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# ON THE NON-DETECTABILITY OF SPIKED LARGE RANDOM TENSORS 

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#### Abstract

This paper addresses the detection of a low rank high-dimensional tensor corrupted by an additive complex Gaussian noise. In the asymptotic regime where all the dimensions of the tensor converge towards $+\infty$ at the same rate, existing results devoted to rank 1 tensors are extended. It is proved that if a certain parameter depending explicitly on the low rank tensor is below a threshold, then the null hypothesis and the presence of the low rank tensor are undistinguishable hypotheses in the sense that no test performs better than a random choice.


## 1. INTRODUCTION

The problem of testing whether an observed $n_{1} \times n_{2}$ matrix $\mathbf{Y}$ is either a zero-mean independent identically distributed Gaussian random matrix $\mathbf{Z}$ with variance $\frac{1}{n_{2}}$, or $\mathbf{X}_{0}+\mathbf{Z}$ (where $\mathbf{X}_{0}$ is a low rank matrix: a useful signal, called also spike) is a fundamental problem arising in numerous applications such as the detection of low-rank multivariate signals or the Gaussian hidden clique problem. When the two dimensions $n_{1}, n_{2}$ converge towards $\infty$ at the same rate, the rank of $\mathbf{X}_{0}$ remaining fixed, the context is this of the so-called additive spiked large random matrix models. Various results on the singular values of $\mathbf{X}_{0}+\mathbf{Z}$ have been established; in particular it is possible to show that the Generalized Likelihood Ratio Test (GLRT) is consistent (i.e. the probability of false alarm and the probability of missed detection both converge towards 0 when $n_{1}, n_{2}$ converge towards $+\infty$ in such a way that $n_{1} / n_{2} \rightarrow c>0$ ) if and only the largest singular value of $\mathbf{X}_{0}$ is above the threshold $c^{1 / 4}$ (see e.g. [12], [3], [2]).

In a number of real life problems, the observation is not a matrix, but a tensor $\mathbf{Y}$ of order $d \geq 3$, i.e. a $d$-dimensional array $\mathbf{Y}=\mathbf{Y}_{i_{1}, i_{2}, \ldots, i_{d}}$ where for each $k=1, \ldots, d, i_{k} \in\left[1, \ldots, n_{k}\right]$. In this context, the generalization of the above matrix hypothesis testing problem becomes: test that the observed order $d \geq 3$ tensor is either a zero-mean independent identically distributed Gaussian random tensor $\mathbf{Z}$, or the sum of $\mathbf{Z}$ and a low rank deterministic tensor

$$
\begin{equation*}
\mathbf{X}_{0}=\sum_{i=1}^{r} \lambda_{i} \mathbf{x}_{0}^{(1, i)} \otimes \mathbf{x}_{0}^{(2, i)} \otimes \ldots \mathbf{x}_{0}^{(d, i)} \tag{1}
\end{equation*}
$$

where $r$ is called the rank of $\mathbf{X}_{0}$. Here $\left(\lambda_{i}\right)_{i=1, \ldots, r}$ are strictly positive real numbers, and for each $i=1, \ldots, r$ and $k=1, \ldots, d, \mathbf{x}_{0}^{(k, i)}$ is a $n_{i} \times 1$ unit norm vector. Recent works (see e.g. [8, 11, 10, 14] ) addressed the detection/estimation of $\mathbf{X}_{0}$ when $r$ is reduced to 1 and when the dimensions $n_{1}, \ldots, n_{d}$ converge towards $\infty$ at the same rate. We also mention that [8] and [14] only considered the case where the rank 1 tensor $\mathbf{X}_{0}$ is symmetric, that is: $n_{1}=n_{2}=$

[^0]$\ldots=n_{d}$ and the $d$ vectors $\left(\mathbf{x}_{0}^{(k, i)}\right)_{i=1, \ldots, d}$ are equal. Since the concept of singular value decomposition cannot be extended to tensors, $a d$ 'hoc statistical strategies have been considered to prove the (non)existence of consistent tests: [11] and [14] $(r=1)$ established that if $\lambda_{1}$ is larger than a certain upper bound, then consistent detection of $\mathbf{X}_{0}$ is possible. In the other direction, [10][14] (again, $r=1$ ) proved that if $\lambda_{1}$ is less than a certain lower bound (which is stricly less than the above upper bound), then $\mathbf{X}_{0}$ is non-detectable in the sense that any test behaves as a random choice between the two hypotheses. This is a remarkable phenomenon because such a behaviour is not observed in the matrix case $(d=2)$ : indeed, in this context, if the largest eigenvalue of $\mathbf{X}_{0}$ is below $c^{1 / 4}$, it is proved in [13] $(r=1)$ that there exist statistical tests having a better performance than a random choice - a result that [10][14] obtained a different way.

