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The 2d-directed spanning forest converges to the Brownian web

David Coupier^{*}, Kumarjit Saha^{†‡}, Anish Sarkar[§], Viet Chi Tran[¶]

June 29, 2018

Abstract

The two-dimensional *directed spanning forest* (DSF) introduced by Baccelli and Bordenave is a planar directed forest whose vertex set is given by a homogeneous Poisson point process \mathcal{N} on \mathbb{R}^2 . If the DSF has direction $-e_y$, the ancestor $h(\mathbf{u})$ of a vertex $\mathbf{u} \in \mathcal{N}$ is the nearest Poisson point (in the L_2 distance) having strictly larger y -coordinate. In this paper we show that the collection of DSF paths, properly scaled, converges in distribution to the Brownian web (BW). See Theorem 2. This verifies a conjecture made by Baccelli and Bordenave in 2007 [3].

A key ingredient for the proof is to control the tail distribution of the coalescence time between two paths of the DSF (Theorem 21). The facts that the DSF spans on a Poisson point process on the plane and that its construction is based on the L^2 distance— which is very natural—destroys all Markov and martingale properties on which the existing literature usually relies for proving convergence of directed forests to the Brownian web. Our proof relies on the construction of clever regeneration events exploiting the particular geometry of the DSF. The distance between two given DSF paths (actually between two *regenerated paths* associated to the original ones) considered at the corresponding regeneration times satisfies a certain Markov property w.r.t. an enhanced filtration, and we can control its hitting time of zero by a new Laplace criterion (Theorem 23).

We then introduce a new criterion (Theorem 27) for the convergence to the BW and its dual, inspired from the wedge condition of Schertzer et al. [24], and show that it is satisfied in our context. We believe that the ideas in this work can be applied to the convergence to the BW of a large variety of directed forests with intricate dependencies..

Finally, the coalescence time estimate for DSF paths is used to quantify the number of semi-infinite paths of the *Radial Spanning Tree* crossing the circle centred at the origin and with radius r (Theorem 3): it is negligible w.r.t. $r^{3/4+\varepsilon}$.

Keywords: stochastic geometry; directed spanning forest; convergence to the Brownian Web; Poisson point processes; geometrical interactions; renewal times.

AMS Classification: 60D05

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1 Introduction and results

The DSF and its conjectured scaling limit.

Let us consider a homogeneous Poisson point process (PPP) \mathcal{N} on \mathbb{R}^2 with intensity $\lambda > 0$ and denote by $\mathbf{x}(i)$, for $i = 1, 2$, the i -th coordinate of $\mathbf{x} \in \mathbb{R}^2$. Let us also denote by $\mathbb{H}^+(l) := \{\mathbf{x} \in \mathbb{R}^2 : \mathbf{x}(2) \geq l\}$ the half plane of points with ordinates greater than $l \in \mathbb{R}$. The *ancestor* of $\mathbf{x} \in \mathcal{N}$ is defined as the closest Poisson point to \mathbf{x} in the half plane $\mathbb{H}^+(\mathbf{x}(2))$:

$$h(\mathbf{x}, \mathcal{N}) := \operatorname{argmin}\{\|\mathbf{y} - \mathbf{x}\|_2 : \mathbf{y} \in \mathcal{N}, \mathbf{y}(2) > \mathbf{x}(2)\} . \quad (1)$$

In most occasions, we drop the second argument for $h(\mathbf{x}, \mathcal{N})$ and merely denote it by $h(\mathbf{x})$. It is useful to observe that for all $\mathbf{x} \in \mathbb{R}^2$, the point $h(\mathbf{x})$ is well defined. The *Directed Spanning Forest* (DSF) with direction $-e_y$ on \mathbb{R}^2 is the random geometric graph \mathfrak{F} with vertex set \mathcal{N} and edge set $E := \{(\mathbf{x}, h(\mathbf{x})) : \mathbf{x} \in \mathcal{N}\}$. Since for any $\mathbf{x} \in \mathcal{N}$, $h(\mathbf{x})$ a.s. denotes a unique Poisson point, the DSF is a directed outgoing-one graph without cycle. This justifies it is called forest. In the sequel, horizontal and vertical axes will be respectively interpreted as space and time axes.

The DSF was introduced in 2007 by Baccelli and Bordenave [3] as a tool to study the asymptotic properties of the *Radial Spanning Tree* (RST) which actually was the main subject of study in [3]. The RST is a tree rooted at the origin O of \mathbb{R}^2 , with vertex set $\mathcal{N} \cup \{O\}$, in which each $\mathbf{x} \in \mathcal{N}$ is connected to the closest Poisson point to \mathbf{x} but inside the open ball $\{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{y}\|_2 < \|\mathbf{x}\|_2\}$. The authors showed that the DSF is an approximation of the RST, in distribution, locally and far from the origin.

However, the DSF appears as truly interesting in itself since it admits beautiful conjectures, already mentioned in [3]. First, is it true that any two given trajectories of the DSF \mathfrak{F} eventually coalesce with probability 1? A trajectory of the DSF is a sequence $(\mathbf{x}, h(\mathbf{x}), h(h(\mathbf{x}))) \dots$ of successive ancestors. In other words, is the DSF a tree? This question was solved in [13] by Coupier and Tran using an efficient percolation technique, namely the Burton and Keane argument [6]. Besides, Baccelli and Bordenave showed that under diffusive scaling, any trajectory of the DSF converges in distribution to a Brownian motion. Then they conjectured a stronger result: the convergence under this diffusive scaling, of the whole forest \mathfrak{F} to the so-called Brownian web (BW).

In this paper we prove this second and stronger conjecture. In fact, we prove a slightly stronger result in the sense that we construct a dual forest and show that under diffusive scaling, the DSF and its dual jointly converge in distribution to the BW and its dual.

A natural strategy to answer these questions would be to exhibit some independence (or Markov) properties in time (i.e. w.r.t. the vertical axis) for any couple of trajectories of the DSF. But this strategy runs up against strong dependencies, due to the construction rule of the DSF \mathfrak{F} , which are of two types: between different trajectories on the one hand and within a single trajectory on the other hand. See Figure 1 for an illustration of these two dependence phenomenons. Indeed, the construction of the ancestor $h(\mathbf{x})$ of \mathbf{x} implies that the semi-ball $B^+(\mathbf{x}, \|\mathbf{x} - h(\mathbf{x})\|_2) := B(\mathbf{x}, \|\mathbf{x} - h(\mathbf{x})\|_2) \cap \mathbb{H}^+(\mathbf{x}(2))$, where $B(\mathbf{x}, r)$ denotes the closed Euclidean ball with radius r , which a.s. overlaps the half-plane $\mathbb{H}^+(h(\mathbf{x})(2))$, is empty of Poisson points. Hence, the ancestor of $h(\mathbf{x})$ cannot belong to the resulting intersection $B^+(\mathbf{x}, \|\mathbf{x} - h(\mathbf{x})\|_2) \cap \mathbb{H}^+(h(\mathbf{x})(2))$. Roughly speaking, the past of a DSF trajectory may influence its future. Furthermore, when the successive ancestors of \mathbf{x} are constructed, the resulting empty region, called the *history set*, may have a complicated shape: it is a union of semi-balls centered at already visited vertices intersected with a proper half plane (we shall be more precise in the sequel). This random region is not necessarily connected and cannot be *a priori* bounded.

In [15], Fontes et al introduced a suitable Polish space to study the BW, characterized its distribution (in Theorem 1 below) and provided criteria ensuring weak convergence to the BW (see Theorem 26 in Section 4.1). Since then, convergence to the BW for various directed forests or navigation schemes have been extensively studied and thence, the BW appeared as the universal scaling limit for a large number of seemingly unrelated models. Let us cite: [5, 19] in the context of coalescing system of independent nonsimple random walks; [11, 14] in the context of drainage networks; [23] for an oriented percolation model; [20]

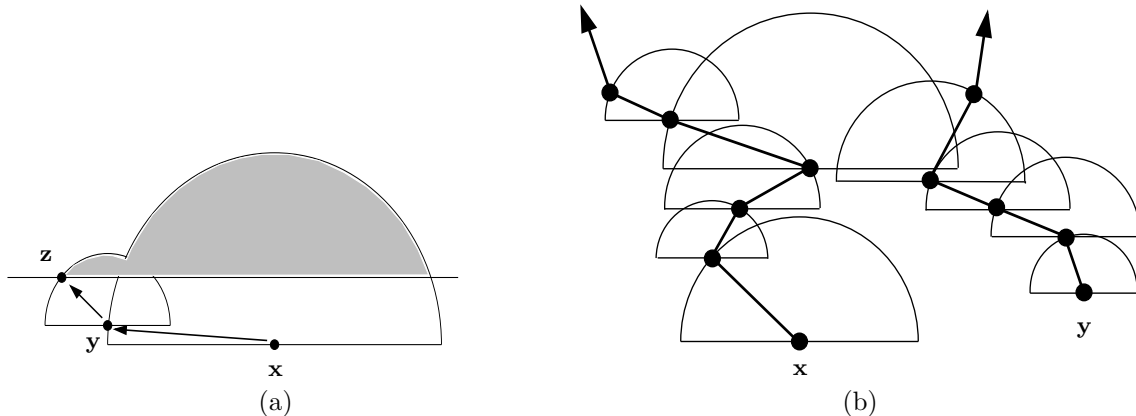


Figure 1: (a) This picture illustrates the dependence phenomenon within a single trajectory and how the past trajectory may influence its next steps. It represents a Poisson point \mathbf{x} and its first two ancestors, i.e. $\mathbf{y} = h(\mathbf{x})$ and $\mathbf{z} = h(h(\mathbf{x}))$, and the two resulting semi-balls. The grey area corresponds to the history set of this trajectory in which the ancestor of \mathbf{z} cannot be. It is worth pointing out here that the (large) empty semi-ball $B^+(\mathbf{x}, \|\mathbf{x} - h(\mathbf{x})\|_2)$ may influence the construction of many ancestors of the initial vertex \mathbf{x} . (b) This second picture illustrates the dependence phenomenon between two DSF trajectories when the resulting semi-balls corresponding to their constructions overlap. This overlapping locally acts as a repulsive effect between trajectories starting at \mathbf{x} and \mathbf{y} .

in connection with Hastings-Levitov planar aggregation models; and [5, 7, 8, 12] in the context of radial systems of coalescing trajectories. In many of these papers, the choice of the ancestor of any vertex \mathbf{x} does not depend on the past, i.e. on what happens below ordinate $\mathbf{x}(2)$, allowing to easily introduce Markov processes and use martingale convergence theorems or Lyapunov functions. As explained above, this is no longer true for the DSF \mathfrak{F} because of complex geometrical dependencies. Recently, several papers [22, 28]–Saha and Sarkar are involved in the first one –have considered modifications of the DSF in order to make the problem more tractable but until this paper, the conjecture of Baccelli and Bordenave remained open.

Existence of the BW and its dual.

The BW was appeared for the first time in the literature in the seminal paper of Arratia [1]. In this work, the author studied the diffusive scaling limit of coalescing simple symmetric random walks starting from every point of $2\mathbb{Z}$ at time 0 and showed that this collection converges to a collection of coalescing Brownian motions starting from every point on \mathbb{R} at time 0. For a general review on the BW see [24] and references therein. Later Tóth and Werner [27] gave a construction of a system of coalescing Brownian motions starting from every point in space-time plane \mathbb{R}^2 and used it to construct the true self-repelling motion.

The framework (topologies, spaces, characterization and convergence criteria) that we will use in this paper have been provided by Fontes et al. in [15]. Let us recall some relevant details. Let \mathbb{R}_c^2 be the completion of the space time plane \mathbb{R}^2 with respect to the metric

$$\rho((x_1, t_1), (x_2, t_2)) := |\tanh(t_1) - \tanh(t_2)| \vee \left| \frac{\tanh(x_1)}{1 + |t_1|} - \frac{\tanh(x_2)}{1 + |t_2|} \right|.$$

As a topological space, \mathbb{R}_c^2 can be identified with the continuous image of $[-\infty, \infty]^2$ under a map that identifies the line $[-\infty, \infty] \times \{\infty\}$ with the point $(*, \infty)$, and the line $[-\infty, \infty] \times \{-\infty\}$ with the point $(*, -\infty)$. We define a path π with starting time $\sigma_\pi \in [-\infty, \infty]$ as a continuous mapping $\pi : [\sigma_\pi, \infty] \rightarrow [-\infty, \infty] \cup \{*\}$ such that $\pi(\infty) = *$ and, when $\sigma_\pi = -\infty$, $\pi(-\infty) = *$. Notice that the mapping $t \mapsto (\pi(t), t) \in (\mathbb{R}_c^2, \rho)$ is continuous on $[\sigma_\pi, \infty]$. We then define Π to be the space of all paths in \mathbb{R}_c^2 with

all possible starting times in $[-\infty, \infty]$. The following metric, for $\pi_1, \pi_2 \in \Pi$

$$d_{\Pi}(\pi_1, \pi_2) := |\tanh(\sigma_{\pi_1}) - \tanh(\sigma_{\pi_2})| \vee \sup_{t \geq \sigma_{\pi_1} \wedge \sigma_{\pi_2}} \left| \frac{\tanh(\pi_1(t \vee \sigma_{\pi_1}))}{1 + |t|} - \frac{\tanh(\pi_2(t \vee \sigma_{\pi_2}))}{1 + |t|} \right|$$

makes Π a complete, separable metric space. The metric d_{Π} is slightly different from the original choice in [15] which is somewhat less natural as explained in [26]. Convergence according to this metric can be described as locally uniform convergence of paths as well as convergence of starting times. Let \mathcal{H} be the space of compact subsets of (Π, d_{Π}) equipped with the Hausdorff metric $d_{\mathcal{H}}$ given by,

$$d_{\mathcal{H}}(K_1, K_2) := \sup_{\pi_1 \in K_1} \inf_{\pi_2 \in K_2} d_{\Pi}(\pi_1, \pi_2) \vee \sup_{\pi_2 \in K_2} \inf_{\pi_1 \in K_1} d_{\Pi}(\pi_1, \pi_2).$$

The couple $(\mathcal{H}, d_{\mathcal{H}})$ is a complete separable metric space. Let also $B_{\mathcal{H}}$ be the Borel σ -algebra on the metric space $(\mathcal{H}, d_{\mathcal{H}})$. The Brownian web \mathcal{W} is then defined and characterized by the following result:

Theorem 1 (Theorem 2.1 of [15]). *There exists an $(\mathcal{H}, B_{\mathcal{H}})$ -valued random variable \mathcal{W} whose distribution is uniquely determined by the following properties:*

- (a) *from any deterministic point $\mathbf{x} \in \mathbb{R}^2$, there is almost surely a unique path $\pi^{\mathbf{x}} \in \mathcal{W}$ starting from \mathbf{x} ;*
- (b) *for a finite set of deterministic points $\mathbf{x}^1, \dots, \mathbf{x}^k \in \mathbb{R}^2$, the collection $(\pi^{\mathbf{x}^1}, \dots, \pi^{\mathbf{x}^k})$ is distributed as coalescing Brownian motions starting from $\mathbf{x}^1, \dots, \mathbf{x}^k$;*
- (c) *for any countable deterministic dense set \mathcal{D} of \mathbb{R}^2 , \mathcal{W} is the closure of $\{\pi^{\mathbf{x}} : \mathbf{x} \in \mathcal{D}\}$ in (Π, d_{Π}) almost surely.*

The above theorem shows that the collection is almost surely determined by countably many coalescing Brownian motions.

To introduce the dual Brownian web $\widehat{\mathcal{W}}$ we need a similar topology on the family of backward paths. As in the definition of Π , let $\widehat{\Pi}$ be the collection of all continuous paths $\widehat{\pi}$ with starting time $\sigma_{\widehat{\pi}} \in [-\infty, \infty]$ such that $\widehat{\pi} : [-\infty, \sigma_{\widehat{\pi}}] \rightarrow [-\infty, \infty] \cup \{*\}$ with $\widehat{\pi}(-\infty) = *$ and, when $\sigma_{\widehat{\pi}} = \infty$, $\widehat{\pi}(\infty) = *$. As earlier $t \mapsto (\widehat{\pi}(t), t)$ is continuous from $[-\infty, \sigma_{\widehat{\pi}}]$ to (\mathbb{R}_c^2, ρ) . We thus equip $\widehat{\Pi}$ with the metric

$$d_{\widehat{\Pi}}(\widehat{\pi}_1, \widehat{\pi}_2) = |\tanh(\sigma_{\widehat{\pi}_1}) - \tanh(\sigma_{\widehat{\pi}_2})| \vee \sup_{t \leq \sigma_{\widehat{\pi}_1} \vee \sigma_{\widehat{\pi}_2}} \left| \frac{\tanh(\widehat{\pi}_1(t \wedge \sigma_{\widehat{\pi}_1}))}{1 + |t|} - \frac{\tanh(\widehat{\pi}_2(t \wedge \sigma_{\widehat{\pi}_2}))}{1 + |t|} \right|$$

making $(\widehat{\Pi}, d_{\widehat{\Pi}})$ a complete, separable metric space. The metric space of compact sets of $\widehat{\Pi}$ is denoted by $(\widehat{\mathcal{H}}, d_{\widehat{\mathcal{H}}})$, where $d_{\widehat{\mathcal{H}}}$ is the Hausdorff metric on $\widehat{\mathcal{H}}$, and let $B_{\widehat{\mathcal{H}}}$ be the corresponding Borel σ -field. The BW and its dual denoted by $(\mathcal{W}, \widehat{\mathcal{W}})$ are a $(\mathcal{H} \times \widehat{\mathcal{H}}, B_{\mathcal{H}} \times B_{\widehat{\mathcal{H}}})$ -valued random variable such that:

- (i) $\widehat{\mathcal{W}}$ is distributed as $-\mathcal{W}$, the BW rotated 180° about the origin;
- (ii) \mathcal{W} and $\widehat{\mathcal{W}}$ uniquely determine each other: $\widehat{\mathcal{W}}$ consists of a collection of coalescing paths running backward in time and that a.s. do not cross the paths of \mathcal{W} , in the sense that for any paths $\pi \in \mathcal{W}$ and $\widehat{\pi} \in \widehat{\mathcal{W}}$ such that $\sigma_{\pi} < \sigma_{\widehat{\pi}}$, we have for all s, t such that $\sigma_{\pi} \leq s < t \leq \sigma_{\widehat{\pi}}$,

$$(\pi(s) - \widehat{\pi}(s))(\pi(t) - \widehat{\pi}(t)) \geq 0. \quad (2)$$

See Schertzer et al. [24, Theorem 2.4]. The interaction between the paths in \mathcal{W} and $\widehat{\mathcal{W}}$ is that of Skorohod reflection (see [25]).

Our convergence theorem and the key ideas of the proof.

Let us return to the DSF. To state our result formally we need to introduce some more notations. From a vertex $\mathbf{u} \in \mathcal{N}$, define $h^0(\mathbf{u}) := \mathbf{u}$ and $h^k(\mathbf{u}) := h(h^{k-1}(\mathbf{u}))$, for $k \geq 1$. Taking the edges $\{(h^{k-1}(\mathbf{u}), h^k(\mathbf{u})) : k \geq 1\}$ to be straight line segments, we parameterize the path started from \mathbf{u} and formed by these edges as the piecewise linear function $\pi^{\mathbf{u}} : [\mathbf{u}(2), \infty) \rightarrow \mathbb{R}$ such that $\pi^{\mathbf{u}}(h^k(\mathbf{u})(2)) := h^k(\mathbf{u})(1)$ for every $k \geq 0$ and $\pi^{\mathbf{u}}(t)$ is linear in the interval $[h^k(\mathbf{u})(2), h^{k+1}(\mathbf{u})(2)]$. The collection of all DSF paths is denoted by $\mathcal{X} := \{\pi^{\mathbf{u}} : \mathbf{u} \in \mathcal{N}\}$.

For given real numbers $\gamma, \sigma > 0$, integer $n \geq 1$ and for a path π with starting time σ_π , the scaled path $\pi_n(\gamma, \sigma) : [\sigma_\pi/n^2\gamma, \infty) \rightarrow [-\infty, \infty]$ is given by

$$\pi_n(\gamma, \sigma)(t) := \frac{\pi(n^2\gamma t)}{n\sigma}. \quad (3)$$

Hence, the scaled path $\pi_n(\gamma, \sigma)$ has the starting time $\sigma_{\pi_n(\gamma, \sigma)} = \sigma_\pi/n^2\gamma$. For each $n \geq 1$, let $\mathcal{X}_n(\gamma, \sigma) := \{\pi_n^{\mathbf{u}}(\gamma, \sigma) : \mathbf{u} \in \mathcal{N}\}$ be the collection of all the scaled paths. The closure $\overline{\mathcal{X}}_n(\gamma, \sigma)$ of $\mathcal{X}_n(\gamma, \sigma)$ in (Π, d_Π) is a $(\mathcal{H}, \mathcal{B}_{\mathcal{H}})$ -valued random variable which a.s. consists of non-crossing paths only. This property will be used in the sequel frequently.

Recall that $\lambda > 0$ is the intensity of the homogeneous PPP \mathcal{N} . Our main result, illustrated by Figure 2, solves the second conjecture of Baccelli and Bordenave [3]:

Theorem 2. *There exist $\sigma = \sigma(\lambda) > 0$ and $\gamma = \gamma(\lambda) > 0$ such that the sequence*

$$\{(\overline{\mathcal{X}}_n(\gamma, \sigma), \overline{\widehat{\mathcal{X}}}_n(\gamma, \sigma)) : n \geq 1\}$$

converges in distribution to $(\mathcal{W}, \widehat{\mathcal{W}})$ as $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H} \times \widehat{\mathcal{H}}})$ -valued random variables as $n \rightarrow \infty$.

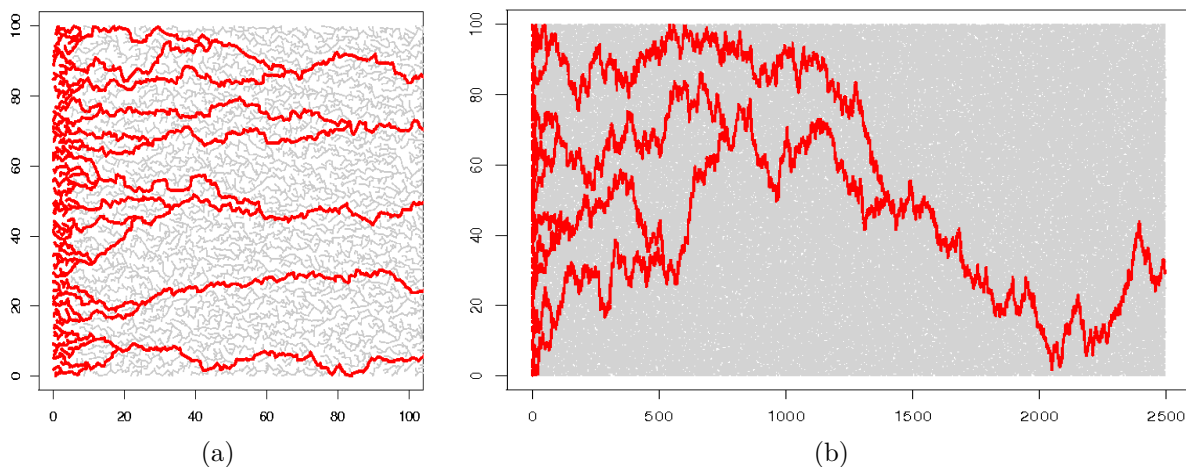


Figure 2: *Simulations of the Directed Spanning Forest with direction $-e_x$ (this direction is chosen for the convenience of the graphical representations). The trajectories coming from vertices with abscissa $0 \leq x \leq 5$ and ordinates $0 \leq y \leq 100$ are represented in bold red lines. These simulations are taken from [13]. On (b), the red paths clearly look like coalescing Brownian motions and they all coalesce before time 1500.*

Our proof actually appears as the combination of three main arguments or ideas described below. First, the criteria ensuring (weak) convergence to the BW have been meaningfully relaxed since the original convergence result in [15], recalled here in Theorem 26 (Section 4.1). Indeed, in the literature [7, 14], the proofs of criterion (B_2) systematically require that the considered forest satisfies some FKG inequality (on its trajectories). But, this strong property becomes difficult to check, or even false, when dependence phenomena arise as it is the case for the DSF. Recently, in the context of non-crossing path

models, Schertzer et al. [24, Theorem 6.6] have replaced criterion (B_2) with a *wedge condition* involving a suitable dual of the considered forest. In this paper, we provide new criteria (Theorem 27), similar in the spirit to [24, Theorem 6.6], in which criterion (B_2) is replaced with the fact that “no limiting primal and dual paths can spend positive Lebesgue time together”. This is condition (iv) of Theorem 27.

The second key tool is a new and general Laplace type argument, stated in Theorem 23, allowing to establish a coalescence time estimate for any couple of trajectories of the DSF (Theorem 21). Obtaining such coalescence time estimate is always a crucial step in the literature on the convergence to the BW. We also think that Theorem 23 is interesting in itself and very robust. In particular, it should provide the required coalescence time estimates for all the drainage network models in the basin of attraction of the BW [7, 8, 14, 22, 28]. See Remark 25 for further details. The coalescence time estimate for the DSF (Theorem 21) plays a central role in the proof of condition (iv) previously cited.

The third main ingredient is a very accurate study, conducted in Section 2, of the joint evolution of DSF trajectories. Exploiting the geometric properties of the DSF, we are able to exhibit some *renewal events* (at some random times) for the joint evolution of trajectories, i.e. some suitable configurations allowing us to recover a Markovian structure (see Proposition 14) w.r.t. a enhanced filtration. Moreover, we show that both time and width of the explored region (by the trajectories) between two consecutive renewal events admit sub-exponentially decays. Put together, these properties allows us to construct a process $\{Y_\ell : \ell \geq 0\}$, defined in (52), satisfying conditions of the Laplace argument, especially a supermartingale condition (Theorem 23 (ii)).

Application to the RST: the highways and byways problem.

In [3, Theorem 2.1], Baccelli and Bordenave also described the semi-infinite paths of the Radial Spanning Tree (RST). In particular, they showed that the (random) number χ_r of semi-infinite paths of the RST crossing the circle \mathcal{C}_r —centered at the origin O and with radius r —tends to infinity with probability 1 as $r \rightarrow \infty$. A natural question is then to specify the growth rate of χ_r w.r.t. the radius r . Since the article of Hammersley and Welsh [16], this question is known as the *highways and byways problem*.

A general method, recently proposed by Coupier [10] and applied to various geometrical random trees, asserts that χ_r is negligible w.r.t. r . Such result for the RST was already known since [4]. Furthermore, this method can be performed whenever the considered tree satisfies the two following conditions (see Section 6 of [10]). First, it can be approximated, locally and far from the origin, by a directed forest— as the DSF approximates the RST. Secondly, the approximating directed forest has to satisfy a suitable coalescence time estimate. Theorem 21 fulfills this last condition for the DSF. Hence, the method developed in Section 6 of [10] applies without major modifications to the RST and leads to the following result:

Theorem 3. *For any $\epsilon > 0$, χ_r and $\mathbb{E}\chi_r$ are a.s. negligible w.r.t. $r^{3/4+\epsilon}$ as r tends to infinity.*

Organization of the paper.

In Section 2.1, a discrete process called the *joint exploration process* is introduced to describe the joint evolution of DSF paths. The dependence structure of this process is encoded with the notion of *history set*. Some particular random times, called *renewal steps* and corresponding to the renewal events mentioned above, are put forward in Section 2.3. At these random times, the joint exploration process can be restarted— we will talk about *regenerated paths*—and thus exhibits some Markov properties (see Proposition 14) w.r.t. a suitable filtration defined in (28). In Proposition 17, the time between two consecutive renewal steps is stochastically dominated by a r.v. with exponential decay. The coalescence time estimate (Theorem 21) is stated in Section 3.1 while the Laplace criteria (Theorem 23) is established and proved in Section 3.2. In Section 4.1, we describe new criteria (Theorem 27) ensuring the weak convergence of a forest and a suitable dual to the BW and its dual, especially condition (iv) which is proved in Section 4.2.2.

