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# ZEROS SETS OF $H^p$ FUNCTIONS IN LINEALLY CONVEX DOMAINS OF FINITE TYPE IN $\mathbb{C}^n$

P. CHARPENTIER & Y. DUPAIN

ABSTRACT. In this note we extend N. Th. Varopoulos result on zero sets of  $H^p$  functions of strictly pseudo-convex domains in  $\mathbb{C}^n$  to lineally convex domains of finite type.

## 1. INTRODUCTION

The study of the zero-sets of holomorphic functions in a given class of a smoothly bounded domain in  $\mathbb{C}^n$  is a very classical problem which has been intensively studied. When  $n = 1$  those sets are characterized, by the Blaschke condition, for the Nevanlinna and Hardy classes, but when  $n \geq 2$ , the situation is more complicated.

Such characterizations are only known for the Nevanlinna under additional hypothesis on the domain: G. M. Henkin ([Hen75]) and H. Skoda ([Sko76]) independently obtained the case of strictly pseudo-convex domains, D. C. Chang A. Nagel and E. Stein ([CNS92]) proved the same result for pseudo-convex domains of finite type in  $\mathbb{C}^2$ , and much later the case of convex domains of finite type ([BCD98, Cum01, DM01]), and, recently, the case of lineally convex domains of finite type were obtained ([CDM14]).

In [Var80] (see also [AC90]) N. Th. Varopoulos proved that, in a strictly pseudo-convex domain, a divisor satisfying a special Carleson condition is always defined by a function in some Hardy space  $H^p(\Omega)$ . Tentatives to generalize this result, for example to convex domains of finite type, were done in [BG99] and [Ngu01], but some gaps in the proofs leave the problem open until a recent paper of W. Alexandre ([Ale17]). In this last paper, the author make a strong use of the estimates of the Bergman metric obtained by Mc. Neal.

In this note, we show that Varopoulos result extends to lineally convex domains of finite type with a classical method using only the anisotropic geometry, described in [Con02], of those domains.

## 2. MAIN RESULTS

Let us first recall the definition of a lineally convex domain:

**Definition 2.1.** A domain  $\Omega$  in  $\mathbb{C}^n$ , with smooth boundary is said to be lineally convex at a point  $p \in \partial\Omega$  if there exists a neighborhood  $W_p$  of  $p$  such that, for all point  $z \in \partial\Omega \cap W_p$ ,

$$(z + T_z^{10}) \cap (\Omega \cap W_p) = \emptyset,$$

where  $T_z^{10}$  is the holomorphic tangent space to  $\partial\Omega$  at the point  $z$ .

In all the paper, we assume that  $\partial\Omega$  is of finite type and lineally convex at every point of  $\partial\Omega$ . We may assume that there exists a  $\mathcal{C}^\infty$  defining function  $\rho$  for  $\Omega$  and a number  $\eta_0 > 0$  such that  $\nabla\rho(z) \neq 0$  at every point of  $W = \{-\eta_0 \leq \rho(z) \leq \eta_0\}$  and the level sets

$$\{z \in W \text{ such that } \rho(z) = \eta\},$$

are lineally convex of finite type.

We thus assume that the defining function  $\rho$  satisfies this hypothesis.

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In the next section, for  $z \in \Omega$  in a fixed small neighborhood  $V$  of  $\partial\Omega$ , we recall the definition of the two fundamental quantities  $\tau(z, u, \delta)$ , for  $0 < \delta \leq \delta_0$ ,  $\delta_0 > 0$  depending only on  $\Omega$ , and  $u$  a non zero complex vector and  $k(z, u) = \frac{\delta_\Omega(z)}{\tau(z, u, \delta_\Omega(z))}$  where  $\delta_\Omega(z)$  denotes the distance of  $z$  to the boundary of  $\Omega$ . The lineal convexity hypothesis implies that  $(z, u) \mapsto k(z, u)$  is a continuous function in  $V \times \mathbb{C}_*^n$  and, for  $0 < \delta_1 < \delta_0$  and  $K < +\infty$ , there exists constants  $c > 0$  and  $C < +\infty$  such that for  $z \in V \cap \{\delta_\Omega(\zeta) \geq \delta_1\}$  and  $\frac{1}{K} \leq |u| \leq K$ ,  $c \leq k(z, u) \leq C$ . So, if  $u(z)$  is a continuous vector field in  $\Omega$ ,  $\frac{1}{K} \leq |u| \leq K$ ,  $k(z, u(z))$  can be extended to  $\Omega$  in a continuous function satisfying  $c \leq k(z, u) \leq C$  in  $\Omega \cap \{\delta_\Omega(\zeta) \geq \delta_1\}$ .