The replica method has been successfully considered [4][9]. In these contributions, the model does not match exactly ours since 1) the spike is assumed symmetric, i.e. $\mathbf{X}_{0}=\sum_{i=1}^{r} \lambda_{i}\left(\mathbf{x}_{0}^{(i)}\right)^{\otimes d}$ and 2) the rows of the matrix $\left(\mathbf{x}_{0}^{(1)}, \ldots, \mathbf{x}_{0}^{(r)}\right)$ are random i.i.d. with a known distribution (the prior). When $r=1$, and the prior is of the Rademacher type, the observed tensor follows the pure p-spin model [15]: in an illuminating contribution [4], a tight threshold when $d \geq 3$ is provided (above which consistent detection is possible and under which any detector performs as a random guess). The case $r \geq 1$ with a general prior is addressed in [9]: there, the estimation of the spike is considered rather than the detection; specifically, the asymptotic performance of the MMSE is computed and an estimation threshold is deduced. This latter is rigorously proved when $r=1$. The threshold is not explicit and intrinsically depends on the prior.

In the present contribution, we follow the methodolodgy of [11] [10][14] and extend it to the general case $r \geq 1$ : though suboptimal (the thresholds provided are not tight in general), the machinery is much lighter than this of the replica method, it provides explicit bounds for the non-detectability and lastly allows one to deal with deterministic spikes. Precisely, we find out a simple sufficient condition on the spike $\mathbf{X}_{0}$ under which $\mathbf{X}_{0}$ is non-detectable. The problem of finding conditions under which the existence of a consistent detection is guaranteed is not addressed here.

## 2. MODEL, NOTATION, AND BACKGROUND

The order- $d$ tensors are complex-valued, and it is assumed that $n_{1}=$ $n_{2}=\ldots=n$ in order to simplify the notations. The set $\otimes^{d} \mathbb{C}^{n}$ is a complex vector-space endowed with the standard scalar product

$$
\forall \mathbf{X}, \mathbf{Y} \in \otimes^{d} \mathbb{C}^{n} \quad\langle\mathbf{X}, \mathbf{Y}\rangle=\sum_{i_{1}, \ldots, i_{d}} \mathbf{X}_{i_{1}, \ldots, i_{d}} \overline{\mathbf{Y}}_{i_{1}, \ldots, i_{d}}
$$

and the Frobenius norm $\|\mathbf{X}\|_{F}=\sqrt{\langle\mathbf{X}, \mathbf{X}\rangle}$.

The spike ("the signal") is assumed to be a tensor of fixed rank $r$ following (1). Along this contribution, $n$ is large or, mathematically, $n \rightarrow \infty$. We hence have for each $n$ a set of $n \times 1$ vectors $\left(\mathbf{x}_{0}^{(k, i)}\right)_{k=1 \ldots d, i=1, \ldots, r}$. For each $k=1, \ldots, d$, we denote by $\chi_{0}^{(k)}$ the $n \times r$ matrix $\chi_{0}^{(k)}=\left(\mathbf{x}_{0}^{(k, 1)}, \ldots, \mathbf{x}_{0}^{(k, r)}\right)$. We impose a non-erratic asymptotic behavior of the spike, and specifically, as all the vectors $\mathbf{x}_{0}^{(k, i)} \in \mathbb{C}^{n \times 1}$ have unit norm, we suppose that for all $i, j,\left\langle\mathbf{x}_{0}^{(k, i)}, \mathbf{x}_{0}^{(k, j)}\right\rangle=\left(\chi_{0}^{(k) *} \chi_{0}^{(k)}\right)_{i, j}$ converges as $n \rightarrow \infty$. The rate of convergence is a technical aspect that is out of the scope of this contribution: we will simply assume that the matrices $\left(\boldsymbol{\chi}_{0}^{(k) *} \boldsymbol{\chi}_{0}^{(k)}\right)_{k=1, \ldots, d}$ do not depend on $n$. We define the SVD of $\boldsymbol{\chi}_{0}^{(k)}$ as $\mathbf{U}_{k}\binom{\boldsymbol{\Sigma}_{k}}{0} \mathbf{V}_{k}^{*}$ for $\mathbf{U}_{k}$ and $\mathbf{V}_{k}$ unitary matrices respectively of size $n \times n$ and $r \times r$ and $\boldsymbol{\Sigma}_{k}$ a diagonal matrix with non-negative entries on the diagonal. $\mathbf{V}_{k}$ and $\boldsymbol{\Sigma}_{k}$ do no depend on $n$ because $\boldsymbol{\chi}_{0}^{(k) *} \boldsymbol{\chi}_{0}^{(k)}=\mathbf{V}_{k} \boldsymbol{\Sigma}_{k}^{2} \mathbf{V}_{k}^{*}$.

We denote by $\mathbf{Z}$ the noise tensor, and assume that its entries are $\mathcal{N}_{\mathbb{C}}(0,1 / n)$ independent identically distributed complex circular Gaussian random variables.