2 Renewal steps and Markov property

This section aims at exhibiting renewal properties for the joint process made up of k trajectories in the DSF. We first define the discrete time joint exploration process of these k trajectories in Section 2.1. At each step n , the *history set* H_n contains all the current information generated by the k trajectories until the n -th step, which influences the next steps. Our main job here is to control this “dependence set”. In Section 2.2, we establish the existence of *good steps* at which the height of the history set is bounded by some constant κ . At these good steps, we can work on the unexplored part of the PPP—which does not overlap the history set—to expect nice configurations ensuring what we will call *renewal steps* (Section 2.3). As remarkable facts, we will show on the one hand that the number of steps between two consecutive renewal steps admits exponentially decaying tails (see Proposition 17) and, on the other hand that these renewal steps act as “breaking points” which allow us to divide the considered trajectories into independent blocks (see Proposition 14).

Several qualitative results of this paper involve constants. For the sake of clarity, we will use C_0 and C_1 to denote two positive constants, whose exact values may change from one line to the other. The important thing is that both C_0 and C_1 are universal constants whose values will depend only on the intensity of PPP, the number k of considered trajectories and the constant κ .

2.1 The discrete time joint exploration process

Let $k \in \mathbb{N}$ be a positive integer. Let us consider k starting points $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbb{R}^2$. In this section, following [22], we define a discrete time process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$ in an inductive way for the joint exploration of the k paths $\pi^{\mathbf{u}_1}, \dots, \pi^{\mathbf{u}_k}$ so that they move together. This discrete time process is the joint exploration process which makes the subject of this section. The sequence $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k)) : n \geq 0\}$ is a representation of the trajectories while $\{H_n : n \geq 0\}$ will be the associated dependence set.

To be consistent with the first move after a renewal step (see Section 2.3) we add an extra and initial information given by the couple $(H_0, \mathcal{N}_0^{\text{extra}})$ where:

- $H_0 = H_0(\mathbf{u}_1, \dots, \mathbf{u}_k)$ is a deterministic compact set in \mathbb{R}^2 whose precise definition will be given later. In the sequel, we will consider the PPP \mathcal{N} conditionally on having no points in H_0 .
- $\mathcal{N}_0^{\text{extra}}$ is a finite set of random points included in H_0 , independent to the PPP \mathcal{N} .

The role of the couple $(H_0, \mathcal{N}_0^{\text{extra}})$ will become clearer in Section 2.3 with the definition of renewal steps. Before this, to understand more easily the joint exploration process, the reader may assume that $H_0 = \mathcal{N}_0^{\text{extra}} = \emptyset$.

Set $g_0(\mathbf{u}_i) = \mathbf{u}_i$ for $i = 1, \dots, k$. In the joint exploration process, only the lowest vertex moves, denoted by W_n^{move} , while the $k-1$ other ones remains unchanged. In case several vertices have the same lower ordinate, we move them one by one starting from the leftmost one. Precisely:

- $W_0^{\text{move}} := \operatorname{argmin}\{\mathbf{w}(1) : \mathbf{w} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\} \text{ and } \mathbf{w}(2) = W_0^{\text{move}}(2)\}$ where $W_0^{\text{move}}(2) := \min\{\mathbf{u}_1(2), \dots, \mathbf{u}_k(2)\}$, and $W_0^{\text{stay}} := \{\mathbf{u}_1, \dots, \mathbf{u}_k\} \setminus W_0^{\text{move}}$;
- For $1 \leq i \leq k$,

$$g_1(\mathbf{u}_i) := \begin{cases} h(g_0(\mathbf{u}_i), (\mathcal{N} \setminus H_0) \cup (W_0^{\text{stay}} \cup \mathcal{N}_0^{\text{extra}})) & \text{if } g_0(\mathbf{u}_i) = W_0^{\text{move}} \\ g_0(\mathbf{u}_i) & \text{otherwise.} \end{cases}$$

After the first step, the history set H_0 is updated into $H_1 = H_1(\mathbf{u}_1, \dots, \mathbf{u}_k)$:

$$H_1 := \left(H_0 \cup \bigcup_{i=1}^k B^+(\mathbf{u}_i, \|g_1(\mathbf{u}_i) - \mathbf{u}_i\|_2) \right) \cap \mathbb{H}^+(W_1^{\text{move}}(2)),$$

where $W_1^{\text{move}}(2) := \min\{g_1(\mathbf{u}_i)(2) : 1 \leq i \leq k\}$.

By induction, given $(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n(\mathbf{u}_1, \dots, \mathbf{u}_k))$, for any $n \geq 1$, let us set

- (i) $W_n^{\text{move}} := \operatorname{argmin}\{\mathbf{w}(1) : \mathbf{w} \in \{g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k)\} \text{ and } \mathbf{w}(2) = W_n^{\text{move}}(2)\}$ where $W_n^{\text{move}}(2) := \min\{g_n(\mathbf{u}_1)(2), \dots, g_n(\mathbf{u}_k)(2)\}$, and $W_n^{\text{stay}} := \{g_n(\mathbf{u}_i) : 1 \leq i \leq k\} \setminus W_n^{\text{move}}$;
- (ii) For $1 \leq i \leq k$,

$$g_{n+1}(\mathbf{u}_i) := \begin{cases} h(g_n(\mathbf{u}_i), (\mathcal{N} \setminus H_n) \cup (W_n^{\text{stay}} \cup \mathcal{N}_0^{\text{extra}})) & \text{if } g_n(\mathbf{u}_i) = W_n^{\text{move}} \\ g_n(\mathbf{u}_i) & \text{otherwise.} \end{cases}$$

When $g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k)$ all have different ordinates– and this is a.s. the case whenever they are points of \mathcal{N} –, W_n^{move} is given by the $g_n(\mathbf{u}_i)$ having the smallest ordinate. When this smallest ordinate is realized by at least two $g_n(\mathbf{u}_i)$'s– this will happen during the renewal event –, then W_n^{move} corresponds to the one having the smallest abscissa.

After the $(n + 1)$ -th move, the new level $W_{n+1}^{\text{move}}(2) := \min\{g_{n+1}(\mathbf{u}_1)(2), \dots, g_{n+1}(\mathbf{u}_k)(2)\}$ allows to define the next history set $H_{n+1} = H_{n+1}(\mathbf{u}_1, \dots, \mathbf{u}_k)$:

$$H_{n+1} := \left(H_n \cup \bigcup_{i=1}^k B^+(g_n(\mathbf{u}_i), \|g_{n+1}(\mathbf{u}_i) - g_n(\mathbf{u}_i)\|_2) \right) \cap \mathbb{H}^+(W_{n+1}^{\text{move}}(2)).$$

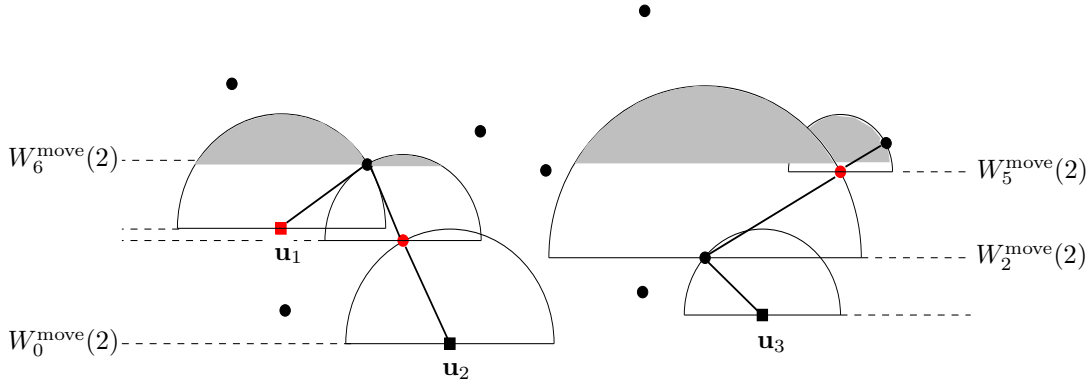


Figure 3: Here are the first 6 steps of the joint process $(g_n(\mathbf{u}_1), g_n(\mathbf{u}_2), g_n(\mathbf{u}_3))_{n \geq 0}$ starting from $\mathbf{u}_1, \mathbf{u}_2, \mathbf{u}_3$ (given by the squares). To simplify the picture, we take $H_0 = \mathcal{N}_0^{\text{extra}} = \emptyset$. The first move concerns \mathbf{u}_2 , i.e. $W_0^{\text{move}} = \mathbf{u}_2$, while the second and third ones concern the trajectory starting at \mathbf{u}_3 . The triplet $(g_3(\mathbf{u}_1), g_3(\mathbf{u}_2), g_3(\mathbf{u}_3))$ is represented by red vertices. After the third step, \mathbf{u}_1 has not moved yet: $g_3(\mathbf{u}_1) = \mathbf{u}_1$. The grey area corresponds to $H_6(\mathbf{u}_1, \mathbf{u}_1, \mathbf{u}_3)$. On both sides of the picture, the levels $W_n^{\text{move}}(2)$, $0 \leq n \leq 6$, are indicated. Of course, the sequence $(W_n^{\text{move}}(2))_{n \geq 0}$ is non-decreasing almost surely.

Let us remark that the moving vertex W_n^{move} necessarily belongs to the (closed) set H_n . Either W_n^{move} belongs to the interior of H_n and this can happen only if W_n^{move} is an element of $\mathcal{N}_0^{\text{extra}}$ or one of the initial points $\{\mathbf{u}_1, \dots, \mathbf{u}_k\}$. Or W_n^{move} is on the boundary of H_n : see Figure 3.

Furthermore, when the ancestor of W_n^{move} equals to an element of W_n^{stay} then two paths among the $\pi^{\mathbf{u}_1}, \dots, \pi^{\mathbf{u}_k}$ coalesce.

Again, the fact that the interior part of H_n avoids the PPP \mathcal{N} provides information on which the next steps of the joint exploration process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$ depend. This kills all direct Markov properties. A tool to deal with this difficulty consists in the use of an auxiliary discrete time process $\{\tilde{g}_n(\mathbf{u}_1), \dots, \tilde{g}_n(\mathbf{u}_k), \tilde{H}_n : n \geq 0\}$ starting from $(\mathbf{u}_1, \dots, \mathbf{u}_k)$. This new process obeys to the same evolution rule than the original one but each move uses a new Poisson point process on \mathbb{R}^2 , independent of those previously used.

Precisely, let us consider a collection $\{\mathcal{N}_n : n \in \mathbb{N}\}$ of i.i.d. Poisson point processes on \mathbb{R}^2 , independent of the original process \mathcal{N} from which $\{g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n : n \geq 0\}$ is defined. Set $\tilde{g}_0(\mathbf{u}_i) = \mathbf{u}_i$ for $1 \leq i \leq k$, $\tilde{H}_0 = H_0$ and consider $\mathcal{N}_0^{\text{extra}} \subset \tilde{H}_0$ as above.

- (i) $\tilde{W}_0^{\text{move}} := \operatorname{argmin}\{\mathbf{w}(1) : \mathbf{w} \in \{\mathbf{u}_1, \dots, \mathbf{u}_k\} \text{ and } \mathbf{w}(2) = \tilde{W}_0^{\text{move}}(2)\}$ where $\tilde{W}_0^{\text{move}}(2) := \min\{\mathbf{u}_1(2), \dots, \mathbf{u}_k(2)\}$, and $\tilde{W}_0^{\text{stay}} := \{\mathbf{u}_1, \dots, \mathbf{u}_k\} \setminus \tilde{W}_0^{\text{move}}$;
- (ii) For $1 \leq i \leq k$,

$$\tilde{g}_1(\mathbf{u}_i) := \begin{cases} h(\tilde{g}_0(\mathbf{u}_i), (\mathcal{N}_1 \setminus \tilde{H}_0) \cup (\tilde{W}_0^{\text{stay}} \cup \mathcal{N}_0^{\text{extra}})) & \text{if } \tilde{g}_0(\mathbf{u}_i) = \tilde{W}_0^{\text{move}} \\ \tilde{g}_0(\mathbf{u}_i) & \text{otherwise.} \end{cases}$$

The history set $\tilde{H}_1 = \tilde{H}_1(\mathbf{u}_1, \dots, \mathbf{u}_k)$ after the first move is defined as:

$$\tilde{H}_1 := \left(\tilde{H}_0 \cup \bigcup_{i=1}^k B^+(\mathbf{u}_i, \|\tilde{g}_1(\mathbf{u}_i) - \mathbf{u}_i\|_2) \right) \cap \mathbb{H}^+(\tilde{W}_1^{\text{move}}(2)),$$

where $\tilde{W}_1^{\text{move}}(2) := \min\{\tilde{g}_1(\mathbf{u}_i)(2) : 1 \leq i \leq k\}$.

Given $(\tilde{g}_n(\mathbf{u}_1), \dots, \tilde{g}_n(\mathbf{u}_k), \tilde{H}_n)$ let

- (i) $\tilde{W}_n^{\text{move}} := \operatorname{argmin}\{\mathbf{w}(1) : \mathbf{w} \in \{\tilde{g}_n(\mathbf{u}_1), \dots, \tilde{g}_n(\mathbf{u}_k)\} \text{ and } \mathbf{w}(2) = \tilde{W}_n^{\text{move}}(2)\}$ where $\tilde{W}_n^{\text{move}}(2) := \min\{\tilde{g}_n(\mathbf{u}_1)(2), \dots, \tilde{g}_n(\mathbf{u}_k)(2)\}$, and $\tilde{W}_n^{\text{stay}} := \{\tilde{g}_n(\mathbf{u}_i) : 1 \leq i \leq k\} \setminus \tilde{W}_n^{\text{move}}$;
- (ii) For $1 \leq i \leq k$,

$$\tilde{g}_{n+1}(\mathbf{u}_i) := \begin{cases} h(\tilde{g}_n(\mathbf{u}_i), (\mathcal{N}_{n+1} \setminus \tilde{H}_n) \cup (\tilde{W}_n^{\text{stay}} \cup \mathcal{N}_0^{\text{extra}})) & \text{if } \tilde{g}_n(\mathbf{u}_i) = \tilde{W}_n^{\text{move}} \\ \tilde{g}_n(\mathbf{u}_i) & \text{otherwise.} \end{cases}$$

The joint history set $\tilde{H}_{n+1} = \tilde{H}_{n+1}(\mathbf{u}_1, \dots, \mathbf{u}_k)$ at the $(n+1)$ -th move is given by:

$$\tilde{H}_{n+1} := \left(\tilde{H}_n \cup \bigcup_{i=1}^k B^+(\tilde{g}_n(\mathbf{u}_i), \|\tilde{g}_{n+1}(\mathbf{u}_i) - \tilde{g}_n(\mathbf{u}_i)\|_2) \right) \cap \mathbb{H}^+(\tilde{W}_{n+1}^{\text{move}}(2)),$$

where $\tilde{W}_{n+1}^{\text{move}}(2) = \min\{\tilde{g}_{n+1}(\mathbf{u}_i)(2) : 1 \leq i \leq k\}$.

The use of independent PPP's at each move will be very useful to exhibit independent r.v.'s in the sequel. This amounts to throwing at each step of the construction a new PPP outside the region already explored, namely the dependence set. This technique was already used in [3] without being clearly stated.

In the sequel, we need to work with a filtration including roughly all the information on the point process until the current step and the initial information given by $\{\mathcal{N} \cap H_0 = \emptyset\}$ and $\mathcal{N}_0^{\text{extra}}$. For any integer n , let

$$\mathcal{F}_n := \sigma\left\{g_l(\mathbf{u}_i), i \in \{1, \dots, k\}, l \leq n, \mathcal{N} \cap H_0 = \emptyset, \mathcal{N}_0^{\text{extra}}\right\}.$$

The next result concludes this section.

Proposition 4. *The joint process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$ is an (\mathcal{F}_n) -Markov chain with state space $(\mathbb{R}^2)^k \times \{A \subseteq \mathbb{R}^2 : A \text{ is compact}\}$. Moreover, the processes $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$ and $\{(\tilde{g}_n(\mathbf{u}_1), \dots, \tilde{g}_n(\mathbf{u}_k), \tilde{H}_n) : n \geq 0\}$ are identically distributed.*

Proof. Let $\{\mathcal{N}_n : n \in \mathbb{N}\}$ be a collection of i.i.d. Poisson processes on \mathbb{R}^2 . Given $(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) = (\mathbf{x}_1, \dots, \mathbf{x}_k, \Lambda_n)$ for some $\mathbf{x}_1, \dots, \mathbf{x}_k \in \mathbb{R}^2$ and $\Lambda_n \subset \mathbb{R}^2$, we observe that the region $\mathbb{H}^+(\min\{\mathbf{x}_i(2) : 1 \leq i \leq k\}) \setminus \Lambda_n$ has not been explored yet and the Poisson point process \mathcal{N} on this region can be replaced with any independent Poisson point process \mathcal{N}_{n+1} . Conditionally to $(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) = (\mathbf{x}_1, \dots, \mathbf{x}_k, \Lambda_n)$, $\{(g_j(\mathbf{u}_1), \dots, g_j(\mathbf{u}_k), H_j) : j < n\}$, $\mathcal{N} \cap H_0 = \emptyset$ and $\mathcal{N}_0^{\text{extra}}$,

$$g_{n+1}(\mathbf{u}_i) \stackrel{d}{=} \begin{cases} h(\mathbf{x}_i, (\mathcal{N}_{n+1} \setminus \Lambda_n) \cup (\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \cup \mathcal{N}_0^{\text{extra}})) & \text{if } \mathbf{x}_i = W_n^{\text{move}} \\ \mathbf{x}_i & \text{otherwise.} \end{cases}$$

For $\mathbf{x}_i = W_n^{\text{move}}$, setting $\mathbf{x}'_i := h(\mathbf{x}_i, (\mathcal{N}_{n+1} \setminus \Lambda_n) \cup (\{\mathbf{x}_1, \dots, \mathbf{x}_k\} \cup \mathcal{N}_0^{\text{extra}}))$, we have

$$H_{n+1} \stackrel{d}{=} (B^+(\mathbf{x}_i, \|\mathbf{x}_i - \mathbf{x}'_i\|_2) \cup \Lambda_n) \cap \mathbb{H}^+(\mathbf{x}'_i(2) \wedge \min\{\mathbf{x}_j(2) : j \neq i\}) .$$

Hence, the original joint exploration process and the auxiliary one have the same transition probabilities. They are identically distributed.

Moreover, conditionally on \mathcal{F}_n , the process $(g_{n+1}(\mathbf{u}_1), \dots, g_{n+1}(\mathbf{u}_k), H_{n+1})$ admits a random mapping representation of the form

$$(g_{n+1}(\mathbf{u}_1), \dots, g_{n+1}(\mathbf{u}_k), H_{n+1}) \stackrel{d}{=} f((g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n), \mathcal{N}_{n+1})$$

for some measurable mapping f , which ensures its Markovian character (see [17]). \square

2.2 Good steps

In this Section 2.2, we will work with the process $\{(\tilde{g}_n(\mathbf{u}_1), \dots, \tilde{g}_n(\mathbf{u}_k), \tilde{H}_n) : n \geq 0\}$ only and for ease of notation we denote this auxiliary process itself by $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$.

Let us define the height of any non empty bounded subset Δ of \mathbb{R}^2 , as

$$L(\Delta) := \sup\{\mathbf{y}(2) - \mathbf{x}(2) : \mathbf{x}, \mathbf{y} \in \Delta\}$$

and $L(\emptyset) = 0$. The goal of this section consists in stating that the height of the history set $L(H_n)$ is regularly smaller than a given positive constant κ which will be specified in (6). This is Proposition 5.

Let us set $\tau_0 = \tau_0(\mathbf{u}_1, \dots, \mathbf{u}_k) = 0$ and for $j \geq 1$,

$$\tau_j = \tau_j(\mathbf{u}_1, \dots, \mathbf{u}_k) := \inf\{kn > \tau_{j-1} : n \geq 1, L(H_{kn}) \leq \kappa \text{ and } W_{kn}^{\text{move}}(2) \geq W_{\tau_{j-1}}^{\text{move}}(2) + \kappa + 1\} . \quad (4)$$

Such a step is called a *good step* of the joint process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$. At a good step, the height of the history set is at most κ . As additional and technical requirements, τ_j has to be a multiple of the number k of trajectories. This condition portends that in the sequel we will consider blocks of k consecutive steps. Moreover, the condition that $W_{\tau_j}^{\text{move}}(2) - W_{\tau_{j-1}}^{\text{move}}(2)$ should be more than $\kappa + 1$ is to ensure that the regions involved by different renewal steps are disjoint. Let us also remark that the τ_j 's are stopping times w.r.t. the filtration $(\mathcal{F}_n)_{n \geq 0}$.

In Section 2.3, we will select some suitable (in some sense) good steps and will call them *renewal steps*. Before this, Proposition 5 states that the number of steps between two consecutive good steps can be stochastically dominated by a r.v. having exponential decay.

To obtain Proposition 5, we make the two following assumptions. First, the initial history set H_0 is such that:

$$H_0 \subset \mathbb{R} \times [W_0^{\text{move}}(2), W_0^{\text{move}}(2) + 1] . \quad (5)$$

In particular, this means that points of the extra set $\mathcal{N}_0^{\text{extra}}$ can be used by the joint exploration process only at the very beginning, i.e. while the moving vertex satisfies $W_n^{\text{move}}(2) \leq W_0^{\text{move}}(2) + 1$. Moreover, the constant κ is chosen such that

$$\max_{i,j} |\mathbf{u}_i(2) - \mathbf{u}_j(2)| \leq \kappa . \quad (6)$$

Notice also that (6) does not prevent the starting points to having the same height, and we can have $\mathbf{u}_1(2) = \dots = \mathbf{u}_k(2)$. Although this does not happen a.s. for the points of the PPP, this will happen in later constructions, when we will deal with renewal steps.

Here is the main result of Section 2.2.

Proposition 5. *Let $j \geq 0$. Under Assumptions (5) and (6), there exists a r.v. T whose distribution does not depend on \mathcal{F}_{τ_j} such that, for all n ,*

$$\mathbb{P}(\tau_{j+1} - \tau_j \geq n \mid \mathcal{F}_{\tau_j}) \leq \mathbb{P}(T \geq n) \leq C_0 e^{-C_1 n} . \quad (7)$$

We will prove Proposition 5 through a sequence of lemmas. To understand how our proof is organized, let us start with describing the evolution of the height of the history set during one step. Two situations may actually occur. If the semi-ball $B^+(W_n^{\text{move}}, \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2)$ created during the $(n+1)$ -th move, does not exceed the horizontal line $\{\mathbf{x} : \mathbf{x}(2) = W_n^{\text{move}}(2) + L(H_n)\}$ then

$$L(H_{n+1}) = L(H_n) - (W_{n+1}^{\text{move}}(2) - W_n^{\text{move}}(2)) .$$

In this case, the height of the history set is decreasing and, on some suitable events (occurring with positive probability), we will be able to quantify how much it is decreasing. See Lemmas 8, 9 and 10.

Otherwise, the new height $L(H_{n+1})$ is realized by the semi-ball $B^+(W_n^{\text{move}}, \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2)$ and

$$L(H_{n+1}) = \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2 - (W_{n+1}^{\text{move}}(2) - W_n^{\text{move}}(2)) .$$

In this second case, the height of the history set may increase or not. A priori, a large distance $\|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2$ should occur with small probability since this would force the PPP to avoid the (large) semi-ball $B^+(W_n^{\text{move}}, \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2)$. However, a large part of that semi-ball can be already covered by the history set H_n , which by definition avoids the PPP. Hence, having a large distance $\|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2$ becomes quite possible. Lemmas 6 and 7 allow us to overcome this obstacle and to control the growth of $L(H_n)$.

In both situations, the sequence $\{L(H_n) : n \geq 0\}$ satisfies the following fundamental and useful induction relation: a.s. and for any n ,

$$L(H_{n+1}) \leq \max\{L(H_n), \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2\} . \quad (8)$$

At the end of this section, we will combine these results in Lemmas 11 and 12 to get Proposition 5.

Part 1: How much $L(H_n)$ is increasing.

Let us introduce some notation. For a real number $l > 0$ and an integer $n \geq 0$, let us set

$$g_n^{\uparrow, l} := W_n^{\text{move}} + (0, l)$$

(recall that W_n^{move} always denotes a single point). Let

$$C_{\pi/4}^{(+)}(\mathbf{0}) := \{re^{i\theta} : r > 0, \theta \in [\pi/2, 3\pi/4]\} \quad \text{and} \quad C_{\pi/4}^{(-)}(\mathbf{0}) := \{re^{i\theta} : r > 0, \theta \in [\pi/4, \pi/2]\}$$

be the two cones with apex $\mathbf{0}$ and making an angle $\pi/4$ with the ordinate axis. We also define, for $\mathbf{x} \in \mathbb{R}^2$ and $i \in \{-, +\}$,

$$C_{\pi/4}^{(i)}(\mathbf{x}) := \mathbf{x} + C_{\pi/4}^{(i)}(\mathbf{0}) \quad \text{and} \quad C_{\pi/2}(\mathbf{x}) := C_{\pi/4}^{(+)}(\mathbf{x}) \cup C_{\pi/4}^{(-)}(\mathbf{x}) .$$

Conditionally to the current configuration $(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n)$, the next lemma exhibits deterministic regions avoiding the history set H_n . Such regions are unexplored yet and will allow us to control how the history set grows (see Lemma 7). Notice that Baccelli and Bordenave used in [3, Lemma 4.2] a similar geometric argument, but which was false. Actually, it is impossible to exhibit a cone, with a positive and deterministic angle and with apex the moving vertex W_n^{move} , which almost surely avoids the history set H_n . To get such property, the cone has to be pushed upward and this is what we do with $g_n^{\uparrow, l}$.

Lemma 6. *Let us consider the exploration process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$ under Assumptions (5) and (6), and assume that one of the two following hypotheses holds:*

- (i) *Either we have $\forall 1 \leq i < i' \leq k, \mathbf{u}_i(2) \neq \mathbf{u}_{i'}(2)$,*
- (ii) *Or $\mathbf{u}_1(2) = \dots = \mathbf{u}_k(2)$ and $\forall 1 \leq i \leq k, h(\mathbf{u}_i) \in B^+(\mathbf{u}_i, 1)$.*

Then, we have for all $n \geq 0$ and for any $l \geq (L(H_n)/2) \vee 1$, the cone $C_{\pi/2}(g_n^{\uparrow, l})$ a.s. avoids the history set H_n , i.e. $C_{\pi/2}(g_n^{\uparrow, l}) \cap H_n = \emptyset$.

Assumption (i) of the lemma corresponds to the case when we start from Poisson points that have a.s. distinct ordinates. But in the sequel, in particular at renewal steps, we will sometimes start paths at deterministic points aligned on the same horizontal line, which justifies our Assumption (ii). The fact that the radii of the half balls in Assumption (ii) are 1 explains the ' $\vee 1$ ' of the condition ' $l \geq (L(H_n)/2) \vee 1$ '.

Remark also that although the unexplored cone $C_{\pi/2}(g_n^{\uparrow, L(H_n)/2 \vee 1})$ avoids the history set H_n , it could contain a starting point $g_n(\mathbf{u}_i) = \mathbf{u}_i$ which has not moved yet and could still be outside H_n .

Proof. Under Assumptions (5) and (ii), the result follows easily. Let us focus on Assumption (i). The proof is based on elementary geometric computation. Fix $l \geq (L(H_n)/2) \vee 1$. Let us first assume that the moving vertex W_n^{move} is not an element of $\mathcal{N}_0^{\text{extra}}$. By construction, this forces W_n^{move} to be on the boundary of H_n . By translation and symmetry, we can assume without loss of generality that $g_n^{\uparrow, l} = (0, 0)$. So $W_n^{\text{move}} = (0, -l)$. The worst case is obtained when the semi-ball B^+ realizing the height $L(H_n)$ is tangent to W_n^{move} with a maximal ordinate, i.e. $B^+ = B^+(A, 2l)$ with $A = (W_n^{\text{move}}(1) + 2l, W_n^{\text{move}}(2)) = (2l, -l)$. If the history set H_n overlaps the cone $C_{\pi/2}(g_n^{\uparrow, l})$ then the point $M = (l/2, l/2)$ has to belong to $B^+(A, 2l)$ since it is the point of the cone which is the closest to A . But $\|A - M\|_2^2 = 18l^2/4 > (2l)^2$.