To state our main result we have to recall the notion of Carleson measure in our context (see [CD]): a bounded measure  $\mu$  in  $\Omega$  is called a Carleson measure if

$$\|\mu\|_{W^1(\Omega)} := \sup_{z \in \partial\Omega, 0 < \varepsilon < \varepsilon_0} \frac{|\mu|(P_\varepsilon(z) \cap \Omega)}{\sigma(P_\varepsilon(z) \cap \partial\Omega)} + |\mu|(\Omega) < +\infty,$$

$\varepsilon_0 = \alpha\delta_0$ , for  $\alpha$  small enough, where  $P_\varepsilon(z)$  is the extremal polydisk defined in the next section and  $\sigma$  the surface measure on  $\partial\Omega$ .  $W^1(\Omega)$  will denote the space of Carleson measures on  $\Omega$ . Then, following ideas initiated in [BCD98] and adapted by W. Alexandre ([Ale17, Definition 1.2]) to non smooth forms we consider the following terminology (see section 4 for more details):

A current  $\vartheta = \sum_{i,j=1}^n \vartheta_{ij} dz_i \wedge d\bar{z}_j$  of degree  $(1, 1)$  and order zero in  $\Omega$  is called a Carleson current (in  $\Omega$ ) if

$$\|\vartheta\|_{W^1(\Omega)} := \sup_{u_1, u_2} \left\| \frac{\delta_\Omega |\vartheta(u_1, u_2)|}{k(\cdot, u_1) k(\cdot, u_2)} \right\|_{W^1(\Omega)} + (\delta_\Omega |\vartheta|)(\Omega) < +\infty,$$

where supremum is taken over all smooth vector fields  $u_1 = (u_1^i)_i$  and  $u_2 = (u_2^i)_i$  never vanishing in  $\Omega$ ,  $|\vartheta(u_1, u_2)|$  is the absolute value of the measure  $\vartheta(u_1, u_2) = \sum_{i,j} \vartheta_{ij} u_1^i \bar{u}_2^j$  and  $k(\cdot, u_k)$  the continuous function  $z \mapsto k(z, u_k)$  defined before.

Similarly, a current  $\omega$  of degree 1 and order zero in  $\Omega$  is called a Carleson current (in  $\Omega$ ) if

$$\|\omega\|_{W^1(\Omega)} := \sup_u \left\| \frac{|\omega(u)|}{k(\cdot, u)} \right\|_{W^1(\Omega)} + |\omega|(\Omega) < +\infty.$$

*Remark.* In the above definitions, the expressions are independent of the modulus of the vector fields. Thus they can always be chosen of modulus one.

**Main Theorem.** *Let  $\Omega$  be a smoothly bounded lineally convex domain of finite type in  $\mathbb{C}^n$ . Let  $X$  be a divisor in  $\Omega$  and  $\vartheta_X$  the associated  $(1, 1)$ -current of integration. Then, if  $\vartheta_X$  is a Carleson current and if the cohomology class of  $X$  in  $H^2(\Omega, \mathbb{Z})$  is zero, there exist  $p > 0$  and  $f$  in the Hardy space  $H^p(\Omega)$  such that  $X$  is the zero set of  $f$ .*

The general scheme of the proof is now standard (see [Var80, AC90, BG99, Ngu01, Ale17]): following Lelong's theory, we have to find a plurisubharmonic function  $u$  such that  $i\partial\bar{\partial}u = \vartheta_X$  satisfying a BMO estimate on  $\partial\Omega$ , and, as in [Var80] (and [Ngu01]), the conclusion will follow the John-Nirenberg theorem ([JN61]). The two main steps are the resolution of the equation  $idw = \vartheta$  with a Carleson estimate and the resolution of the  $\bar{\partial}_b$ -equation with a BMO estimate on the boundary of  $\Omega$ :

**Theorem 2.1.** *Let  $\vartheta$  be a closed Carleson current of degree  $(1, 1)$  and order 0 in  $\Omega$  such that his canonical cohomology class in  $H^2(\Omega : \mathbb{C})$  is zero. Then there exists a Carleson current of degree 1 and order 0  $\omega$  satisfying  $d\omega = \vartheta$ . Furthermore, if  $\vartheta$  is real,  $\omega$  can be chosen real.*

*Remark.* This theorem could have been stated in the more general context of geometrically separated domains  $\Omega$  introduced in [CD14]. But, as we cannot prove the Main Theorem in that case, and because the technical details of the proof would be much more complicated, we restrict us to the case of lineally convex domains of finite type.