In the following, we consider the alternative $\mathcal{H}_{0}: \mathbf{Y}=\mathbf{Z}$ versus $\mathcal{H}_{1}: \mathbf{Y}=\mathbf{X}_{0}+\mathbf{Z}$. We denote by $p_{1, n}(\mathbf{y})$ the probability probability density of $\mathbf{Y}$ under $\mathcal{H}_{1}$ and $p_{0, n}(\mathbf{y})$ the density of $\mathbf{Y}$ under $\mathcal{H}_{0} . \Lambda(\mathbf{Y})=\frac{p_{1}(\mathbf{Y})}{p_{0}(\mathbf{Y})}$ is the likelihood ratio and we denote by $\mathbb{E}_{0}$ the expectation under $\mathcal{H}_{0}$. We now recall the fundamental information geometry results used in [10] in order to address the detection problem. The following properties are well known (see also [1] section 3):

- (i) If $\mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]$ is bounded, then no consistent detection test exists.
- (ii) If moreover $\mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]=1+o(1)$, then the total variation distance between $p_{0, n}$ and $p_{1, n}$ converges towards 0 , and no test performs better than a decision at random.
Therefore, the computation of the second order moment of $\Lambda(\mathbf{Y})$ under $p_{0, n}$ may provide insights on the detection. We however notice that conditions (i) and (ii) are only sufficient. In particular, if $\lim \sup _{n} \mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]=+\infty$, nothing can be inferred on the behaviour of the detection problem when $n \rightarrow+\infty$.


## 3. EXPRESSION OF THE SECOND-ORDER MOMENT.

The density of $\mathbf{Z}$, seen as a collection of $n^{d}$ complex-valued random variables, is obviously $p_{0, n}(\mathbf{z})=\kappa_{n} \exp \left(-n\|\mathbf{z}\|_{F}^{2}\right)$ where $\kappa_{n}=\left(\frac{n}{\pi}\right)^{n^{d}}$. On the one hand, we notice that the second-order moment approach is not suited to the deterministic model of the spike as presented previously. Indeed, in this case $\mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]$ has the simple expression $\exp \left(2 n\left\|\mathbf{X}_{0}\right\|_{F}^{2}\right)$ and always diverges. On the other hand, the noise tensor shows an invariance property: if $\boldsymbol{\Theta}_{1}, \ldots, \boldsymbol{\Theta}_{d}$ are unitary $n \times n$ matrices, then the density of the mode products $\left(\boldsymbol{\Theta}_{1} \otimes \boldsymbol{\Theta}_{2} \ldots \otimes \boldsymbol{\Theta}_{d}\right) \mathbf{Z}$ equals this of $\mathbf{Z}$. For $d=2$, the notation $\left(\boldsymbol{\Theta}_{1} \otimes \boldsymbol{\Theta}_{2}\right) \mathbf{Z}$ simply means $\boldsymbol{\Theta}_{1} \mathbf{Z} \boldsymbol{\Theta}_{2}$ and for a general $d$, $\left(\left(\boldsymbol{\Theta}_{1} \otimes \boldsymbol{\Theta}_{2} \ldots \otimes \boldsymbol{\Theta}_{d}\right) \mathbf{Z}\right)_{i_{1}, \ldots, i_{d}}$ is

$$
\sum_{\ell_{1}, \ldots, \ell_{d}}\left(\boldsymbol{\Theta}_{1}\right)_{i_{1}, \ell_{1}}\left(\boldsymbol{\Theta}_{2}\right)_{i_{2}, \ell_{2}} \ldots\left(\boldsymbol{\Theta}_{d}\right)_{i_{d}, \ell_{d}} \mathbf{Z}_{\ell_{1}, \ldots, \ell_{d}}
$$

We hence modify the data according to the procedure: we pick i.i.d. complex Haar samples $\boldsymbol{\Theta}_{1}, \ldots, \boldsymbol{\Theta}_{d}$ and consider the "new" data tensor defined as $\left(\boldsymbol{\Theta}_{1} \otimes \boldsymbol{\Theta}_{2} \ldots \otimes \boldsymbol{\Theta}_{d}\right) \mathbf{Y}$. This does not affect the distribution of the noise, but this amounts to assume an artificial prior
on the spike. Indeed, the vectors $\mathbf{x}_{0}^{(k, i)}$ are replaced by $\boldsymbol{\Theta}_{k} \mathbf{x}_{0}^{(k, i)}$. They are all uniformly distributed on the unit sphere of $\mathbb{C}^{n}$ and for $k \neq l$, vectors $\boldsymbol{\Theta}_{k} \mathbf{x}_{0}^{(k, i)}$ and $\boldsymbol{\Theta}_{l} \mathbf{x}_{0}^{(l, j)}$ are independent for each $i, j$. However, vectors $\left(\boldsymbol{\Theta}_{k} \mathbf{x}_{0}^{(k, i)}\right)_{i=1, \ldots, r}$ are not independent. In the following, the data and the noise tensors after this procedure are still denoted respectively by $\mathbf{Y}$ and $\mathbf{Z}$. This transformation of the spike is an extension of a trick used in Section III.C of [10].