When W_n^{move} belongs to the interior of H_n , i.e. when $W_n^{\text{move}} \in \mathcal{N}_0^{\text{extra}} \subset H_0$, the cone $C_{\pi/2}(g_n^{\uparrow, l})$ could overlap H_0 (and then possibly H_n). But this situation is forbidden since $l \geq (L(H_n)/2) \vee 1 \geq 1 \geq L(H_0)$ by (5). \square

For $n \geq 0$, we denote by ζ_{n+1} the distance between $g_n^{\uparrow, L(H_n)/2 \vee 1}$ and its nearest Poisson point inside the unexplored cone $C_{\pi/2}(g_n^{\uparrow, L(H_n)/2 \vee 1})$:

$$\zeta_{n+1} := \inf \left\{ \|g_n^{\uparrow, L(H_n)/2 \vee 1} - \mathbf{x}\|_2 : \mathbf{x} \in \mathcal{N}_{n+1} \cap C_{\pi/2}(g_n^{\uparrow, L(H_n)/2 \vee 1}) \right\} . \quad (9)$$

As we will consider blocks of k consecutive steps in the sequel, let us introduce for $n \geq 0$

$$X_{n+1} := \sum_{j=1}^k (\lfloor 2\zeta_{kn+j} \rfloor + 2) . \quad (10)$$

The next result says that when the height of the history set increases between steps kn and $k(n+1)$ then the new height, i.e., $L(H_{k(n+1)})$, is bounded from above by the r.v. X_{n+1} which admits an exponential tail.

Lemma 7. *Using the previous notations:*

- (i) *For all $n \geq 0$, the following inequality holds with probability 1:*

$$L(H_{k(n+1)}) \mathbf{1}_{\{L(H_{k(n+1)}) > L(H_{kn})\}} \leq X_{n+1} .$$

- (ii) *The r.v.'s $\{X_{n+1} : n \geq 0\}$ are i.i.d. and satisfy $\forall n, m \in \mathbb{N}$,*

$$\mathbb{P}(X_{n+1} > m) \leq C_0 e^{-C_1 m} . \quad (11)$$

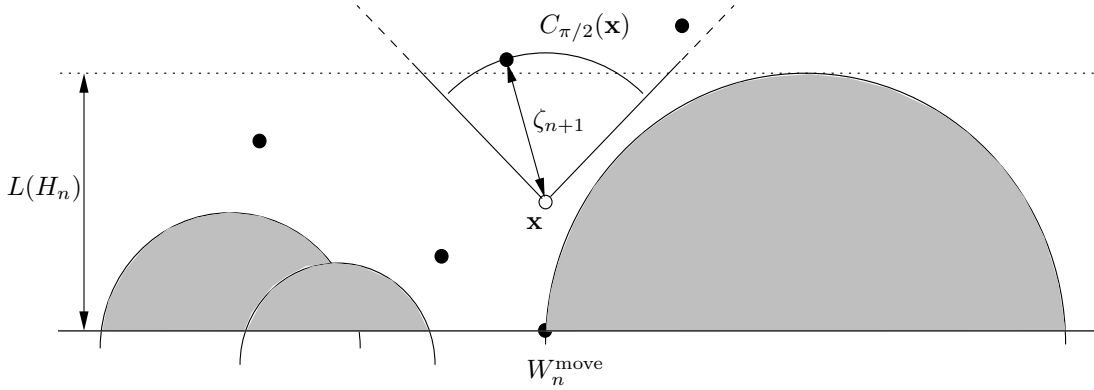


Figure 4: Black vertices are Poisson points. The gray area corresponds to the history set H_n . The white point—denoted by \mathbf{x} —is $g_n^{\uparrow, L(H_n)/2\vee 1}$. The cone with apex \mathbf{x} and bisector the ordinate axis is the unexplored cone and avoids the history set H_n .

Proof. We are going to state that a.s.

$$L(H_{n+1}) \leq \max\{L(H_n), \lfloor 2\zeta_{n+1} \rfloor + 2\} \quad (12)$$

which leads to Item (i) for $k = 1$. The proof of Item (i) for any $k \geq 1$ follows by applying (12) k times. Let us first consider the case $L(H_n) < 2$. Then, denoting by \mathbf{y} the element of the unexplored cone realizing the r.v. ζ_{n+1} , it follows:

$$\begin{aligned} \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2 &\leq \|\mathbf{y} - W_n^{\text{move}}\|_2 \\ &\leq \zeta_{n+1} + \|g_n^{\uparrow, L(H_n)/2\vee 1} - W_n^{\text{move}}\|_2 \\ &= \zeta_{n+1} + 1 \\ &\leq \lfloor 2\zeta_{n+1} \rfloor + 2 = X_{n+1}. \end{aligned}$$

We then conclude using the fundamental induction relation (8).

Now, consider the case $L(H_n) \geq 2$ and assume that $L(H_{n+1}) > L(H_n)$. By (8), we must have $\|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2 \geq L(H_{n+1}) > L(H_n)$ which ensures that

$$B^+(g_n^{\uparrow, L(H_n)/2}, \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2/2) \subseteq B^+(W_n^{\text{move}}, \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2).$$

Since the semi-ball $B^+(W_n^{\text{move}}, \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2)$ avoids \mathcal{N}_{n+1} , it follows:

$$\zeta_{n+1} \geq \|h(W_n^{\text{move}}) - W_n^{\text{move}}\|_2/2 > L(H_{n+1})/2.$$

Then, $L(H_{n+1})$ is smaller than $\lfloor 2\zeta_{n+1} \rfloor + 1$ and (12) follows.

Item (ii) is mainly based on the independence between random variables ζ_{n+1} , $n \in \mathbb{N}$, which is due to the fact that independent PPP's are used for each step of the joint process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$. Moreover, by Lemma 6, the ζ_{n+1} 's are (independent and) identically distributed and admit an exponential tail. The same holds for the X_{n+1} 's. \square

Part 2: How much $L(H_n)$ is decreasing.

Now let us show that $(L(H_n))_{n \geq 0}$ is submitted to a ‘negative drift’ so that the sequence regularly returns to small values. Precisely, with positive probability, the ordinate of the moving vertex increases of at least 1 between the kn -th and $k(n+1)$ -th steps. This also allows to control the number of steps

needed for the ordinate of the moving vertex to reach a distance at least $\kappa + 1$ from the last good step.

For $\mathbf{x} \in \mathbb{R}^2$ and for $w, l > 0$, the rectangle with width $2w$ and height l , whose base is centered at \mathbf{x} , is denoted by

$$\text{Rec}(\mathbf{x}; w, l) := \mathbf{x} + [-w, w] \times [0, l] .$$

Thus we set

$$l_n := \inf\{l \geq 0 : \text{Area}(\text{Rec}(g_n^{\uparrow,1}; 1, l) \setminus H_n) \geq 1/2\} . \quad (13)$$

In other words, l_n is the random height of the rectangle centered at $W_n^{\text{move}} + (0, 1)$ with width 2 so that the area of its unexplored part becomes at least $1/2$. The justification of the constant $1/2$ in the definition of l_n will appear in the proof of Lemma 9. Besides, the overlap of $\text{Rec}(g_n^{\uparrow,1}; 1, L(H_n)/2 \vee 1)$ with the unexplored cone $C_{\pi/2}(g_n^{\uparrow, L(H_n)/2 \vee 1})$ has area 1. Thanks to Lemma 6, this means that a.s.

$$l_n \leq \frac{L(H_n)}{2} \vee 1 . \quad (14)$$

For any integer $n \geq 0$, let I_{n+1} be the indicator random variable defined as

$$I_{n+1} := \mathbf{1}_{\{(\text{Rec}(g_n^{\uparrow,1}; 1, l_n) \setminus H_n) \cap \mathcal{N}_{n+1} \neq \emptyset \text{ and } \text{Rec}(W_n^{\text{move}}; 5, 1) \cap \mathcal{N}_{n+1} = \emptyset\}} .$$

Let us now explain the ideas behind Lemmas 8, 9 and 10. First notice that $\text{Rec}(g_n^{\uparrow,1}; 1, l_n) \setminus H_n$ and $\text{Rec}(W_n^{\text{move}}; 5, 1)$ are two disjoint regions with area $1/2$ and 10 respectively. So the events indicated by the I_{n+1} 's all occur with the same fixed positive probability, denoted by p_0 in Lemma 8. Such an event will be pleasant in the sense that, provided there is no points of W_n^{stay} in the horizontal rectangle $\text{Rec}(W_n^{\text{move}}; 5, 1)$, the ancestor $h(W_n^{\text{move}})$ advances by at least 1 in ordinate w.r.t. W_n^{move} . Combining this with (14) should force the height of the history set to decrease by at least 1 during the $(n+1)$ -th move. However, it can happen that some points of W_n^{stay} are in $\text{Rec}(W_n^{\text{move}}; 5, 1)$ (or $\text{Rec}(g_n^{\uparrow,1}; 1, l_n)$) as illustrated in Figure 5. In this case, $h(W_n^{\text{move}}) \in W_n^{\text{stay}}$ and the increment $h(W_n^{\text{move}})(2) - W_n^{\text{move}}(2)$ cannot be bounded from below. But, this situation corresponds to the coalescence of two paths among the $\pi^{\mathbf{u}_1}, \dots, \pi^{\mathbf{u}_k}$. Here is the reason why we consider blocks of k consecutive steps: on the event $\{\prod_{j=1}^k I_{kn+j} = 1\}$ where such events occur between the kn -th and the $k(n+1)$ -th steps, the ordinate of the current moving vertex has to advance by at least 1 (Lemma 9) and the history set decreases by at least 1 (Lemma 10).

Lemma 8. *Let $p_0 := (1 - e^{-\lambda/2})e^{-10\lambda} > 0$ where λ denotes the (common) intensity of the Poisson point processes. Then, for any $n \geq 0$,*

$$\mathbb{P}(I_n = 1) = p_0 \quad \text{and} \quad \mathbb{P}\left(\prod_{j=1}^k I_{kn+j} = 1\right) = p_0^k .$$

Proof. Recall that the process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$ is an \mathcal{F}_n -Markov chain. Then,

$$\begin{aligned} \mathbb{P}(I_{n+1} = 1) &= \mathbb{E}(\mathbb{E}(I_{n+1} | \mathcal{F}_n)) \\ &= \mathbb{E}(\mathbb{P}((\text{Rec}(g_n^{\uparrow,1}; 1, l_n) \setminus H_n) \cap \mathcal{N}_{n+1} \neq \emptyset \text{ and } \text{Rec}(W_n^{\text{move}}; 5, 1) \cap \mathcal{N}_{n+1} = \emptyset | \mathcal{F}_n)) \\ &= \mathbb{E}(\mathbb{P}((\text{Rec}(g_n^{\uparrow,1}; 1, l_n) \setminus H_n) \cap \mathcal{N}_{n+1} \neq \emptyset | \mathcal{F}_n)) \mathbb{P}(\text{Rec}(W_n^{\text{move}}; 5, 1) \cap \mathcal{N}_{n+1} = \emptyset | \mathcal{F}_n)) \\ &= (1 - e^{-\lambda/2})e^{-10\lambda} (= p_0) \end{aligned}$$

because the random set $\text{Rec}(g_n^{\uparrow,1}; 1, l_n) \setminus H_n$ is a.s. of area $1/2$. Thus,

$$\begin{aligned} \mathbb{P}\left(\prod_{j=1}^k I_{kn+j} = 1\right) &= \mathbb{E}\left(\mathbb{E}\left(\prod_{j=1}^k I_{kn+j} \mid \mathcal{F}_{k(n+1)-1}\right)\right) \\ &= \mathbb{E}\left(\prod_{j=1}^{k-1} I_{kn+j} \mathbb{E}(I_{k(n+1)} \mid \mathcal{F}_{k(n+1)-1})\right) \\ &= p_0 \mathbb{E}\left(\prod_{j=1}^{k-1} I_{kn+j}\right) = p_0^k . \end{aligned}$$

□

Lemma 9. *On the event $\{\prod_{j=1}^k I_{kn+j} = 1\}$, the ordinate of the moving vertex has to increase by at least 1 between the kn -th and the $k(n+1)$ -th steps with probability 1:*

$$W_{k(n+1)}^{\text{move}}(2) \geq W_{kn}^{\text{move}}(2) + 1 . \quad (15)$$

Proof. Let us first prove it for only one path, i.e. $k = 1$. On the event $\{I_{n+1} = 1\}$, the rectangle $\text{Rec}(g_n^{\uparrow,1}; 1, l_n)$ contains at least one Poisson point. So, $X := h(W_n^{\text{move}}) = W_{n+1}^{\text{move}}$ belongs to $B^+(W_n^{\text{move}}, l_n + 2)$. Let us prove that $X(2) \geq W_n^{\text{move}}(2) + 1$. This is clear by definition of $\{I_{n+1} = 1\}$ whenever $l_n \leq 3$. We can now focus on the case $l_n > 3$. Without loss of generality we assume that $W_n^{\text{move}} = (0, 0)$. Since $k = 1$ (and so $W_n^{\text{stay}} = \emptyset$), it suffices to prove that the set

$$U := \{\mathbf{x} \in \mathbb{R}^2 : 1 \leq |\mathbf{x}(1)| \leq l_n + 2 \text{ and } 0 \leq \mathbf{x}(2) \leq 1\}$$

is included in H_n . To do it, let us remark that both points $A := (1, l_n + 1/2)$ and $B := (-1, l_n + 1/2)$ belong to the history set H_n . Otherwise, the region $\text{Rec}(g_n^{\uparrow,1}; 1, l_n) \setminus H_n$ would contain at least one of the two rectangles $[B(1), 0] \times [l_n + 1/2, l_n + 1]$ or $[0, A(1)] \times [l_n + 1/2, l_n + 1]$, each of area $1/2$, which is impossible by definition of l_n (recall (13)). Now, it is not difficult to check that any semi-ball $B^+(g_m(\mathbf{u}_1), \cdot)$, for $0 \leq m \leq n - 1$, which contains A but not $W_n^{\text{move}} = (0, 0)$ in its interior, also contains the strip $[1, l_n + 2] \times [0, 1]$ when $l_n > 3$. By symmetry, the same holds for the left part of U .

It remains to prove (15) for any $k \geq 2$. If $W_{kn+1}^{\text{move}}(2)$ is already larger than $W_{kn}^{\text{move}}(2) + 1$ then this is also the case for $W_{k(n+1)}^{\text{move}}(2)$. Otherwise, the ancestor of W_{kn}^{move} coincides with an element of W_{kn}^{stay} : this is the tricky situation described in Figure 5. Actually the worst case is the following: $W_{kn}^{\text{move}}, \dots, W_{k(n+1)-2}^{\text{move}}$ are $k - 1$ different vertices which have all merged with $W_{k(n+1)-1}^{\text{move}}$ during the $k - 1$ last steps. In other words, the k paths starting from $\mathbf{u}_1, \dots, \mathbf{u}_k$ were still disjoint at the kn -th step but have all coalesced $k - 1$ steps after. Then, it remains to apply the argument for $k = 1$ to the only remaining path, i.e. to $W_{k(n+1)-1}^{\text{move}}$:

$$W_{k(n+1)}^{\text{move}}(2) = h(W_{k(n+1)-1}^{\text{move}})(2) \geq W_{k(n+1)-1}^{\text{move}}(2) + 1 \geq W_{kn}^{\text{move}}(2) + 1 .$$

It is worth pointing out here that the above tricky situation—see also Figure 5—justifies that we consider blocks of k steps when defining the τ_j 's. □

Lemma 9 leads to the next result which provides a ‘drift condition’. Precisely, on the event $\{\prod_{j=1}^k I_{kn+j} = 1\}$, the height of the history set between the steps kn and $k(n+1)$ has to decrease by at least 1 if it is larger than κ .

Lemma 10. *Without loss of generality we can now assume that κ (the constant appearing in the definition of the τ_j 's (4)) is an integer larger than 6. For any $n \geq 0$, on the event $\{\prod_{j=1}^k I_{kn+j} = 1\}$, the following inequality holds almost surely:*

$$L(H_{k(n+1)}) \leq \max\{L(H_{kn}) - 1, \kappa\} .$$

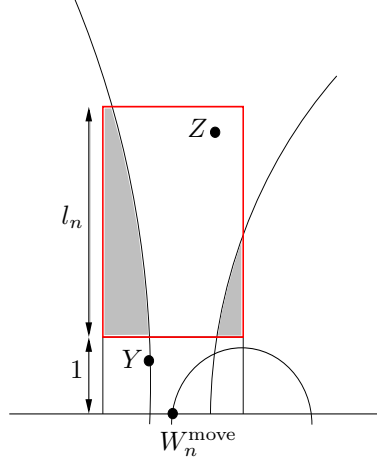


Figure 5: The red rectangle represents $\text{Rec}(g_n^{\uparrow,1}; 1, l_n)$, where $g_n^{\uparrow,1} = W_n^{move} + (0, 1)$, which is partially covered by the history set H_n (the gray regions). By definition of l_n the area of $\text{Rec}(g_n^{\uparrow,1}; 1, l_n) \setminus H_n$ is equal to $1/2$. The vertex Y is an element of W_n^{stay} while Z is a Poisson point of N_{n+1} . This picture illustrates the tricky situation occurring in the proof of Lemma 9: although $I_{n+1} = 1$, $W_{n+1}^{move}(2)$ is not larger than $W_n^{move}(2) + 1$ since $h(W_n^{move}) = Y$.

Proof. Let $n \geq 0$. Let us first assume that $L(H_{kn}) \leq \kappa$. If $I_{kn+1} = 1$ there is a Poisson point in $\text{Rec}(g_{kn}^{\uparrow,1}; 1, l_{kn})$. So, by (14),

$$\|h(W_{kn}^{move}) - W_{kn}^{move}\|_2 \leq l_{kn} + 2 \leq L(H_{kn})/2 \vee 1 + 2 \leq \kappa$$

since $\kappa \geq 6$. By (8), we deduce that $L(H_{kn+1})$ is also smaller than κ . By induction, the same holds for $L(H_{k(n+1)})$.

From now on, let us assume that $L(H_{kn}) \geq \kappa$. Two cases must be distinguished. If none of the semi-balls

$$B^+(W_{kn+j}^{move}, \|h(W_{kn+j}^{move}) - W_{kn+j}^{move}\|_2), j = 0, \dots, k-1,$$

generated between the $(kn+1)$ -th step and the $k(n+1)$ -th step exceed the horizontal line $\{\mathbf{x} : \mathbf{x}(2) = W_{kn}^{move}(2) + L(H_{kn})\}$ then

$$L(H_{k(n+1)}) \leq L(H_{kn}) - (W_{k(n+1)}^{move}(2) - W_{kn}^{move}(2))$$

which is smaller than $L(H_{kn}) - 1$ thanks to Lemma 9. Otherwise, we necessarily have

$$L(H_{k(n+1)}) \leq \max_{0 \leq j \leq k-1} \|h(W_{kn+j}^{move}) - W_{kn+j}^{move}\|_2. \quad (16)$$

Combining $I_{kn+1} = 1$ and (14), we get

$$\|h(W_{kn}^{move}) - W_{kn}^{move}\|_2 \leq l_{kn} + 2 \leq L(H_{kn})/2 + 2 \leq L(H_{kn}) - 1$$

whenever $L(H_{kn}) \geq \kappa \geq 6$. Here is the justification for the choice of $\kappa \geq 6$. This and (8) imply that $L(H_{kn+1})$ is smaller than $L(H_{kn})$. Then, $I_{kn+2} = 1$ and (14) imply

$$\|h(W_{kn+1}^{move}) - W_{kn+1}^{move}\|_2 \leq L(H_{kn+1})/2 + 2 \leq L(H_{kn})/2 + 2 \leq L(H_{kn}) - 1$$

since $L(H_{kn}) \geq \kappa$. By induction, we have on $\{\prod_{j=1}^k I_{kn+j} = 1\}$ that, for all $j \in \{0, \dots, k-1\}$, $L(H_{kn+j}) \leq L(H_{kn})$ and $\|h(W_{kn+j}^{move}) - W_{kn+j}^{move}\|_2 \leq L(H_{kn}) - 1$. Hence:

$$\max_{0 \leq j \leq k-1} \|h(W_{kn+j}^{move}) - W_{kn+j}^{move}\|_2 \leq L(H_{kn}) - 1,$$

which by (16) concludes the proof. \square

Part 3: Synthesis.

Now we are going to define a discrete time integer-valued process $\{M_n = M_n(\mathbf{u}_1, \dots, \mathbf{u}_k) : n \geq 1\}$ whose role is to dominate the height of the history set. Let us define it in an inductive way. Set $M_0 := \kappa$, where κ is the integer defined in (4). For $n \geq 0$ given M_n , we define M_{n+1} as follows:

$$M_{n+1} := \begin{cases} \max\{M_n - 1, \kappa\} & \text{if } \prod_{j=1}^{(\kappa+1)^k} I_{(\kappa+1)kn+j} = 1, \\ \max\{M_n, \sum_{j=1}^{(\kappa+1)^k} X_{(\kappa+1)n+j}, (\kappa+1)\} & \text{otherwise,} \end{cases} \quad (17)$$

where the r.v.'s X_n are defined in (10). The sequence $\{M_n : n \geq 1\}$ is a Markov chain with state space $\mathbb{N} \setminus \{1, \dots, \kappa - 1\} = \{0\} \cup \{\kappa, \dots\}$. Let τ^M be the hitting time of κ :

$$\tau^M := \inf\{n \geq 1 : M_n = \kappa\}. \quad (18)$$

The r.v. M_n is built in order to dominate the height $L(H_{(\kappa+1)kn})$. Moreover, by construction, the chain (M_n) returns to κ only if the event $\{\prod_{j=1}^{(\kappa+1)^k} I_{(\kappa+1)kn+j} = 1\}$ occurs. This guarantees a progress of at least $\kappa + 1$ for the ordinate of the moving vertex by Lemma 9, which corresponds to the last condition defining the good steps in (4).

Lemma 11. *The inequality $L(H_{(\kappa+1)kn}) \leq M_n$ holds a.s. for all $n \geq 0$. As a consequence, the random time τ_1 defined in (4) satisfies a.s. $\tau_1 \leq (\kappa + 1)k\tau^M$.*

Proof. Let us first prove $L(H_{(\kappa+1)kn}) \leq M_n$ by induction. By (5) we have $L(H_0) \leq 1 \leq \kappa = M_0$. Assume that $L(H_{(\kappa+1)kn}) \leq M_n$ for some $n \geq 0$. Either $\prod_{j=1}^{(\kappa+1)^k} I_{(\kappa+1)kn+j} = 1$ then by Lemma 10,

$$L(H_{(\kappa+1)kn+k}) \leq \max\{L(H_{(\kappa+1)kn}) - 1, \kappa\} \leq \max\{M_n - 1, \kappa\} = M_{n+1}.$$

We can easily iterate the argument, still applying Lemma 10:

$$L(H_{(\kappa+1)kn+2k}) \leq \max\{L(H_{(\kappa+1)kn+k}) - 1, \kappa\} \leq M_{n+1}$$

to finally get $L(H_{(\kappa+1)k(n+1)}) \leq M_{n+1}$. Otherwise, by Lemma 7 (i),

$$\begin{aligned} L(H_{(\kappa+1)k(n+1)}) &\leq \max\{L(H_{(\kappa+1)kn+\kappa k}), X_{(\kappa+1)n+(\kappa+1)}\} \\ &\leq \max\{L(H_{(\kappa+1)kn+(\kappa-1)k}), X_{(\kappa+1)n+\kappa} + X_{(\kappa+1)n+(\kappa+1)}\} \\ &\leq \max\{L(H_{(\kappa+1)kn}), X_{(\kappa+1)n+1} + \dots + X_{(\kappa+1)n+(\kappa+1)}\} \\ &\leq M_{n+1}. \end{aligned}$$

This completes the proof by induction. As a consequence, at the $(\kappa + 1)k\tau^M$ -th step the height $L(H_{(\kappa+1)k\tau^M})$ is smaller than κ . Moreover, by construction of the chain (M_n) , there exists an integer $m' < \tau^M$ such that $\prod_{j=1}^{(\kappa+1)^k} I_{(\kappa+1)km'+j} = 1$. By Lemma 9, applied $(\kappa + 1)$ times, this implies

$$W_{(\kappa+1)k\tau^M}^{\text{move}}(2) \geq W_{(\kappa+1)k(m'+1)}^{\text{move}}(2) \geq W_{(\kappa+1)km'}^{\text{move}}(2) + (\kappa + 1) \geq W_0^{\text{move}}(2) + (\kappa + 1).$$

We finally get $\tau_1 \leq (\kappa + 1)k\tau^M$. □

A similar proof to the one of Lemma 2.6 in [22] leads to the next result.

Lemma 12. *For any $n \in \mathbb{N}$ we have*

$$\mathbb{P}(\tau^M > n) \leq C_0 e^{-C_1 n}.$$

Proof. Thanks to [2, Proposition 5.5, Chapter 1] (see also [18, Chap. 15]) it is enough to show that there exists a function $f : \mathbb{N} \rightarrow \mathbb{R}^+$, an integer n_0 and real numbers $r > 1$, $\delta > 0$ such that

- $f(l) > \delta$ for any $l \in \mathbb{N}$;
- $\mathbb{E}[f(M_1)|M_0 = l] < \infty$ for any $l \leq n_0$;
- and $\mathbb{E}[f(M_1)|M_0 = l] \leq f(l)/r$ for any $l > n_0$.

Indeed, this implies the existence of some $r > 1$ such that $\mathbb{E}(r^{\tau^M(n_0)}|M_0 = n_0) < \infty$ where $\tau^M(n_0) := \inf\{n \geq 1 : M_n \in [0, n_0]\}$. In other words, the hitting time $\tau^M(n_0)$ admits an exponential moment. Finally, Lemma 12 follows from the fact that for any $l \leq n_0$, starting from l , $p_0^{(\kappa+1)kn_0} > 0$ gives a lower bound for the probability that the chain hits the state κ within the next n_0 steps where p_0 is defined in Lemma 8.

We take $f : \mathbb{N} \rightarrow \mathbb{R}$ to be $f(l) := e^{\alpha l}$ where $\alpha > 0$ is small enough so that $\mathbb{E}[e^{\alpha Y}] < \infty$ with $Y := \max\{X_1 + \dots + X_{(\kappa+1)}, (\kappa+1)\}$. This is possible by Lemma 7. So, for any $l \leq n_0$,

$$\mathbb{E}[f(M_1)|M_0 = l] \leq e^{\alpha n_0} \mathbb{E}[e^{\alpha(M_1 - M_0)}|M_0 = l] \leq e^{\alpha n_0} \mathbb{E}[e^{\alpha Y}] < \infty .$$

Then, pick $r > 1$ such that $e^{-\alpha} p_0^{(\kappa+1)k} + (1 - p_0^{(\kappa+1)k}) < 1/r$. Using (17), we can write for $l \geq n_0 > \kappa$:

$$\begin{aligned} \mathbb{E}[e^{\alpha(M_1 - M_0)}|M_0 = l] &= \mathbb{E}[e^{\alpha(M_0 - 1 - M_0)} \mathbf{1}_{\prod_{j=1}^{(\kappa+1)k} I_j = 1} | M_0 = l] + \mathbb{E}[e^{\alpha(\max(M_0, Y) - M_0)} \mathbf{1}_{\prod_{j=1}^{(\kappa+1)k} I_j = 0} | M_0 = l] \\ &\leq e^{-\alpha} p_0^{(\kappa+1)k} + (1 - p_0^{(\kappa+1)k}) + e^{-\alpha l} \mathbb{E}[\mathbf{1}_{\{Y > l\}} e^{\alpha Y}] \\ &< 1/r', \end{aligned}$$

for n_0 large enough an $r' \in (1, r)$. This completes the proof. \square

We are now able to prove Proposition 5. As suggested by Lemma 11, the dominating r.v. T occurring in Proposition 5 is given by $(\kappa + 1)k\tau^M$.

Proof of Prop. 5. Let us first start with the case $j = 0$. Lemmas 11 and 12 ensure that

$$\mathbb{P}(\tau_1 - \tau_0 \geq n \mid \mathcal{F}_{\tau_0}) = \mathbb{P}(\tau_1 \geq n \mid \mathcal{F}_0) \leq \mathbb{P}((\kappa + 1)k\tau^M \geq n) \leq C_0 e^{-C_1 n}$$

for suitable positive constants C_0, C_1 . So, $\tau_1 - \tau_0$ is stochastically dominated by $T := (\kappa + 1)k\tau^M$.

