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May 17, 2018

# THE $p$ -ADIC VARIATION OF THE GROSS-KOHNEN-ZAGIER THEOREM

MATTEO LONGO, MARC-HUBERT NICOLE

ABSTRACT. We relate  $p$ -adic families of Jacobi forms to Big Heegner points constructed by B. Howard, in the spirit of the Gross-Kohnen-Zagier theorem. We view this as a  $\mathrm{GL}(2)$  instance of a  $p$ -adic Kudla program.

## 1. INTRODUCTION

In their seminal paper [GKZ87], Gross, Kohnen and Zagier showed that Heegner points are generating series of a Jacobi form arising from a theta lift. Suppose that  $E/\mathbb{Q}$  is an elliptic curve of conductor  $M$ . We assume that the sign of the functional equation of  $L(E/\mathbb{Q}, s)$  at  $s = 1$  is  $-1$ . Then it is well known ([GZ86], [Kol88a], [Kol88b]) that if the value  $L'(E/\mathbb{Q}, s)_{s=1}$  of the first derivative of  $L(E/\mathbb{Q}, s)$  at  $s = 1$  is non-zero, then the  $\mathbb{Q}$ -vector space  $E(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{Q}$  is one-dimensional. Moreover, one can show that this vector space is generated by a Heegner point  $P_K$  attached to a quadratic imaginary field  $K$  in which all primes  $\ell \mid N$  are split and such that  $L'(E/K_D, s)_{s=1} \neq 0$ . As the imaginary quadratic field  $K$  varies, it is then a natural question to investigate the relative positions of the points  $P_K$  on this one-dimensional line. The theorem of Gross-Kohnen-Zagier gives an answer to this question. For each discriminant  $D$  and each residue class  $r \pmod{2N}$  subject to the condition that  $D \equiv r^2 \pmod{4N}$ , one defines a Heegner point  $P_{D,r}$  (so the point  $P_K$  considered above corresponds to one of these points, for a suitable choice of the pair  $(D, r)$ ). Then [GKZ87, Thm. C] shows that the relative positions of the points  $P_{D,r}$ , at least under the condition that  $(M, D) = 1$ , are encoded by the  $(D, r)$ -th Fourier coefficient of the Jacobi form  $\phi_{f_E}$  coming from the theta lifting of  $f_E$ , where  $f_E$  is the weight 2 newform of level  $\Gamma_0(M)$  associated with  $E$  by modularity; one may express briefly this relation by saying that Heegner points are generating series for Jacobi forms, and actually [GKZ87] formulates an *ideal statement* in which the above relation is conjecturally extended to all coefficients of  $\phi_{f_E}$ , including therefore those  $D$  such that  $(D, M) \neq 1$ . Several generalizations are available in the literature, especially by Borcherds [Bor99], using singular theta liftings, and Yuan-Zhang-Zhang [YZZ09] using a multiplicity one theorem for automorphic representations. In particular, these works complete the work of [GKZ87] by essentially proving the ideal statement alluded to above.

The purpose of this paper is to investigate a variant in families of the GKZ theorem, in which all objects are made to vary via  $p$ -adic interpolation. We regard this as evidence for a  $p$ -adic Kudla program for  $\mathrm{GL}(2)$  and groups that are closely related via theta liftings.

More precisely, pick a positive even integer  $2k_0$ , and an ordinary  $p$ -stabilized newform  $f_{2k_0}$  of level  $\Gamma_0(Np)$ , trivial character and weight  $2k_0$ , where  $N \geq 1$  is an odd integer and  $p \nmid N$  a prime number bigger or equal to 5. Let  $\mathcal{X}$  be the group of continuous endomorphisms of  $\mathbb{Z}_p^\times$ ; we view  $\mathbb{Z}$  inside  $\mathcal{X}$  via the map  $k \mapsto [x \mapsto x^{k-2}]$ . Let

$$f_\infty(\kappa) = \sum_{n \geq 1} a_n(\kappa) q^n$$

be the Hida family passing through  $f_{2k_0}$ , where  $\kappa \mapsto a_n(\kappa)$  are  $p$ -adic analytic functions defined in a neighborhood  $U$  of  $2k_0$  in  $\mathcal{X}$ ; if we denote by  $\mathcal{A}_{2k_0}$  the subring of  $\mathbb{Q}_p[[X]]$  consisting of power series converging in a neighborhood of  $2k_0$ , then  $a_n \in \mathcal{A}_{2k_0}$ . For each even integer  $2k \in U$ , let  $f_{2k}$  be the weight  $2k$  modular form with trivial character appearing in the Hida family. The form  $f_{2k}$  is then an ordinary  $p$ -stabilized form of weight  $2k$  and level  $\Gamma_0(Np)$ , and

$f_{2k_0}$  is the form we started with above, justifying the slight abuse of notation. If  $f_{2k}$  is not a newform (which is always the case if  $k > 1$ ), then we write  $f_{2k}^\sharp$  for the ordinary newform of weight  $2k$  and level  $\Gamma_0(N)$  whose  $p$ -stabilization is  $f_{2k}$ . For each of the forms  $f_{2k}^\sharp$  we have an associated Jacobi form

$$\mathcal{S}_{D_0, r_0}(f_{2k}^\sharp) = \sum_{D=r^2-4Nn} c_{f_{2k}^\sharp}(n, r) q^n \zeta^r$$

of weight  $k+1$ , index  $N$  having the same Hecke eigenvalues as  $f_{2k}^\sharp$ ; this form depends on the choice of a pair of integers  $(D_0, r_0)$  such that  $D_0$  is a fundamental discriminant which we assume to be prime to  $p$  and  $D_0 \equiv r_0^2 \pmod{4N}$ . To be clear, in the above formula the sum is over all negative discriminants  $D$ , and the coefficients  $c_{f_{2k}^\sharp}(n, r)$  only depend on the residue class of  $r \pmod{2N}$ . The first observation we make is that these coefficients  $c_{f_{2k}^\sharp}(n, r)$  can be interpolated, up to Euler factors, by  $p$ -adic analytic functions. We show that a suitable linear combination  $\mathcal{L}_{n,r} \in \mathcal{A}_{2k_0}$  of the square root  $p$ -adic  $L$ -functions attached to genus characters of real quadratic fields, introduced in [BD09], [BD07] and studied extensively in [Sha08], [GSS16], [LV14], interpolates Fourier-Jacobi coefficients  $c_{f_{2k}^\sharp}(n, r)$ . For all positive even integers  $2k$  in a sufficiently small neighbourhood of  $2k_0$ , and for all  $D = r^2 - 4nN$  such that  $p \nmid D$ , we have

$$(1) \quad \mathcal{L}_{n,r}(2k) \stackrel{\bullet}{=} c_{f_{2k}^\sharp}(n, r)$$

where  $\stackrel{\bullet}{=}$  means equality up to an explicit Euler factor and a suitable  $p$ -adic period, both independent of  $(n, r)$  and non-vanishing. This is our first result, corresponding to Theorem 3.1 below.

On the other hand, B. Howard constructed in [How07b] an analogue of the Heegner point  $P_K$  in the setting of Hida families. To explain this, recall that attached to  $f_\infty$  we have a Big Galois representation  $\mathbb{T}^\dagger$  interpolating self-dual twists of the Deligne representation attached to  $f_{2k}$  for  $2k$  a positive even integer. We assume that the residual representation  $\mathbb{T}^\dagger/\mathfrak{m}_{\mathcal{R}}\mathbb{T}^\dagger$  (where  $\mathfrak{m}_{\mathcal{R}}$  is the maximal ideal of  $\mathcal{R}$ ) is absolutely irreducible and  $p$ -distinguished. If we denote by  $\mathcal{R}$  the branch of the Hida-Hecke algebra corresponding to  $f_\infty$  (which is a complete noetherian integral domain, finite over  $\Lambda = \mathbb{Z}_p[[1+p\mathbb{Z}_p]]$ ), then  $\mathbb{T}^\dagger$  is a free  $\mathcal{R}$ -module of rank 2 equipped with a continuous action of  $G_{\mathbb{Q}} = \text{Gal}(\overline{\mathbb{Q}}/\mathbb{Q})$  and specialization maps  $\mathbb{T}^\dagger \mapsto \mathbb{T}_\kappa^\dagger$  for each  $\kappa \in \mathcal{X}$ , such that when  $\kappa = 2k$  is a positive even integer,  $\mathbb{T}_{2k}^\dagger$  is isomorphic to the self-dual twist of the Deligne representation attached to  $f_{2k}$ . Let  $\text{Sel}(\mathbb{Q}, \mathbb{T}^\dagger) \subseteq H^1(G_{\mathbb{Q}}, \mathbb{T}^\dagger)$  be the Greenberg Selmer group attached to the Galois representation  $\mathbb{T}^\dagger$ . We require that the generic sign of the functional equation of the  $L$ -functions of  $f_{2k}$  is  $-1$ , and that the central critical value of their first derivative is generically non-vanishing, cf. Assumptions 4.1 and 4.2 below. Under these conditions, which might be viewed as analogues to those in [GKZ87], the  $\mathcal{R}$ -module  $\text{Sel}(\mathbb{Q}, \mathbb{T}^\dagger)$  is finitely generated of rank 1. Howard constructs certain cohomology classes

$$\mathfrak{Z}_{D,r}^{\text{How}} \in \text{Sel}(\mathbb{Q}, \mathbb{T}^\dagger),$$

taking inverse limits of norm-compatible sequences of Heegner points in towers of modular curves. Therefore it is a natural question to relate the positions of the points  $\mathfrak{Z}_{D,r}^{\text{How}}$  in the 1-dimensional  $\mathcal{K}$ -vector space  $\text{Sel}_{\mathcal{K}}(\mathbb{Q}, \mathbb{T}^\dagger) = \text{Sel}(\mathbb{Q}, \mathbb{T}^\dagger) \otimes_{\mathcal{R}} \mathcal{K}$ , where  $\mathcal{K} = \text{Frac}(\mathcal{R})$  is the fraction field of  $\mathcal{R}$ . To describe our second result, we also require that the height pairing between Heegner cycles is positive definite, cf. Assumption 4.5 below. Define