We are now in position to give a closed-form expression of the second-order moment of $\boldsymbol{\Lambda}(\mathbf{Y})$. We have $p_{1, n}(\mathbf{Y})=$ $\mathbb{E}_{X}\left[p_{0, n}(\mathbf{Y}-\mathbf{X})\right]$ where $\mathbb{E}_{X}$ is the mathematical expectation over the distribution of the spike, or equivalently over the Haar matrices $\left(\boldsymbol{\Theta}_{k}\right)_{k=1, \ldots, d}$. It holds that

$$
\begin{aligned}
& \mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]=\mathbb{E}_{X, X^{\prime}}\left[\exp \left(2 n \mathfrak{R}\left\langle\mathbf{X}, \mathbf{X}^{\prime}\right\rangle\right)\right]= \\
& \mathbb{E}_{X, X^{\prime}}\left[\exp \left(2 n \mathfrak{R} \sum_{i, j=1}^{r} \lambda_{i} \lambda_{j} \prod_{k=1}^{d}\left\langle\left(\boldsymbol{\Theta}_{k}^{\prime}\right)^{*} \boldsymbol{\Theta}_{k} \mathbf{x}_{0}^{(k, i)}, \mathbf{x}_{0}^{(k, j)}\right\rangle\right)\right]
\end{aligned}
$$

where $\mathbb{E}_{X, X^{\prime}}$ is over independent copies $\mathbf{X}, \mathbf{X}^{\prime}$ of the spike associated respectively with $\left(\boldsymbol{\Theta}_{k}\right)_{k=1, \ldots, d}$ and $\left(\boldsymbol{\Theta}_{k}^{\prime}\right)_{k=1, \ldots, d}$. $\mathfrak{\Re}$ stands for the real part. As $\boldsymbol{\Theta}_{k}$ and $\boldsymbol{\Theta}_{k}^{\prime}$ are Haar and independent, then $\left(\boldsymbol{\Theta}_{k}^{\prime}\right)^{*} \boldsymbol{\Theta}_{k}$ is also Haar distributed and $\mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]=$ $\mathbb{E}[\exp (2 n \eta)]$, where the expectation is over the i.i.d. Haar matrices $\boldsymbol{\Theta}_{1}, \boldsymbol{\Theta}_{2}, \ldots, \boldsymbol{\Theta}_{d}$ and

$$
\begin{equation*}
\eta=\mathfrak{R} \sum_{i, j=1}^{r} \lambda_{i} \lambda_{j} \prod_{k=1}^{d} \underbrace{\left\langle\boldsymbol{\Theta}_{k} \mathbf{x}_{0}^{(k, i)}, \mathbf{x}_{0}^{(k, j)}\right\rangle}_{\xi_{k}^{(i, j)}} . \tag{2}
\end{equation*}
$$

$\eta$ may be factored as $\eta=\mathfrak{R}\left[\boldsymbol{\lambda}^{T}\left(\odot_{k=1}^{d}\left(\boldsymbol{\chi}_{0}^{(k) *} \boldsymbol{\Theta}_{k} \boldsymbol{\chi}_{0}^{(k)}\right)\right) \boldsymbol{\lambda}\right]$. In the latter equation, $\odot$ stands for the Hadamard product of matrices. The ultimate simplification comes from the SVD of $\chi_{0}^{(k)}$ :

$$
\boldsymbol{\chi}_{0}^{(k) *} \boldsymbol{\Theta}_{k} \boldsymbol{\chi}_{0}^{(k)}=\mathbf{V}_{k}\left(\begin{array}{cc}
\boldsymbol{\Sigma}_{k} & \mathbf{0}
\end{array}\right) \mathbf{U}_{k}^{*} \boldsymbol{\Theta}_{k} \mathbf{U}_{k}\binom{\boldsymbol{\Sigma}_{k}}{\mathbf{0}} \boldsymbol{\Sigma}_{k} \mathbf{V}_{k}^{*}
$$

Firstly, $\mathbf{U}_{k}^{*} \boldsymbol{\Theta}_{k} \mathbf{U}_{k}$ has the same distribution as $\boldsymbol{\Theta}_{k}$; secondly, we may associate with any $\boldsymbol{\Theta}_{k}$ its upper $r \times r$ block, that we will denote $\boldsymbol{\Psi}_{k}$. As a conclusion, we may express $\eta$ as

$$
\begin{equation*}
\eta=\mathfrak{R}\left[\boldsymbol{\lambda}^{T}\left(\odot_{k=1}^{d}\left(\mathbf{V}_{k} \boldsymbol{\Sigma}_{k} \mathbf{\Psi}_{k} \boldsymbol{\Sigma} \mathbf{V}_{k}^{*}\right)\right) \boldsymbol{\lambda}\right] \tag{3}
\end{equation*}
$$

## 4. EXTENDING KNOWN RESULTS

When $r=1$, Montanari et al. [10] found a bound on the parameter $\lambda_{1}$ ensuring that $\mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]$ is bounded. In this case, $\eta$ has a simple expression since $\eta=\lambda^{2} \mathfrak{R} \prod_{k=1}^{d} \xi_{k}$ where the $\left(\xi_{k}\right)_{k=1, \ldots, d}$ are i.i.d. distributed as the first component of a uniform vector of the unit sphere of $\mathbb{C}^{n}$. As in [10], we introduce

$$
\begin{equation*}
\beta_{d}^{2 \text { nd }}=\sqrt{\min _{u \in[0,1]}-\frac{1}{u^{d}} \log \left(1-u^{2}\right)} \tag{4}
\end{equation*}
$$

Adapting the result of the aforementioned article to the complexcircular context is straight-forward:
Theorem 1 (case $\mathrm{r}=1$ (Montanari et al.)). Let $\xi_{1}, \ldots, \xi_{d}$ be i.i.d. distributed as the first component of a vector uniformly distributed on the unit sphere of $\mathbb{C}^{n}$.