Then, the idea consists in working conditionally to \mathcal{F}_{τ_1} and applying the previous strategy (i.e. Lemmas 6 to 12) to the “new starting configuration” $(g_{\tau_1}(\mathbf{u}_1), \dots, g_{\tau_1}(\mathbf{u}_k), H_{\tau_1})$. Remark that this strategy, only based on assumptions (5) and (6), also works when (5) is replaced with

$$L(H_0) \leq \kappa \text{ and } \mathcal{N}_0^{\text{extra}} = \emptyset . \quad (19)$$

Actually, the only place where the hypothesis $L(H_0) \leq 1$ is used, appears in the proof of Lemma 6 when W_n^{move} belongs to the extra set $\mathcal{N}_0^{\text{extra}}$. This case does not exist under (19).

Now, conditionally to \mathcal{F}_{τ_1} , the configuration $(g_{\tau_1}(\mathbf{u}_1), \dots, g_{\tau_1}(\mathbf{u}_k), H_{\tau_1})$ satisfies by definition

$$\max_{i,j} |g_{\tau_1}(\mathbf{u}_i)(2) - g_{\tau_1}(\mathbf{u}_j)(2)| \leq L(H_{\tau_1}) \leq \kappa$$

and $\mathcal{N}_0^{\text{extra}} = \emptyset$ since the extra set (for $\mathbf{u}_1, \dots, \mathbf{u}_k$) lies below the level $W_{\tau_1}^{\text{move}}(2)$. Then, from step τ_1 onwards, we can dominate the height of the history set by a new Markov chain, say (M'_n) , built as in (17) and distributed as (M_n) . Hence, the increment $\tau_2 - \tau_1$ is stochastically dominated by a r.v. $(\kappa + 1)k\tau^{M'}$ where $\tau^{M'}$ is the hitting time of κ for the chain (M'_n) . Furthermore, remark that $\tau^{M'}$ and τ^M are identically distributed on the one hand, and on the other hand, $(M'_n)^-$ and then the hitting time $\tau^{M'}$ only uses PPP's \mathcal{N}_n for $n > \tau_1$ while \mathcal{F}_{τ_1} depends on \mathcal{N}_n for $n \leq \tau_1$ and $\mathcal{N}_0^{\text{extra}}$. We thus deduce the result for any j similarly. \square

2.3 Renewal steps

Let us consider k starting points $\mathbf{u}_1, \dots, \mathbf{u}_k$ in \mathbb{R}^2 (which do not need to be all different) and an initial information $(H_0, \mathcal{N}_0^{\text{extra}})$ such that assumptions (5) and (6) hold. The goal of this section consists in extracting from the sequence of good steps $\{(g_{\tau_j}(\mathbf{u}_1), \dots, g_{\tau_j}(\mathbf{u}_k), H_{\tau_j}) : j \geq 1\}$ a subsequence of what we will call *renewal steps* such that:

1. These renewal steps allow us to split the k trajectories of the DSF starting from $\mathbf{u}_1, \dots, \mathbf{u}_k$ into disjoint excursions which are conditionally independent from each other (Proposition 14);
2. The number of steps of the exploration process between two consecutive renewal steps admits exponentially decaying tails (Proposition 17).

These properties will be used several times in the sequel to check the convergence criteria toward the Brownian Web.

In this section, we work on the original process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$ based on a unique PPP \mathcal{N} , and not on the representation with a sequence of i.i.d. PPP's $\{\mathcal{N}_n : n \in \mathbb{N}\}$ any more. As often as possible, the letter n is used for indexing the steps of the exploration process, j for the good steps and ℓ for the renewal steps.

Let us start with describing the first renewal step. Consider the j -th good step with $j \geq 1$. Let us set for any $1 \leq i \leq k$,

$$g_{\tau_j}^\downarrow(\mathbf{u}_i) := (g_{\tau_j}(\mathbf{u}_i)(1), W_{\tau_j}^{\text{move}}(2)) \quad \text{and} \quad g_{\tau_j}^\uparrow(\mathbf{u}_i) := (g_{\tau_j}(\mathbf{u}_i)(1), W_{\tau_j}^{\text{move}}(2) + \kappa)$$

respectively the projections of $g_{\tau_j}(\mathbf{u}_i)$ onto the horizontal axes with ordinates $W_{\tau_j}^{\text{move}}(2)$ and $W_{\tau_j}^{\text{move}}(2) + \kappa$. The $g_{\tau_j}(\mathbf{u}_i)$'s lie in the horizontal strip delimited by these two axes. Let us introduce the event $A_j = A_j(\mathbf{u}_1, \dots, \mathbf{u}_k)$ associated to the j -th good step defined as follows:

$$A_j := \bigcap_{i=1}^k \left\{ \text{Card}((B^+(g_{\tau_j}^\downarrow(\mathbf{u}_i), \kappa + 1) \setminus H_{\tau_j}) \cap \mathcal{N}) = \text{Card}(B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1) \cap \mathcal{N}) = 1 \right\}. \quad (20)$$

In other words, A_j says that each set $B^+(g_{\tau_j}^\downarrow(\mathbf{u}_i), \kappa + 1) \setminus H_{\tau_j}$ contains a unique Poisson point which actually lies in $B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1)$. Remark that these Poisson points (i.e. those involved by A_j) are not necessarily distinct when the semi-balls $B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1)$'s have non-empty intersections.

When the event A_j occurs for the first time, the corresponding good step is called a *renewal step*. Let $\beta = \beta(\mathbf{u}_1, \dots, \mathbf{u}_k)$ be the index of the first renewal step:

$$\beta := \inf \{n > 0 : \exists j \geq 0, n = \tau_j \text{ and } A_j \text{ occurs}\}.$$

Since the event A_j depends only on the PPP \mathcal{N} inside the horizontal strip delimited by the ordinates $W_{\tau_j}^{\text{move}}(2)$ and $W_{\tau_j}^{\text{move}}(2) + \kappa + 1$ and above the history set H_{τ_j} , it does not belong to the σ -field \mathcal{F}_{τ_j} . So β is not a (\mathcal{F}_n) -stopping time.

The random integer β is almost surely finite.

Lemma 13. *There exists $p_1 = p_1(k, \kappa, \lambda) > 0$ such that for all $j \geq 1$,*

$$\mathbb{P}(A_j \mid \mathcal{F}_{\tau_j}) \geq p_1. \quad (21)$$

Moreover, β is stochastically dominated by τ_G where G is a geometric r.v. with parameter p_1 .

Proof. Assume (21) and consider $j \geq 2$. Recall that the stopping times τ_{j-1} and τ_j have been defined such that $W_{\tau_j}^{\text{move}}(2) - W_{\tau_{j-1}}^{\text{move}}(2) \geq \kappa + 1$. This means that events A_{j-1} and A_j concern disjoint subsets of \mathcal{N} . Hence, conditionally to \mathcal{F}_{τ_j} , the event A_j is independent from A_1, \dots, A_{j-1} . Thus,

$$\mathbb{P}(\beta > \tau_j) = \mathbb{P}(A_j^c \cap \dots \cap A_1^c) \geq (1 - p_1)^j = \mathbb{P}(G > j) = \mathbb{P}(\tau_G > \tau_j)$$

where G is a geometric r.v. with parameter p_1 . So, β is stochastically dominated by τ_G .

It then remains to prove (21). Instead of considering a subset of $\{1, \dots, k\}$, we assume here that all the $g_{\tau_j}(u_i)$'s are different. The same holds for the $g_{\tau_j}^\downarrow(u_i)$'s and the $g_{\tau_j}^\uparrow(u_i)$'s. Let us set

$$F := \bigcup_{1 \leq i \leq k} (B^+(g_{\tau_j}^\downarrow(\mathbf{u}_i), \kappa + 1) \setminus B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1)) .$$

The probability that $F \cap \mathcal{N}$ is empty is easily bounded from below by a positive constant depending on k, κ and the intensity λ of the PPP. But it requires tedious geometric computations to show that each $B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1)$ contains exactly one Poisson point with positive probability, because the $g_{\tau_j}(u_i)$'s can be very close to each other. So, we give the main arguments and skip the details.

Let $\epsilon > 0$. If

$$\min_{i \neq i'} |g_{\tau_j}^\uparrow(\mathbf{u}_i)(1) - g_{\tau_j}^\uparrow(\mathbf{u}_{i'})(1)| \geq \epsilon \quad (22)$$

then it is possible to exhibit deterministic (conditionally to \mathcal{F}_{τ_j}) regions $\Lambda_1, \dots, \Lambda_k$ such that their areas are equal to some constant $c(\epsilon) > 0$, they do not overlap F and each Λ_i satisfies

$$\Lambda_i \subset B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1) \setminus \left(\bigcup_{i' \neq i} B^+(g_{\tau_j}^\uparrow(\mathbf{u}_{i'}), 1) \right) .$$

From there, putting exactly one Poisson point inside each Λ_i , it is not difficult to conclude. When (22) is no longer true, we split $\{1, \dots, k\}$ into disjoint subsets in which consecutive elements $g_{\tau_j}^\uparrow(\mathbf{u}_i)$ and $g_{\tau_j}^\uparrow(\mathbf{u}_{i+1})$ are at distance smaller than ϵ . We treat these subsets separately. Let us consider the first of them, say $\{g_{\tau_j}^\uparrow(\mathbf{u}_1), \dots, g_{\tau_j}^\uparrow(\mathbf{u}_s)\}$ with $1 \leq s \leq k$ and $|g_{\tau_j}^\uparrow(\mathbf{u}_i)(1) - g_{\tau_j}^\uparrow(\mathbf{u}_{i+1})(1)| < \epsilon$ for $1 \leq i \leq s-1$. For $\epsilon > 0$ small enough (depending on κ and k), the semi-balls $B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1)$, $1 \leq i \leq s$, have a non-empty intersection. Hence, we can exhibit a deterministic region $\Lambda^{(1)}$ with area $c'(\epsilon) > 0$ such that

$$\Lambda^{(1)} \subset \bigcap_{1 \leq i \leq s} B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1) \text{ and } \Lambda^{(1)} \cap \left(F \cup \left(\bigcup_{i > s} B^+(g_{\tau_j}^\uparrow(\mathbf{u}_i), 1) \right) \right) = \emptyset .$$

Thus we repeat the argument with the remaining set $\{g_{\tau_j}^\uparrow(\mathbf{u}_{s+1}), \dots, g_{\tau_j}^\uparrow(\mathbf{u}_k)\}$ (possibly empty). We finally exhibit disjoint subsets $\Lambda^{(1)}, \dots, \Lambda^{(n)}$ with good properties. Putting exactly one Poisson point inside each of them, it is then not difficult to conclude. \square

Let us now investigate the situation at the (first) renewal step β . Let $k_0^{(1)}$ be the number of remaining paths at this stage (some paths may have already coalesced):

$$k_0^{(1)} := \text{Card}(\{g_\beta(\mathbf{u}_1), \dots, g_\beta(\mathbf{u}_k)\}) .$$

Let also denote by $U_i^{(1)}$, $i = 1, \dots, k$ (not necessarily all different) the Poisson points involved by the occurrence of A_j , with $\beta = \tau_j$. Let us define the joint exploration process

$$\{(g_n(g_\beta^\uparrow(\mathbf{u}_1)), \dots, g_n(g_\beta^\uparrow(\mathbf{u}_k)), H_n(g_\beta^\uparrow(\mathbf{u}_1), \dots, g_\beta^\uparrow(\mathbf{u}_k))) : n \geq 0\}$$

restarted from $(g_\beta^\uparrow(\mathbf{u}_1), \dots, g_\beta^\uparrow(\mathbf{u}_k))$ as the *regenerated paths*. By opposition we will call the sequence

$$\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n(\mathbf{u}_1, \dots, \mathbf{u}_k)) : n \geq \beta\}$$

the *original paths* (but considered from step β). By definition of A_j , the ancestor of $g_\beta^\uparrow(\mathbf{u}_i)$ is $U_i^{(1)}$, $h(g_\beta^\uparrow(\mathbf{u}_i)) = U_i^{(1)}$, for any i since

$$B^+(g_\beta^\uparrow(\mathbf{u}_i), \|g_\beta^\uparrow(\mathbf{u}_i) - U_i^{(1)}\|_2) \subset B^+(g_\beta^\downarrow(\mathbf{u}_i), \kappa + 1) . \quad (23)$$

The second inclusion

$$B^+(g_\beta(\mathbf{u}_i), \|g_\beta(\mathbf{u}_i) - U_i^{(1)}\|_2) \subset B^+(g_\beta^\downarrow(\mathbf{u}_i), \kappa + 1) \quad (24)$$

should also imply that $h(g_\beta(\mathbf{u}_i)) = U_i^{(1)}$. But this actually depends on the proximity of points of W_β^{stay} as depicted in Figure 6. This is the reason why we distinguish two cases below. Let us point out here the role of the event A_j : when it occurs, inclusions (23) and (24) force both vertices $g_\beta(\mathbf{u}_i)$ and $g_\beta^\uparrow(\mathbf{u}_i)$ to be connected to $U_i^{(1)}$ provided that there is no element of W_β^{stay} nearby.

- First consider the case where any two balls $B^+(g_\beta^\downarrow(\mathbf{u}_i), \kappa + 1)$ and $B^+(g_\beta^\downarrow(\mathbf{u}_j), \kappa + 1)$ do not overlap whenever $g_\beta(\mathbf{u}_i)$ and $g_\beta(\mathbf{u}_j)$ are different: they are far enough from each other. In this case, for any i , the ancestor of $g_\beta(\mathbf{u}_i)$ is also $U_i^{(1)}$:

$$h(g_\beta(\mathbf{u}_i)) = h(g_\beta^\uparrow(\mathbf{u}_i)) = U_i^{(1)} .$$

Then, from step $\beta + k_0^{(1)}$ the regenerated and original processes coincide. The $k_0^{(1)}$ first moves may indeed be in different orders since the regenerated paths start from points with the same ordinate and move one by one starting from the leftmost path.

- Otherwise, assume that $B^+(g_\beta^\downarrow(\mathbf{u}_i), \kappa + 1)$ and $B^+(g_\beta^\downarrow(\mathbf{u}_j), \kappa + 1)$ overlap for some indices i, j with $g_\beta(\mathbf{u}_i) \neq g_\beta(\mathbf{u}_j)$. A tricky situation, described in Figure 6, may then happen. The vertex $g_\beta(\mathbf{u}_i)$ for instance is connected to $g_\beta(\mathbf{u}_j)$ and the single trajectory $(h^n(g_\beta(\mathbf{u}_i)))_{n \geq 0}$ avoids the Poisson point $U_i^{(1)}$. So, the regenerated path $(g_n(g_\beta^\uparrow(\mathbf{u}_i)))_{n \geq 0}$ does not coincide with the original path $(g_n(\mathbf{u}_i))_{n \geq \beta}$ at least for the first steps (they can coalesce afterward). In this case which actually corresponds to the coalescence of two trajectories, the set of original paths constitutes a subset of the regenerated ones.

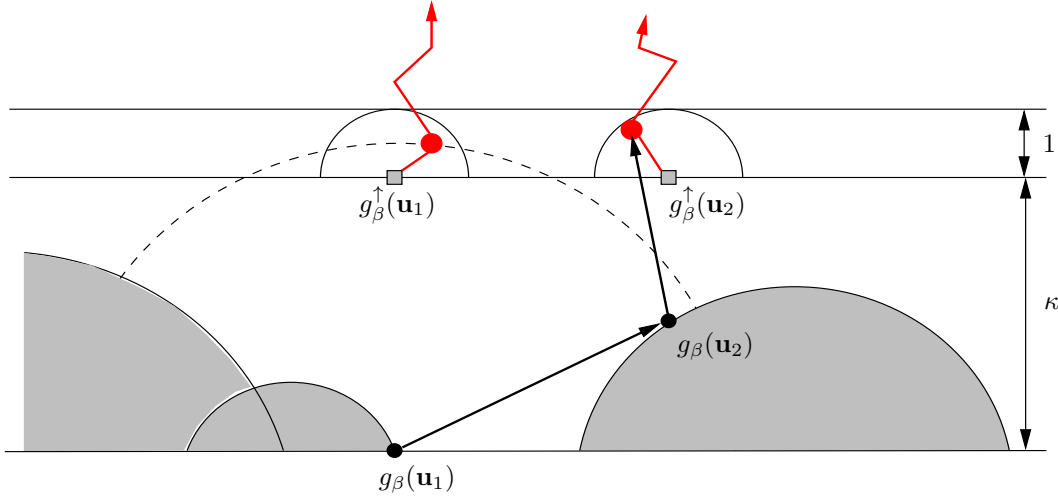


Figure 6: This picture represents the first renewal step of the joint exploration process $\{(g_n(\mathbf{u}_1), g_n(\mathbf{u}_2), H_n) : n \geq 0\}$ (with only $k = 2$ trajectories): the event A_j occurs where $\beta = \tau_j$. The gray squares represent $g_\beta^\uparrow(\mathbf{u}_1)$ and $g_\beta^\uparrow(\mathbf{u}_2)$ whose ancestors are respectively $U_1^{(1)}$ and $U_2^{(1)}$ (the red points). The regenerated paths $\{(g_n(g_\beta^\uparrow(\mathbf{u}_1)), g_n(g_\beta^\uparrow(\mathbf{u}_2)), H_n(g_\beta^\uparrow(\mathbf{u}_1), g_\beta^\uparrow(\mathbf{u}_2))) : n \geq 0\}$ are in red. This picture also illustrates the tricky situation described after (24). The vertex $g_\beta(\mathbf{u}_1)$ is connected to $g_\beta(\mathbf{u}_2) \in W_\beta^{\text{stay}}$ and not to $U_1^{(1)}$, i.e. the two trajectories of the DSF starting at \mathbf{u}_1 and \mathbf{u}_2 merge at the $(\beta + 1)$ -th step. The regenerated paths and the original paths differ since $(g_2(g_\beta^\uparrow(\mathbf{u}_1)), g_2(g_\beta^\uparrow(\mathbf{u}_2))) = (U_1^{(1)}, U_2^{(1)})$ whereas $g_{\beta+2}(\mathbf{u}_1) = g_{\beta+2}(\mathbf{u}_2) = U_2^{(1)}$.

In both situations, the two following statements hold. The next renewal step (after β) for the regenerated paths will be also a renewal step for the original ones and the coalescence of regenerated paths implies the coalescence of original ones. This control of the original paths by the regenerated ones allows us to work only with these latter paths, from which we will be able to extract a Markov chain. See Proposition 14 for details.

Let us now remark that the starting configuration for the regenerated paths meets the conditions invoked in Sections 2.1 and 2.2. The starting vertices $g_\beta^\uparrow(\mathbf{u}_1), \dots, g_\beta^\uparrow(\mathbf{u}_k)$ all have the same ordinate (and some of them can be equal). The initial and extra information is given by the couple $(H_0^{(1)}, \mathcal{N}_0^{\text{extra},(1)})$ where the history set

$$H_0^{(1)} := \left(\bigcup_{i=1}^k B^+(g_\beta^\downarrow(\mathbf{u}_i)), \kappa + 1 \right) \cap \mathbb{H}^+(g_\beta^\uparrow(\mathbf{u}_1)(2))$$

is included in the strip $\mathbb{R} \times [g_\beta^\uparrow(\mathbf{u}_1)(2), g_\beta^\uparrow(\mathbf{u}_1)(2) + 1]$ and contains a collection of random points given by

$$\mathcal{N}_0^{\text{extra},(1)} := \{U_1^{(1)}, \dots, U_k^{(1)}\}$$

(actually, $\mathcal{N}_0^{\text{extra},(1)}$ admits no more than $k_0^{(1)}$ points). See Figure 7.

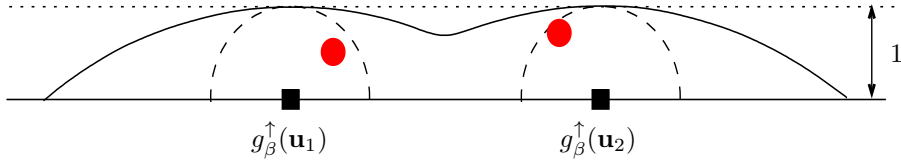


Figure 7: This picture represents the starting configuration for the regenerated paths $\{(g_n(g_\beta^\uparrow(\mathbf{u}_1)), g_n(g_\beta^\uparrow(\mathbf{u}_2)), H_n(g_\beta^\uparrow(\mathbf{u}_1), g_\beta^\uparrow(\mathbf{u}_2))) : n \geq 0\}$ corresponding to the realization depicted in Figure 6. Conditionally to \mathcal{G}_1 (see (29)), the initial history set is given by the deterministic and compact subset $H_0^{(1)} = (B^+(g_\beta^\downarrow(\mathbf{u}_1)), \kappa + 1) \cup B^+(g_\beta^\downarrow(\mathbf{u}_2)), \kappa + 1) \cap \mathbb{H}^+(g_\beta^\uparrow(\mathbf{u}_1)(2))$ and the extra points by the random finite set $\mathcal{N}_0^{\text{extra},(1)} = \{U_1^{(1)}, U_2^{(1)}\}$ – the red points.

Hence, the regenerated paths

$$\{(g_n(g_\beta^\uparrow(\mathbf{u}_1)), \dots, g_n(g_\beta^\uparrow(\mathbf{u}_k)), H_n(g_\beta^\uparrow(\mathbf{u}_1), \dots, g_\beta^\uparrow(\mathbf{u}_k))) : n \geq 0\}$$

are obtained as the joint exploration process starting from $(g_\beta^\uparrow(\mathbf{u}_1), \dots, g_\beta^\uparrow(\mathbf{u}_k))$ and $(H_0^{(1)}, \mathcal{N}_0^{\text{extra},(1)})$, and built on the PPP $\mathcal{N} \setminus H_0^{(1)}$. This allows us to recursively define a sequence of renewal steps. Let $\mathbf{u}_i^{(0)} = \mathbf{u}_i$ for all i and $\beta_1 = \beta(\mathbf{u}_1^{(0)}, \dots, \mathbf{u}_k^{(0)})$. Given $(\mathbf{u}_1^{(\ell-1)}, \dots, \mathbf{u}_k^{(\ell-1)})$ for some $\ell \geq 1$, we set:

$$\beta_\ell := \beta(\mathbf{u}_1^{(\ell-1)}, \dots, \mathbf{u}_k^{(\ell-1)}) \quad (25)$$

$$\mathbf{u}_i^{(\ell)} := g_{\beta_\ell}^\uparrow(\mathbf{u}_i^{(\ell-1)}), \text{ for } i \in \{1, \dots, k\}$$

$$H_0^{(\ell)} := \left(\bigcup_{i=1}^k B^+(g_{\beta_\ell}^\downarrow(\mathbf{u}_i^{(\ell-1)}), \kappa + 1) \right) \cap \mathbb{H}^+(g_{\beta_\ell}^\uparrow(\mathbf{u}_1^{(\ell-1)})(2))$$

$$\mathcal{N}_0^{\text{extra},(\ell)} := \{U_1^{(\ell)}, \dots, U_k^{(\ell)}\}, \quad (26)$$

where the point $U_i^{(\ell)}$ denotes the unique point of \mathcal{N} in $B^+(\mathbf{u}_i^{(\ell)}, 1)$. The random integer β_ℓ corresponds to the number of steps between the $(\ell - 1)$ -th and the ℓ -th renewal steps. It is defined as the number of steps until the first renewal event of the $(\ell - 1)$ -th regenerated process started from $\{\mathbf{u}_1^{(\ell-1)}, \dots, \mathbf{u}_k^{(\ell-1)}\}$

with initial history set $H_0^{(\ell-1)}$ and extra points $\mathcal{N}_0^{\text{extra},(\ell-1)}$. Let us define by $(\varrho_\ell)_{\ell \geq 0}$ the sequence of renewal steps: $\varrho_0 = 0$ and for any $\ell \geq 1$,

$$\varrho_\ell := \sum_{1 \leq m \leq \ell} \beta_m. \quad (27)$$

Proposition 14 enounces the properties of renewal steps and regenerated paths described previously and claims that the embedded process $\{(\mathbf{u}_1^{(\ell)}, \dots, \mathbf{u}_k^{(\ell)}) : \ell \geq 1\}$ defines a Markov chain with respect to a proper filtration. This is the key for establishing estimates on the coalescence time of two paths and then convergence of the DSF to the BW.

For any $\ell \geq 0$ and $j \geq 0$, let us define

$$\mathcal{F}_{\tau_j}^{(\ell)} := \sigma\left(\{g_n(\mathbf{u}_i^{(\ell)}), i \in \{1, \dots, k\}, n \leq \tau_j(\mathbf{u}_1^{(\ell)}, \dots, \mathbf{u}_k^{(\ell)})\}, \mathcal{N} \cap H_0^{(\ell)} = \emptyset, \mathcal{N}_0^{\text{extra},(\ell)}, \{\mathbf{1}_{A_{j'}}, j' \leq j\}\right). \quad (28)$$

The filtration $\{\mathcal{F}_{\tau_j}^{(\ell)} : j \geq 0\}$ contains the information brought by the regenerated paths started at $(\mathbf{u}_1^{(\ell)}, \dots, \mathbf{u}_k^{(\ell)})$ until its j -th good step, and the information on whether the events $A_{j'}$, for $j' \leq j$, occur or not. Actually, for each $\ell \geq 1$, the r.v. β_ℓ is a stopping time w.r.t. the new filtration $\{\mathcal{F}_{\tau_j}^{(\ell-1)} : j \geq 0\}$. We can then introduce, for $\ell \geq 1$,

$$\mathcal{G}_\ell := \sigma\left\{\mathcal{F}_{\beta_m}^{(m-1)}, 1 \leq m \leq \ell\right\} \quad (29)$$

the σ -field corresponding to the information carried by the regenerated paths from $(\mathbf{u}_1, \dots, \mathbf{u}_k)$ to the ℓ -th renewal step as constructed above. The family $\{\mathcal{G}_\ell : \ell \geq 1\}$ constitutes a filtration w.r.t. which the sequence of renewal states $\{(\mathbf{u}_1^{(\ell)}, \dots, \mathbf{u}_k^{(\ell)}) : \ell \geq 1\}$ is a Markov chain.

Proposition 14. *With the above notation, the following holds:*

(i) *Let $\ell \geq 1$ and assume that for any $1 \leq i, j \leq k$, the semi-balls $B^+(g_{\beta_\ell}(\mathbf{u}_i^{(\ell-1)}), \kappa+1)$ and $B^+(g_{\beta_\ell}(\mathbf{u}_j^{(\ell-1)}), \kappa+1)$ do not overlap whenever $g_{\beta_\ell}(\mathbf{u}_i^{(\ell-1)}) \neq g_{\beta_\ell}(\mathbf{u}_j^{(\ell-1)})$. Then, conditionally to \mathcal{G}_ℓ , the paths*

$$\{(g_{\beta_\ell+n}(\mathbf{u}_1^{(\ell-1)}), \dots, g_{\beta_\ell+n}(\mathbf{u}_k^{(\ell-1)})) : n \geq k_0^{(\ell)}\}$$

a.s. coincide with the regenerated paths

$$\{(g_n(\mathbf{u}_1^{(\ell)}), \dots, g_n(\mathbf{u}_k^{(\ell)})) : n \geq k_0^{(\ell)}\}$$

with initial information $(H_0^{(\ell)}, \mathcal{N}_0^{\text{extra},(\ell)})$ where $k_0^{(\ell)} := \text{Card}(\{g_{\beta_\ell}(\mathbf{u}_i^{(\ell-1)}), \dots, g_{\beta_\ell}(\mathbf{u}_j^{(\ell-1)})\})$ denotes the number of remaining paths at the ℓ -th renewal step.