The second step is based on the proof of [CD, Theorem 2.4]:

**Theorem 2.2.** *There exists a constant  $C > 0$  such that, for all  $\bar{\partial}$ -closed Carleson current of degree  $(0, 1)$  and order 0  $\omega$  on  $\Omega$ , there exists a solution of the equation  $\bar{\partial}_b u = \omega$  such that  $\|u\|_{BMO(\partial\Omega)} \leq C \|\omega\|_{W^1(\Omega)}$ .*

Note that in [CD] this last result is stated, for smooth forms, with  $\|\omega(\zeta)\|_k d\lambda\|_{W^1}$  instead of  $\|\omega\|_{W^1(\Omega)}$ , but we will see in section 4 that, for smooth forms, these two quantities are equivalent.

Theorem 2.1 is proved in section 5: after a regularization procedure we will essentially follow the general scheme developed in [AC90] (see also [Sko76] and [Var80]), the technical part being a strong modification of the calculus made in [BG99].

Theorem 2.2 is proved in section 6: once again, after a convenient regularization we use the methods developed in [CD] and in [Sko76].

### 3. GEOMETRY OF LINEALLY CONVEX DOMAINS OF FINITE TYPE

The anisotropic geometry of lineally convex domains of finite type is described in [Con02]. Let us just recall the basic estimates (from [CD]) we will use in the next section.

For  $\zeta$  close to  $\partial\Omega$  and  $\varepsilon \leq \varepsilon_0$ ,  $\varepsilon_0$  small, define, for all non zero vector  $v$ ,

$$(3.1) \quad \tau(\zeta, v, \varepsilon) = \sup \{c \text{ such that } \rho(\zeta + \lambda v) - \rho(\zeta) < \varepsilon, \forall \lambda \in \mathbb{C}, |\lambda| < c\}.$$

Note that the lineal convexity hypothesis implies that the function  $(\zeta, \varepsilon) \mapsto \tau(\zeta, v, \varepsilon)$  is smooth. In particular,  $\zeta \mapsto \tau(\zeta, v, \delta_\Omega(\zeta))$  is a smooth function. The pseudo-balls  $\mathcal{B}_\varepsilon(\zeta) = \mathcal{B}(\zeta, \varepsilon)$  (for  $\zeta$  close to the boundary of  $\Omega$ ) of the homogeneous space associated to the anisotropic geometry of  $\Omega$  are

$$(3.2) \quad \mathcal{B}_\varepsilon(\zeta) = \{\xi = \zeta + \lambda u \text{ with } |u| = 1 \text{ and } |\lambda| < c_0 \tau(\zeta, u, \varepsilon)\}$$

where  $c_0$  is chosen sufficiently small depending only on the defining function  $\rho$  of  $\Omega$  and we define

$$d(\zeta, z) = \inf \{\varepsilon \text{ such that } z \in \mathcal{B}_\varepsilon(\zeta)\}.$$

Let  $\zeta$  and  $\varepsilon$  be fixed. Then, an orthonormal basis  $(v_1, v_2, \dots, v_n)$  is called  $(\zeta, \varepsilon)$ -*extremal* (or  $\varepsilon$ -*extremal*, or simply *extremal*) if  $v_1$  is the complex normal (to  $\rho$ ) at  $\zeta$ , and, for  $i > 1$ ,  $v_i$  belongs to the orthogonal space of the vector space generated by  $(v_1, \dots, v_{i-1})$  and minimizes  $\tau(\zeta, v, \varepsilon)$  in the unit sphere of that space. In association to an extremal basis, we denote

$$(3.3) \quad \tau(\zeta, v_i, \varepsilon) = \tau_i(\zeta, \varepsilon).$$

Then we defined polydiscs  $AP_\varepsilon(\zeta)$  by

$$(3.4) \quad AP_\varepsilon(\zeta) = \left\{ z = \zeta + \sum_{k=1}^n \lambda_k v_k \text{ such that } |\lambda_k| \leq c_0 A \tau_k(\zeta, \varepsilon) \right\}.$$