If $\lambda_{1}<\sqrt{\frac{d}{2}} \beta_{d}^{2 n d}$ then $\mathbb{E}_{0}\left[\exp \left(2 n \lambda_{1}^{2} \Re \prod_{k=1}^{d} \xi_{k}\right)\right]$ is bounded; moreover, if $d>2$, the above expectation is $1+o(1)$.

This non-obvious result may be used in order to derive a condition ensuring that hypotheses $\mathcal{H}_{0}$ and $\mathcal{H}_{1}$ are indistinguishable when $r>1$. In this respect, recall the expansion (2). Thanks to the Hölder inequality, $\mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]$ is upper bounded by (see (2) for the definition of $\xi_{k}^{(i, j)}$ )

$$
\begin{equation*}
\prod_{i, j=1}^{r} \mathbb{E}^{1 / p_{i, j}}\left[\exp \left(2 n p_{i, j} \lambda_{i} \lambda_{j} \Re \prod_{k=1}^{d} \xi_{k}^{(i, j)}\right)\right] \tag{5}
\end{equation*}
$$

for any non-negative numbers $p_{i, j}$ such that $\sum_{i, j} \frac{1}{p_{i, j}}=1$. For fixed $i, j$, we notice that the random variables $\left(\xi_{k}^{(i, j)}\right)_{k=1, \ldots, d}$ verify the condition of Theorem 1. Any of the expectations in (5) are hence upper-bounded when $n \rightarrow \infty$ provided that, for all $i, j: p_{i, j} \lambda_{i} \lambda_{j}<$ $\frac{d}{2}\left(\beta_{d}^{\text {nnd }}\right)^{2}$. Choosing eventually $p_{i, j}=\frac{\left(\sum_{p} \lambda_{p}\right)^{2}}{\lambda_{i} \lambda_{j}}$, we deduce

Theorem 2 (case $r \geq 1$ - extension of Theorem 1 ). If $\sum_{i=1}^{r} \lambda_{i}<$ $\sqrt{\frac{d}{2}} \beta_{d}^{2 n d}$ then $\mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]$ is bounded. If moreover $d>2$, we have $\mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]=1+o(1)$ and the hypotheses $\mathcal{H}_{0}$ and $\mathcal{H}_{1}$ are indistinguishable.

Remark 3. Due to the use of the Hölder inequality, Theorem 2 is suboptimum in general. The inequality is patently an equality when $\forall k, i, j, \mathbf{x}_{0}^{(k, i)}=\mathbf{x}_{0}^{(k, j)}$,i.e. the spike has rank $r=1$ and amplitude $\sum_{i=1}^{r} \lambda_{i}$.

## 5. A TIGHTER BOUND

The main result of our contribution is the following
Theorem 4 (case $r \geq 1$ ). We define $\eta_{\max }$ as

$$
\begin{equation*}
\eta_{\max }=\boldsymbol{\lambda}\left(\odot_{k=1}^{d} \boldsymbol{\chi}_{0}^{(k) *} \boldsymbol{\chi}_{0}^{(k)}\right) \boldsymbol{\lambda} \tag{6}
\end{equation*}
$$

If $\sqrt{\eta_{\max }}<\sqrt{\frac{d}{2}} \beta_{d}^{2 n d}$ then, for $d>2, \mathbb{E}_{0}\left[\Lambda(\mathbf{Y})^{2}\right]=1+o(1)$.
Before providing elements of the proof of the above result, we may briefly justify why the bound in Theorem 4 is tighter than this of Theorem 2, whatever the choice of $\boldsymbol{\lambda}$. On the one hand, indeed, $\left(\sum_{i} \lambda_{i}\right)^{2}=\boldsymbol{\lambda}^{T} \mathbf{J} \boldsymbol{\lambda}$ where $\mathbf{J}$ is the $r \times r$ matrix having all its entries equal to 1 . On the other hand, all the vectors $\mathbf{x}_{0}^{(k, i)}$ are normalized and consequently, any of the diagonal entries of $\boldsymbol{\chi}_{0}^{(k) *} \boldsymbol{\chi}_{0}^{(k)}$ equals 1 and for any $i \neq j,\left|\left(\boldsymbol{\chi}_{0}^{(k) *} \boldsymbol{\chi}_{0}^{(k)}\right)_{i, j}\right| \leq 1$. This proves that $\left(\sum_{i} \lambda_{i}\right)^{2}-\eta_{\max }=\boldsymbol{\lambda}^{T}\left(\mathbf{J}-\odot_{k=1}^{d} \boldsymbol{\chi}_{0}^{(k) *} \boldsymbol{\chi}_{0}^{(k)}\right) \boldsymbol{\lambda} \geq 0$.