(ii) *Let $\ell \geq 1$. Conditionally to \mathcal{G}_ℓ and for any $n \geq k_0^{(\ell)}$, the set $\{g_{\beta_\ell+n}(\mathbf{u}_1^{(\ell-1)}), \dots, g_{\beta_\ell+n}(\mathbf{u}_k^{(\ell-1)})\}$ is a.s. included in $\{g_n(\mathbf{u}_1^{(\ell)}), \dots, g_n(\mathbf{u}_k^{(\ell)})\}$ where $\{(g_n(\mathbf{u}_1^{(\ell)}), \dots, g_n(\mathbf{u}_k^{(\ell)})) : n \geq k_0^{(\ell)}\}$ denotes the regenerated paths with initial information $(H_0^{(\ell)}, \mathcal{N}_0^{\text{extra},(\ell)})$.*

(iii) *The coalescence time of the $g_n(\mathbf{u}_i)$'s is bounded from above by the coalescence time of the $\mathbf{u}_i^{(\ell)}$'s: a.s.*

$$\min\{n \geq 0 : g_n(\mathbf{u}_1) = \dots = g_n(\mathbf{u}_k)\} \leq \min\{\beta_1 + \dots + \beta_\ell : \ell \geq 1 \text{ and } \mathbf{u}_1^{(\ell)} = \dots = \mathbf{u}_k^{(\ell)}\}.$$

(iv) *The process $\{(\mathbf{u}_1^{(\ell)}, \dots, \mathbf{u}_k^{(\ell)}) : \ell \geq 1\}$ defines a (\mathcal{G}_ℓ) -Markov chain.*

Proof. Items (i) and (ii) have been already explained in the case $\ell = 1$. This is exactly the same in the general case $\ell \geq 1$. A strict inclusion in Item (ii) corresponds to the occurrence of a tricky situation as depicted in Figure 6. Item (iii) is consequence of (ii). For the proof of (iv), we notice that the construction of the $(\ell+1)$ -th regenerated paths depends only on the initial positions $(\mathbf{u}_1^{(\ell)}, \dots, \mathbf{u}_k^{(\ell)})$, on the knowledge that the balls $B^+(g_{\beta_\ell}(\mathbf{u}_i^{(\ell-1)}), \kappa+1) \cap \mathbb{H}^+(\mathbf{u}_1^{(\ell)}(2))$ contain each only one point in $B^+(\mathbf{u}_i^{(\ell)}, 1)$ and on the PPP \mathcal{N} on the complement of $\mathbb{H}^+(\mathbf{u}_1^{(\ell)}(2)) \setminus H_0^{(\ell)}$. Therefore, using similar arguments as in Proposition 4, it follows that $(\mathbf{u}_1^{(\ell)}, \dots, \mathbf{u}_k^{(\ell)})$ admits a random mapping representation and hence is a Markov chain (see [17]). \square

This section ends with three consequences of the above constructions which will be often used in the sequel.

When there is a single trajectory, i.e. $k = 1$, there is no problem due to the set W_n^{stay} which is then empty, and the regenerated path always coincides with the original one (except at the renewal steps). Hence the sequence $\{\mathbf{u}_1^{(\ell)}(1) : \ell \geq 1\}$ admits the following properties:

Proposition 15. *Assume $k = 1$. The process $\{\mathbf{u}_1^{(\ell)}(1) : \ell \geq 1\}$ is a random walk with i.i.d. symmetric increments whose distribution does not depend upon the starting point \mathbf{u}_1 .*

Note that the distribution of $(\mathbf{u}_1^{(1)}(1) - \mathbf{u}_1^{(0)}(1))$ is different since the initial conditions are different. Combined with Proposition 17 below, this result will allow us to prove the convergence of any given (normalized) trajectory of the DSF to a standard Brownian motion in Section 4.2.1.

Proof. Conditionally to \mathcal{G}_ℓ , the path started from $\mathbf{u}_1^{(\ell)}$ can be determined by the point process in $\mathbb{H}^+(\mathbf{u}_1^{(\ell)}(2))$, conditioned on having exactly one point in $B^+(\mathbf{u}_1^{(\ell)} - (0, \kappa), \kappa + 1) \cap \mathbb{H}^+(\mathbf{u}_1^{(\ell)}(2))$ which actually lies in $B^+(\mathbf{u}_1^{(\ell)}, 1)$. Doing a construction similar to the one of the auxiliary exploration process in Section 2.1 by re-sampling the PPP \mathcal{N} after each regeneration time allows us to prove the result.

Moreover, thanks to the translation invariance of the DSF, the abscissa increments

$$\mathbf{u}_1^{(\ell+1)}(1) - \mathbf{u}_1^{(\ell)}(1), \ell \geq 0,$$

are i.i.d. with the same distribution as $g_\beta(\mathbf{0})(1)$ started with a PPP conditioned on having exactly one point in $B^+((0, -\kappa), \kappa + 1) \cap \mathbb{H}^+(0)$ which actually lies in $B^+(\mathbf{0}, 1)$. It is then easily seen that the distributions of the increments are symmetric and do not depend on the starting point \mathbf{u}_1 . \square

Proposition 15 does not hold for multiple trajectories. In particular, the symmetry argument ensuring that $\mathbf{u}_1^{(\ell+1)}(1) - \mathbf{u}_1^{(\ell)}(1)$ has a symmetric distribution fails whenever $k \geq 2$. The presence of (at least) another path with its associated history creates non-symmetric geometrical interactions.

Because of translation invariance of our model, an immediate corollary of Proposition 14 is the following. Corollary 16 will be used in Section 3 to get coalescence time estimate for two trajectories.

Corollary 16. *For $k = 2$, the process $\{\mathbf{u}_2^{(\ell)}(1) - \mathbf{u}_1^{(\ell)}(1) : \ell \geq 1\}$ is a Markov chain with absorbing state 0.*

Finally, we prove that the total number of steps $\beta_{\ell+1} = \varrho_{\ell+1} - \varrho_\ell$ between two successive renewal steps has an exponential tail distribution. This result will be used many times in the sequel.

Proposition 17. *Let $\ell \geq 0$. There exists a r.v. \mathcal{R} whose distribution does not depend on \mathcal{G}_ℓ such that for all $n \geq 1$,*

$$\mathbb{P}(\beta_{\ell+1} \geq n \mid \mathcal{G}_\ell) \leq \mathbb{P}(\mathcal{R} \geq n) \leq C_0 e^{-C_1 n}. \quad (30)$$

Proof. We work conditionally to \mathcal{G}_ℓ , for $\ell \geq 0$. Let us denote by τ_j the j -th good step after the ℓ -th renewal step. By Lemma 13, $\beta_{\ell+1}$ is stochastically dominated by $\sum_{j=1}^G (\tau_j - \tau_{j-1})$ where G is a geometric r.v. with parameter $p_1 > 0$. Moreover, each increment $\tau_j - \tau_{j-1}$ is stochastically dominated by a copy, say T_j , of the r.v. T given by Proposition 5 such that the $\{T_j : j \geq 1\}$ are independent r.v.'s with exponentially decaying tails. As a consequence, $\beta_{\ell+1}$ is stochastically dominated by

$$\mathcal{R} := \sum_{j=1}^G T_j$$

whose distribution does not depend on \mathcal{G}_ℓ since this is already the case for G and the T_j 's.

Now, let $\vartheta > 0$ small enough so that $\mathbb{E}(e^{\vartheta T_1}) < \infty$. Then, for any constant $c > 0$,

$$\begin{aligned} \mathbb{P}(\mathcal{R} \geq n) &\leq \mathbb{P}\left(\sum_{j=1}^{\lfloor cn \rfloor} T_j \geq n\right) + \mathbb{P}(G > \lfloor cn \rfloor) \\ &\leq e^{-\vartheta n} \mathbb{E}(e^{\vartheta T_1})^{\lfloor cn \rfloor} + (1 - p_1)^{\lfloor cn \rfloor} . \end{aligned}$$

from which we conclude by choosing $c = c(\vartheta)$ sufficiently small. \square

Remark 18. *Most of arguments developed in the two previous sections could be extended with slight modifications to higher dimensions. Then, they would lead to the following statements with probability 1: the DSF is still a tree in dimension $d = 3$, i.e. all the paths eventually coalesce, and is made up of a countable number of disjoint trees when $d \geq 4$. However, in any dimension, the DSF should almost surely contain no bi-infinite path.*

2.4 Additional tools

2.4.1 Size of the renewal blocks

Let $\mathbf{u}_1, \dots, \mathbf{u}_k$ be starting points and $\ell \geq 1$. In this section, our goal is to exhibit random rectangles containing the trajectories of the joint exploration process $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k)) : n \geq 0\}$ between the $(\ell - 1)$ -th and the ℓ -th renewal steps.

These trajectories constitute a subset of the regenerated paths by Proposition 14. So, given the vertices $\mathbf{u}_1^{(\ell-1)}, \dots, \mathbf{u}_k^{(\ell-1)}$ (some of them being equal), we define

$$W_\ell = W_\ell(\mathbf{u}_1^{(\ell-1)}, \dots, \mathbf{u}_k^{(\ell-1)}) := \sum_{m=0}^{\beta_\ell - 1} \|W_m^{\text{move}} - h(W_m^{\text{move}})\|_2 , \quad (31)$$

where W_m^{move} denotes the moving vertex of the m -th step of the regenerated process starting from $(\mathbf{u}_1^{(\ell-1)}, \dots, \mathbf{u}_k^{(\ell-1)})$. The random variable W_ℓ is such that, for any $1 \leq i \leq k$, the random set

$$\bigcup_{m=0}^{\beta_\ell - 1} B^+ \left(g_m(\mathbf{u}_i^{(\ell-1)}), \|g_m(\mathbf{u}_i^{(\ell-1)}) - g_{m+1}(\mathbf{u}_i^{(\ell-1)})\|_2 \right) \subset \mathbf{u}_i^{(\ell-1)} + [-W_\ell, W_\ell] \times [0, W_\ell] , \quad (32)$$

where the union in the l.h.s. is made up of semi-balls created by the path starting at $\mathbf{u}_i^{(\ell-1)}$ until the next renewal step. The rectangle in the r.h.s. is called *renewal block*. This is the reason why W_ℓ is termed as the *size* of these renewal blocks.

It is important to remark that the regenerated path starting from $\mathbf{u}_i^{(\ell-1)}$ till the next renewal step only depends on the random set in the l.h.s. of (32). Hence, this regenerated path is not altered by any change of the PPP outside the renewal block $\mathbf{u}_i^{(\ell-1)} + [-W_\ell, W_\ell] \times [0, W_\ell]$. Such argument will be used in the proof of Theorem 21 (precisely in Lemma 24 of Section 3.3) and in Section 4.2.1 to state that two paths far from each other evolve almost independently.

Proposition 19 states that the sizes W_ℓ 's are stochastically dominated by another random variable \mathcal{W} with sub-exponential tail.

Proposition 19. *Let $\ell \geq 1$. There exists a r.v. \mathcal{W} whose distribution does not depend on $\mathcal{G}_{\ell-1}$ such that for all $n \geq 1$,*

$$\mathbb{P}(W_\ell \geq n \mid \mathcal{G}_{\ell-1}) \leq \mathbb{P}(\mathcal{W} \geq n) \leq C_0 e^{-C_1 n^{1/2}} . \quad (33)$$

With more work, it is possible to show that the distribution of W_ℓ admits an exponentially decaying tail (and not sub-exponential), but (33) will be sufficient.

Proof. Let $\ell \geq 1$. Conditionally to $\mathcal{G}_{\ell-1}$, we work with the regenerated process

$$\{(g_n(\mathbf{u}_1^{(\ell-1)}), \dots, g_n(\mathbf{u}_k^{(\ell-1)})) : n \geq 0\}.$$

Let us recall the definition of the random variables $\{\zeta_{m+1} : m \geq 0\}$, in (9) and Figure 4, which are i.i.d. with exponentially decaying tails. Let us now show by recursion that, for any $m \geq 0$,

$$\|W_m^{\text{move}} - h(W_m^{\text{move}})\|_2 \leq \max_{0 \leq n \leq m} \lfloor 2\zeta_{n+1} \rfloor + 2. \quad (34)$$

First, (34) holds when $m = 0$. Indeed, when $\ell = 1$ this is already done in the proof of Lemma 7 and when $\ell \geq 2$, by definition, $W_0^{\text{move}} = \mathbf{u}_1^{(\ell-1)}$ and $\|\mathbf{u}_1^{(\ell-1)} - h(\mathbf{u}_1^{(\ell-1)})\|_2$ is smaller than 1.

Assume that (34) holds for $m \geq 0$. Either $\|W_{m+1}^{\text{move}} - h(W_{m+1}^{\text{move}})\|_2 \leq \max_{0 \leq n \leq m} \|W_n^{\text{move}} - h(W_n^{\text{move}})\|_2$ and the proof is finished. Or $\|W_{m+1}^{\text{move}} - h(W_{m+1}^{\text{move}})\|_2 > \max_{0 \leq n \leq m} \|W_n^{\text{move}} - h(W_n^{\text{move}})\|_2$. The fundamental relation (8) gives

$$\begin{aligned} L(H_{m+1}) &\leq \max\{L(H_m), \|W_m^{\text{move}} - h(W_m^{\text{move}})\|_2\} \\ &\leq \max\left\{1, \max_{0 \leq n \leq m} \|W_n^{\text{move}} - h(W_n^{\text{move}})\|_2\right\} \end{aligned}$$

by induction and since $L(H_0) \leq 1$ by the definition of a renewal step. Hence we have in this case that

$$L(H_{m+1}) \leq \max\{1, \|W_{m+1}^{\text{move}} - h(W_{m+1}^{\text{move}})\|_2\},$$

which forces, via similar arguments to those developed in the proof of Lemma 7 (i), that

$$\|W_{m+1}^{\text{move}} - h(W_{m+1}^{\text{move}})\|_2 \leq \lfloor 2\zeta_{m+2} \rfloor + 2.$$

This concludes the proof of (34).

By Proposition 17, β_ℓ is stochastically dominated by an integer-valued r.v. \mathcal{R} with exponentially decaying tails. Then, using (34), the dominating random variable \mathcal{W} is defined as

$$\mathcal{W} := \sum_{m=0}^{\mathcal{R}} \left(\max_{0 \leq n \leq m} \lfloor 2\zeta_{n+1} \rfloor + 2 \right)$$

whose distribution does not depend on $\mathcal{G}_{\ell-1}$.

The fact that the r.v.'s $\{\max_{n \leq m} \lfloor 2\zeta_{n+1} \rfloor + 2 : m \in \mathbb{N}\}$ are not identically distributed prevents us to immediately obtain exponential decay for the distribution of \mathcal{W} . So we content ourself with the following computation leading to sub-exponential decay. First,

$$\mathbb{P}(\mathcal{W} \geq n) \leq \mathbb{P}\left(\sum_{m=0}^{\lfloor n^{1/2} \rfloor} \max_{l \leq m} (\lfloor 2\zeta_{l+1} \rfloor + 2) \geq n\right) + \mathbb{P}(\mathcal{R} \geq n^{1/2}). \quad (35)$$

The second term of the l.h.s. of (35) is bounded from above by $C_0 e^{-C_1 n^{1/2}}$ thanks to Proposition 17 while the first one is treated as follows:

$$\begin{aligned} \mathbb{P}\left(\sum_{m=0}^{\lfloor n^{1/2} \rfloor} \max_{l \leq m} (\lfloor 2\zeta_{l+1} \rfloor + 2) \geq n\right) &\leq \mathbb{P}\left(\bigcup_{m \leq \lfloor n^{1/2} \rfloor} \{\lfloor 2\zeta_{m+1} \rfloor + 2 \geq n^{1/2} - 1\}\right) \\ &\leq (\lfloor n^{1/2} \rfloor + 1) \mathbb{P}\left(\lfloor 2\zeta_1 \rfloor + 2 \geq n^{1/2} - 1\right) \\ &\leq (\lfloor n^{1/2} \rfloor + 1) C_0 e^{-C_1 (n^{1/2} - 1)} \end{aligned}$$

by (11). We conclude by adjusting the constants $C_0, C_1 > 0$. \square

2.4.2 From number of steps to real time

In this section we focus on the original paths $\{(g_n(\mathbf{u}_1), \dots, g_n(\mathbf{u}_k), H_n) : n \geq 0\}$. Recall that $W_n^{\text{move}}(2) = \min\{g_n(\mathbf{u}_1)(2), \dots, g_n(\mathbf{u}_k)(2)\}$, for any n . Hence, for any $t \geq W_0^{\text{move}}(2)$, let us define $n(t)$ as the unique integer n such that

$$W_n^{\text{move}}(2) \leq t < W_{n+1}^{\text{move}}(2). \quad (36)$$

The integer $n(t)$ is the number of steps required for the moving vertex to progress more than t in ordinate. The next result states that, with high probability, $n(t)$ grows at least linearly with t :

Proposition 20. *For any constant $c > 0$ small enough and for any $t > W_0^{\text{move}}(2)$, we have*

$$\mathbb{P}(n(t) < c(t - W_0^{\text{move}}(2))) \leq C_0 e^{-C_1(t - W_0^{\text{move}}(2))}.$$

Proof. By translation invariance of the DSF, we can assume in this proof that $W_0^{\text{move}}(2) = 0$. Let us start with a sequence of positive r.v.'s $(Y_n)_{n \geq 1}$ defined as follows: for any $n \geq 1$,

$$\sum_{m=1}^n Y_m := \max\{g_n(\mathbf{u}_1)(2), \dots, g_n(\mathbf{u}_k)(2)\}.$$

See Figure 8 for an illustration.

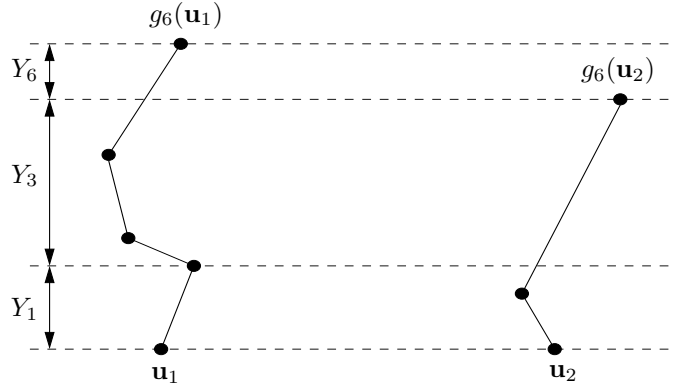


Figure 8: This picture represents $k = 2$ trajectories starting from \mathbf{u}_1 and \mathbf{u}_2 (with the same ordinate) until step $n = 6$. At the first step, \mathbf{u}_1 moves and $g_1(\mathbf{u}_1)$ realizes the height Y_1 . Thus, $Y_2 = 0$ since $g_2(\mathbf{u}_2)$ is below $g_2(\mathbf{u}_1)$, and the sum $Y_1 + Y_2 + Y_3$ is realized by $g_3(\mathbf{u}_2)$. For the same reason Y_4 and Y_5 are null.

From the definition of $n(t)$, we can write

$$\sum_{m=1}^{n(t)+1} Y_m = \max\{g_{n(t)+1}(\mathbf{u}_1)(2), \dots, g_{n(t)+1}(\mathbf{u}_k)(2)\} \geq W_{n(t)+1}^{\text{move}}(2) \geq t, \quad (37)$$

for any t . As a consequence, the inequality $n(t) < ct$ forces the sum $\sum_{m \leq \lfloor ct \rfloor + 1} Y_m$ to be larger than t . This is the reason why we aim at bounding from above the sum of Y_m 's by i.i.d. r.v.'s. We claim that for any $n \geq 1$ and with probability 1,

$$\sum_{m=1}^n Y_m \leq \sum_{m=0}^n (\lfloor 2\zeta_m \rfloor + 2) \mathbf{1}_{B_m} + 2\kappa \quad (38)$$

where $\mathbf{1}_{B_0} := 1$ and for $m \geq 0$, the event B_{m+1} is defined as $B_{m+1} := \{\|h(W_m^{\text{move}}) - W_m^{\text{move}}\|_2 > L(H_m)\}$. Recall also that the r.v.'s ζ_m , $m \geq 1$, are defined in (9) and $\zeta_0 := 0$.

Inequality (38) allows us to write

$$\mathbb{P}(n(t) < ct) \leq \mathbb{P}\left(\sum_{m=0}^{\lfloor ct \rfloor + 1} (\lfloor 2\zeta_m \rfloor + 2)\mathbf{1}_{B_m} + 2\kappa \geq t\right)$$

from which it is not difficult to conclude with c small enough and using that the $\{\zeta_m : m \geq 0\}$ are i.i.d. with exponential decay.

It then remains to prove (38). Let us first show that

$$\sum_{m=1}^n Y_m = \max\{g_n(\mathbf{u}_1)(2), \dots, g_n(\mathbf{u}_k)(2)\} \leq W_n^{\text{move}}(2) + L(H_n) + \kappa. \quad (39)$$

To have $\max_i g_n(\mathbf{u}_i)(2)$ larger than $W_n^{\text{move}}(2) + L(H_n)$, it is required that this maximum is realized by a starting point $\mathbf{u}_i = g_n(\mathbf{u}_i)$ which has not moved yet. But assumption (6) on the \mathbf{u}_i 's then gives $\max_i g_n(\mathbf{u}_i)(2) \leq \kappa$. And (39) follows. If we can show that the inequality

$$W_n^{\text{move}}(2) + L(H_n) \leq \sum_{m=0}^n (\lfloor 2\zeta_m \rfloor + 2)\mathbf{1}_{B_m} + \kappa \quad (40)$$

holds a.s. for any integer $n \geq 0$, then (38) is proved. We show (40) by induction over $n \geq 0$. The case $n = 0$ is obvious. Assume that (40) holds for a given integer $n \geq 0$. If the event B_{n+1} does not occur, then by induction,

$$W_{n+1}^{\text{move}}(2) + L(H_{n+1}) = W_n^{\text{move}}(2) + L(H_n) \leq \sum_{m=0}^n (\lfloor 2\zeta_m \rfloor + 2)\mathbf{1}_{B_m} + \kappa = \sum_{m=0}^{n+1} (\lfloor 2\zeta_m \rfloor + 2)\mathbf{1}_{B_m} + \kappa.$$

Let us now focus on the case $\mathbf{1}_{B_{n+1}} = 1$. We will use the following statement, stated in the proof of Lemma 7. On the event B_{n+1} ,

$$\max\{L(H_{n+1}), \|W_n^{\text{move}}(2) - h(W_n^{\text{move}}(2))\|_2\} \leq \lfloor 2\zeta_{n+1} \rfloor + 2. \quad (41)$$

Let us distinguish two cases.

- Assume the set W_n^{stay} contains an element which is below the level $W_n^{\text{move}}(2) + L(H_n)$. Then

$$W_{n+1}^{\text{move}}(2) \leq W_n^{\text{move}}(2) + L(H_n).$$

We get:

$$W_{n+1}^{\text{move}}(2) + L(H_{n+1}) \leq W_n^{\text{move}}(2) + L(H_n) + L(H_{n+1})\mathbf{1}_{B_{n+1}} \leq \sum_{m=0}^{n+1} (\lfloor 2\zeta_m \rfloor + 2)\mathbf{1}_{B_m} + \kappa$$

by the recurrence hypothesis and (41).

- Otherwise, all the elements of W_n^{stay} (if they exist) are above the level $W_n^{\text{move}}(2) + L(H_n)$. This situation may only occur in the two following cases. Either $W_n^{\text{stay}} = \emptyset$ which means that at step n , only one trajectory remains. Then

$$W_{n+1}^{\text{move}}(2) + L(H_{n+1}) = W_n^{\text{move}}(2) + \|W_n^{\text{move}}(2) - h(W_n^{\text{move}}(2))\|_2$$

on the event B_{n+1} , and we conclude as previously by the recurrence hypothesis and (41). Or $W_n^{\text{stay}} \neq \emptyset$ and then it is made up of starting vertices \mathbf{u}_i 's. Assumption (6) ensures that $W_{n+1}^{\text{move}}(2)$ is smaller than κ . We conclude easily. □

3 Tail distribution for the coalescence time of two paths

3.1 Main result

Let $\mathbf{u}_1, \mathbf{u}_2$ be two starting points in \mathbb{R}^2 that we can assume without loss of generality to be such that $\mathbf{u}_1(1) < \mathbf{u}_2(1)$ and $\mathbf{u}_1(2) = \mathbf{u}_2(2) = 0$. As explained in the introduction, a key result for proving the convergence of the DSF to the BW, lies in a precise estimate for the tail distribution of the coalescence time of two paths of the DSF:

$$T(\mathbf{u}_1, \mathbf{u}_2) := \inf\{t \geq 0 : \pi^{\mathbf{u}_1}(t) = \pi^{\mathbf{u}_2}(t)\} \quad (42)$$

where $\pi^{\mathbf{u}_i} = (\pi^{\mathbf{u}_i}(t))_{t \geq 0}$ denotes the parametrization of the path $\pi^{\mathbf{u}_i}$. This random time is known to be almost surely finite [13]. Our strategy to get such estimate is to work with the Markov chain $\{Z_\ell : \ell \geq 0\}$ defined from the positions at the renewal steps, that has non-negative values and admits 0 as absorbing state:

$$\forall \ell \geq 0, Z_\ell := \mathbf{u}_2^{(\ell)}(1) - \mathbf{u}_1^{(\ell)}(1)$$

(see Corollary 16). Let us denote by $\nu = \nu(\mathbf{u}_1, \mathbf{u}_2)$ the number of (renewal) steps required by the process $\{Z_\ell : \ell \geq 1\}$ to hit 0:

$$\nu := \inf\{\ell \geq 1 : Z_\ell = 0\} . \quad (43)$$

Proposition 14 (iii) asserts that the ordinate $T_\nu := \mathbf{u}_1^{(\nu)}(2) - \mathbf{u}_1^{(0)}(2) = \mathbf{u}_1^{(\nu)}(2)$ gives an upper bound for the coalescence time $T(\mathbf{u}_1, \mathbf{u}_2)$ of the two paths $\pi^{\mathbf{u}_1}$ and $\pi^{\mathbf{u}_2}$. Here is the expected result:

Theorem 21. *Assume $\mathbf{u}_2(1) - \mathbf{u}_1(1) \geq 1$. There exists a constant $C_0 > 0$ which does not depend on $\mathbf{u}_1, \mathbf{u}_2$ such that, for any $t > 0$,*

$$\mathbb{P}(T(\mathbf{u}_1, \mathbf{u}_2) > t | Z_0 = (\mathbf{u}_2(1) - \mathbf{u}_1(1))) \leq \mathbb{P}(T_\nu > t | Z_0 = (\mathbf{u}_2(1) - \mathbf{u}_1(1))) \leq \frac{C_0(\mathbf{u}_2(1) - \mathbf{u}_1(1))}{\sqrt{t}} .$$

To establish Theorem 21, it is classical (see e.g. [2, Prop. 5.3]) to first focus on the random time

$$\nu^{(m_0)} = \nu^{(m_0)}(\mathbf{u}_1, \mathbf{u}_2) := \inf\{\ell \geq 1 : Z_\ell \leq m_0\}$$

at which (Z_ℓ) comes into the segment $[0, m_0]$ for the first time. We will establish the following intermediate estimate on $\nu^{(m_0)}$:

Proposition 22. *There exist positive constants m_0, C_0 such that for any real number $z > 0$ and any integer n ,*

$$\mathbb{P}(\nu^{(m_0)} > n | Z_0 = z) \leq \frac{C_0 z}{\sqrt{n}} . \quad (44)$$

The proof of Proposition 22 relies on a general result providing estimates of tail decay for hitting times of processes that are *not* necessarily Markov: see Theorem 23 below. This result, which has its own interest, will be proved in Section 3.2. In Section 3.3 we show how Theorem 23 can be applied to our context and how it leads to Proposition 22.

This section ends with the proof of Theorem 21 from Proposition 22.

Proof. The proof is divided in two parts. We first show how Proposition 22 implies the following intermediate inequality: for any real number $z \geq 1$ and any integer n ,

$$\mathbb{P}(\nu > n | Z_0 = z) \leq \frac{C_0 z}{\sqrt{n}} \quad (45)$$

where ν is defined in (43), from which we will deduce Theorem 21.