$$\mathcal{Z}_{n,r}(\kappa) = (2D)^{\frac{\kappa-2}{4}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r}^{\text{How}}(\kappa)$$

for  $D = r^2 - 4Nn$  and  $\kappa \in \mathcal{X}$ ; here  $2u_D$  is the number of units in  $\mathbb{Q}(\sqrt{D})$  and, as above, all primes dividing  $D = r^2 - 4Nn$  are required to split in  $K_D$ . Suppose  $2k_0 \equiv 2 \pmod{p-1}$ . Let  $\mathcal{M}_{2k_0}$  be the fraction field of  $\mathcal{A}_{2k_0}$ . There is a canonical map  $\mathcal{K} \rightarrow \mathcal{M}_{2k_0}$ , and we may

define  $\text{Sel}_{\mathcal{M}_{2k_0}}(\mathbb{Q}, \mathbb{T}^\dagger) = \text{Sel}_{\mathcal{K}}(\mathbb{Q}, \mathbb{T}^\dagger) \otimes_{\mathcal{K}} \mathcal{M}_{2k_0}$ . Our second result, under Assumptions 4.1, 4.2, 4.5 discussed above, shows that there exists an element  $\Phi^{\text{ét}} \in \text{Sel}_{\mathcal{M}_{2k_0}}(\mathbb{Q}, \mathbb{T}^\dagger)$  such that in a sufficiently small connected neighborhood of  $2k_0$  in  $\mathcal{X}$  and for all  $D = r^2 - 4nN$  such that  $p$  splits in  $\mathbb{Q}(\sqrt{D})$  we have

$$(2) \quad \mathcal{Z}_{n,r} = \mathcal{L}_{n,r} \cdot \Phi^{\text{ét}}.$$

This corresponds to Theorem 5.1 below. Combining (1) and (2), we see that Big Heegner points  $\mathcal{Z}_{n,r}$  interpolate the coefficients  $c_{f_{2k}^\#}(n, r)$ , up to Euler factors, which we view as a  $p$ -adic analogue of the Gross-Kohnen-Zagier theorem, as well as a first step towards a more general  $p$ -adic Kudla program.

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## 2. THETA LIFTS

In this section, we recall the formalism of theta liftings relating elliptic and Jacobi cuspforms, following [EZ85], [GKZ87].

Fix an odd integer  $N \geq 1$  and  $k$  a positive even integer  $2k$ . Denote  $S_{2k}(\Gamma_0(N))$  the complex vector space of cuspforms of weight  $2k$  and level  $\Gamma_0(N)$  and  $J_{k+1, M}^{\text{cusp}}$  the complex vector space of Jacobi cuspforms of weight  $k+1$  and index  $N$  (see [EZ85, Ch. I, §1] for a definition). For  $f \in S_{2k}(\Gamma_0(N))$  and  $\phi \in J_{k+1, N}^{\text{cusp}}$  we write the corresponding  $q$ -expansions  $f(z) = \sum_{n \geq 1} a_n q^n$  and  $(q, \zeta)$ -expansions

$$\phi(\tau, z) = \sum_{r^2 - 4Nn < 0} c(n, r) q^n \zeta^r,$$

where  $q = e^{2\pi i \tau}$  and  $\zeta = e^{2\pi i z}$  and the second sum is over all pairs  $(n, r)$  of integers such that  $r^2 - 4Nn < 0$ .

**Definition 2.1.** A level  $N$  index pair is a pair  $(D, r)$  of integers consisting of a negative discriminant  $D$  of an integral quadratic form  $Q = [a, b, c]$  such that  $D \equiv r^2 \pmod{4N}$ . A level  $N$  index pair is said to be fundamental if  $D$  is a fundamental discriminant.

The Fourier expansion of forms  $\phi \in J_{k+1, N}^{\text{cusp}}$  is enumerated by level  $N$  index pairs, explaining the terminology. Since  $N$  is fixed throughout the paper, unless otherwise stated we simply call index pairs or fundamental index pairs the pairs  $(D, r)$  in Definition 2.1. If  $(D, r)$  is a index pair, then we usually denote  $n$  the integer such that  $D^2 = r^2 - 4Nn$ . The spaces  $S_{2k}(\Gamma_0(N))$  and  $J_{k+1, N}^{\text{cusp}}$  are equipped with the action of standard Hecke operators  $T(m)$  and  $T_J(m)$  respectively, for integers  $m \geq 1$ . We recall the formula from [EZ85, Thm. 4.5] for the action of  $T_J(m)$  on Jacobi cuspforms when  $(N, m) = 1$ . If  $D = r^2 - 4Nn$  is a fundamental discriminant,  $\phi = \sum_{n,r} c(n, r) q^n \zeta^r$  belongs to  $J_{k+1, N}^{\text{cusp}}$  and  $\phi|T_J(m) = \sum_{n,r} c^*(n, r) q^n \zeta^r$ , then we have

$$(3) \quad c^*(n, r) = \sum_{d|m} d^{k-1} \left( \frac{D}{d} \right) c \left( \frac{nm^2}{d^2}, \frac{rm}{d} \right).$$

For any ring  $R$ , let  $\mathcal{P}_{k-2}(R)$  denotes the  $R$ -module of homogeneous polynomials in 2 variables of degree  $k-2$  with coefficients in  $R$ , equipped with the right action of the semigroup  $M_2(R)$  defined by the formula

$$(4) \quad (F|\gamma)(X, Y) = F(aX + bY, cX + dY)$$

for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ . Let  $\mathcal{V}_{k-2}(R)$  denote the  $R$ -linear dual of  $\mathcal{P}_{k-2}(R)$ , equipped with the left  $M_2(R)$ -action induced from that on  $\mathcal{P}_{k-2}(R)$ .

Let  $f$  be a cuspform of integral even weight  $2k$  and level  $\Gamma_0(N)$ . To  $f$  we may associate the modular symbol  $\tilde{I}_f \in \text{Symb}_{\Gamma_0(N)}(\mathcal{V}_{2k-2}(\mathbb{C}))$  by the integration formula

$$\tilde{I}_f\{r \rightarrow s\}(P) = 2\pi i \int_r^s f(z)P(z, 1)dz.$$

Here, for any congruence subgroup  $\Gamma \subseteq \text{SL}_2(\mathbb{Z})$  and any  $\mathbb{Z}[\text{GL}_2(\mathbb{Q}) \cap M_2(\mathbb{Z})]$ -module  $M$ , we denote  $\text{Symb}_{\Gamma}(M)$  the group of  $\Gamma$ -invariant modular symbols with values in  $M$  (cf. [GS93, (4.1)]). The matrix  $\begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$  normalizes  $\Gamma_0(N)$  and hence induces an involution on the space of modular symbols  $\text{Symb}_{\Gamma_0(N)}(\mathcal{V}_{2k-2}(\mathbb{C}))$ ; for each  $\varepsilon \in \{\pm 1\}$ , we denote  $\tilde{I}_f^\varepsilon$  the  $\varepsilon$ -eigencomponents of  $\tilde{I}_f$  with respect to this involution. It is known that there are complex periods  $\Omega_f^\varepsilon$  such that

$$I_f^\varepsilon = \frac{\tilde{I}_f^\varepsilon}{\Omega_f^\varepsilon}$$

belong to  $\text{Symb}_{\Gamma_0(N)}(\mathcal{V}_{2k-2}(F_f))$ , where  $F_f$  is the extension of  $\mathbb{Q}$  generated by the Fourier coefficients of  $f$ . These periods can be chosen so that the Petersson norm  $\langle f, f \rangle$  equals the product  $\Omega_f^+ \cdot \Omega_f^-$ ; note that the  $\Omega_f^\varepsilon$  are well-defined only up to multiplication by non-zero factors in  $F_f^\times$ .

For each integer  $\Delta$ , let  $\mathcal{Q}_\Delta$  be the set of integral quadratic forms

$$Q = [a, b, c] = ax^2 + bxy + cy^2$$

of discriminant  $\Delta$  and, for any integer  $\rho$ , let  $\mathcal{Q}_{N,\Delta,\rho}$  denote the subset of  $\mathcal{Q}_\Delta$  consisting of integral binary quadratic forms  $Q = [a, b, c]$  of discriminant  $\Delta$  such that  $b \equiv \rho \pmod{2N}$  and  $a \equiv 0 \pmod{N}$ . Let  $\mathcal{Q}_{N,\Delta,\rho}^0$  be the subset of  $\mathcal{Q}_{N,\Delta,\rho}$  consisting of forms which are  $\Gamma_0(N)$ -primitive, i.e., those  $Q \in \mathcal{Q}_{N,\Delta,\rho}$  which can be written as  $Q = [Na, b, c]$  with  $(a, b, c) = 1$ . These sets are equipped with the right action of  $\text{SL}_2(\mathbb{Z})$  described in (4).

Fix a fundamental index pair  $(D_0, r_0)$ . For any index pair  $(D, r)$ , let  $\mathcal{F}_{D_0, r_0}^{(D, r)}(N)$  be the set of integral binary quadratic forms  $Q = [a, b, c]$  modulo the right action of  $\Gamma_0(N)$  described in (4), such that:

- $\delta_Q = b^2 - 4ac = D_0 D$ ;
- $b \equiv -r_0 r \pmod{2N}$ ;
- $a \equiv 0 \pmod{N}$ .

Let  $Q \mapsto \chi_{D_0}(Q)$  be the generalized genus character attached to  $D_0$  defined in [GKZ87, Prop. 1]. We recall the definition for  $Q \in \mathcal{Q}_{N,\Delta,\rho}$ . If  $Q = \ell \cdot Q'$  for some form  $Q' \in \mathcal{Q}_{N,\Delta,\rho}^0$ , then define  $\chi_{D_0}(Q) = \left(\frac{D_0}{\ell}\right) \cdot \chi_{D_0}(Q')$ , so it is enough to define it on  $\Gamma_0(N)$ -primitive forms. Fix  $Q \in \mathcal{Q}_{N,\Delta,\rho}^0$ . If  $(a/N, b, c, D_0) = 1$ , then pick any factorization  $N = m_1 \cdot m_2$  with  $m_1 > 0$ ,  $m_2 > 0$  and any integer  $n$  coprime with  $D_0$  represented by the quadratic form  $[a/m_1, b, cm_2]$ ; then put  $\chi_{D_0}(Q) = \left(\frac{D_0}{n}\right)$ . If  $(a/N, b, c, D_0) \neq 1$ , then put  $\chi_{D_0}(Q) = 0$ .