We provide the key elements of the proof of Theorem 4. Remind that we are looking for a condition on the spike under which $\mathbb{E}[\exp (2 n \eta)]$ is bounded. Evidently, the divergence may occur only when $\eta>0$. We hence consider $E_{1}=\mathbb{E}\left[\exp (2 n \eta) \mathbf{1}_{\eta>\epsilon}\right]$ and $E_{2}=\mathbb{E}\left[\exp (2 n \eta) \mathbf{1}_{\eta \leq \epsilon}\right]$, and prove that under the condition $\sqrt{\eta_{\max }}<\sqrt{\frac{d}{2}} \beta_{d}^{2 \text { nd }}$, for a certain small enough $\epsilon, E_{1}=o(1)$ (for $d \geq 2$ ) and that $E_{2}=1+o(1)$ (for $d>2$ ).

The $E_{1}$ term. It is clear that the boundedness of the integral $E_{1}$ is achieved when $\eta$ rarely deviates from 0 . As remarked in [10], the natural machinery to consider to understand $E_{1}$ is this of the Large Deviation Principle (LDP). In essence, if $\eta$ follows the LDP with rate $n$, there can be found a certain non-negative function called Good Rate Function (GRF) $I_{\eta}$ such that for any Borel set $A$ of $\mathbb{R}$,
$\frac{1}{n} \log \mathbb{P}(\eta \in A)$ converges towards $\sup _{x \in A}-I_{\eta}(x)$. The existence of a GRF allows one to analyze the asymptotic behaviour of integral $E_{1}$. Indeed, the Varadhan lemma (see Theorem 4.3 .1 in [6]) states that $\frac{1}{n} \log \mathbb{E}\left[\exp (2 n \eta) \mathbf{1}_{\eta>\epsilon}\right] \rightarrow \sup _{x>\epsilon}\left(2 x-I_{\eta}(x)\right)$ and hence the $E_{1}$ term converges towards 0 when $\sup _{x>\epsilon}\left(2 x-I_{\eta}(x)\right)<0$.

We thus justify that $\eta$ follows a Large Deviation Principle with rate $n$, and we compute a lower bound of its GRF. For this, we use that for each $k$, the random matrix $\boldsymbol{\Psi}_{k}$ defined in (3) follows a LDP with rate $n$ and that its GRF at the parameter $\psi \in \mathbb{C}^{r \times r}$ (we may evidently take $\|\boldsymbol{\psi}\|_{2} \leq 1$ ) is $\log \operatorname{det}\left(\mathbf{I}_{r}-\boldsymbol{\psi}^{*} \boldsymbol{\psi}\right)$ (see Theorem 3-6 in [7]). $\eta$ is a function of the i.i.d. matrices $\left(\boldsymbol{\Psi}_{k}\right)_{k=1, \ldots, d}$. Therefore, the contraction principle (see Theorem 4.2.1 in [6]) ensures that $\eta$ follows a LDP with rate $n$ and GRF $I_{\eta}$ given, for each real $x$ in the range of $\eta$, as the solution of the optimization problem:

$$
\max _{\forall k 0 \leq \alpha_{k} \leq 1} \max _{\substack{\forall k\left\|\boldsymbol{\psi}_{k}\right\|=\alpha_{k} \\ \eta\left(\boldsymbol{\psi}_{1}, \ldots, \boldsymbol{\psi}_{d}\right)=x}} \sum_{k=1}^{d} \log \operatorname{det}\left(\mathbf{I}_{r}-\boldsymbol{\psi}_{k}^{*} \boldsymbol{\psi}_{k}\right) .
$$

When $d \geq 3$, the solution of this optimization problem cannot apparently be expressed in closed form. We thus just provide a lower bound of $I_{\eta}(x)$. When $d=2$, it is possible to evaluate $I_{\eta}(x)$ : see [5].
Proposition 5. For each $x \in \mathbb{R}$, it holds that

$$
\begin{equation*}
I_{\eta}(x) \geq-d \log \left(1-\left(\frac{|x|}{\eta_{\max }}\right)^{2 / d}\right) \tag{8}
\end{equation*}
$$

where the right-hand side should be understood as $+\infty$ if $|x| \geq$ $\eta_{\text {max }}$.

In order to establish Proposition 5, we use the following algebraic result whose proof is omitted.
Lemma 6. For any matrices $\left(\mathbf{A}_{k}\right)_{k=1, \ldots, d} \in \mathbb{C}^{r \times r}$ and vector $\boldsymbol{\lambda} \in$ $\mathbb{R}^{r}$, the supremum of $\left|\boldsymbol{\lambda}^{T} \odot_{k=1}^{d}\left(\mathbf{A}_{k} \boldsymbol{\psi}_{k} \mathbf{A}_{k}^{*}\right) \boldsymbol{\lambda}\right|$ over $r \times r$ matrices $\boldsymbol{\psi}_{k}$ such that for all $k:\left\|\boldsymbol{\psi}_{k}\right\|_{2}=\alpha_{k}$ is

$$
\left(\prod_{k=1}^{d} \alpha_{k}\right) \boldsymbol{\lambda}^{T}\left(\odot_{k=1}^{d}\left(\mathbf{A}_{k} \mathbf{A}_{k}^{*}\right)\right) \boldsymbol{\lambda}
$$