Let us prove (45). Let us first start with some remarks. Assume that Z_ℓ is in $[0, m_0]$ for some index $\ell \geq 0$. The probability for hitting 0 in one step, i.e. $Z_{\ell+1} = 0$, is uniformly bounded from below by some constant $p(m_0) > 0$ (which does not depend on ℓ) with respect to the position of Z_ℓ in $[0, m_0]$. Otherwise, $Z_{\ell+1}$ belongs to $[0, m_0 + 2\mathcal{W}]$ where the r.v. \mathcal{W} bounds the maximal deviation for one path between two consecutive renewal steps (see Section 2.4.1). Since the paths of the DSF cannot cross, the return time to $[0, m_0]$ starting from $Z_{\ell+1}$ is then dominated by the same return time but starting from $m_0 + 2\mathcal{W}$. These remarks allow us to stochastically dominate the random time ν for the Markov chain (Z_ℓ) to hit 0, starting from $Z_0 = z$, by a geometric number (with parameter $p(m_0)$) of i.i.d. return times to $[0, m_0]$ starting from $\max\{z, m_0 + 2\mathcal{W}_i\}$ where $(\mathcal{W}_i)_{i \geq 1}$ are i.i.d. copies of \mathcal{W} . Then, classic computation and Proposition 22 imply that, for any $n > 0$,

$$\mathbb{P}(\nu > n | Z_0 = 1) \leq \frac{C_0 \max\{1, m_0 + 2\mathbb{E}\mathcal{W}\}}{\sqrt{n}}. \quad (46)$$

To get (45), we proceed as in [19] whose argument is based on the following idea. For $z \geq 1$, if the paths of the DSF starting at 0 and z have not coalesced yet at step n then this is also true for the paths starting at i and $i + 1$ for at least one integer $0 \leq i \leq \lfloor z \rfloor - 1$, or for the paths starting at $\lfloor z \rfloor$ and z . Then, applying inequality (46), we obtain (45).

Now, let us prove Theorem 21. When (Z_ℓ) hits 0 for the first time, at the ν -th (renewal) step, then the two paths of the regenerated process started at $(\mathbf{u}_1^{(\nu-1)}, \mathbf{u}_2^{(\nu-1)})$ have already coalesced at step β_ν . Thanks to Proposition 14, they also coincide with the original paths, and

$$g_{\beta_\nu}(\mathbf{u}_1^{(\nu-1)}) = g_{\beta_\nu}(\mathbf{u}_2^{(\nu-1)}) = g_m(\mathbf{u}_1) = g_m(\mathbf{u}_2)$$

for some m such that $m \leq \sum_{\ell=1}^\nu \beta_\ell$. If $T_\nu > t$ then the integer m is also larger than the number $n(t)$ of steps of the joint exploration process $\{(g_n(\mathbf{u}_1), g_n(\mathbf{u}_2), H_n) : n \geq 0\}$ at level t . See (36) for a precise definition. So,

$$\mathbb{P}(T_\nu > t) \leq \mathbb{P}\left(\sum_{\ell=1}^\nu \beta_\ell \geq \lfloor ct \rfloor\right) + \mathbb{P}(n(t) < \lfloor ct \rfloor). \quad (47)$$

Choosing the constant $c > 0$ as in Proposition 20, the second term of the r.h.s. of (47) is bounded by $C_0 e^{-C_1 t}$. Recall that the r.v.'s β_ℓ are i.i.d. with exponential tail. Hence, it is not difficult to obtain

$$\mathbb{P}\left(\sum_{\ell=1}^{\lfloor c't \rfloor} \beta_\ell \geq \lfloor ct \rfloor\right) \leq C_0 e^{-C_1 t}$$

for a constant $c' > 0$ small enough. To sum up, we have:

$$\mathbb{P}(T_\nu > t) \leq \mathbb{P}(\nu > \lfloor c't \rfloor) + C_0 e^{-C_1 t}$$

from which we conclude using (45). \square

3.2 A general result for upper bounding hitting time tails

In this section we obtain tail decay for the hitting time of 0 for certain class of processes which need not to be Markov.

Theorem 23. *Let $\{Y_\ell : \ell \geq 0\}$ be a $\{\mathcal{G}_\ell : \ell \geq 0\}$ adapted stochastic process taking values in \mathbb{R}_+ with a.s. $Y_0 > 0$. Let $\nu^Y := \inf\{\ell \geq 1 : Y_\ell = 0\}$ be the first return time to 0. Let us assume that:*

- (i) ν^Y is almost surely finite.
- (ii) For any $\ell \geq 0$, a.s. $\mathbb{E}[Y_{\ell+1} - Y_\ell | \mathcal{G}_\ell] \leq 0$.

(iii) There exist constants $C_0, C_1 > 0$ such that for any $\ell \geq 0$, a.s. on the event $\{Y_\ell > 0\}$, we have

$$\mathbb{E}[(Y_{\ell+1} - Y_\ell)^2 \mid \mathcal{G}_\ell] \geq C_0 \quad \text{and} \quad \mathbb{E}[|Y_{\ell+1} - Y_\ell|^3 \mid \mathcal{G}_\ell] \leq C_1 .$$

Then there exists a constant $C_2 > 0$ such that for any $y > 0$ and any integer n ,

$$\mathbb{P}(\nu^Y > n \mid Y_0 = y) \leq \frac{C_2 y}{\sqrt{n}} .$$

Proof. The proof is divided into three steps. In the sequel, we will use the notation \mathbb{P}_y (resp. \mathbb{E}_y) for $\mathbb{P}(\cdot \mid Y_0 = y)$ (resp. $\mathbb{E}(\cdot \mid Y_0 = y)$).

Step 1: Assume that there exist constants $C_3, \theta_0 > 0$ such that for all $0 < \theta < \theta_0$

$$\mathbb{E}_y(\exp(-C_3 \theta^2 \nu^Y)) \geq \exp(-\theta y) . \quad (48)$$

Using that $x \mapsto 1 - \exp(-C_3 \theta^2 x)$ is a non-decreasing function for any $\theta > 0$, the Markov inequality and (48), we get:

$$\mathbb{P}_y(\nu^Y > n) \leq \frac{\mathbb{E}_y(1 - \exp(-C_3 \theta^2 \nu^Y))}{1 - \exp(-C_3 \theta^2 n)} \leq \frac{1 - \exp(-\theta y)}{1 - \exp(-C_3 \theta^2 n)} ,$$

provided that $\theta < \theta_0$. Hence, for $\theta = 1/\sqrt{n}$ with $n > 1/\theta_0^2$,

$$\mathbb{P}_y(\nu^Y > n) \leq \frac{1 - \exp(-y/\sqrt{n})}{1 - \exp(-C_3)} \leq \frac{y}{\sqrt{n}(1 - \exp(-C_3))} ,$$

which is the announced result with $C_2 = (1 - \exp(-C_3))^{-1}$.

Step 2: It remains to prove the estimate (48) on the Laplace transform of ν^Y . To do it, we use martingale techniques. For $\theta > 0$ and $j \geq 0$, let us set

$$\psi_{\theta,j} := \mathbb{E}(\exp(-\theta(Y_{j+1} - Y_j)) \mid \mathcal{G}_j) .$$

Thus we define a discrete time process as follows: $Z_0 := \exp(-\theta Y_0) = \exp(-\theta y)$ \mathbb{P}_y -a.s. and for $\ell \geq 1$,

$$Z_\ell := \frac{\exp(-\theta Y_\ell)}{\prod_{j=0}^{\ell-1} \psi_{\theta,j}} .$$

This process is a $\{\mathcal{G}_\ell : \ell \geq 0\}$ -martingale since

$$\begin{aligned} \mathbb{E}(Z_{\ell+1} \mid \mathcal{G}_\ell) &= \mathbb{E}\left[\frac{\exp(-\theta(Y_{\ell+1} - Y_\ell)) \exp(-\theta Y_\ell)}{\prod_{j=0}^{\ell} \psi_{\theta,j}} \mid \mathcal{G}_\ell\right] \\ &= \frac{Z_\ell}{\psi_{\theta,\ell}} \mathbb{E}[\exp(-\theta(Y_{\ell+1} - Y_\ell)) \mid \mathcal{G}_\ell] \\ &= Z_\ell . \end{aligned}$$

Then, $(Z_{\ell \wedge \nu^Y})_{\ell \geq 0}$ is also a non-negative martingale and for any $\ell \geq 0$,

$$\mathbb{E}_y(Z_{\ell \wedge \nu^Y}) = \mathbb{E}_y(Z_0) = \exp(-\theta y) . \quad (49)$$

For the moment, let us assume that there exist constants $C_3, \theta_0 > 0$ such that for all $\theta \in (0, \theta_0)$ and for all index ℓ ,

$$\exp(-\theta Y_{\ell \wedge \nu^Y} - (\ell \wedge \nu^Y) C_3 \theta^2) \geq Z_{\ell \wedge \nu^Y} . \quad (50)$$

Then, we are able to prove (48). Let $\theta \in (0, \theta_0)$. By hypothesis the return time ν^Y is almost surely finite and $Y_{\nu^Y} = 0$. So the following limit holds with probability 1:

$$\lim_{\ell \rightarrow \infty} \exp(-\theta Y_{\ell \wedge \nu^Y} - (\ell \wedge \nu^Y) C_3 \theta^2) = \exp(-\nu^Y C_3 \theta^2).$$

Let us apply the dominated convergence theorem. Indeed $\exp(-\theta Y_{\ell \wedge \nu^Y} - (\ell \wedge \nu^Y) C_3 \theta^2)$ is smaller than 1 for all ℓ since Y_ℓ is non-negative. Hence, with (49) and (50), we can write:

$$\begin{aligned} \mathbb{E}_y(\exp(-C_3 \theta^2 \nu^Y)) &= \lim_{\ell \rightarrow \infty} \mathbb{E}_y(\exp(-\theta Y_{\ell \wedge \nu^Y} - (\ell \wedge \nu^Y) C_3 \theta^2)) \\ &\geq \limsup_{\ell \rightarrow \infty} \mathbb{E}_y(Z_{\ell \wedge \nu^Y}) \\ &= \exp(-\theta y). \end{aligned}$$

Step 3: If there exist constants $C_3, \theta_0 > 0$ such that, for any $\theta \in (0, \theta_0)$ and any $j \in \{0, 1, \dots, \ell \wedge \nu^Y - 1\}$,

$$\log(\psi_{\theta, j}) \geq C_3 \theta^2 \tag{51}$$

then $\sum_{j=0}^{\ell \wedge \nu^Y - 1} \log(\psi_{\theta, j}) \geq (\ell \wedge \nu^Y) C_3 \theta^2$ from which (50) easily follows. To do it, let us combine the inequality $\exp(x) \geq 1 + x + x^2/2 + x^3/6$ with hypotheses (ii) and (iii):

$$\begin{aligned} \psi_{\theta, j} &= \mathbb{E}(e^{-\theta(Y_{j+1} - Y_j)} \mid \mathcal{G}_j) \\ &\geq 1 - \theta \mathbb{E}(Y_{j+1} - Y_j \mid \mathcal{G}_j) + \frac{\theta^2}{2} \mathbb{E}((Y_{j+1} - Y_j)^2 \mid \mathcal{G}_j) - \frac{\theta^3}{6} \mathbb{E}((Y_{j+1} - Y_j)^3 \mid \mathcal{G}_j) \\ &\geq 1 + C_0 \frac{\theta^2}{2} - C_1 \frac{\theta^3}{6}, \end{aligned}$$

for any θ and any index $j \in \{0, 1, \dots, \ell \wedge \nu^Y - 1\}$ (so that $Y_j > 0$). The constants C_0, C_1 do not depend on j . The function $\theta \in [0, \infty) \mapsto 1 + C_0 \theta^2/2 - C_1 \theta^3/6$ is continuous, equal to 1 at $\theta = 0$ and increasing on the neighborhood of this point. Hence, it is possible to pick $\theta_0 > 0$ such that for all $0 < \theta < \theta_0$, $1 < 1 + C_0 \theta^2/2 - C_1 \theta^3/6 < 2$. Since $\log(x) \geq (x - 1)/2$ for $x \in (1, 2)$, we obtain for any $0 < \theta < \theta_0$,

$$\frac{1}{\theta^2} \log(\psi_{\theta, j}) \geq \frac{1}{\theta^2} \log(1 + C_0 \frac{\theta^2}{2} - C_1 \frac{\theta^3}{6}) \geq \frac{C_0}{4} - \frac{C_1 \theta}{12}.$$

We then deduce (51) for $\theta_0 > 0$ small enough and $C_3 = C_0/8$. \square

3.3 Proof of Proposition 22

The goal of this section is to prove Proposition 22 thanks to the general result stated in the previous section, namely Theorem 23.

It is not clear at all that the process $\{Z_\ell : \ell \geq 0\}$, where $Z_\ell = \mathbf{u}_2^{(\ell)}(1) - \mathbf{u}_1^{(\ell)}(1)$, satisfies condition (ii) of Theorem 23. So the strategy consists in considering instead a function of this process, denoted here by $\{Y_\ell : \ell \geq 0\}$ and defined for any $\ell \geq 0$ by

$$Y_\ell := f(Z_\ell) \mathbf{1}_{\nu(m_0) > \ell}, \tag{52}$$

where $\nu(m_0) = \inf\{\ell \geq 1 : Z_\ell \leq m_0\}$ and f is the following function.

$$\begin{aligned} f : \mathbb{R}_+ &\rightarrow \mathbb{R}_+ \\ x &\mapsto f(x) = \left(\frac{1}{2} + \frac{1}{x+2}\right)x. \end{aligned} \tag{53}$$

The choice of f has to make easier the verifications of hypotheses (i) – (iii) of Theorem 23 (similar ideas are used in [2, Prop. 5.3] for example). The function f is differentiable with $f'(x) = 1/2 + 2/(x+2)^2$. It is

a strictly increasing and concave bijection from \mathbb{R}_+ to \mathbb{R}_+ . It also satisfies $f(0) = 0$ and $x/2 < f(x) < x$ for any x .

We then deduce that $\{Y_\ell : \ell \geq 0\}$ is a non-negative (\mathcal{G}_ℓ) -Markov chain with absorbing state 0 by Corollary 16, where the filtration $\{\mathcal{G}_\ell : \ell \geq 0\}$ is defined in Section 2.3. Moreover, the hitting time of 0, $\nu^Y = \inf\{\ell \geq 1 : Y_\ell = 0\}$, satisfies for any $z > 0$,

$$\mathbb{P}(\nu^{(m_0)} > n \mid Z_0 = z) = \mathbb{P}(\nu^Y > n \mid Y_0 = f(z)) .$$

Let us prove that the process $\{Y_\ell : \ell \geq 0\}$ satisfies the hypotheses of Theorem 23. The first one is easy to check. The condition $\nu^Y = \infty$ forces $Z_\ell = \mathbf{u}_2^{(\ell)}(1) - \mathbf{u}_1^{(\ell)}(1)$ to be larger than m_0 for any ℓ . Provided $m_0 \geq 2(\kappa + 1)$, the sequence of regenerated paths coincides with original paths $\pi^{\mathbf{u}_1}$ and $\pi^{\mathbf{u}_2}$. Recall that original paths may bifurcate (and then coalesce) from regenerated paths only during a renewal step and only if $Z_\ell < 2(\kappa + 1)$. See Section 2.3. Then, $\nu^Y = \infty$ prevents the coalescence of $\pi^{\mathbf{u}_1}$ and $\pi^{\mathbf{u}_2}$. But any two paths of the DSF eventually coalesce almost surely: this is the main result of Coupier and Tran [13]. So, for $m_0 \geq 2(\kappa + 1)$, the event $\{\nu^Y < \infty\}$ occurs with probability 1.

Let us now check hypothesis (iii). From the derivative of f , we have the inequality $|Y_{\ell+1} - Y_\ell| \geq \frac{1}{2}|Z_{\ell+1} - Z_\ell|$. Hence,

$$\begin{aligned} \mathbb{E}[(Y_{\ell+1} - Y_\ell)^2 \mid \mathcal{G}_\ell] &= \mathbb{E}[(Y_{\ell+1} - Y_\ell)^2 \mid Y_\ell] \geq \frac{1}{4}\mathbb{E}[(Z_{\ell+1} - Z_\ell)^2 \mid Z_\ell] \\ &\geq \frac{1}{4}\mathbb{E}[(Z_{\ell+1} - Z_\ell)^2 \mathbf{1}_{\{|Z_{\ell+1} - Z_\ell| \geq 1\}} \mid Z_\ell] \\ &\geq \frac{1}{4}\mathbb{P}(|Z_{\ell+1} - Z_\ell| \geq 1 \mid Z_\ell) . \end{aligned}$$

On the event $\{Y_\ell > 0\} \subset \{Z_\ell > 0\}$, it is not difficult to convince oneself that the conditional probability $\mathbb{P}(|Z_{\ell+1} - Z_\ell| \geq 1 \mid Z_\ell)$ is strictly positive (suitable configurations are easy to build). For third moment, we have

$$\mathbb{E}[|Y_{\ell+1} - Y_\ell|^3 \mid \mathcal{G}_\ell] \leq \mathbb{E}[|Z_{\ell+1} - Z_\ell|^3 \mid \mathcal{G}_\ell] \leq \mathbb{E}[(2W_{\ell+1})^3 \mid \mathcal{G}_\ell] \leq 8\mathbb{E}[\mathcal{W}^3] < \infty ,$$

where $W_{\ell+1}$ is the size of the ℓ -th renewal block— see (31) —and \mathcal{W} is the r.v. appearing in Proposition 19 with sub-exponential tail distribution.

It then remains to check that $\{Y_\ell : \ell \geq 1\}$ satisfies condition (ii) which actually means that $\{Y_\ell : \ell \geq 1\}$ is a super-martingale. Let $m > 0$. Let us first write:

$$\begin{aligned} \mathbb{E}[(f(Z_{\ell+1}) - f(Z_\ell))\mathbf{1}_{\{W_{\ell+1} \geq m/2\}} \mid Z_\ell = m] &\leq \mathbb{E}[|Z_{\ell+1} - Z_\ell|\mathbf{1}_{\{W_{\ell+1} \geq m/2\}} \mid Z_\ell = m] \\ &\leq 2\mathbb{E}[W_{\ell+1}\mathbf{1}_{\{W_{\ell+1} \geq m/2\}} \mid Z_\ell = m] \\ &\leq C_0\mathbb{E}[\mathcal{W}^2]^{1/2} e^{-C_1(m/2)^{1/2}} \end{aligned}$$

thanks to Proposition 19. Let us consider the increment $I_{\ell+1} := Z_{\ell+1} - Z_\ell$. Conditionally to $Z_\ell = m$,

$$f(Z_{\ell+1}) - f(Z_\ell) = f(m + I_{\ell+1}) - f(m) \leq f'(m)I_{\ell+1} - \frac{2}{(m+2)^3}I_{\ell+1}^2 + \frac{f^{(3)}(m)}{3!}I_{\ell+1}^3 \quad (54)$$

using Taylor's expansion with $f^{(2)}(x) = -4/(x+2)^3$ and $f^{(4)}(x) = -48/(x+2)^5 < 0$ for all $x > 0$. A remarkable fact is that the distribution of the increment $I_{\ell+1}$ is symmetric while paths $\pi^{\mathbf{u}_1}$ and $\pi^{\mathbf{u}_2}$ do not interact between the ℓ -th and $(\ell+1)$ -th renewal steps.

Lemma 24. *For any $\ell \geq 0$, conditionally to $\{Z_\ell = m\}$ and on the event $\{W_{\ell+1} < m/2\}$, the distribution of $I_{\ell+1}$ is symmetric, i.e. $I_{\ell+1} \stackrel{(d)}{=} -I_{\ell+1}$.*

Lemma 24 will be proved at the end of the section. From now on, for any odd integer s ,

$$\mathbb{E}[I_{\ell+1}^s \mathbf{1}_{\{W_{\ell+1} < m/2\}} \mid Z_\ell = m] = 0 .$$

From the expansion (54), we then obtain:

$$\mathbb{E}[(f(Z_{\ell+1}) - f(Z_\ell))\mathbf{1}_{\{W_{\ell+1} < m/2\}} \mid Z_\ell = m] \leq -\frac{2}{(m+2)^3} \mathbb{E}[I_{\ell+1}^2 \mathbf{1}_{\{W_{\ell+1} < m/2\}} \mid Z_\ell = m] .$$

For $m \geq 2(\kappa + 1)$ (so that there is no interaction between the two regenerated paths at the ℓ -th step), it is not difficult to observe that, by creating suitable configurations, the probability $\mathbb{P}(I_{\ell+1} \mathbf{1}_{\{W_{\ell+1} < m/2\}} \geq 1 \mid Z_\ell = m)$ is bounded from below by a positive constant, say $\alpha > 0$. So, for any $m \geq 2(\kappa + 1)$,

$$\mathbb{E}[I_{\ell+1}^2 \mathbf{1}_{\{W_{\ell+1} < m/2\}} \mid Z_\ell = m] \geq \alpha .$$

Gathering the previous inequalities, for all $m \geq 2(\kappa + 1)$, we have

$$\begin{aligned} \mathbb{E}[Y_{\ell+1} - Y_\ell \mid Y_\ell = f(m)] &= \mathbb{E}[f(Z_{\ell+1}) - f(Z_\ell) \mid Z_\ell = m] \\ &\leq -\frac{2\alpha}{(m+2)^3} + C_0 \mathbb{E}[\mathcal{W}^2]^{1/2} e^{-C_1(m/2)^{1/2}} . \end{aligned}$$

So, there exists m_0 large enough such that for any $m \geq m_0$, the conditional expectation $\mathbb{E}[Y_{\ell+1} - Y_\ell \mid Y_\ell = f(m)]$ is negative. For this suitable value m_0 , the process $\{Y_\ell : \ell \geq 1\}$ satisfies condition (ii) of Theorem 23.

Proof of Lemma 24. Let $\ell \geq 0$ and let us consider the regenerated paths starting from $(\mathbf{u}_1^{(\ell)}, \mathbf{u}_2^{(\ell)})$ with $Z_\ell = \mathbf{u}_2^{(\ell)}(1) - \mathbf{u}_1^{(\ell)}(1) = m$. As in Section 2.1, recall that these regenerated paths can be constructed with resampled Poisson point processes $\{\mathcal{N}_{\varrho_\ell+n} : n \geq 1\}$ without changing their distribution, where ϱ_ℓ is defined in (27) by $\varrho_\ell = \sum_1^\ell \beta_m$. We construct a new sequence of point processes in the following way: for each $n \geq 1$,

1. the realizations of $\mathcal{N}_{\varrho_\ell+n}$ inside the rectangles the rectangles $R_1 = \mathbf{u}_1^{(\ell)} + [-m/2, m/2] \times [0, m/2]$ and $R_2 = \mathbf{u}_2^{(\ell)} + [-m/2, m/2] \times [0, m/2]$ are interchanged,
2. while the realization of $\mathcal{N}_{\varrho_\ell+n}$ outside these rectangles is kept as it is.

From the properties of Poisson process, we observe that the newly constructed sequence of point processes has the same distribution as $\{\mathcal{N}_{\varrho_\ell+n} : n \geq 1\}$.

Now, we restrict our attention to the event $\{W_{\ell+1} \leq m/2\}$ and consider the “new” regenerated paths starting from $(\mathbf{u}_1^{(\ell)}, \mathbf{u}_2^{(\ell)})$ but using the new sequence of Poisson point processes. For these “new” regenerated paths, remark that the number of steps (and the time) until the next renewal step and the size of the corresponding renewal block have not changed. Moreover, the increments of each regenerated path between the ℓ -th and the $(\ell + 1)$ -th renewal steps, i.e. $\mathbf{u}_i^{(\ell+1)}(1) - \mathbf{u}_i^{(\ell)}(1)$ for $i = 1, 2$, have been interchanged. This means that the increment $I_{\ell+1}$ is become $-I_{\ell+1}$. \square

Remark 25. *Let us end this section with a final remark. Similar arguments as above show that the conditions of Theorem 23 hold for other drainage network models which are also in the basin of attraction of the BW [7, 8, 14, 22, 28]. Actually both models studied in [8, 28] have crossing paths and by taking the distance between two paths, this method should give the required coalescing time tail estimates for these models also. This method is very robust and requires only two properties:*

1. *When the paths are sufficiently far away, they behave like independent random walks;*
2. *The construction of a path depends only on the point configuration in a region whose width decays exponentially.*

4 Convergence to the Brownian web

Recall that the couple $(\mathcal{X}_n, \widehat{\mathcal{X}}_n)$ denotes the scaled DSF and its scaled dual forest, this latter being precisely defined thereafter. This section is devoted to the proof of our main result, namely Theorem 2, stating that the sequence $\{(\mathcal{X}_n, \widehat{\mathcal{X}}_n) : n \geq 1\}$ converges in distribution to the BW and its dual $(\mathcal{W}, \widehat{\mathcal{W}})$. Let us also recall that with a slight abuse of notation, the closure of any element X in (Π, d_Π) or $(\widehat{\Pi}, d_{\widehat{\Pi}})$ will be still denoted by X .

Because of the intricate dependencies of the DSF model, we are not able to apply the earlier techniques available in the literature, as Theorem 26 below, in order to obtain Theorem 2. This is the reason why we provide in Section 4.1 new convergence criteria (Theorem 27) regarding joint convergence to $(\mathcal{W}, \widehat{\mathcal{W}})$ for non-crossing path models. Let us mention here that ideas sustaining this result are already present in [21] (Section 2.3). In Section 4.2, we use results obtained in Sections 2 and 3 to show that the sequence $\{(\mathcal{X}_n, \widehat{\mathcal{X}}_n) : n \geq 1\}$ satisfies the conditions of Theorem 27.

4.1 Convergence criteria for non-crossing path models

Let us recall and comment the first convergence criteria to the BW, provided by Fontes et al. [15], in order to motivate new convergence criteria given in Theorem 27. This section focuses on non-crossing path models. The reader may refer to [24] for a very complete overview on the topic.

Let $\Xi \subset \Pi$. For $t > 0$ and $t_0, a, b \in \mathbb{R}$ with $a < b$, consider the counting random variable $\eta_\Xi(t_0, t; a, b)$ defined as

$$\eta_\Xi(t_0, t; a, b) := \text{Card}\{\pi(t_0 + t) : \pi \in \Xi, \sigma_\pi \leq t_0 \text{ and } \pi(t_0) \in [a, b]\} \quad (55)$$

which considers all paths in Ξ , born before t_0 , that intersect $[a, b]$ at time t_0 and counts the number of different positions these paths occupy at time $t_0 + t$. In Theorem 2.2 of [15], Fontes et al. provided the following convergence criteria.

Theorem 26 (Theorem 2.2 of [15]). *Let $\{\Xi_n : n \in \mathbb{N}\}$ be a sequence of $(\mathcal{H}, B_{\mathcal{H}})$ valued random variables with non-crossing paths. Assume that the following conditions hold:*

(I_1) *Fix a deterministic countable dense set \mathcal{D} of \mathbb{R}^2 . For each $\mathbf{x} \in \mathcal{D}$, there exists $\pi_n^{\mathbf{x}} \in \Xi_n$ such that for any finite set of points $\mathbf{x}^1, \dots, \mathbf{x}^k \in \mathcal{D}$, as $n \rightarrow \infty$, we have $(\pi_n^{\mathbf{x}^1}, \dots, \pi_n^{\mathbf{x}^k})$ converges in distribution to $(W^{\mathbf{x}^1}, \dots, W^{\mathbf{x}^k})$, where $(W^{\mathbf{x}^1}, \dots, W^{\mathbf{x}^k})$ denotes coalescing Brownian motions starting from the points $\mathbf{x}_1, \dots, \mathbf{x}_k$.*

(B_1) *For all $t > 0$, $\limsup_{n \rightarrow \infty} \sup_{(a, t_0) \in \mathbb{R}^2} \mathbb{P}(\eta_{\Xi_n}(t_0, t; a, a + \epsilon) \geq 2) \rightarrow 0$ as $\epsilon \downarrow 0$.*

(B_2) *For all $t > 0$, $\frac{1}{\epsilon} \limsup_{n \rightarrow \infty} \sup_{(a, t_0) \in \mathbb{R}^2} \mathbb{P}(\eta_{\Xi_n}(t_0, t; a, a + \epsilon) \geq 3) \rightarrow 0$ as $\epsilon \downarrow 0$.*

Then Ξ_n converges in distribution to the standard Brownian web \mathcal{W} as $n \rightarrow \infty$.

Let us first mention that for a sequence of $(\mathcal{H}, B_{\mathcal{H}})$ -valued random variables $\{\Xi_n : n \in \mathbb{N}\}$ with non-crossing paths, Criterion (I_1) implies tightness (see Proposition B.2 in the Appendix of [15] or Proposition 6.4 in [24]) and hence subsequential limit(s) always exists. Moreover, Criterion (B_1) has in fact been shown to be redundant with (I_1) for non-crossing path models (see Theorem 6.5 of [24]). Combining (I_1) with Theorem 1, we obtain that any such subsequential limit Ξ a.s. contains a random subset which is distributed as the standard BW \mathcal{W} .