For  $Q \in \mathcal{Q}_{N,\Delta,\rho}$ , denote  $\gamma_Q$  the oriented geodesic in the upper half plane attached to  $Q$  (cf. [GKZ87, p. 518]); recall that  $\gamma_Q$  is defined as the image of the semicircle  $a|z|^2 + bx + c = 0$  (for  $z = x + iy$ ) oriented from  $r_Q = \frac{-b - \sqrt{\delta_Q}}{2a}$  to  $s_Q = \frac{-b + \sqrt{\delta_Q}}{2a}$  if  $a \neq 0$ , in the remaining case when  $a = 0$ , it is defined as the vertical line oriented from  $r_Q = -c/b$  to  $s_Q = i\infty$  if  $b > 0$  and the vertical line oriented from  $r_Q = i\infty$  to  $s_Q = -c/b$  if  $b < 0$ . In the previous notation, if  $Q$  is a representative of a class in  $\mathcal{F}_{D_0, r_0}^{(D, r)}(N)$ , then  $Q$  belongs to  $\mathcal{Q}_{M, D_0 D, -r_0 r}$ . In particular, we may consider the genus character  $Q \mapsto \chi_{D_0}(Q)$  for all classes  $Q$  in  $\mathcal{F}_{D_0, r_0}^{(D, r)}(N)$ . For  $D = r^2 - 4Nn$ ,

define

$$(5) \quad c_f(n, r) = \sum_{Q \in \mathcal{F}_{D_0, r_0}^{(D, r)}(N)} \chi_{D_0}(Q) \cdot I_f^- \{r_Q \rightarrow s_Q\} (Q^{k-1})$$

and set

$$\mathcal{S}_{D_0, r_0}(f) = \sum_{r^2 - 4Nn < 0} c_f(n, r) q^n \zeta^r.$$

Let  $S_{2k}^-(\Gamma_0(N))$  be the subspace of  $S_{2k}(\Gamma_0(N))$  consisting of forms whose  $L$ -function admits a functional equation with sign  $-1$ . The association  $f \mapsto \mathcal{S}_{D_0, r_0}(f)$  gives a  $\mathbb{C}$ -linear map, called theta lifting,

$$\mathcal{S}_{D_0, r_0} : S_{2k}^-(\Gamma_0(N)) \longrightarrow J_{k+1, N}^{\text{cusp}}$$

which is equivariant for the action of Hecke operators on both spaces, and such that  $c(n, r)$  belong to  $F_f$  for all  $n, r$ ; see [EZ85, Ch. II] for details. The restriction of  $\mathcal{S}_{D_0, r_0}$  to newforms of  $S_{2k}^-(\Gamma_0(N))$  is an isomorphism onto the image, and for different choices of  $(D_0, r_0)$  we get multiples of the same Jacobi form.

### 3. $p$ -ADIC ANALYTIC THETA LIFTS

In this section we construct a  $p$ -adic analytic family of Jacobi forms using the  $p$ -adic variation of Shintani integrals. This family interpolates Fourier-Jacobi coefficients of newforms of level  $\Gamma_0(N)$  whose  $p$ -stabilizations belong to a fixed Hida family (in contrast with [LN18], where  $\Lambda$ -adic families directly interpolate arithmetic specializations in Hida families). The results of this section are, as in [LN18], consequences of Stevens and Hida's works on half-integral weight modular forms [Ste94], [Hid95] via the formalism of theta liftings and the work of Greenberg-Stevens [GS93] and Bertolini-Darmon [BD07] on measure-valued modular symbols. Fix throughout a prime number  $p \nmid N$ ,  $p \geq 5$ .

We first set up the notation for Hida families. Let  $\Gamma = 1 + p\mathbb{Z}_p$  and  $\Lambda = \mathcal{O}[[\Gamma]]$  its Iwasawa algebra, where  $\mathcal{O}$  is the valuation ring of a finite field extension of  $\mathbb{Q}_p$ . We fix an isomorphism  $\Lambda \simeq \mathcal{O}[[X]]$ . Let

$$\mathcal{X} = \text{Hom}_{\text{cont}}(\mathbb{Z}_p^\times, \mathbb{Z}_p^\times)$$

be the group of continuous group  $\mathbb{Z}_p^\times$ -valued homomorphisms of  $\mathbb{Z}_p^\times$ . Embed  $\mathbb{Z}$  in  $\mathcal{X}$  by  $k \mapsto [x \mapsto x^{k-2}]$ ; if we equip  $\mathcal{X}$  with the rigid analytic topology, then  $\mathbb{Z} \subseteq \mathcal{X}$  is rigid Zariski dense. We see elements  $a \in \Lambda$  as functions on  $\mathcal{X}$  by  $a(\kappa) = \varphi_\kappa(a)$  for all  $\kappa \in \mathcal{X}$ , where  $\varphi_\kappa : \Lambda \rightarrow \mathcal{O}$  denotes the  $\mathcal{O}$ -linear extension of  $\kappa$ .

Recall the fixed  $p$ -stabilized form  $f_0$  of trivial nebentype and weight  $k_0$  and  $\mathcal{R}$  be the branch of the Hida family passing through  $f_0$ . Then  $\mathcal{R}$  is an integral noetherian domain, finite over  $\Lambda$ ; we denote  $\mathcal{X}(\mathcal{R})$  the set of continuous homomorphisms of  $\mathcal{O}$ -algebras  $\mathcal{R} \rightarrow \mathbb{Q}_p$ , and  $\mathcal{X}^{\text{arith}}(\mathcal{R})$  the subset of arithmetic points (see [How07b, Def. 2.1.1] for its definition). For each positive even integer  $2k \in \mathcal{X}$ , using that  $\mathcal{R}/\Lambda$  is unramified at the point  $\varphi_{2k}$ , we obtain a unique arithmetic point  $\tilde{\varphi}_{2k}$  lying over  $\varphi_{2k}$ . Hida theory shows that there exists a formal power series  $F_\infty = \sum_{n \geq 1} A_n q^n \in \mathcal{R}[[q]]$  such that for each  $\varphi \in \mathcal{X}^{\text{arith}}(\mathcal{R})$  the power series  $F_\varphi = \sum_{n \geq 1} \varphi(A_n) q^n$  is the  $q$ -expansion of a  $p$ -stabilized newform, and  $F_{\tilde{\varphi}_{2k_0}} = f_0$ .

Fix an even positive integer  $2k_0$  and let  $\mathcal{A}_{2k_0}$  be the ring of power series in  $\mathbb{Q}_p[[X]]$  which converge in a neighborhood of  $2k_0$ . If  $\mathcal{R}_{2k_0}$  denotes the localisation of  $\mathcal{R}$  at  $\tilde{\varphi}_{2k_0}$ , then, using that  $\mathcal{A}_{2k_0}$  is Henselian, we obtain a canonical morphism  $\psi_{2k_0} : \mathcal{R}_{2k_0} \rightarrow \mathcal{A}_{2k_0}$  ([GS93, (2.7)]), and therefore, using the localisation map, we have a canonical map  $\mathcal{R} \rightarrow \mathcal{A}_{2k_0}$ . If  $\mathcal{K}$  and  $\mathcal{M}_{2k_0}$  are the fraction fields of  $\mathcal{R}$  and  $\mathcal{A}_{2k_0}$ , respectively, then we obtain a map, still denoted  $\psi_{2k_0} : \mathcal{K} \rightarrow \mathcal{M}_{2k_0}$ . The domain of convergence about  $2k_0$  is the intersections of the discs of

convergence of  $\psi_{2k_0}(a)$  for  $a \in \mathcal{R}$ . We denote  $U_{2k_0}$  the domain of convergence, which we may assume to be connected. We let

$$f_\infty = \sum_{n \geq 1} a_n q^n$$

be the image of  $F_\infty$  via  $\psi_{2k_0}$ ; so  $a_n = \psi_{2k_0}(A_n)$  are rigid analytic functions converging in the domain of convergence about  $2k_0$ , such that for each even positive integer  $2k \in U_{2k_0}$  the power series  $\sum_{n \geq 1} a_n(2k)q^n$  is the Fourier expansion of an ordinary  $p$ -stabilized modular form  $f_{2k} \in S_k(\Gamma_0(Np))$ .

We now describe universal measure-valued modular symbols, following [GS93] and [BD07]. Let  $\mathbb{D}_*$  denote the  $\mathcal{O}$ -module of  $\mathcal{O}$ -valued measures on  $\mathbb{Z}_p^2$  which are supported on the set of primitive vectors  $(\mathbb{Z}_p^2)'$  of  $\mathbb{Z}_p^2$ . The  $\mathcal{O}$ -module  $\mathbb{D}_*$  is equipped with the action induced by the action of  $\mathrm{GL}_2(\mathbb{Z}_p)$  on  $\mathbb{Z}_p^2$  by  $(x, y) \mapsto (ax + by, cx + dy)$  for  $\gamma = \begin{pmatrix} a & b \\ c & d \end{pmatrix} \in \mathrm{GL}_2(\mathbb{Z}_p)$ , and a structure of  $\mathcal{O}[[\mathbb{Z}_p^\times]]$ -module induced by the scalar action of  $\mathbb{Z}_p^\times$  on  $\mathbb{Z}_p^2$  (cf. [Ste94, §5], [BD09, §2.2]); in particular,  $\mathbb{D}_*$  is also equipped with a structure of  $\Lambda$ -module.

Let  $\Gamma_0(p\mathbb{Z}_p)$  denote the subgroup of  $\mathrm{GL}_2(\mathbb{Z}_p)$  consisting of matrices which are upper triangular modulo  $p$ . The group  $\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{D}_*)$  is equipped with an action of Hecke operators, including the Hecke operator at  $p$ , denoted  $U(p)$ , and we denote

$$\mathbb{W} = \mathrm{Symb}_{\Gamma_0(N)}^{\mathrm{ord}}(\mathbb{D}_*)$$

the ordinary subspace of  $\mathrm{Symb}_{\Gamma_0(N)}(\mathbb{D}_*)$  for the action of  $U(p)$ ; see [GS93, (2.2), (2.3)] for details and definitions.