The immediate consequence of this lemma is that the random variable $\eta$ is bounded and $|\eta| \leq \eta_{\max }$ where $\eta_{\max }$ is given by (6). Moreover, take a set of matrices $\boldsymbol{\psi}_{k}$ such that $\left\|\boldsymbol{\psi}_{k}\right\|_{2}=\alpha_{k} \in[0,1]$; then by Lemma $6,\left|\eta\left(\boldsymbol{\psi}_{1}, \ldots, \boldsymbol{\psi}_{d}\right)\right| \leq\left(\prod_{k} \alpha_{k}\right) \eta_{\max }$ hence the optimization (7) is to be carried out only on the set of matrice $\psi_{k}$ such that $\prod_{k} \alpha_{k} \geq \frac{|x|}{\eta_{\max }}$. On the other hand, one may use the generous bound $\log \operatorname{det}\left(\mathbf{I}_{r}-\boldsymbol{\psi}_{k}^{*} \boldsymbol{\psi}_{k}\right) \leq \log \left(1-\left\|\boldsymbol{\psi}_{k}\right\|_{2}^{2}\right)$ and finally prove that

$$
-I_{\eta}(x) \leq \max _{\Pi_{k} \alpha_{k} \geq \frac{|x|}{\eta_{\max }}} \sum_{k=1}^{d} \log \left(1-\alpha_{k}^{2}\right)
$$

The supremum of the r.h.s. of this equation is achieved for balanced $\alpha_{k}$ and we immediately obtain (8). This completes the proof of Proposition 5.

Using Proposition 5 and setting $u=\left(\frac{|x|}{\eta_{\max }}\right)^{1 / d}$, we obtain immediately that for each $\delta>0$ :

$$
\frac{1}{n} \log E_{1}<\sup _{\tilde{\epsilon} \leq u \leq 1}\left[2 u^{d}\left(\eta_{\max }+\frac{d}{2} \frac{1}{u^{d}} \log \left(1-u^{2}\right)\right)\right]+\delta
$$

for $n$ large enough, where $\tilde{\epsilon}=\left(\epsilon / \eta_{\max }\right)^{1 / d}$. Recalling (4) and choosing $\delta$ small enough, we deduce that the condition $\eta_{\max }<$ $\frac{d}{2}\left(\beta_{d}^{2 \text { nd }}\right)^{2}$ implies that $E_{1} \rightarrow 0$. This holds for any order $d \geq 2$.

The $E_{2}$ term. The Varadhan lemma may be invoked: but its conclusion, namely $\frac{1}{n} \log E_{2} \rightarrow 0$, says nothing on the boundedness of $E_{2}$. We have, however

$$
\begin{aligned}
E_{2}= & \int_{0}^{\infty} \mathbb{P}(\exp (2 n \eta) \geq t \text { and } \eta \leq \epsilon) \mathrm{d} t \\
= & \int_{-\infty}^{0} \mathbb{P}(\eta \geq u \text { and } \eta \leq \epsilon) 2 n \exp (2 n u) \mathrm{d} u+ \\
& \int_{0}^{\epsilon} \mathbb{P}(\eta \geq u \text { and } \eta \leq \epsilon) 2 n \exp (2 n u) \mathrm{d} u \\
\leq & \mathbb{P}(\eta \leq \epsilon)+\int_{0}^{\epsilon} \mathbb{P}(\eta \geq u) 2 n \exp (2 n u) \mathrm{d} u
\end{aligned}
$$

A weak consequence of the LDP on $\eta$ is the concentration of $\eta$ around 0 , namely $\mathbb{P}(\eta \leq \epsilon)=1-\mathbb{P}(\eta>\epsilon)=1-o(1)$. We recall the expanded expression for $\eta$ : see (2). Notice that $\eta \geq u$ implies that at least one of the $r^{2}$ terms of this expansion is at least equal to $\frac{u}{r^{2}}$. By the union bound, and the fact that $\Re \prod_{k=1}^{d} \xi_{k}^{(i, j)} \leq \prod_{k=1}^{d}\left|\xi_{k}^{(i, j)}\right|$ we deduce that $\mathbb{P}(\eta \geq u) \leq$ $\sum_{i, j=1}^{r} \mathbb{P}\left(\prod_{k=1}^{d}\left|\xi_{k}^{(i, j)}\right| \geq \frac{u}{r^{2} \lambda_{i} \lambda_{j}}\right)$. Invoking again the union bound and noticing that for fixed $i, j,\left(\xi_{k}^{(i, j)}\right)_{k=1, \ldots, d}$ have the same distribution, we deduce that

$$
\mathbb{P}(\eta \geq u) \leq d \sum_{i, j=1}^{r} \mathbb{P}\left(\left|\xi_{k}^{(i, j)}\right| \geq\left(\frac{u}{r^{2} \lambda_{i} \lambda_{j}}\right)^{1 / d}\right)
$$