There are several approaches to state the (a.s.) upper bound $\Xi \subset \mathcal{W}$. Criterion (B_2) is often verified by applying an FKG type correlation inequality together with a bound on the distribution of the coalescence time between two paths. However, FKG is a strong property that is not satisfied by most models (especially for models with interactions). This strategy seems really hard to carry out in the DSF context. Hence, new criteria have been suggested to replace (B_2): let us mention for instance Criterion (E) proposed by Newman et al [19]. See also Theorem 6.3 of [24]. Always in [24], Schertzer et al have given in Theorem 6.6 a new criterion replacing (B_2), called the *wedge condition*. Our convergence

result (Theorem 27 below) appears as a generalization of Theorem 6.6 of [24] by considering the joint convergence of $\{(\Xi_n, \widehat{\Xi}_n) : n \geq 1\}$ to the BW and its dual. Here, $\widehat{\Xi}_n$ merely denotes a $(\widehat{\mathcal{H}}, \widehat{\mathcal{B}}_{\widehat{\mathcal{H}}})$ -valued random variable made up of paths running backward in time. Theorem 27 also replaces the wedge condition by the fact that no limiting primal and dual paths can spend positive Lebesgue time together: this is condition (iv). We believe that Theorem 27 is robust and can be applied for studying convergence to the BW for a large variety of models with non-crossing paths.

Theorem 27. *Let $\{(\Xi_n, \widehat{\Xi}_n) : n \geq 1\}$ be a sequence of $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H} \times \widehat{\mathcal{H}}})$ -valued random variables with non-crossing paths only, satisfying the following assumptions:*

- (i) *For each $n \geq 1$, paths in Ξ_n do not cross (backward) paths in $\widehat{\Xi}_n$ almost surely: there does not exist any $\pi \in \Xi_n$, $\widehat{\pi} \in \widehat{\Xi}_n$ and $t_1, t_2 \in (\sigma_\pi, \sigma_{\widehat{\pi}})$ such that $(\widehat{\pi}(t_1) - \pi(t_1))(\widehat{\pi}(t_2) - \pi(t_2)) < 0$ almost surely.*
- (ii) *$\{\Xi_n : n \in \mathbb{N}\}$ satisfies (I_1) .*
- (iii) *The collection $\{(\widehat{\pi}_n(\sigma_{\widehat{\pi}_n}), \sigma_{\widehat{\pi}_n}) : \widehat{\pi}_n \in \widehat{\Xi}_n, n \geq 1\}$ of all starting points of the scaled backward paths is dense in \mathbb{R}^2 .*
- (iv) *For any sub sequential limit $(\mathcal{Z}, \widehat{\mathcal{Z}})$ of $\{(\Xi_n, \widehat{\Xi}_n) : n \in \mathbb{N}\}$, paths of \mathcal{Z} do not spend positive Lebesgue measure time together with the paths of $\widehat{\mathcal{Z}}$: there does not exist any $\pi \in \mathcal{Z}$, $\widehat{\pi} \in \widehat{\mathcal{Z}}$ and t_1, t_2 with $\sigma_\pi \leq t_1 < t_2 \leq \sigma_{\widehat{\pi}}$ such that $\pi(t) = \widehat{\pi}(t)$ for all $t \in [t_1, t_2]$ almost surely.*

Then $(\Xi_n, \widehat{\Xi}_n)$ converges in distribution $(\mathcal{W}, \widehat{\mathcal{W}})$ as $n \rightarrow \infty$.

This section ends with the proof of Theorem 27.

Proof. As mentioned in Section 6.2 of [24], conditions (i) and (ii) imply that the sequence $\{(\Xi_n, \widehat{\Xi}_n) : n \geq 1\}$ is jointly tight and then subsequential limit(s) always exists. Let $(\mathcal{Z}, \widehat{\mathcal{Z}})$ be one of them. Our goal is to identify the distribution of this limiting value with $(\mathcal{W}, \widehat{\mathcal{W}})$.

As the sequence $\{\Xi_n : n \geq 1\}$ satisfies (I_1) , for any $(x, t) \in \mathbb{Q}^2$, there a.s. exists a path $\pi^{(x,t)}$ in \mathcal{Z} starting from the point (x, t) and distributed as a Brownian motion starting from x at time t . Because of the non-crossing paths property of the limit \mathcal{Z} — which inherits this property from Ξ_n (condition (i))—, similar arguments as in the proof of Proposition 3.1 of [15] ensure that $\pi^{(x,t)}$ is a.s. the only path in \mathcal{Z} starting at (x, t) . This means that $\mathcal{Z}_{\mathbb{Q}^2}$ is distributed as a collection of coalescing Brownian motions:

$$\mathcal{Z}_{\mathbb{Q}^2} \stackrel{d}{=} \mathcal{W}_{\mathbb{Q}^2} . \quad (56)$$

(See also discussions in Section 6.2 of [24]).

In order to assert that the closure \mathcal{Z} of $\mathcal{Z}_{\mathbb{Q}^2}$ in (Π, d_Π) is a standard Brownian web, we have to prove that \mathcal{Z} contains no more paths than \mathcal{W} . This is the role of the wedge condition and Theorem 6.6 of [24]. Let us first introduce some notation. For any backward paths $\widehat{\pi}^l$ and $\widehat{\pi}^r$ in $\widehat{\Pi}$ that are ordered with $\widehat{\pi}^l(s) < \widehat{\pi}^r(s)$ at time $s := \min\{\sigma_{\widehat{\pi}^r}, \sigma_{\widehat{\pi}^l}\}$, we define $T(\widehat{\pi}^l, \widehat{\pi}^r) := \sup\{t < s : \widehat{\pi}^l(t) = \widehat{\pi}^r(t)\}$ (possibly equal to $-\infty$) as the first hitting time of $\widehat{\pi}^l$ and $\widehat{\pi}^r$ (which will actually be the coalescing time of these paths). The *wedge with left boundary $\widehat{\pi}^l$ and right boundary $\widehat{\pi}^r$* is the following open set of \mathbb{R}^2 :

$$A(\widehat{\pi}^l, \widehat{\pi}^r) := \{(y, u) \in \mathbb{R}^2 : T(\widehat{\pi}^l, \widehat{\pi}^r) < u < s \text{ and } \widehat{\pi}^l(u) < y < \widehat{\pi}^r(u)\} . \quad (57)$$

A path $\pi \in \Pi$, is said to *enter the wedge $A(\widehat{\pi}^l, \widehat{\pi}^r)$ from outside* if there exist t_1, t_2 with $\sigma_\pi < t_1 < t_2$ such that $(\pi(t_1), t_1) \notin \bar{A}$ and $(\pi(t_2), t_2) \in A$, where \bar{A} denotes the closure of A in \mathbb{R}^2 . The *bottom point* of $A(\widehat{\pi}^l, \widehat{\pi}^r)$ is $(\widehat{\pi}^l(T(\widehat{\pi}^l, \widehat{\pi}^r)), T(\widehat{\pi}^l, \widehat{\pi}^r)) = (\widehat{\pi}^r(T(\widehat{\pi}^l, \widehat{\pi}^r)), T(\widehat{\pi}^l, \widehat{\pi}^r))$. Now, the *wedge condition* claims that

a.s. no path in \mathcal{Z} enters any wedge of $\widehat{\mathcal{Z}}_{\mathbb{Q}^2}$ from outside.

This is Criterion (U) of [24]. The wedge condition combined with (i), (ii) and (iii) implies (Theorem 6.6 of [24]) that Ξ_n converges in distribution to \mathcal{W} as n tends to infinity, i.e. \mathcal{Z} is distributed as \mathcal{W} . By condition (i), primal and dual paths do not cross with probability 1. Hence, the only way for a path π in \mathcal{Z} to enter a wedge of $\widehat{\mathcal{Z}}_{\mathbb{Q}^2}$ from outside is through its bottom point by spending a time of positive Lebesgue measure with the dual path started from the bottom point of the wedge. But this is forbidden by condition (iv). So the wedge condition holds and \mathcal{Z} is distributed as \mathcal{W} .

Next, we focus on the dual paths in $\widehat{\mathcal{Z}}$. From condition (iii), it follows that for any $(x, t) \in \mathbb{Q}^2$, a.s. there exists a backward path $\widehat{\pi}^{(x,t)}$ in $\widehat{\mathcal{Z}}$ starting from (x, t) . Since paths in \mathcal{Z} and $\widehat{\mathcal{Z}}$ do not cross, the position of $\widehat{\pi}^{(x,t)}$ at the rational time $s < t$ can be specified as follows:

$$\widehat{\pi}^{(x,t)}(s) = \sup\{y \in \mathbb{Q} : \pi^{(y,s)} \in \mathcal{Z}_{\mathbb{Q}^2}, \pi^{(y,s)}(t) < x\} = \inf\{y \in \mathbb{Q} : \pi^{(y,s)} \in \mathcal{Z}_{\mathbb{Q}^2}, \pi^{(y,s)}(t) > x\} .$$

For this, the fact that the path $\pi^{(y,s)}$ hits the point (x, t) with zero probability is used. Hence, dual paths in $\widehat{\mathcal{Z}}_{\mathbb{Q}^2}$ are uniquely determined by the forward paths in $\mathcal{Z}_{\mathbb{Q}^2}$. The characterization of the double Brownian web $(\mathcal{W}, \widehat{\mathcal{W}})$ ensures that the dual paths in $\widehat{\mathcal{W}}_{\mathbb{Q}^2}$ are also a.s. uniquely determined by the forward paths in $\mathcal{W}_{\mathbb{Q}^2}$ (see Theorem 2.4 of [24]). We then deduce from (56) that

$$\widehat{\mathcal{Z}}_{\mathbb{Q}^2} \stackrel{d}{=} \widehat{\mathcal{W}}_{\mathbb{Q}^2} .$$

As previously, we can conclude using conditions (i) and (iv) that a.s. paths of $\widehat{\mathcal{Z}}$ do not enter any wedge in $\mathcal{W}_{\mathbb{Q}^2} = \mathcal{Z}_{\mathbb{Q}^2}$ from outside. We then conclude thanks to the next result which is a slight variant of Theorem 1.9 of [26] (see also Theorem 3.9 in [24] and the following remark), whose proof is omitted here:

Lemma 28. *Let $(\mathcal{W}, \widehat{\mathcal{Z}})$ be a $(\mathcal{H} \times \widehat{\mathcal{H}}, \mathcal{B}_{\mathcal{H} \times \widehat{\mathcal{H}}})$ -valued random variable with \mathcal{W} denoting the Brownian web such that a.s. paths of $\widehat{\mathcal{Z}}$ do not enter any wedge in $\mathcal{W}_{\mathbb{Q}^2}$ from outside. Then, we have*

$$\widehat{\mathcal{Z}} \stackrel{d}{=} \widehat{\mathcal{W}} .$$

This completes the proof of Theorem 27: the distribution of the subsequential limit $(\mathcal{Z}, \widehat{\mathcal{Z}})$ is identified as $(\mathcal{W}, \widehat{\mathcal{W}})$. \square

Remark 29. *In [13], it was proved that a.s. there is no bi-infinite path in the DSF, i.e. the DSF contains no path which is also infinite in the backward sense. It was also asked whether the non-existence of bi-infinite path in the DSF could be proved using some duality argument. The dual construction and joint convergence to the double Brownian web $(\mathcal{W}, \widehat{\mathcal{W}})$ stated here gives a positive answer to this question.*

4.2 Verification of conditions of Theorem 27

In this section, we show that the sequence of diffusively scaled path families $\{(\mathcal{X}_n, \widehat{\mathcal{X}}_n) : n \geq 1\}$ obtained from the DSF and its dual forest satisfies the conditions in Theorem 27.

It is time to specify a dual forest $\widehat{\mathfrak{F}}$ to the DSF \mathfrak{F} . We start with the dual vertex set \widehat{V} . For any $(x, t) \in \mathbb{R}^2$, let $(x, t)_r \in \mathcal{N}$ be the unique Poisson point such that

- $(x, t)_r(2) < t$, $h((x, t)_r)(2) \geq t$ and $\pi^{(x,t)_r}(t) > x$ where $\pi^{(x,t)_r}$ denotes the path in \mathcal{X} starting from $(x, t)_r$;
- there is no path $\pi \in \mathcal{X}$ with $\sigma_\pi < t$ and $\pi(t) \in (x, \pi^{(x,t)_r}(t))$.

Hence, $\pi^{(x,t)_r}$ is the nearest path in \mathcal{X} to the right of (x, t) starting strictly before time t . It is useful to observe that $\pi^{(x,t)_r}$ is defined for any $(x, t) \in \mathbb{R}^2$. Similarly, $\pi^{(x,t)_l}$ denotes the nearest path to the left of

(x, t) which starts strictly before time t . Now, for each $(x, t) \in \mathcal{N}$ the nearest left and right dual vertices are respectively defined as

$$\widehat{r}_{(x,t)} := ((x + \pi^{(x,t)r}(t))/2, t) \quad \text{and} \quad \widehat{l}_{(x,t)} := ((x + \pi^{(x,t)l}(t))/2, t) .$$

Then, the dual vertex set \widehat{V} is given by $\widehat{V} := \{\widehat{r}_{(x,t)}, \widehat{l}_{(x,t)} : (x, t) \in \mathcal{N}\}$.

Next, let us define the dual ancestor $\widehat{h}(y, s) = \widehat{h}(y, s, \mathcal{N})$ of $(y, s) \in \widehat{V}$ as the unique vertex in \widehat{V} given by

$$\widehat{h}(y, s) := \begin{cases} \widehat{l}_{(y,s)_r} & \text{if } (y, s)_r(2) > (y, s)_l(2) \\ \widehat{r}_{(y,s)_l} & \text{otherwise.} \end{cases}$$

The dual edge set is then $\widehat{E} := \{\langle (y, s), \widehat{h}(y, s) \rangle : (y, s) \in \widehat{V}\}$. Clearly, each dual vertex has exactly one outgoing edge which goes in the downward direction. Hence, the dual graph $\widehat{\mathfrak{F}} := (\widehat{V}, \widehat{E})$ does not contain any cycle. This forest is entirely determined from \mathfrak{F} without extra randomness. We obtain a dual (or backward) path $\widehat{\pi}^{(y,s)} \in \widehat{\Pi}$ starting at (y, s) , by linearly joining the successive $\widehat{h}(\cdot)$ steps. Thus, $\widehat{\mathcal{X}} := \{\widehat{\pi}^{(y,s)} : (y, s) \in \widehat{V}\}$ denotes the collection of all dual paths obtained from $\widehat{\mathfrak{F}}$.

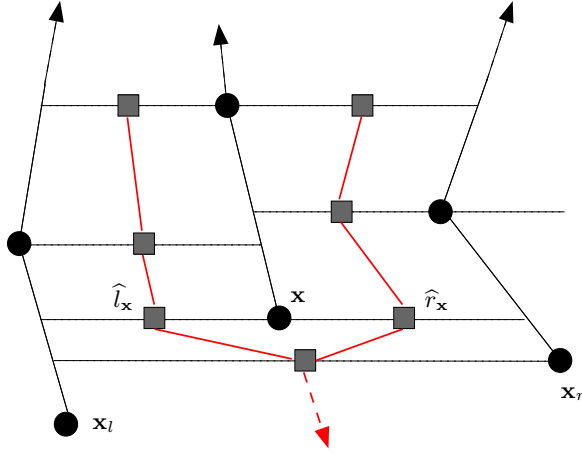


Figure 9: Here is a picture of the DSF \mathfrak{F} (in upward direction) and its dual forest $\widehat{\mathfrak{F}}$ (in downward direction). Vertices of the DSF are black circles whereas dual vertices are grey squares. In particular, the vertex \mathbf{x} produces two dual vertices $\widehat{l}_{\mathbf{x}}$ and $\widehat{r}_{\mathbf{x}}$. On this picture, $(\widehat{r}_{\mathbf{x}})_r = \mathbf{x}_r$ and $(\widehat{r}_{\mathbf{x}})_l = \mathbf{x}_l$ with $\mathbf{x}_r(2) > \mathbf{x}_l(2)$: this implies that $\widehat{h}(\widehat{r}_{\mathbf{x}}) = \widehat{l}_{\mathbf{x}_r}$. The same is true for $\widehat{l}_{\mathbf{x}}$.

Let us recall that $\mathcal{X}_n = \mathcal{X}_n(\gamma, \sigma)$ for $\gamma, \sigma > 0$ and $n \geq 1$, is the collection of n -th order diffusively scaled paths. See (3). In the same way, we define $\widehat{\mathcal{X}}_n = \widehat{\mathcal{X}}_n(\gamma, \sigma)$ as the collection of diffusively scaled dual paths. For any dual path $\widehat{\pi}$ with starting time $\sigma_{\widehat{\pi}}$, the scaled dual path $\widehat{\pi}_n(\gamma, \sigma) : [-\infty, \sigma_{\widehat{\pi}}/n^2\gamma] \rightarrow [-\infty, \infty]$ is given by

$$\widehat{\pi}_n(\gamma, \sigma)(t) := \widehat{\pi}(n^2\gamma t)/n\sigma . \quad (58)$$

For each $n \geq 1$, the closure of $\widehat{\mathcal{X}}_n$ in $(\widehat{\Pi}, d_{\widehat{\Pi}})$, still denoted by $\widehat{\mathcal{X}}_n$, is a $(\widehat{\mathcal{H}}, \mathcal{B}_{\widehat{\mathcal{H}}})$ -valued random variable.

Conditions (i) and (iii) of Theorem 27 hold by construction. Indeed, paths of \mathcal{X} do not cross (backward) paths of $\widehat{\mathcal{X}}$ with probability 1. The same holds for the scaled sets \mathcal{X}_n and $\widehat{\mathcal{X}}_n$. Moreover, the collection $\{(\widehat{\pi}_n(\sigma_{\widehat{\pi}_n}), \sigma_{\widehat{\pi}_n}) : \widehat{\pi}_n \in \widehat{\Xi}_n, n \geq 1\}$ of all starting points of the scaled backward paths is obviously dense in \mathbb{R}^2 .

The next two sections are respectively devoted to the proofs of conditions (ii) and (iv). This will conclude the proof of Theorem 2.

4.2.1 Verification of condition (ii)

Let us prove that the diffusively scaled sequence $\{\mathcal{X}_n : n \geq 1\}$ satisfies condition (ii), i.e. Criterion (I_1) of Theorem 26. The main ingredients on which (I_1) is based have been stated in Section 2. On the one hand, multiple paths of the DSF can be simultaneously broken down into independent pieces through renewals steps (Proposition 14). On the other hand, both time (or number of steps) and size of renewal block between two consecutive renewal steps admit sub-exponentially decaying tails (Propositions 17 and 19). Thenceforth, to get (I_1) , we follow the strategy of Ferrari et al [14], also used in [22]. The proof here is very similar to that of [22] (see Section 5.1) but in a continuous setting. For this reason we only provide the main steps so that the reader may understand the method without referring to [22].

Let us first focus on a single path, $\pi^{\mathbf{0}}$ starting at the origin $\mathbf{u}_1^{(0)} = \mathbf{0}$. Let $\{\mathbf{u}_1^{(\ell)} : \ell \geq 0\}$ be the sequence of renewal vertices allowing to break down $\pi^{\mathbf{0}}$ into independent pieces. Let us scale $\pi^{\mathbf{0}}$ into $\pi_n^{\mathbf{0}}$ as in (3) with

$$\sigma := (\text{Var}(\mathbf{u}_1^{(1)}(1) - \mathbf{u}_1^{(0)}(1)))^{1/2} \quad \text{and} \quad \gamma := \mathbb{E}(\mathbf{u}_1^{(1)}(2) - \mathbf{u}_1^{(0)}(2)) .$$

The parameters σ and γ depend on λ , k and κ . From now on, the diffusively scaled sequence $\{\mathcal{X}_n : n \geq 1\}$ is considered w.r.t. these parameters, but for ease of writing, we drop (γ, σ) from our notation. Results of Section 2 allow an application of Donsker's invariance principle to show that $\pi_n^{\mathbf{0}}$ converges in distribution in (Π, d_{Π}) to $B^{\mathbf{0}}$ a standard Brownian motion started at $\mathbf{0}$.

Thus we obtain that, for any sequences (\mathbf{v}_n) and (\mathbf{w}_n) such that $\mathbf{v}_n(2) = \mathbf{w}_n(2) = 0$, $\mathbf{w}_n(1) < 0 < \mathbf{v}_n(1)$ with $(\mathbf{v}_n(1) - \mathbf{w}_n(1))/n \rightarrow 0$, the couple $(\pi_n^{\mathbf{w}_n}, \pi_n^{\mathbf{v}_n})$ converges in distribution (in the suitable product metric space) to $(B^{\mathbf{0}}, B^{\mathbf{0}})$. This result means that whenever two paths are close to each other, precisely within a $o(n)$ distance, then they will quickly coalesce. Although we can deal without it (see e.g. [14]), this is directly implied by the estimated on the coalescing time that we have established at Theorem 21: for any $t > 0$, $\mathbb{P}(T(\mathbf{v}_n, \mathbf{w}_n) > n^2 \gamma t) = o_n(1)$.

For showing the joint convergence of multiple paths, we use the fact that paths behave (almost) independently when they are separated by a large distance (roughly, at least of order n). This is possible since the size of renewal blocks between two consecutive renewal steps admits sub-exponentially decaying tails. Hence, distributions of two paths far enough from each other can be realized using independent PPP's. Thus, when paths come close to each other, they coalesce very quickly as indicated just above.

This strategy dealing with dependent paths, originally introduced in [14], has been modified later to treat the case of long range interactions in [7] and [22]. We again emphasize the fact that the dependency structure of the DSF model is much more complicated compared to models previously cited.

The main change w.r.t. the proof in Section 5.1 of [22] concerns Proposition 5.4 which estimates the horizontal deviations of a path in terms of the height of the rectangle on which the configuration is known. Here is the result corresponding to our setting.

Proposition 30. *Let $0 < \beta < \alpha$. Consider the rectangle $R := [-m^\beta, m^\beta] \times [0, m^\beta]$ for some $m \geq 1$. Let $\pi^{\mathbf{0}}$ be the path of the DSF starting at $\mathbf{0}$. Then,*

$$\mathbb{P}\left(\sup_{0 \leq s \leq m^\beta} |\pi^{\mathbf{0}}(s)| \geq 3m^\alpha \mid \mathcal{N} \cap R\right) \leq C_0 \exp(-C_1 m^{\frac{\alpha-\beta}{2}}) .$$

Proof. Let \mathcal{N}' be another PPP independent of \mathcal{N} . We consider two paths, say $\pi^{(2m^\alpha, 0)}$ and $\pi_{\text{new}}^{(2m^\alpha, 0)}$, both starting from $(2m^\alpha, 0)$, and using respectively the PPP's \mathcal{N} and $(\mathcal{N}' \cap R) \cup (\mathcal{N} \cap R^c)$. In other words, for the path $\pi_{\text{new}}^{(2m^\alpha, 0)}$, the PPP inside the rectangle R has been re-sampled. Since both paths $\pi^{\mathbf{0}}$ and $\pi^{(2m^\alpha, 0)}$ are constructed with the same PPP \mathcal{N} , the non-crossing path property applies and gives:

$$\sup_{0 \leq s \leq m^\beta} \pi^{\mathbf{0}}(s) \geq 3m^\alpha \Rightarrow \sup_{0 \leq s \leq m^\beta} \pi^{(2m^\alpha, 0)}(s) \geq 3m^\alpha .$$

Now, let us consider the sequence $(W_j)_{j \geq 1}$ of sizes of renewal blocks associated with the single path $\pi_{\text{new}}^{(2m^\alpha, 0)}$. By construction, it does not depend on the configuration $\mathcal{N} \cap R$. After each renewal step, the

y -ordinate of the moving vertex increases by at least $\kappa \geq 6$ and hence the path $\pi_{\text{new}}^{(2m^\alpha, 0)}$ can admit at most $\lfloor m^\beta \rfloor$ renewal steps before crossing the horizontal line $\{\mathbf{x} : \mathbf{x}(2) = m^\beta\}$. So, on the event

$$A := \left\{ \sum_{j=1}^{\lfloor m^\beta \rfloor} W_j \leq m^\alpha \right\},$$

$\pi_{\text{new}}^{(2m^\alpha, 0)}$ cannot exit the rectangle $[m^\alpha, 3m^\alpha] \times [0, m^\beta]$. Moreover, on A , the paths $\pi^{(2m^\alpha, 0)}$ and $\pi_{\text{new}}^{(2m^\alpha, 0)}$ must agree over time interval $[0, m^\beta]$. We can then write:

$$\begin{aligned} \mathbb{P}\left(\sup_{0 \leq s \leq m^\beta} \pi^0(s) \geq 3m^\alpha \mid \mathcal{N} \cap R \right) &\leq \mathbb{P}\left(\sup_{0 \leq s \leq m^\beta} \pi^{(2m^\alpha, 0)}(s) \geq 3m^\alpha \mid \mathcal{N} \cap R \right) \\ &\leq \mathbb{P}\left(\sup_{0 \leq s \leq m^\beta} \pi_{\text{new}}^{(2m^\alpha, 0)}(s) \geq 3m^\alpha, A \mid \mathcal{N} \cap R \right) + \mathbb{P}(A^c \mid \mathcal{N} \cap R) \\ &= \mathbb{P}(A^c \mid \mathcal{N} \cap R) = \mathbb{P}(A^c). \end{aligned}$$

We conclude using Proposition 19:

$$\mathbb{P}(A^c) \leq \lfloor m^\beta \rfloor \mathbb{P}(\mathcal{W} \geq m^{\alpha-\beta}) \leq C_0 \exp(-C_1 m^{\frac{\alpha-\beta}{2}}),$$

for suitable positive constants C_0, C_1 .

Similar argument using paths starting from the point $(-2m^\alpha, 0)$ completes the proof. \square

4.2.2 Verification of condition (iv)

To show condition (iv), we mainly follow the proof of Theorem 2.9 in [21], which was in a discrete setting. As a key ingredient, the coalescence time estimate (Theorem 21) will be used in the proof of Lemma 31 below.

Let $(\mathcal{Z}, \widehat{\mathcal{Z}})$ be any subsequential limit of $\{(\mathcal{X}_n, \widehat{\mathcal{X}}_n) : n \geq 1\}$. Instead of working with a subsequence, we may assume—for ease of notation—that the sequence $\{(\mathcal{X}_n, \widehat{\mathcal{X}}_n) : n \geq 1\}$ converges to $(\mathcal{Z}, \widehat{\mathcal{Z}})$ almost surely in the $(\mathcal{H} \times \widehat{\mathcal{H}}, d_{\mathcal{H} \times \widehat{\mathcal{H}}})$ metric space.

We have to prove that, with probability 1, paths in \mathcal{Z} do not spend positive Lebesgue measure time together with the dual paths in $\widehat{\mathcal{Z}}$. This means that for any $\delta > 0$ and any integer $m \geq 1$, the probability of the event

$$A(\delta, m) := \left\{ \begin{array}{l} \exists \text{ paths } \pi \in \mathcal{Z}, \widehat{\pi} \in \widehat{\mathcal{Z}} \text{ and } t_0 \in \mathbb{R} \text{ s.t. } -m < \sigma_\pi < t_0 < t_0 + \delta < \sigma_{\widehat{\pi}} < m \\ \text{and } -m < \pi(t) = \widehat{\pi}(t) < m \text{ for all } t \in [t_0, t_0 + \delta] \end{array} \right\}$$

has to be 0.