For any integer  $k \in \mathcal{X}$ , we have a  $\Gamma_0(p\mathbb{Z}_p)$ -equivariant homomorphism  $\rho_k : \mathbb{D}_* \rightarrow \mathcal{V}_{k-2}(\mathbb{C}_p)$  defined by the formula

$$\rho_k(\mu)(P) = \int_{\mathbb{Z}_p \times \mathbb{Z}_p^\times} P(x, y) d\mu(x, y)$$

which gives rise to an homomorphism, denoted by the same symbol,

$$\rho_k : \mathbb{W} \longrightarrow \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{V}_{k-2}(\mathbb{C}_p)).$$

We may then define  $\mathbb{W}_{\mathcal{A}_{2k_0}} = \mathbb{W} \otimes_{\Lambda} \mathcal{A}_{2k_0}$  and consider the extension of  $\rho_k$ , still denoted by the same symbol,

$$\rho_k : \mathbb{W}_{\mathcal{A}_{2k_0}} \longrightarrow \mathrm{Symb}_{\Gamma_0(N)}(\mathcal{V}_{k-2}(\mathbb{C}_p)).$$

By [GS93, Thm. 5.13], there exists a connected neighborhood  $U_{2k_0}$  of  $2k_0$  in  $\mathcal{X}$  and an element  $\Phi_{2k_0} \in \mathbb{W}_{\mathcal{A}_{2k_0}}$  such that for all positive even integers  $2k \in U_{2k_0}$  with  $k \geq 1$  we have

$$(6) \quad \rho_k(\Phi_{2k_0}) = \lambda(k) \cdot I_{f_{2k}}$$

where  $\lambda(k) \in F_{f_{2k}}$ , the field extension of  $\mathbb{Q}_p$  generated by the Fourier coefficients of  $f_{2k}$ , and  $\lambda(k_0) = 1$ .

Fix a fundamental index pair  $(D_0, r_0)$  so that  $p \nmid D_0$ , and an index pair  $(D, r)$  with  $p \nmid D$ . Fix a system of representatives  $\mathcal{R}_{D_0, r_0}^{(D, r)}(N)$  of  $\mathcal{F}_{D_0, r_0}^{(D, r)}(N)$  so that each form  $[a, b, c]$  in  $\mathcal{R}_{D_0, r_0}^{(D, r)}(N)$  satisfies  $p \nmid a$  and  $p \nmid c$ . Such a system can easily be obtained up to multiplying quadratic forms by matrices of the form  $\begin{pmatrix} 1 & i \\ 0 & 1 \end{pmatrix}$  and  $\begin{pmatrix} 1 & 0 \\ N_i & 1 \end{pmatrix}$ , for suitable integers  $i$ . Fix a matrix  $Q = [a, b, c]$  in  $\mathcal{R}_{D_0, r_0}^{(D, r)}(N)$ . Let  $\Delta = D_0 D$ . If  $p$  is inert in  $K_\Delta$ , we define  $X_Q = (\mathbb{Z}_p^2)'$ . If  $p$  is split in  $K_\Delta$ , then  $Q(X, Y)$  splits into the product  $a \cdot q_1(X, Y) \cdot q_2(X, Y)$  where  $q_1(X, Y) = X - \alpha Y$  and  $q_2(X, Y) = X - \beta Y$  are two linear forms in  $\mathbb{Z}_p[X, Y]$ . Define the elements  $v_1 = (\alpha, 1)$  and  $v_2 = (\beta, 1)$  and put  $X_Q = \mathbb{Z}_p^\times \cdot v_1 + \mathbb{Z}_p^\times v_2$ . Since  $p \nmid ac$ , then  $\mathbb{Z}_p v_1 + \mathbb{Z}_p v_2 = \mathbb{Z}_p^2$ . Both in the split and inert cases, it is easy to show that  $Q(x, y) \in \mathbb{Z}_p^\times$  for all  $(x, y) \in X_Q$ .

Define for  $\kappa \in U_{2k_0}$  and  $Q \in \mathcal{R}_{D_0, r_0}^{(D, r)}(N)$ ,

$$\mathcal{L}_Q(\kappa) = \int_{X_Q} \omega(Q(x, y))^{k_0-1} \langle Q(x, y) \rangle^{\frac{k-2}{2}} d\Phi_{2k_0} \{r_Q \rightarrow s_Q\}(x, y).$$

The next result exploits the interpolation formulas of  $\mathcal{L}_Q(\kappa)$ . Put

$$\mathcal{E}_p = \begin{cases} \left(1 - \frac{p^{k-1}}{a_p(2k)}\right)^2 & \text{if } p \text{ is split in } K_\Delta, \\ 1 - \frac{p^{2k-2}}{a_p^2(2k)} & \text{if } p \text{ is inert in } K_\Delta. \end{cases}$$

Then, for all even positive integers  $2k \in U_{2k_0}$  with  $k > 1$  and all quadratic forms  $Q$  in  $\mathcal{R}_{D_0, r_0}^{(D, r)}(N)$  we have

$$(7) \quad \mathcal{L}_Q(2k) = \lambda(k) \cdot \mathcal{E}_p \cdot I_{f_{2k}^\#} \{r_Q \rightarrow s_Q\}.$$

In the inert case, (7) is [BD09, Prop. 2.4]. In the split case, a proof of this result can be found in [Sha08, Prop. 3.3.1]; see also [GSS16, Sec. 5.1], [LV14, Prop. 4.24], [LM18, Prop. 3.3].

Suppose that both  $D_0$  and  $D$  are fundamental discriminants. With the usual convention that  $D = r^2 - 4nN$ , put

$$(8) \quad \mathcal{L}_{n,r}(\kappa) = \sum_{Q \in \mathcal{R}_{D_0, r_0}^{(D, r)}(N)} \chi_{D_0}(Q) \cdot \mathcal{L}_Q(\kappa).$$

The function  $\mathcal{L}_{n,r}(\kappa)$  is called the square-root  $p$ -adic  $L$ -function attached to the genus character  $\chi_{D_0}$  of the real quadratic field  $K_\Delta$ , since its square for  $\kappa = 2k$  an even positive integer in  $U_{2k_0}$ , interpolates special values  $L(f_k^\#/K_\Delta, \chi_{D_0}, k)$ . More precisely, setting  $L_{n,r} = \mathcal{L}_{n,r}^2$ , we have for all even positive integers  $2k \in U_{2k_0}$ ,

$$L_{n,r}(2k) = \lambda(k)^2 \cdot \mathcal{E}_p^2 \cdot \Delta^{k-1} \cdot L^{\text{alg}}(f_{2k}^\#/K_\Delta, \chi_{D_0}, k)$$

where the algebraic part of the special value of the  $L$ -series of  $f_{2k}^\#$  twisted by  $\chi_{D_0}$  is defined as

$$L^{\text{alg}}(f_{2k}^\#/K_\Delta, \chi_{D_0}, k) = \frac{(k-1)!^2 \sqrt{\Delta}}{(2\pi i)^{2k-2} \cdot (\Omega_{f_{2k}^\#}^-)^2} \cdot L(f_{2k}^\#/K, \chi_{D_0}, k).$$

In the inert case, the above formula is proved in [BD09, Thm. 3.5], while in the split case the reader may consult [Sha08, Thm. A, Rmk. 3.3.3], [GSS16, Pop. 5.5] or [LV14, Prop.4.25].

**Theorem 3.1.** *Fix a fundamental index pair  $(D_0, r_0)$  and let  $2k_0$  be a positive even integer. Then for all index pair  $(D, r)$  such that  $p \nmid \Delta = D_0 D$  and for all even positive integers  $2k \in U_{2k_0}$ , we have*

$$\mathcal{L}_{n,r}(2k) = \lambda(k) \cdot \mathcal{E}_p^2 \cdot c_{f_{2k}^\#}(n, r),$$

*Proof.* This is an immediate combination of (5) and (8).  $\square$

#### 4. BIG HEEGNER POINTS AND JACOBI FORMS

**4.1. Big Heegner points.** Let  $\mathbb{T}$  denote Hida's big Galois representation, and  $\mathbb{T}^\dagger$  its critical twist, introduced in [How07b, Def. 2.1.3]. Recall that  $\mathbb{T}^\dagger$  is a free  $\mathcal{R}$ -module of rank 2 equipped with a continuous  $G_{\mathbb{Q}} := \text{Gal}(\bar{\mathbb{Q}}/\mathbb{Q})$  action, such that for each arithmetic point  $\varphi \in \mathcal{X}^{\text{arith}}(\mathcal{R})$ , the  $G_{\mathbb{Q}}$ -representation

$$V_\varphi^\dagger := \mathbb{T}^\dagger \otimes_{\mathcal{R}} L_\varphi$$

is the self-dual twist of the Deligne representation attached to the module form  $F_\varphi$ ; here  $L_\varphi$  is a finite field extension of  $\mathbb{Q}_p$ , and the tensor product is taken with respect to  $\varphi : \mathcal{R} \rightarrow L_\varphi$ . Assume that the semi-simple residual  $G_{\mathbb{Q}}$ -representation  $\mathbb{T}^\dagger/\mathfrak{m}_{\mathcal{R}}\mathbb{T}^\dagger$ , where  $\mathfrak{m}_{\mathcal{R}}$  is the maximal



ideal of  $\mathcal{R}$ , is absolutely irreducible and  $p$ -distinguished. The definition of  $\mathbb{T}^\dagger$  depends on the choice of a critical character

$$\Theta : G_{\mathbb{Q}} \longrightarrow \Lambda^\times,$$

and that there are two possible choices (see [How07b, Def. 2.1.3 and Rem. 2.1.4]), related by the quadratic character  $x \mapsto \left(\frac{p}{x}\right)$  of conductor  $p$ ; more precisely, if we write the cyclotomic character  $\epsilon_{\text{cyc}}$  as the product  $\epsilon_{\text{tame}} \cdot \epsilon_{\text{wild}}$  with  $\epsilon_{\text{tame}}$  taking values in  $(\mathbb{Z}/p\mathbb{Z})^\times$  and  $\epsilon_{\text{wild}}$  taking values in  $1 + p\mathbb{Z}_p$ , then the two possible critical characters are defined to be

$$\Theta = \left(\frac{p}{-}\right)^a \cdot \epsilon_{\text{tame}}^{\frac{k_0-2}{2}} \cdot \left[\epsilon_{\text{wild}}^{1/2}\right]$$

for  $a = 0$  or  $a = 1$ , where  $\epsilon_{\text{wild}}^{1/2}$  is the unique square root of  $\epsilon_{\text{wild}}$  taking values in  $1 + p\mathbb{Z}_p$ , and  $x \mapsto [x]$  is the inclusion of group-like elements  $1 + p\mathbb{Z}_p \hookrightarrow \Lambda$ ; here recall that  $k_0$  is the weight of our fixed form  $f_0$  through which the Hida family  $f_\infty$  passes. We fix the choice of  $\Theta$  for  $a = 0$  as prescribed in [Cas13, (4.1) and Rem. 4.1].