Now, the density of $\xi_{k}^{(i, j)}$ is in polar coordinates $\frac{n-1}{\pi}\left(1-r^{2}\right)^{n-2}$ hence, choosing $\epsilon$ such that $\epsilon \leq r^{2} \max _{i, j} \lambda_{i} \lambda_{j}$ :

$$
\mathbb{P}\left(\left|\xi_{k}^{(i, j)}\right| \geq\left(\frac{u}{r^{2} \lambda_{i} \lambda_{j}}\right)^{1 / d}\right)=\left(1-\left(\frac{u}{r^{2} \lambda_{i} \lambda_{j}}\right)^{2 / d}\right)^{n-1}
$$

For any $0 \leq x<1, \log (1-x) \leq-x$, hence

$$
E_{2} \leq d \sum_{i, j} 2 n \int_{0}^{\epsilon} \exp \left(-(n-1)\left(\frac{u}{r^{2} \lambda_{i} \lambda_{j}}\right)^{2 / d}+2 n u\right) d u
$$

When $d>2$, it is always possible to determine $\epsilon$ sufficiently small such that $-(n-1)\left(\frac{u}{r^{2} \lambda_{i} \lambda_{j}}\right)^{2 / d}+2 n u \leq-\frac{n-1}{2}\left(\frac{u}{r^{2} \lambda_{i} \lambda_{j}}\right)^{2 / d}$. This implies that, for such an $\epsilon$, we have

$$
E_{2} \leq d^{2} r^{2} n\left(\frac{2}{n-1}\right)^{d / 2} \sum_{i, j} \lambda_{i} \lambda_{j} \int_{0}^{\infty} v^{d / 2-1} \exp (-v) d v
$$

The r.h.s. is of course $o(1)$ since $d>2$.
Remark 7. The bound $\sqrt{\eta_{\max }}<\sqrt{\frac{d}{2}} \beta_{d}^{2 \text { nd }}$ guarantees the nondetectability but it is not tight in general because, in order to study the asymptotics of $E_{1}$, we replaced the true GRF $I_{\eta}$ by the lower bound (8). Based on the loose inequality $\log \operatorname{det}\left(\mathbf{I}_{r}-\boldsymbol{\psi}_{k}^{*} \boldsymbol{\psi}_{k}\right) \leq$ $\log \operatorname{det}\left(1-\left\|\boldsymbol{\psi}_{k}\right\|_{2}^{2}\right)$, (8) may not be very accurate. It is easy to check that the equality is reached in (8) when all the matrices $\left(\chi_{0}^{(k)}\right)_{k=1, \ldots, d}$ are rank 1 , i.e. if the rank of $\mathbf{X}_{0}$ is equal to 1 . Therefore, the lower bound (8) of $I_{\eta}$ is all the better as all the matrices $\chi_{0}^{(k)}$ are close to being rank 1 matrices. This suggests that, conversely, the bound (8) is likely to be loose when matrices $\left(\chi_{0}^{(k)}\right)_{k=1, \ldots, d}$ are close to be orthogonal. As an illustration, we would like to consider experimental results. For a given configuration of the spike, we have chosen at random the matrices $\psi_{k}$
with $\left\|\boldsymbol{\psi}_{k}\right\| \leq 1$. For each trial, we plot the points of coordinates $x=\eta\left(\boldsymbol{\psi}_{1}, \ldots, \boldsymbol{\psi}_{d}\right)$ and $y=\sum_{k=1}^{d} \log \operatorname{det}\left(\mathbf{I}_{r}-\boldsymbol{\psi}_{k}^{*} \boldsymbol{\psi}_{k}\right)$ and we obtain a cloud the upper envelope of which is a representation of the true GRF of $\eta$; for comparison, we have plotted the graph of the function defined by the lower bound (8). We have chosen $r=2$, $d=3$, and two configurations of the spike: in the first one, all the matrices $\chi_{k}$ have orthogonal columns (top graph of 1), in the second one, the eigenvalues of $\boldsymbol{\chi}_{k}^{*} \boldsymbol{\chi}_{k}$ are the same for $k=1,2$ equal to 1.8 and 0.2 (bottom graph of 1 ).


Fig. 1. $-I_{\eta}$ (seen as the upper envelope of the point constellation) and our upper bound

Remark 8. In the specific case $d=2$, it is possible to compute in closed-form the exact GRF $I_{\eta}$ of $\eta$, and to establish the following result: if $\mu_{\max }\left(\mathbf{X}_{0} \mathbf{X}_{0}^{*}\right)<\beta_{2}^{\text {nd }}=1$ (here, $\mu_{\max }$ denotes the largest eigenvalue), then $E_{1}$ converges towards 0 and $E_{2}$ is bounded: see [5].

## 6. CONCLUSION

In this paper, we have addressed the detection problem of a rank $r$ high-dimensional tensor $\mathbf{X}_{0}$. We have generalized the results of [10] to the case where $r>1$, and established that if the parameter $\eta_{\max }$ defined by (6) is less that parameter $\beta_{2}^{2 \text { nd }}$ introduced in [10], the low rank tensor is undetectable. This condition is based on the lower bound (8) of the GRF $I_{\eta}$ which is however not tight in general. It is thus relevant to try to improve this bound in a future work.

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