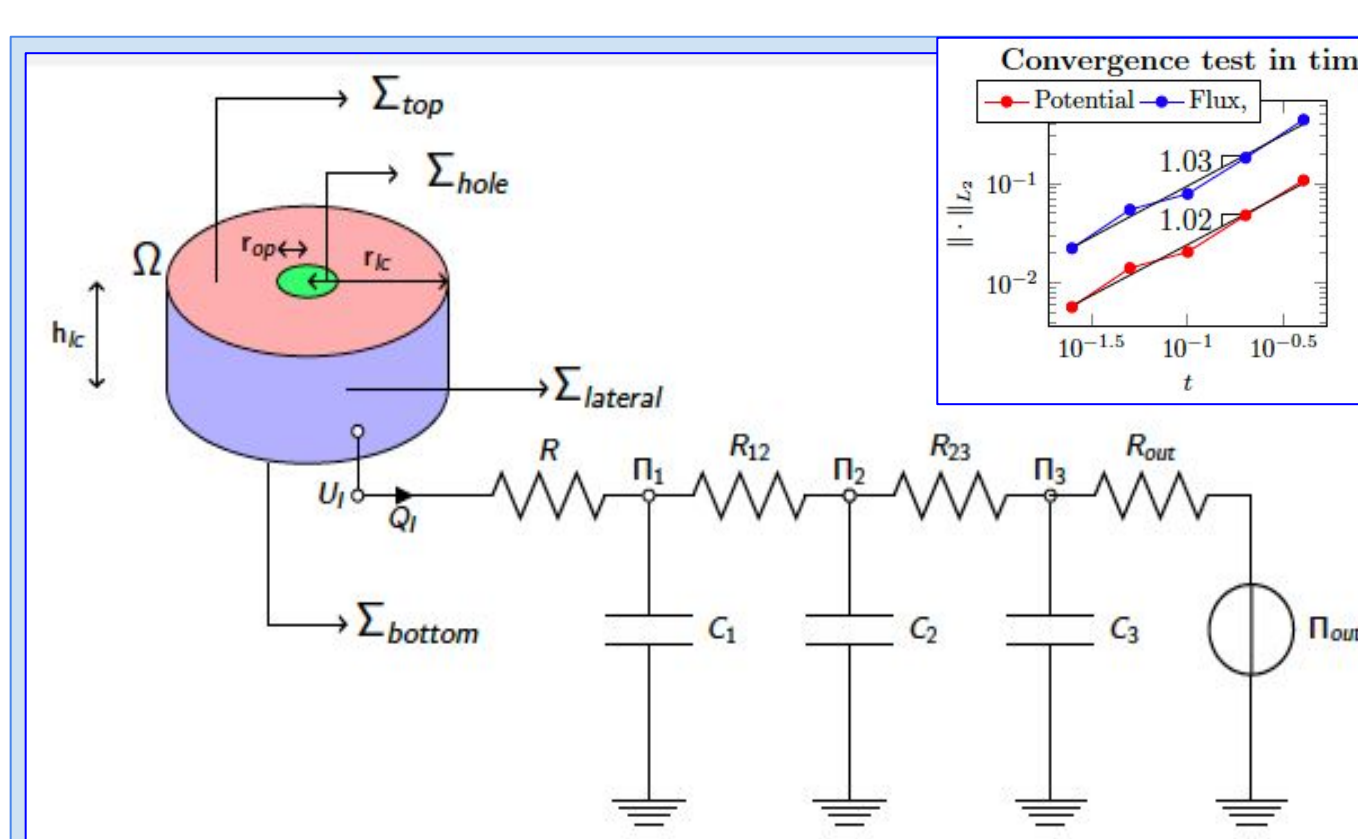


Fig. 7

6. Perspectives

We study the **perfusion of the lamina cribrosa**, a collagen structure located in the optic nerve head that plays a crucial role in several diseases such as glaucoma.

These pilot investigations support the idea that **hemodynamic models should have a systemic view on the circulation**. Specifically, our results^[1] suggest that a localized systemic variation can have a significant effect on the whole system.



7. Conclusions

We have formulated and implemented a numerical algorithm combining **HDG discretization with an unconditionally stable operator splitting method** for multiscale problems in order to simulate the physiological behavior of blood flow in the ocular posterior segment.

More generally, the algorithm might be utilized for other problems arising in **engineering and life sciences**, where a **robust and efficient coupling** between a 3d poro-elastic model and a 0d fluid flow circuit described by a system of ODEs is required.

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