For any extension  $H$  of  $\mathbb{Q}$ , denote by  $\text{Sel}(H, \mathbb{T}^\dagger)$  the strict Greenberg Selmer group, whose definition by means of the ordinary filtration of  $\mathbb{T}$  can be found in [How07b, Def. 2.4.2].

Let  $K_D$  be an imaginary quadratic field of discriminant  $D < 0$ ; denote by  $\mathcal{O}_D$  its ring of algebraic integers and  $H_D$  its Hilbert class field. Suppose that  $D$  is a square mod  $2N$ . Fix a residue class  $r \pmod{2N}$  such that  $r^2 \equiv D \pmod{4N}$  and consider the integral  $\mathcal{O}_D$ -ideal  $\mathfrak{N} = \left(N, \frac{r+\sqrt{D}}{2}\right)$ ; then  $\mathcal{O}_D/\mathfrak{N} \cong \mathbb{Z}/N\mathbb{Z}$ . Also, take a generator  $\omega_D$  of  $\mathcal{O}_K/\mathbb{Z} \simeq \mathbb{Z}$  so that the imaginary part of  $\omega_D$  (viewed as complex number via our fixed embedding  $\bar{\mathbb{Q}} \hookrightarrow \mathbb{C}$ ) is positive. Depending on these choices, B. Howard in [How07b, §2.2] constructs a point

$$\mathfrak{X}_{D,r}^{\text{How}} \in H^1(H_D, \mathbb{T}^\dagger)$$

the Big Heegner point of conductor 1 associated to the quadratic imaginary field  $K_D$  (see [How07b, Def. 2.2.3]); the notation, slightly different from loc. cit., reflects the dependence on  $D$  and  $r$ . Finally, define  $\mathfrak{Z}_{D,r}^{\text{How}} \in H^1(K_D, \mathbb{T}^\dagger)$  to be the image of  $\mathfrak{X}_{D,r}^{\text{How}}$  via the corestriction map. Recall that all primes dividing  $N$  are split in  $K_D$ , and then by [How07b, Prop. 2.4.5] we have  $\mathfrak{Z}_{D,r}^{\text{How}} \in \text{Sel}(K_D, \mathbb{T}^\dagger)$ .

Let  $\varphi \in \mathcal{X}^{\text{arith}}(\mathcal{R})$  and let  $\chi \mapsto L_p(F_\varphi, \chi)$  denote the Mazur-Tate-Teitelbaum  $p$ -adic  $L$ -function of  $F_\varphi$ , viewed as a function on characters  $\chi : \mathbb{Z}_p^\times \rightarrow \bar{\mathbb{Q}}_p^\times$ . The functional equation satisfied by this function is

$$L_p(F_\varphi, \chi) = -w\chi^{-1}(-N)\Theta_\varphi(-N) \cdot L_p(F_\varphi, \chi^{-1}[\cdot]_\varphi),$$

where  $\Theta_\varphi$  is the composition of the chosen critical character  $\Theta$  with  $\varphi$  and  $[\cdot]_\varphi$  is the composition of the tautological character  $[\cdot] : \mathbb{Z}_p^\times \hookrightarrow \Lambda \hookrightarrow \mathcal{R}$  with  $\varphi$ ; here  $w = \pm 1$ , and is independent of  $\varphi$  (but it depends on the choice of  $\Theta$ ) and satisfies the equation  $L_p(F_\varphi, \Theta_\varphi) = -wL_p(F_\varphi, \Theta_\varphi)$  (see [How07b, Prop. 2.3.6]).

In this paper we will work under the following assumption on the sign of the functional equation of the Mazur-Tate-Teitelbaum  $p$ -adic  $L$ -function:

**Assumption 4.1.**  $w = 1$ .

We now discuss the analytic condition that we will assume in this paper to obtain our results. First, let  $\theta : \mathbb{Z}_p^\times \rightarrow \mathcal{R}^\times$  be the character obtained by factoring  $\Theta$  through the  $p$ -cyclotomic extension  $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$  (where  $\mu_{p^\infty}$  is the group of  $p$ -power roots of unity) and identifying  $\text{Gal}(\mathbb{Q}(\mu_{p^\infty})/\mathbb{Q})$  with  $\mathbb{Z}_p^\times$  via the cyclotomic character  $\chi_{\text{cyc}}$ ; thus we have the relation  $\Theta = \theta \circ \chi_{\text{cyc}}$ . Decompose  $\mathbb{Z}_p^\times \simeq \Delta \times \Gamma$  with  $\Gamma = 1 + p\mathbb{Z}_p$  and  $\Delta = (\mathbb{Z}/p\mathbb{Z})^\times$  is identified with the group  $\mu_{p-1}$  of  $p-1$ -roots of unity via the Teichmüller character, which we denote by  $\omega$  as usual. The character  $\theta : \mathbb{Z}_p^\times \rightarrow \mathcal{R}^\times$  satisfies the condition

$$\theta_\varphi(\delta\gamma) = \omega^{k_0-1}(\delta) \cdot \psi^{1/2}(\gamma) \cdot \gamma^{k-1}$$

for all arithmetic morphisms  $\varphi$  of weight  $2k$  and wild character  $\psi$ .

Following the terminology in [How07a, Def. 2], we say that an arithmetic morphism  $\varphi \in \mathcal{X}^{\text{arith}}(\mathcal{R})$  of weight 2 is generic for  $\theta$  if one of the following conditions hold:

- (1)  $F_\varphi$  has non-trivial nebentype;
- (2)  $F_\varphi$  is the  $p$ -stabilization of a newform in  $S_2(\Gamma_0(N))$ , and  $\theta_\varphi$  is trivial;
- (3)  $F_\varphi$  is a newform of level  $Np$  and  $\theta_\varphi = \omega^{(p-1)/2}$ .

For any  $\varphi \in \mathcal{X}^{\text{arith}}(\mathcal{R})$ , define the character  $\chi_{D,\varphi} : \mathbb{A}_{K_D}^\times \rightarrow \bar{\mathbb{Q}}_p^\times$  by the formula

$$\chi_{D,\varphi}(x) := \Theta_\varphi(\text{art}_{\mathbb{Q}}(\text{N}_{K_D/\mathbb{Q}}(x)));$$

here  $\text{art}_{\mathbb{Q}}$  is the arithmetically normalized Artin map of class field theory, and  $\text{N}_{K_D/\mathbb{Q}}$  is the norm map. For an arithmetic morphism  $\varphi \in \mathcal{X}^{\text{arith}}(\mathcal{R})$  which is generic for  $\theta$ , we say that  $(F_\varphi, \chi_{D,\varphi})$  has analytic rank 1 if

$$\text{ord}_{s=1} L(F_\varphi, \chi_{D,\varphi}, s) = 1.$$

We will work under the following analytic condition:

**Assumption 4.2.** There exists a fundamental discriminant  $D_1$  and a weight two prime  $\varphi_1$  which is generic for  $\theta$  such that  $(F_{\varphi_1}, \chi_{D_1, \varphi_1})$  has analytic rank equal to one.

We fix such a pair  $(D_1, \varphi_1)$ . Also fix a residue class  $r_1 \pmod{2N}$  such that  $D_1 \equiv r_1^2 \pmod{4N}$ . As a consequence of this assumption, we see by [How07a, Cor. 5] that  $(F_{\varphi_1}, \chi_{D_1, \varphi_1})$  has analytic rank 1 for all generic arithmetic morphisms  $\varphi$ , except possibly a finite number of them, that  $\mathfrak{Z}_{D_1, r_1}^{\text{How}}$  is non- $\mathcal{R}$ -torsion, and

$$\text{rank}_{\mathcal{R}} \left( \text{Sel}(\mathbb{Q}, \mathbb{T}^\dagger) \right) = 1.$$

Further, if we denote by  $\mathfrak{Z}_{D_1, r_1}^{\text{How}}(\varphi)$  the image via localization map  $\mathbb{T}^\dagger \rightarrow V_\varphi^\dagger$  of  $\mathfrak{Z}_{D_1, r_1}^{\text{How}}$  in the Nekovář's extended Bloch-Kato Selmer group  $\tilde{H}_f^1(\mathbb{Q}, V_\varphi^\dagger)$  attached to the representation  $V_\varphi^\dagger$  (introduced in [Nek06]), we see that  $\mathfrak{Z}_{D_1, r_1}^{\text{How}}(\varphi)$  is non-zero for all arithmetic morphisms  $\varphi$ , except possibly a finite number of them; also, it follows from the discussion in [How07b, §2.4] and [Nek06, §9] that, for all except a finite number of arithmetic primes,  $\mathfrak{Z}_{D_1, r_1}^{\text{How}}(\varphi)$  belongs to the Bloch-Kato Selmer group  $H_f^1(\mathbb{Q}, V_\varphi^\dagger)$ , which, again for all arithmetic primes except a finite number of them, is equal to the strict Greenberg Selmer group  $\text{Sel}(\mathbb{Q}, V_\varphi^\dagger)$  of  $V_\varphi^\dagger$  (see [BK90] and [Gre91] for details on the definitions of these Selmer groups, or [How07b, §2.4]).