To show that $\mathbb{P}(A(\delta, m)) = 0$, we introduce a generic event $B_n^\epsilon(\delta, m)$ defined as follows. Given an integer $m \geq 1$ and $\delta, \epsilon > 0$,

$$B_n^\epsilon(\delta, m) := \left\{ \begin{array}{l} \exists \text{ paths } \pi_1^n, \pi_2^n, \pi_3^n \in \mathcal{X}_n \text{ s.t. } \sigma_{\pi_1^n}, \sigma_{\pi_2^n} \leq 0, \sigma_{\pi_3^n} \leq \delta \text{ and } \pi_1^n(0), \pi_1^n(\delta) \in [-m, m] \\ \text{with } |\pi_1^n(0) - \pi_2^n(0)| < \epsilon \text{ but } \pi_1^n(\delta) \neq \pi_2^n(\delta) \\ \text{and with } |\pi_1^n(\delta) - \pi_3^n(\delta)| < \epsilon \text{ but } \pi_1^n(2\delta) \neq \pi_3^n(2\delta) \end{array} \right\}.$$

The event $B_n^\epsilon(\delta, m)$ means that there exists a path π_1^n localized in $[-m, m]$ at time 0 as well as at time δ which is approached (within distance ϵ) by two paths π_2^n and π_3^n respectively at times 0 and δ while still being different from them respectively at time δ and 2δ . Thanks to the coalescence time estimate (Theorem 21), the following lemma, proved at the end of the section, shows that $B_n^\epsilon(\delta, m)$ has a small probability:

Lemma 31. *For any integer $m \geq 1$, real numbers $\epsilon, \delta > 0$, there exists a constant $C_0(\delta, m) > 0$ (only depending on δ and m) such that for all large n ,*

$$\mathbb{P}(B_n^\epsilon(\delta, m)) \leq C_0(\delta, m) \epsilon .$$

Let us now explain how Lemma 31 allows us to conclude. For $j = 1, \dots, \lfloor \frac{6m}{\delta} \rfloor$, let us set $t^j := -m + (j\delta)/3$ and

$$B_n^\epsilon(\delta, m; j) := \left\{ \begin{array}{l} \exists \text{ paths } \pi_1^n, \pi_2^n, \pi_3^n \in \mathcal{X}_n \text{ s.t. } \sigma_{\pi_1^n}, \sigma_{\pi_2^n} \leq t^j, \sigma_{\pi_3^n} \leq t^{j+1} \text{ and} \\ \pi_1^n(t^j), \pi_1^n(t^{j+1}) \in [-2m, 2m] \text{ with } |\pi_1^n(t^j) - \pi_2^n(t^j)| < 4\epsilon \\ \text{but } \pi_1^n(t^{j+1}) \neq \pi_2^n(t^{j+1}) \text{ and with } |\pi_1^n(t^{j+1}) - \pi_3^n(t^{j+1})| < 4\epsilon \\ \text{but } \pi_1^n(t^{j+2}) \neq \pi_3^n(t^{j+2}) \end{array} \right\} .$$

The event $B_n^\epsilon(\delta, m; j)$ corresponds to the event $B_n^{4\epsilon}(\delta/3, 2m)$ considered in Lemma 31, and shifted up by t^j . Hence, by the translation invariance property of the DSF and Lemma 31:

$$\mathbb{P}(B_n^\epsilon(\delta, m; j)) = \mathbb{P}(B_n^{4\epsilon}(\delta/3, 2m)) \leq 4C_0(\delta/3, 2m) \epsilon$$

for all n large enough. The expected result will follow from:

$$A(\delta, m) \subset \liminf_{n \rightarrow \infty} \bigcup_{j=1}^{\lfloor \frac{6m}{\delta} \rfloor} B_n^\epsilon(\delta, m; j) , \quad (59)$$

since we then have:

$$\begin{aligned} \mathbb{P}(A(\delta, m)) &\leq \limsup_{\epsilon \rightarrow 0} \mathbb{P}\left(\liminf_{n \rightarrow \infty} \bigcup_{j=1}^{\lfloor \frac{6m}{\delta} \rfloor} B_n^\epsilon(\delta, m; j)\right) \\ &\leq \limsup_{\epsilon \rightarrow 0} \liminf_{n \rightarrow \infty} \sum_{j=1}^{\lfloor \frac{6m}{\delta} \rfloor} \mathbb{P}(B_n^\epsilon(\delta, m; j)) \\ &\leq \limsup_{\epsilon \rightarrow 0} \frac{6m}{\delta} 4C_0(\delta/3, 2m) \epsilon = 0 . \end{aligned}$$

It then remains to prove (59). Let us work on the event $A(\delta, m)$, and consider $\pi \in \mathcal{Z}$, $\widehat{\pi} \in \widehat{\mathcal{Z}}$ and $t_0 \in (\sigma_\pi, \sigma_{\widehat{\pi}})$ as in its definition. Since the limit \mathcal{Z} satisfies (I_1) , it a.s. contains, for any $(x, s) \in \mathbb{Q}^2$, a path $\pi^{(x, s)}$ starting from (x, s) (and distributed as a Brownian motion starting from x at time s). Choose $0 < \epsilon_1 = \epsilon_1(\omega) < \epsilon$ such that for all $\pi \in \mathcal{W}$ with $(\sigma_\pi, \sigma_\pi) \in [-m, m] \times [-m, m]$, we have

$$\sup\{|\pi(t) - \pi(s)| : \sigma_\pi \leq t \leq s \leq (t + \epsilon_1) \wedge m\} \leq \epsilon .$$

As the Brownian web a.s. takes values in \mathcal{H} , (random) $\epsilon_1 > 0$ always exists. Then, by continuity of the paths, we can find two paths in \mathcal{Z} , say π_- and π_+ , born before time $t_0 - \epsilon_1$ and such that

$$\pi(t_0) - 2\epsilon \leq \pi_-(t_0) \leq \pi(t_0) - \epsilon \leq \pi(t_0) + \epsilon \leq \pi_+(t_0) \leq \pi(t_0) + 2\epsilon .$$

The convergence of $(\mathcal{X}_n, \widehat{\mathcal{X}}_n)$ to $(\mathcal{Z}, \widehat{\mathcal{Z}})$ for the Hausdorff distance on $\mathcal{H} \times \widehat{\mathcal{H}}$ means that, for all n large enough, we can find π_-^n, π^n and π_+^n in \mathcal{X}_n starting before time t_0 , approximating resp. π_-, π and π_+ , and also $\widehat{\pi}^n$ in $\widehat{\mathcal{X}}_n$ approximating $\widehat{\pi}$ in the sense that

$$\max \left\{ |\sigma_{\pi_-} - \sigma_{\pi_-^n}|, |\pi_-(\sigma_{\pi_-}) - \pi_-^n(\sigma_{\pi_-^n})|, \sup_{t \in [t_0, t_0 + \delta]} |\pi_-(t) - \pi_-^n(t)| \right\} < \frac{\epsilon_1}{2}$$

(idem for the three other paths). Condition (i) says that paths in \mathcal{X}_n do not cross (backward) paths in $\widehat{\mathcal{X}}_n$ almost surely. Let us first assume that $\pi^n(t) < \widehat{\pi}^n(t)$ on the whole time interval $[t_0, t_0 + \delta]$. It

then follows that $\pi^n(t_0) < \widehat{\pi}^n(t_0) < \pi_+^n(t_0)$ and, still by condition (i), the backward path $\widehat{\pi}^n$ prevents paths π^n and π_+^n to coalesce before time $t_0 + \delta$. Let $\pi_1^n := \pi^n$ and $\pi_2^n := \pi_+^n$, we have $\sigma_{\pi_1^n}, \sigma_{\pi_2^n} \leq t_0$, $\pi_1^n(t_0) \in [-2m, 2m]$, $|\pi_1^n(t_0) - \pi_2^n(t_0)| < 4\epsilon$ and $\pi_1^n(t_0 + \delta/3) \neq \pi_2^n(t_0 + \delta/3)$. In the case $\pi^n(t) > \widehat{\pi}^n(t)$ on $[t_0, t_0 + \delta]$, we choose $\pi_2^n := \pi_-^n$. With a similar proof, we can show the existence of a third path π_3^n satisfying the requirements of $B_n^\epsilon(\delta, m; j)$ by approximating π by two paths π_-' and π_+' on $[t_0 - \delta, t_0 + \delta]$. This completes the proof of (59). \square

Let us end with the proof of Lemma 31 which is close to that of Lemma 2.11 of [21]. Both results are mainly based on the coalescence tail estimate (Theorem 21). With respect to Lemma 2.11 of [21] two additional difficulties appear here: paths of the DSF are non-Markovian and constructed on a Poisson point process. Proposition 19 will help us to control this long range dependence.

Proof of Lemma 31. Let $\delta, \epsilon > 0$ be (small) real numbers and m be a positive real number. Fix $\theta > 0$ small. It is enough to show that for all large n , we have

$$\mathbb{P}(B_n^\epsilon(\delta, m)) \leq C_0(\delta, m)\epsilon + \theta.$$

Choose $0 < 2\beta < \alpha < 1$ and consider $\pi^{(0,0)}$, the DSF path starting from $(0, 0)$. Note that $\pi^{(0,0)}$ is always defined, which need not be in \mathcal{X} . Set $M = M(\delta) \geq m$ such that

$$\mathbb{P}(\max\{|\pi^{(0,0)}(t)| : t \in [0, n^2\gamma\delta + 2n^\beta]\} > n\sigma M) < \theta/4. \quad (60)$$

Define the event $D_n^{(1)}$ as

$$D_n^{(1)} = \left\{ \begin{aligned} &\pi^{(-n\sigma(m+M), -2n^\beta)}(t) \in (-n\sigma(m+2M), -n\sigma m) \text{ and} \\ &\pi^{(n\sigma(m+M), -2n^\beta)}(t) \in (n\sigma m, n\sigma(m+2M)) \text{ for all } t \in [-2n^\beta, n^2\gamma\delta]. \end{aligned} \right. \quad (61)$$

Because of the non-crossing nature of the paths of the DSF, it follows that on the event $D_n^{(1)}$, only the paths starting before time 0 and sandwiched between the paths $\pi^{(-n\sigma(m+M), -2n^\beta)}$ and $\pi^{(n\sigma(m+M), -2n^\beta)}$ are eligible to be the paths π_1^n and π_2^n of $B_n^\epsilon(\delta, m)$.

Because of the translation homogeneity of our model, (60) implies that for all sufficiently large n we have $\mathbb{P}((D_n^{(1)})^c) < \theta/2$. Let us emphasize that the event $D_n^{(1)}$ does not depend on ϵ and hence neither does θ . The constant $\theta > 0$ being chosen arbitrarily, it suffices to show that $\mathbb{P}(B_n^\epsilon \cap D_n^{(1)}) \leq C_0(\delta, m)\epsilon$ to prove Lemma 31.

Next we will show that (with high probability) it is enough to consider the (unscaled) paths starting from Poisson vertices in a ‘thin’ rectangular strip

$$S := [-n\sigma(m+M), n\sigma(m+M)] \times [-2n^\beta, 0],$$

and that the construction of these paths until they cross the line $\{\mathbf{x} : \mathbf{x}(2) = n^2\gamma\delta\}$ is independent of the point process in $\mathbb{H}^+(n^2\gamma\delta + 2n^\beta)$.

Define the boxes of side length n^β and with lower sides on the lines $\{\mathbf{x} : \mathbf{x}(2) = -2n^\beta\}$ and $\{\mathbf{x} : \mathbf{x}(2) = -n^\beta\}$. These boxes are given for $0 \leq j \leq \lfloor 2n\sigma(m+M)/n^\beta \rfloor$ by

$$\begin{aligned} R_l^{(1)}(j) &:= [-n\sigma(m+M) + jn^\beta, -n\sigma(m+M) + (j+1)n^\beta] \times [-2n^\beta, -n^\beta] \text{ and} \\ R_l^{(2)}(j) &:= [-n\sigma(m+M) + jn^\beta, -n\sigma(m+M) + (j+1)n^\beta] \times [-n^\beta, 0]. \end{aligned}$$

Define the event $D_n^{(2)}$ as

$$D_n^{(2)} := \bigcap_{j=0}^{\lfloor 2n\sigma(m+M)/n^\beta \rfloor} (\{R_l^{(1)}(j) \cap \mathcal{N} \neq \emptyset\} \cap \{R_l^{(2)}(j) \cap \mathcal{N} \neq \emptyset\}).$$

It is not difficult to see that $\lim_{n \rightarrow \infty} \mathbb{P}((D_n^{(2)})^c) = 0$. On the event $B_n^\epsilon(\delta, m) \cap D_n^{(1)} \cap D_n^{(2)}$, the scaled paths π_1^n and π_2^n considered in $B_n^\epsilon(\delta, m)$ must start from Poisson vertices in the rectangular strip S .

Next we show that on an event with high probability, the construction of the paths starting from S till they cross the line $\{\mathbf{x} : \mathbf{x}(2) = n^2\gamma\delta\}$ is independent of the point process $\mathcal{N} \cap \mathbb{H}^+(n^2\gamma\delta + 2n^\beta)$. In order to do that, we consider boxes with side length n^β with lower sides on the line $\{\mathbf{x} : \mathbf{x}(2) = n^2\gamma\delta\}$ defined for $0 \leq j \leq \lfloor 2n\sigma(m + 2M)/n^\beta \rfloor$ by

$$R_u(j) := [-n\sigma(m + 2M) + jn^\beta, -n\sigma(m + 2M) + (j + 1)n^\beta] \times [n^2\gamma\delta, n^2\gamma\delta + n^\beta].$$

The event $D_n^{(3)}$ is defined as

$$D_n^{(3)} := \bigcap_{j=0}^{\lfloor 2n\sigma(m+2M)/n^\beta \rfloor} \{R_u(j) \cap \mathcal{N} \neq \emptyset\}.$$

Again we have $\lim_{n \rightarrow \infty} \mathbb{P}((D_n^{(3)})^c) = 0$ and henceforth we can concentrate only on $B_n^\epsilon(\delta, m) \cap (\bigcap_{i=1}^3 D_n^{(i)})$. The reason for introducing the event $D_n^{(3)}$ is that, on this event in order to cross the line $\{\mathbf{x} : \mathbf{x}(2) = n^2\gamma\delta\}$, the paths starting from the rectangular strip S do not explore the point process $\mathcal{N} \cap \mathbb{H}^+(n^2\gamma\delta + 2n^\beta)$.

For $\lfloor -n\sigma m \rfloor - 1 \leq k \leq \lfloor n\sigma m \rfloor$, we consider the event $B_n^\epsilon(\delta, m) \cap \{n\sigma\pi_1^n(\delta) \in [k, k + 1]\}$. From the non-crossing nature of paths, it follows that we must have $\pi^{(k-n\sigma\epsilon, n^2\gamma\delta)}(2n^2\gamma\delta) \neq \pi^{(k+n\sigma\epsilon, n^2\gamma\delta)}(2n^2\gamma\delta)$, because these paths are separated by π_1^n and π_2^n . Recall that $0 < 2\beta < \alpha < 1$. For $k \in \mathbb{Z}$, define the event $F_n(k)$ as

$$F_n(k) := \left\{ k - n\sigma\epsilon - n^\alpha \leq \pi^{(k-n\sigma\epsilon, n^2\gamma\delta)}(n^2\gamma\delta + 2n^\beta) \leq \pi^{(k+n\sigma\epsilon, n^2\gamma\delta)}(n^2\gamma\delta + 2n^\beta) \leq k + n\sigma\epsilon + n^\alpha \right\}.$$

The event $F_n(k)$ asks that paths starting at $(k - n\sigma\epsilon, n^2\gamma\delta)$ and $(k + n\sigma\epsilon, n^2\gamma\delta)$ do not fluctuate too much till time $n^2\gamma\delta + 2n^\beta$. We mentioned earlier that, on the event $\bigcap_{i=1}^3 D_n^{(i)}$, the DSF paths starting from Poisson vertices in the rectangular strip S do not explore the point process $\mathcal{N} \cap \mathbb{H}^+(n^2\gamma\delta + 2n^\beta)$ before crossing the line $\{\mathbf{x} : \mathbf{x}(2) = n^2\gamma\delta\}$. Remark that on the event $F_n(k)^c$, at least one of the two paths starting from $(k - n\sigma\epsilon, n^2\gamma\delta)$ and $(k + n\sigma\epsilon, n^2\gamma\delta)$ admits fluctuations larger than n^α on the time interval $[n^2\gamma\delta, n^2\gamma\delta + 2n^\beta]$. By Proposition 30, this has a probability smaller than $C_0 e^{-C_1 n^{(\alpha-\beta)/2}}$. This gives that for any $\lfloor -n\sigma m \rfloor - 1 \leq k \leq \lfloor n\sigma m \rfloor$, the probability of the event

$$(F_n(k))^c \cap \{n\sigma\pi_1^n(\delta) \in [k, k + 1]\} \cap B_n^\epsilon(\delta, m) \cap (\bigcap_{i=1}^3 D_n^{(i)})$$

decays to 0 sub-exponentially and uniformly in k .

Define the event $G_n^\epsilon(\delta, m)$ as

$$G_n^\epsilon(\delta, m) := \left\{ \begin{array}{l} \exists \text{ paths } \pi_1^n, \pi_2^n \in \mathcal{X}_n \text{ s.t. } \sigma_{\pi_1^n}, \sigma_{\pi_2^n} \leq 0 \text{ and } \pi_1^n(0), \pi_1^n(\delta) \in [-m, m] \\ \text{with } |\pi_1^n(0) - \pi_2^n(0)| < \epsilon \text{ but } \pi_1^n(\delta) \neq \pi_2^n(\delta) \end{array} \right\}.$$

On the event $F_n(k) \cap \{n\sigma\pi_1^n(\delta) \in [k, k + 1]\} \cap B_n^\epsilon(\delta, m) \cap (\bigcap_{i=1}^3 D_n^{(i)})$, the non-crossing path property forces the paths starting at $(k - n\sigma\epsilon - n^\alpha, n^2\gamma\delta + 2n^\beta)$ and $(k + n\sigma\epsilon + n^\alpha, n^2\gamma\delta + 2n^\beta)$ to be still different at time $2n^2\gamma\delta$. So,

$$\begin{aligned} & \mathbb{P}(F_n(k) \cap \{n\sigma\pi_1^n(\delta) \in [k, k + 1]\} \cap B_n^\epsilon(\delta, m) \cap (\bigcap_{i=1}^3 D_n^{(i)})) \\ & \leq \mathbb{P}(\{\pi^{(k-n\sigma\epsilon-n^\alpha, n^2\gamma\delta+2n^\beta)}(2n^2\gamma\delta) \neq \pi^{(k+n\sigma\epsilon+n^\alpha, n^2\gamma\delta+2n^\beta)}(2n^2\gamma\delta)\} \cap \\ & \quad \{n\sigma\pi_1^n(\delta) \in [k, k + 1]\} \cap G_n^\epsilon(\delta, m) \cap (\bigcap_{i=1}^3 D_n^{(i)})). \end{aligned} \tag{62}$$

Observe that the event $\{\pi^{(k-n\sigma\epsilon-n^\alpha, n^2\gamma\delta+2n^\beta)}(2n^2\gamma\delta) \neq \pi^{(k+n\sigma\epsilon+n^\alpha, n^2\gamma\delta+2n^\beta)}(2n^2\gamma\delta)\}$ depends on the point process $\mathcal{N} \cap \mathbb{H}^+(n^2\gamma\delta + 2n^\beta)$ and the event $\{n\sigma\pi_1^n(\delta) \in [k, k + 1]\} \cap G_n^\epsilon(\delta, m) \cap (\bigcap_{i=1}^3 D_n^{(i)})$ does

not depend on the point process $\mathcal{N} \cap \mathbb{H}^+(n^2\gamma\delta + 2n^\beta)$. Hence we have independence of the two events and (62) becomes,

$$\begin{aligned} & \mathbb{P}\left(\{\pi^{(k-n\sigma\epsilon-n^\alpha, n^2\gamma\delta+2n^\beta)}(2n^2\gamma\delta) \neq \pi^{(k+n\sigma\epsilon+n^\alpha, n^2\gamma\delta+2n^\beta)}(2n^2\gamma\delta)\}\right) \\ & \quad \times \mathbb{P}\left(\{n\sigma\pi_1^n(\delta) \in [k, k+1]\} \cap B_n^\epsilon(\delta, m) \cap (\cap_{i=1}^3 D_n^{(i)})\right). \end{aligned}$$

By use of Theorem 21,

$$\mathbb{P}\left(\pi^{(k-n\sigma\epsilon-n^\alpha, n^2\gamma\delta+2n^\beta)}(2n^2\gamma\delta) \neq \pi^{(k+n\sigma\epsilon+n^\alpha, n^2\gamma\delta+2n^\beta)}(2n^2\gamma\delta)\right) \leq \frac{C_0(2n\sigma\epsilon + 2n^\alpha)}{\sqrt{n^2\gamma\delta - 2n^\beta}} \leq C_0\epsilon$$

where $C_0 = C_0(\alpha, \beta, \sigma, \gamma, \delta) > 0$ is a constant. As the events $\{n\sigma\pi_1^n(\delta) \in [k, k+1]\}$ are disjoint for different $k \in \mathbb{Z}$, it follows:

$$\sum_{k=\lfloor -n\sigma m \rfloor - 1}^{\lfloor n\sigma m \rfloor} \mathbb{P}\left(\{n\sigma\pi_1^n(\delta) \in [k, k+1]\} \cap G_n^\epsilon(\delta, m) \cap (\cap_{i=1}^3 D_n^{(i)})\right) \leq \mathbb{P}(G_n^\epsilon(\delta, m)).$$

In order to estimate the probability of the event $G_n^\epsilon(\delta, m)$, for $\lfloor -n\sigma m \rfloor - 1 \leq l \leq \lfloor n\sigma m \rfloor$ we define another event

$$H^\epsilon(\delta, m, l) := \{\pi^{(l,0)}(n^2\gamma\delta) \neq \pi^{(l+1,0)}(n^2\gamma\delta)\}.$$

Non-crossing nature of DSF paths gives us that $G_n^\epsilon(\delta, m) \subset \bigcup_{l=\lfloor -n\sigma m \rfloor - 1}^{\lfloor n\sigma m \rfloor} H^\epsilon(\delta, m, l)$. To observe this inclusion relation, consider the event $B_n^\epsilon \cap \{\pi_1^n(0) > \pi_2^n(0)\}$ and choose the (random) integer l such that $l := \min\{i \in \{\lfloor -n\sigma m \rfloor - 1, \dots, \lfloor n\sigma m \rfloor\} : \pi^{(i,0)}(n^2\gamma\delta) \neq \pi^{(i+1,0)}(n^2\gamma\delta)\}$. Translation homogeneity of our model and use of Theorem 21 give that for all $\lfloor -n\sigma m \rfloor - 1 \leq l \leq \lfloor n\sigma m \rfloor$

$$\mathbb{P}(H^\epsilon(\delta, m, l)) \leq \frac{C_0}{\sqrt{n^2\gamma\delta}}.$$

Hence, for all large n we have

$$\begin{aligned} \mathbb{P}(B^\epsilon(\delta, m)) & \leq \theta + C_0\epsilon\mathbb{P}(G_n^\epsilon(\delta, m)) \leq \theta + C_0\epsilon \times (2n\sigma m)\mathbb{P}(H^\epsilon(\delta, m, l)) \\ & \leq \theta + C_0(m, \delta)\epsilon. \end{aligned}$$

Since $\theta > 0$ is chosen arbitrarily, this completes the proof. \square

5 Theorem 3: a sketch of the proof

Recall that the Radial Spanning Tree (RST), initially introduced in [3], is a tree rooted at the origin O with vertex set $\mathcal{N} \cup \{O\}$ in which each vertex $\mathbf{x} \in \mathcal{N}$ is connected to the closest Poisson point to \mathbf{x} but inside the open ball $\{\mathbf{y} \in \mathbb{R}^2 : \|\mathbf{y}\|_2 < \|\mathbf{x}\|_2\}$. Theorem 2.1 of [3] states that the RST a.s. admits semi-infinite paths in each direction $\theta \in [0, 2\pi)$. In particular, the (random) number χ_r of semi-infinite paths of the RST crossing the circle \mathcal{C}_r with radius r , tends to infinity with probability 1. Theorem 3 claims that

$$\mathbb{E}\chi_r = o(r^{3/4+\epsilon}), \quad (63)$$

for any $\epsilon > 0$. Actually, our strategy to prove (63) has been already developed in Section 6 of [10] for a similar geometric random tree called the *Radial Poisson Tree*. So we will only focus here on the (minor) changes w.r.t. [10].

By isotropy, it is sufficient to prove that, for $0 < \alpha < 1/4$, $\mathbb{E}\chi_r(0, 2r^\alpha)$ tends to 0 as $r \rightarrow \infty$ where $\chi_r(0, 2r^\alpha)$ counts the intersection points between the semi-infinite paths of the RST and the arc of the

circle \mathcal{C}_r , centred at $(r, 0)$ and with length $2r^\alpha$. Approximating the RST around $(r, 0)$ by the DSF with direction e_x (especially using Lemma 3.4 of [3] instead of Lemma 6.4 of [10]) we show that

$$\limsup_{r \rightarrow \infty} \mathbb{E}\chi_r(0, 2r^\alpha) \leq \limsup_{r \rightarrow \infty} \mathbb{E}\eta_r(\alpha, \beta, \varepsilon) \quad (64)$$

where $\varepsilon, \beta > 0$ are such that $\alpha < \beta/2$ and $\beta + \varepsilon < 1/2$, and where $\eta_r(\alpha, \beta, \varepsilon)$ counts the intersection points between the vertical segment $\{r\} \times [-r^\alpha, r^\alpha]$ and paths of the DSF starting from the outside of the rectangle $[r, r + r^\beta] \times [-r^{\beta/2+\varepsilon}, r^{\beta/2+\varepsilon}]$.

Controlling with high probability the deviations of DSF paths (with Theorem 4.10 of [3] instead of Lemma 6.6 of [10]), (64) also holds if paths counted by $\eta_r(\alpha, \beta, \varepsilon)$ are assumed to cross the right side of the corresponding rectangle, i.e. the vertical segment $\{r + r^\beta\} \times [-r^{\beta/2+\varepsilon}, r^{\beta/2+\varepsilon}]$. Thus, the invariant translation property of the DSF leads to

$$\limsup_{r \rightarrow \infty} \mathbb{E}\chi_r(0, 2r^\alpha) \leq \limsup_{r \rightarrow \infty} \mathbb{E}\tilde{\eta}_r(\alpha, \beta) \quad (65)$$

where $\tilde{\eta}_r(\alpha, \beta)$ is defined as the number of intersection points between the vertical axis $\{r\} \times \mathbb{R}$ and DSF paths crossing the segment $\{r + r^\beta\} \times [-r^\alpha, r^\alpha]$. For $i = \lfloor -r^\alpha \rfloor, \dots, \lfloor r^\alpha \rfloor$, let us denote by γ_i the DSF path starting at the deterministic point $(r + r^\beta, i)$ and by Y_i the number of edges crossing the vertical unit segment $\{r + r^\beta\} \times [i, i + 1)$. Then, a.s.

$$\tilde{\eta}_r(\alpha, \beta) \leq 1 + \sum_{i=\lfloor -r^\alpha \rfloor}^{\lfloor r^\alpha \rfloor} (Y_i + 1) \mathbf{1}_{\{\gamma_i \neq \gamma_{i+1} \text{ at time } r\}}, \quad (66)$$

where the event $\{\gamma_i \neq \gamma_{i+1} \text{ at time } r\}$ means that paths γ_i and γ_{i+1} are still disjoint when they cross the vertical axis $\{r\} \times \mathbb{R}$. Since $\alpha < \beta/2$, one can find parameters $p, q > 1$ such that $\alpha < \beta/(2p)$ and $1/p + 1/q = 1$. Then, the Hölder's inequality combined with our coalescence time estimate (Theorem 21) gives:

$$\begin{aligned} \mathbb{E}\tilde{\eta}_r(\alpha, \beta) &\leq 1 + 3r^\alpha \mathbb{E}(Y_0 + 1) \mathbf{1}_{\{\gamma_0 \neq \gamma_1 \text{ at time } r\}} \\ &\leq 1 + 3r^\alpha (\mathbb{E}(Y_0 + 1)^q)^{1/q} \mathbb{P}(\gamma_0 \neq \gamma_1 \text{ at time } r)^{1/p} \\ &\leq 1 + 3(\mathbb{E}(Y_0 + 1)^q)^{1/q} \frac{C_0^{1/p} r^\alpha}{r^{\beta/(2p)}} \end{aligned}$$

which tends to 1 as $r \rightarrow \infty$. Above, we have used the fact that Y_0 admits moments of all orders. Thus, we conclude as in [10].

Finally, the proof of the almost sure convergence of $\chi_r/r^{3/4+\varepsilon}$ to 0 follows from the convergence in expectation using the same arguments as in Section 7 of [10].

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