**4.2. The GKZ Theorem for Heegner cycles.** The aim of this subsection is to review results of Hui Xue [Xue10] extending the Gross-Kohnen-Zagier Theorem [GKZ87, Thm. C] to modular forms of higher weight.

Fix in this section a newform  $f$  of even weight  $2k$  and level  $\Gamma_0(N)$  with  $p \nmid N$ . Let  $K_D$  be a quadratic imaginary field, of discriminant  $D$ , with ring of algebraic integers  $\mathcal{O}_{K_D}$ , and assume all primes dividing  $Np$  are split in  $K$ . Finally, assume that  $w = 1$ , where  $w$  is the sign of the functional equation of  $L(f, s)$ .

Fix a residue class  $r \pmod{2N}$  such that  $r^2 \equiv D \pmod{4N}$ . Let  $z_{D,r}$  be the solution in the upper half plane  $\mathcal{H}$  of the equation  $az^2 + bz + c = 0$ , where  $Q = [a, b, c] \in \mathcal{Q}_{N, D, r}$ . Then  $x_{D,r} = [z_{D,r}]$  is a Heegner point of conductor 1 in  $X_0(N)$ , namely, it represents an isogeny of elliptic curves with complex multiplication by  $\mathcal{O}_{K_D}$  and cyclic kernel of order  $N$ . If  $H_D$  denotes as above the Hilbert class field of  $K_D$ , there are  $h_D = [H_D : K_D]$  such points, permuted transitively by  $G_D = \text{Gal}(H_D/K_D)$ . To  $x_{D,r}$  we may attach a codimension  $k$  in the Chow group of the  $(2k-1)$ -dimensional Kuga-Sato variety  $W_{2k-2}$ , which is rational over  $H_D$ , as described in [Nek92, Sec. 5]. Briefly, one starts by considering the elliptic curve  $E_{x_{D,r}}$  equipped with a cyclic subgroup of order  $N$ , corresponding to  $x_{D,r}$  via the moduli

interpretation, and let  $\Gamma$  be the graph of the multiplication by  $\sqrt{D}$  on  $E \times E$ ; we then consider the cycle

$$Z(x_{D,r}) = \Gamma - E \times \{0\} - D(\{0\} \times E)$$

in  $E \times E$  and define

$$W(x_{D,r}) = \sum_{g \in \Sigma_{2k-2}} \text{sgn}(g)g \left( Z(x_{D,r})^{k-1} \right)$$

where the action of the symmetric group  $\Sigma_{2k-2}$  on  $2k-2$  letters on  $E^{2k-2}$  is via permutation. Thanks to the canonical desingularization, this defines a cycle, denoted by  $W_{D,r}^{\text{Heeg}}$ , of codimension  $k$  in the Chow group of the  $(2k-1)$ -dimensional Kuga-Sato variety  $W_{2k-2}$ , which is rational over  $H_D$ . Adopting a standard notation for Chow groups, we write  $X_{D,r} \in \text{CH}^k(W_{2k-2})_0(H_D)$ . We finally define  $X_{D,r}^{\text{Heeg}} = \sum_{\sigma \in G_D} W_{D,r}^{\text{Heeg}}$  and, again in a standard notation, we write  $X_{D,r}^{\text{Heeg}} \in \text{CH}^k(W_{2k-2})_0(K_D)$ . The Heegner cycle considered in [Zha97a, §2.4] and [Xue10, §2], which we denote by  $S_{D,r}^{\text{Heeg}}$ , is the multiple of  $X_{D,r}^{\text{Heeg}}$  such that the self-intersection of  $S_{D,r}^{\text{Heeg}}$  is  $(-1)^{k-1}$ . Then

$$S_{D,r}^{\text{Heeg}} \in \text{CH}^k(W_{2k-2})_0(K_D) \otimes_{\mathbb{Q}} \mathbb{R}$$

and, since the self-intersection of  $Z(x_{D,r})$  is  $-2D$  (see for example [Nek95, §(3.1)]) in this vector space we have  $S_{D,r}^{\text{Heeg}} = X_{D,r}^{\text{Heeg}} \otimes |2D|^{-\frac{k-1}{2}}$ , where we make the choice of square root of  $|2D|$  in  $\mathbb{R}$  to be the positive one.

Let  $x \mapsto \bar{x}$  denote the action of the non-trivial element  $\tau_D$  of  $\text{Gal}(K_D/\mathbb{Q})$  on the Chow group of  $K_D$ -rational cycles, and define

$$\left( S_{D,r}^{\text{Heeg}} \right)^* := S_{D,r}^{\text{Heeg}} + \overline{S_{D,r}^{\text{Heeg}}}.$$

Denote by  $\text{Heeg}_k(X_0(N))$  the  $\mathbb{Z}$ -submodule of  $\text{CH}^k(W_{2k-2})_0(K_D) \otimes_{\mathbb{Q}} \mathbb{R}$  generated by the elements  $\left( S_{D,r}^{\text{Heeg}} \right)^*$  as  $D$  varies. Let  $\mathbb{T}_N$  be the standard Hecke algebra (over  $\mathbb{Z}$ ) of level  $\Gamma_0(N)$ , and for any  $\mathbb{T}_N \otimes_{\mathbb{Z}} \mathbb{Q}$ -module  $M$  let  $M_f$  denote its  $f$ -isotypical component. Let finally

$$\left( S_{D,r}^{\text{Heeg}} \right)_f^* \in \text{Heeg}_k(X_0(N))_f$$

be the  $f$ -isotypical components of  $\left( S_{D,r}^{\text{Heeg}} \right)^*$ . The assumption  $w = 1$  implies then that the image  $S_{D,r,f}^{\text{Heeg}}$  of  $S_{D,r}^{\text{Heeg}}$  in  $(\text{CH}^k(W_{2k-2})_0(K_D) \otimes_{\mathbb{Z}} \mathbb{R})_f$  belongs to  $(\text{CH}^k(W_{2k-2})_0(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R})_f$ , and therefore  $\left( S_{D,r}^{\text{Heeg}} \right)_f^* = 2 \cdot S_{D,r,f}^{\text{Heeg}}$ .

Let

$$\phi_f(\tau, z) = \sum_{r^2 \leq 4Nn} c_f(n, r) q^n \zeta^r$$

be the Jacobi form corresponding to  $f$  under the Skoruppa-Zagier correspondence [SZ88], where as usual  $q = e^{2\pi i\tau}$  and  $\zeta = e^{2\pi iz}$ .

Let

$$(9) \quad \langle \cdot, \cdot \rangle_{\mathbb{R}} : \text{Heeg}_k(X_0(N)) \otimes_{\mathbb{Z}} \mathbb{R} \times \text{Heeg}_k(X_0(N)) \otimes_{\mathbb{Z}} \mathbb{R} \longrightarrow \mathbb{R}$$

be the restriction to  $\text{Heeg}_k(X_0(N)) \otimes_{\mathbb{Z}} \mathbb{R}$  of the height pairing defined via arithmetic intersection theory by Gillet-Soulé [GS90]. Choose  $s_f^* \in (\text{Heeg}_k(X_0(N))(\mathbb{Q})) \otimes_{\mathbb{Z}} \mathbb{R}$  such that

$$\langle s_f^*, s_f^* \rangle = \frac{(2k-2)!N^{k-1}}{2^{2k-1}\pi^k(k-1)! \|\phi_f\|^2} L'(f, k)$$

where  $\|\phi_f\|$  is the norm of  $\phi_f$  in the space of Jacobi forms equipped with the Petersson scalar product.

**Theorem 4.3** (Xue). *Assume the height pairing  $\langle, \rangle_{\mathbb{R}}$  in (9) is positive definite. For all fundamental discriminants  $D$  which are coprime with  $2N$  we have*

$$(D)^{\frac{k-1}{2}} \cdot \left(S_{D,r}^{\text{Heeg}}\right)_f^* = c_f \left(\frac{r^2 - D}{4N}, r\right) \cdot s_f^*.$$

**4.3. Specialization of Big Heegner points.** The aim of this subsection is to review the explicit comparison result between Howard Big Heegner points and Heegner cycles that could be found conditionally in Castella's thesis [Cas13], announced in Castella-Hsieh [CH17] and proved explicitly in [Cas17] under a number of arithmetic conditions, the most prominent being that primes  $p$  split in the imaginary quadratic field  $K_D$ .

Let  $\varphi \in \mathcal{X}^{\text{arith}}(\mathcal{R})$  be an arithmetic point of trivial character and weight  $2k$ , and let  $f_{2k}^{\sharp}$  be the form whose  $p$ -stabilization is  $F_{\varphi}$ . For any field extension  $L/\mathbb{Q}$ , we may consider the étale Abel-Jacobi map

$$\Phi_{W_{2k-2},L}^{\text{ét}} : \text{CH}^k(W_{2k-2})(L) \longrightarrow H^1(L, V_{\varphi}^{\dagger}).$$

Let  $X_{D,r,2k}^{\text{Heeg}}$  be the image of  $X_{D,r}^{\text{Heeg}}$  in  $(\text{CH}^k(W_{2k-2})_0(K_D))_{f_{2k}^{\sharp}}$  and define

$$\mathfrak{Z}_{D,r,2k}^{\text{Heeg}} := \Phi_{W_{2k-2},\mathbb{Q}}^{\text{ét}} \left( X_{D,r,2k}^{\text{Heeg}} \right).$$

Denote  $\mathfrak{Z}_{D,r,2k}^{\text{How}} = \varphi \left( \mathfrak{Z}_{D,r}^{\text{How}} \right)$ . The following result is due to Castella, cf. [Cas17, Thm 5.5]:

**Theorem 4.4** (Castella). *Let  $\varphi_0$  be an arithmetic point in  $\mathcal{X}^{\text{arith}}(\mathcal{R})$  with trivial nebentype and even weight  $2k_0 \equiv 2 \pmod{p-1}$ . If  $p$  is split in  $K_D$ , then for any arithmetic point  $\varphi$  of even integer  $2k > 2$  and trivial character, with  $2k \equiv 2k_0 \pmod{2(p-1)}$  we have*

$$\mathfrak{Z}_{D,r,2k}^{\text{How}} = \frac{\left(1 - \frac{p^{k-1}}{\varphi(A_p)}\right)^2}{u_D(4|D|)^{\frac{k-1}{2}}} \cdot \mathfrak{Z}_{D,r,2k}^{\text{Heeg}}$$

as elements in  $H^1(\mathbb{Q}, V_{\kappa}^{\dagger})$ , where  $u_D = |\mathcal{O}_{K_D}^{\times}|/2$ .

Since  $S_{D,r}^{\text{Heeg}}$  is equal to  $X_{D,r}^{\text{Heeg}} \otimes (2D)^{-\frac{k-1}{2}}$ , it follows immediately from Theorem 4.4 that for all  $\varphi$  as in the above theorem we have

$$(10) \quad 2^{\frac{k-1}{2}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\text{How}} = \left(1 - \frac{p^{k-1}}{a_p(2k)}\right)^2 \cdot \Phi_{W_{2k-2},\mathbb{Q}}^{\text{ét}} \left( S_{D,r}^{\text{Heeg}} \right)_{f_{2k}^{\sharp}}$$

as elements in the  $\bar{\mathbb{Q}}_p$ -vector space  $H^1(\mathbb{Q}, V_{f_{2k}^{\sharp}}^{\dagger}) \otimes_{\mathbb{Q}} \bar{\mathbb{Q}}_p$ , where we fix an embedding  $\bar{\mathbb{Q}} \hookrightarrow \bar{\mathbb{Q}}_p$  and, with a slight abuse of notation, we write  $\Phi_{W_{2k-2},\mathbb{Q}}^{\text{ét}}$  for the  $\bar{\mathbb{Q}}_p$ -linear extension of  $\Phi_{W_{2k-2},\mathbb{Q}}^{\text{ét}}$ ; here we use the fact that the element  $S_{D,r}^{\text{Heeg}}$  belongs to  $\text{CH}^k(W_{2k-2})_0(\mathbb{Q}) \otimes_{\mathbb{Z}} \bar{\mathbb{Q}}$  and not merely in  $\text{CH}^k(W_{2k-2})_0(\mathbb{Q}) \otimes_{\mathbb{Z}} \mathbb{R}$ .

**4.4. The  $\Lambda$ -adic GKZ Theorem.** Suppose from now on that Assumptions 4.1 and 4.2 are satisfied. Since our main results depend on the positivity of the height pairing in (9), we introduce a third assumption:

**Assumption 4.5.** The height pairing  $\langle, \rangle_{\mathbb{R}}$  in (9) is positive definite, for each even integer  $k \geq 2$ .

*Remark 4.6.* The validity of Assumption 4.5 is a consequence of the Bloch-Beilinson conjectures: see [Blo84], [Bl87], [GS93], and [Zha97b, Conj. 1.1.1 and 1.3.1]. It should be possible to remove this assumption, as suggested in [Xue10], using Borcherds's approach [Bor99], [Bor98] to the GKZ theorem. Actually, by Zemel [Zem15] pursued a variant of this idea in the context of higher weight modular forms, and therefore it seems possible that a combination of [Zem15] and [Xue10] would establish Theorem 4.3 unconditionally. However, it is not a priori clear that the different Heegner cycles used in [Zem15] and [Xue10] can be directly compared, so for this paper we prefer to assume the validity of a standard conjecture instead.

**Proposition 4.7.** *Suppose that Assumptions 4.1, 4.2 and 4.5 are satisfied. Let  $\varphi_0$  be a fixed arithmetic point with trivial character and weight  $2k_0 \equiv 2 \pmod{p-1}$ . Choose a fundamental index  $N$  pair  $(D_0, r_0)$  such that  $p$  is split in  $K_{D_0}$ . Then for any arithmetic point  $\varphi$  of even positive integer  $2k \equiv 2k_0 \pmod{p-1}$  and trivial character, and any index  $N$  pair  $(D, r)$  with  $(D, Np) = 1$  and  $p$  split in  $K_D$ , we have*

$$(2D)^{\frac{k-1}{2}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\text{How}} = \left(1 - \frac{p^{k-1}}{a_p(2k)}\right)^2 \cdot c_{f_{2k}^\#} \left(\frac{r^2 - D}{4N}, r\right) \cdot \Phi_{W_{2k-2}, \mathbb{Q}}^{\acute{e}t} \left(s_{f_{2k}^\#}^*\right)$$

where  $c_{f_{2k}^\#}(n, r)$  for the Fourier-Jacobi coefficients of  $\mathcal{S}_{D_0, r_0}(f_{2k}^\#)$

*Proof.* As a preliminary observation, note that Assumption 4.1 ensures that the sign of the functional equation of the  $L$ -function of  $f_{2k}^\#$  is  $-1$  (this sign is the same as the sign of the function equation of the  $L$ -function of  $f_{2k}$ ; see for example [How07a, Prop. 4]). If the index pair  $(D, r)$  is fundamental, the result is an obvious consequence of Theorem 4.3 and Equation (10), so suppose that  $D$  is not fundamental, but still  $(Np, D) = 1$ . In this case, we may argue as in [GKZ87, p. 558]. We start with the above equation for  $D$  fundamental, and fix an integer  $m$  prime to  $Np$ . Multiply both sides by  $a_m(2k)$ , the coefficient  $q^m$  in  $f_{2k}^\#$ . Since  $a_m(2k)$  is the eigenvalue of  $T(m)$  acting on  $f_{2k}^\#$ , the left-hand side is  $(2D)^{\frac{k-1}{2}} \cdot T(m) \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\text{How}}$ , while in the right-hand side we substitute the factor  $c_{f_{2k}^\#}(n, r)$  with the coefficient  $c_{f_{2k}^\#}^*(n, r)$  of  $q^n \zeta^r$  in  $(\mathcal{S}_{D_0, r_0}(f_{2k}^\#)) | T_J(m)$  (here as usual  $D = r^2 - 4Nn$ ). The formula in [GKZ87, p. 508] (top of the page) shows that

$$a_m(2k) \cdot \mathfrak{Z}_{D,r,2k}^{\text{How}} = T_J(m) \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\text{How}} = \sum_{d|m} d^{k-1} \left(\frac{D}{d}\right) 2u_D \cdot \mathfrak{Z}_{\frac{Dm^2}{d^2}, \frac{rm}{d}, k}^{\text{How}}$$

and by the equation for the action of Hecke operators in Section 2, we also have

$$a_m(\kappa) \cdot c_{f_{2k}^\#}(n, r) = c_{f_{2k}^\#}^*(n, r) = \sum_{d|m} d^{k-1} \left(\frac{D}{d}\right) c_{f_{2k}^\#}(n, r) \left(\frac{nm^2}{d^2}, \frac{rm}{d}\right).$$

It follows by induction from the case of fundamental index pair that the equality in the statement remains true if  $(D, r)$  is replaced by  $(Dm^2, rm)$ . Now each integer  $R$  satisfying the congruence  $R^2 \equiv Dm^2 \pmod{4N}$ , also satisfies the congruence  $R \equiv rm \pmod{2N}$  for some integer  $r$  with  $r^2 \equiv D \pmod{4N}$ , which implies the result.  $\square$

*Remark 4.8.* Tracing back the relation between Big Heegner points and Heegner cycles, one can deduce the main result of [Xue10] for non-fundamental discriminants as well. However, a more direct proof can be also obtained working directly with Heegner cycles and following the same argument using [Zha97a, Prop. 2.4.2].

*Remark 4.9.* Observe that the product on the right hand side of Proposition 4.7 does not depend on the choice of the fundamental index  $N$  pair  $(D_0, r_0)$ , even if the last two factors of this product depend on it.

5. THE  $p$ -ADIC GKZ THEOREM

**5.1. The main result.** We suppose in this section that Assumptions 4.1, 4.2 and 4.5 are satisfied. Fix a fundamental index pair  $(D_0, r_0)$  such that  $p$  split in  $K_{D_0}$  and write as above

$$\mathcal{S}_{D_0, r_0}(f_{2k}^\sharp) = \sum_{r^2 \leq 4Nn} c_{f_{2k}^\sharp}(n, r) q^n \zeta^r$$

for an even positive integer  $2k$ . Fix an even positive integer  $2k_0 \equiv 2 \pmod{p-1}$ . Recall the domain of convergence  $U_{2k_0}$  about  $2k_0$  introduced in Section 4.4, and fix a connected neighborhood  $U \subseteq U_{2k_0}$  in  $\mathcal{X}$  such that  $\lambda(k) \neq 0$  for all  $k$  in  $U_{2k_0}$ . Define for  $\kappa \in U_{2k_0}$  and  $D = r^2 - 4nN$ ,

$$\mathcal{Z}_{n,r}(\kappa) = (2D)^{\frac{\kappa-2}{4}} \cdot 2u_D \cdot \tilde{\varphi}_{2k}(\mathfrak{Z}_{D,r}^{\text{How}}).$$

Recall the notation  $\text{Sel}_{\mathcal{K}}(\mathbb{Q}, \mathbb{T}^\dagger) = \text{Sel}(\mathbb{Q}, \mathbb{T}^\dagger) \otimes_{\mathcal{R}} \mathcal{K}$  and

$$\text{Sel}_{\mathcal{M}_{2k_0}}(\mathbb{Q}, \mathbb{T}^\dagger) = \text{Sel}_{\mathcal{K}}(\mathbb{Q}, \mathbb{T}^\dagger) \otimes_{\mathcal{K}} \mathcal{M}_{2k_0},$$

where for the second tensor product we use the map  $\mathcal{K} \rightarrow \mathcal{M}_{2k_0}$  whose construction is recalled in Section 3.

**Theorem 5.1.** *There exists  $\Phi^{\acute{e}t}$  in  $\text{Sel}_{\mathcal{M}_{2k_0}}(\mathbb{Q}, \mathbb{T})$  such that, in a sufficiently small neighborhood of  $2k_0$ , we have  $\mathcal{Z}_{n,r} = \mathcal{L}_{n,r} \cdot \Phi^{\acute{e}t}$ .*

*Proof.* Combining Proposition 4.7 with Theorem 3.1 (in the split case) we obtain

$$\lambda(k) \cdot \mathcal{Z}_{n,r}(2k) = \mathcal{L}_{n,r}(2k) \cdot \Phi_{W_{2k-2}, \mathbb{Q}}^{\acute{e}t} \left( s_{f_{2k}^\sharp}^* \right).$$

The result follows by setting  $\Phi^{\acute{e}t}(\kappa) = \frac{\mathcal{Z}_{n_0, r_0}(\kappa)}{\mathcal{L}_{n_0, r_0}(\kappa)}$  for any index pair with  $\mathcal{L}_{n_0, r_0}(2k_0) \neq 0$ , and using the density of the set of even positive integers in  $\mathcal{X}$ .  $\square$

*Remark 5.2.* The paper [Cas17] shows that Big Heegner points are closely related to  $p$ -adic  $L$ -functions à la Bertolini-Darmon-Prasanna, and (2) shows still another relation of this nature. However, note that the proof of (2) makes an essential use of Castella's result.

**5.2. Twisted Gross-Kohlen-Zagier.** In this subsection we discuss the general conjecture suggested by the results obtained so far.

Combining Theorems 3.1 and 5.1 proves the formula

$$(11) \quad (2D)^{\frac{2k-2}{4}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r,2k}^{\text{How}} = c_{f_{2k}^\sharp}(n, r) \cdot \frac{(2D_0)^{\frac{2k-2}{4}} \cdot 2u_{D_0} \cdot \mathfrak{Z}_{D_0, r_0, 2k}^{\text{How}}}{c_{f_{2k}^\sharp}(n_0, r_0)}$$

for each even positive integer  $2k \equiv 2k_0 \equiv 2 \pmod{p-1}$ , and all discriminants  $D$  with  $(D, N) = 1$  and  $p$  split in  $K_D$ , where we chose an index pair  $(D_0, r_0)$  such that  $\mathcal{L}_{n_0, r_0}(2k_0) \neq 0$  and  $p$  splits in  $K_{D_0}$ .

In [LN18] we construct a  $p$ -adic family of Jacobi forms  $\mathbb{S}_{D_0, r_0}(\tilde{\varphi}) = \sum_{n,r} c_{n,r}(\tilde{\varphi}) q^n \zeta^r$  defined for  $\tilde{\varphi}$  in the metaplectic covering  $\tilde{\mathcal{X}}(\mathcal{R})$  of the weight space  $\mathcal{X}(\mathcal{R})$ , such that the specialisation of  $\mathbb{S}_{D_0, r_0}(\tilde{\varphi})$  at arithmetic points  $\tilde{\varphi}$  lying over arithmetic points  $\varphi \in \mathcal{X}^{\text{arith}}(\mathcal{R})$  interpolates certain theta lifts of the classical forms  $F_\varphi$ , see [LN18, Thm. 5.5] for details. In particular, [LN18, Thm. 5.8] shows the equation

$$(12) \quad c_{n,r}(\tilde{\varphi}) = \lambda(k) \cdot \varphi(A_p) \cdot \mathcal{E}_2 \cdot c_{f_{2k}^\sharp}(n, r)$$

where

$$\mathcal{E}_2 = \left( 1 - \frac{2p^{k-1}}{a_p(\kappa)} - \frac{p^{2k-2}}{a_p^2(\kappa)} \right),$$

for each arithmetic point  $\varphi \in \mathcal{X}(\mathcal{R})$  of trivial character and weight  $2k$ , where  $\tilde{\varphi}$  is the lift of  $\varphi$  to the metaplectic covering  $\tilde{\mathcal{X}}(\mathcal{R})$  of  $\mathcal{X}(\mathcal{R})$  having trivial character (see [LN18, Thm. 5.8]).

Combining (11) and (12), we obtain for each arithmetic morphism  $\varphi \in \mathcal{X}(\mathcal{R})$  of trivial character and weight  $2k \equiv 2k_0 \equiv 2 \pmod{p-1}$  with  $2k > 2$  (so that  $\mathcal{E}_2 \neq 0$ ), the formula

$$(13) \quad (2D)^{\frac{2k-2}{4}} \cdot 2u_D \cdot \mathfrak{Z}_{D,r}^{\text{How}}(\varphi) = c_{n,r}(\tilde{\varphi}) \cdot (2D_0)^{\frac{2k-2}{4}} \cdot 2u_{D_0} \cdot \mathfrak{Z}_{D_0,r_0}^{\text{How}}(\varphi)$$

where we write  $\mathfrak{Z}_{D,r}^{\text{How}}(\varphi)$  for the specialisation of  $\mathfrak{Z}_{D,r}^{\text{How}}$  at  $\varphi$ .

Assume now that  $k_0$  is congruent to 2 modulo 4. Then there are two distinct ways to  $p$ -adically interpolate the term  $(2D)^{\frac{2k-2}{4}}$  appearing in (13), by choosing a square root of the critical character  $\Theta$  used to define Big Heegner points. Indeed, we may define the square roots  $\theta_a^{1/2} : \mathbb{Z}_p^\times \rightarrow \Lambda^\times$  for  $a \in \{0, 1\}$  of the critical character  $\theta$  by

$$\theta_a^{1/2} = \left(\frac{-}{p}\right)^a \cdot \epsilon_{\text{tame}}^{\frac{2k_0-2}{4}} \cdot \left[\epsilon_{\text{wild}}^{1/4}\right]$$

where  $\epsilon_{\text{wild}}^{1/4}$  is the unique square root of  $\epsilon_{\text{wild}}^{1/2}$  taking values in  $1 + p\mathbb{Z}_p$ . Since  $k_0$  is congruent to 2 modulo 4, the expression  $\epsilon_{\text{tame}}^{\frac{k_0-2}{4}}$  is without ambiguity. For each  $\varphi \in \mathcal{X}(\mathcal{R})$  we let  $\theta_{a,\varphi}^{1/2}$  be the composition of  $\theta_a^{1/2}$  with the restriction of  $\varphi$  to  $\Lambda^\times$ . On the  $\mathcal{O}$ -module  $\Lambda$  we define a new structure of  $\Lambda$ -algebra  $\sigma : \Lambda \rightarrow \Lambda$  given by the map  $\sigma(t) = t^2$ , and we denote this new  $\Lambda$ -algebra by  $\tilde{\Lambda}$ . For any  $\tilde{\varphi} \in \mathcal{X}(\tilde{\mathcal{R}})$  choose  $a(\tilde{\varphi}) \in \{0, 1\}$  such that the restriction of  $\tilde{\varphi}$  to  $\tilde{\Lambda}$  is equal to  $\theta_{a(\tilde{\varphi}),\varphi}^{1/2}$ . Then, if  $\varphi$  has trivial character and weight  $2k$ , and  $\tilde{\varphi}$  is the lift of  $\varphi$  with trivial character as above, we have  $\theta_{a(\tilde{\varphi}),\varphi}^{1/2}(2D) = (2D)^{\frac{2k-2}{4}}$ . It makes then sense to define for any  $\tilde{\varphi} \in \mathcal{X}(\tilde{\mathcal{R}})$ , the element

$$\mathcal{Z}_{D,r}(\tilde{\varphi}) = \theta_{a(\tilde{\varphi}),\varphi}^{1/2}(2D) \cdot 2u_D \cdot \mathfrak{Z}_{D,r}^{\text{How}}(\varphi)$$

where  $\varphi = \pi(\tilde{\varphi})$  and  $\pi : \tilde{\mathcal{X}}(\mathcal{R}) \rightarrow \mathcal{X}(\mathcal{R})$  is the covering map (for the choice of  $\varphi = \varphi_{2k}$  with trivial character and  $\tilde{\varphi}$  the lift of  $\varphi$  with trivial character, this coincides with the element  $\mathcal{Z}_{n,r}(2k)$  defined above, so the new notation is consistent with the old one). Then (13) reads as

$$(14) \quad \mathcal{Z}_{D,r}(\tilde{\varphi}) = c_{n,r}(\tilde{\varphi}) \cdot \mathcal{Z}_{D_0,r_0}(\tilde{\varphi})$$

We remark that this formula holds under the restrictive conditions that  $\varphi$  has trivial character and weight  $2k \equiv 2k_0 \equiv 2 \pmod{p-1}$ ,  $2k_0 \equiv 2 \pmod{4}$ ,  $2k > 2$ ,  $p$  splits in  $D$  (and  $D_0$ ), and  $\tilde{\varphi}$  the lift of  $\varphi$  with trivial character. However, (14) makes sense for all arithmetic primes  $\tilde{\varphi} \in \tilde{\mathcal{X}}(\mathcal{R})$  and even if  $p$  is inert in  $D$  (at least under the condition that  $2k_0 \equiv 2 \pmod{4}$ ), and it might be viewed as a fuller analogue of the GKZ Theorem over a larger portion of the whole weight space. In particular, if  $\varphi$  is a weight two arithmetic primes (with possibly non-trivial character), the statement (14) would imply a twisted Gross-Kohnen-Zagier Theorem.

*Remark 5.1.* Moreover, we would also like to indicate that our work might shed some light on the main theorem of [BDP14, Thm. 1.4]; we thank F. Castella for pointing this out to us. There, it is shown that the cohomology class  $\Phi^{\text{ét}}(\Delta_t)$  of a generalized Heegner cycle  $\Delta_t$  is equal to  $m_{D,t} \cdot \sqrt{D} \cdot \delta(P_D)$ , where  $P_D$  is a point independent of  $t$ , and  $m_{D,t} \in \mathbb{Z}$  satisfies:

$$m_{D,t}^2 = \frac{2t!(2\pi\sqrt{-D})^t}{\Omega^{2t+1}} L(\psi^{2t+1}, t+1),$$

where  $\Omega(A)$  is some complex period, and  $L(\psi^{2t+1}, t+1)$  a Hasse-Weil  $L$ -series attached to a Hecke character  $\psi^{2t+1}$ . This suggests, in line with the philosophy of the original GKZ theorem, that the coefficients  $m_{D,t}$  might have some relationship with Fourier coefficients of a Jacobi form lifting a Hecke theta series.

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