$P_\varepsilon(\zeta)$  being the corresponding polydisc with  $A = 1$  and we also define

$$d_1(\zeta, z) = \inf \{\varepsilon \text{ such that } z \in P_\varepsilon(\zeta)\}.$$

*Remark.* Note that there are neither uniqueness of the extremal basis  $(v_1, v_2, \dots, v_n)$  nor of associated polydisk  $P_\varepsilon(\zeta)$ . However the functions  $\tau_i$  and the polydiscs associated to two different  $(\zeta, \varepsilon)$ -extremal basis are equivalent. Thus in all the paper  $P_\varepsilon(\zeta) = P(\zeta, \varepsilon)$  will denote a polydisk associated to any  $(\zeta, \varepsilon)$ -extremal basis and  $\tau_i(\zeta, \varepsilon)$  the radius of  $P_\varepsilon(\zeta)$ .

The fundamental result here is that  $d$  and  $d_1$  are equivalent pseudo-distance which means that there exists a constant  $K$  and,  $\forall \alpha > 0$ , constants  $c(\alpha)$  and  $C(\alpha)$  such that

$$(3.5) \quad \text{for } \zeta \in P_\varepsilon(z), P_\varepsilon(z) \subset P_{K\varepsilon}(\zeta),$$

and

$$(3.6) \quad c(\alpha)P_\varepsilon(\zeta) \subset P_{\alpha\varepsilon}(\zeta) \subset C(\alpha)P_\varepsilon(\zeta) \text{ and } P_{c(\alpha)\varepsilon}(\zeta) \subset \alpha P_\varepsilon(\zeta) \subset P_{C(\alpha)\varepsilon}(\zeta).$$

Moreover the pseudo-balls  $B_\varepsilon$  and the polydisks  $P_\varepsilon$  are equivalent in the sense that there exists a constant  $K > 0$  depending only on  $\Omega$  such that

$$(3.7) \quad \frac{1}{K}P_\varepsilon(\zeta) \subset \mathcal{B}_\varepsilon(\zeta) \subset KP_\varepsilon(\zeta),$$

so

$$d(\zeta, z) \simeq d_1(\zeta, z).$$

Let us recall for  $\zeta$  close to  $\partial\Omega$  and  $\varepsilon > 0$  small, other basic properties of this geometry (see [Con02] and [CDM14]):

**Lemma 3.1.** (1) *Let  $w = (w_1, \dots, w_n)$  be an orthonormal system of coordinates centered at  $\zeta$ . Then*

$$\left| \frac{\partial^{|\alpha+\beta|} \rho(\zeta)}{\partial w^\alpha \partial \bar{w}^\beta} \right| \lesssim \frac{\varepsilon}{\prod_i \tau(\zeta, w_i, \varepsilon)^{\alpha_i + \beta_i}}, \quad |\alpha + \beta| \geq 1.$$

(2) *If  $(v_1, \dots, v_n)$  is a  $(\zeta, \varepsilon)$ -extremal basis and  $\gamma = \sum_1^n a_j v_j \neq 0$ , then*

$$\frac{1}{\tau(\zeta, \gamma, \varepsilon)} \simeq \sum_{j=1}^n \frac{|a_j|}{\tau_j(\zeta, \varepsilon)}.$$

(3) *If  $v$  is a unit vector then:*

(a)  $z = \zeta + \lambda v \in P_\varepsilon(\zeta)$  implies  $|\lambda| \lesssim \tau(\zeta, v, \varepsilon)$ ,

(b)  $z = \zeta + \lambda v$  with  $|\lambda| \leq \tau(\zeta, v, \varepsilon)$  implies  $z \in CP_\varepsilon(\zeta)$ .

(4) *If  $v$  is the unit complex normal vector, then  $\tau(\zeta, v, \varepsilon) = \varepsilon$  and if  $v$  is any unit vector and  $\lambda \geq 1$ ,*

$$(3.8) \quad \lambda^{1/m} \tau(\zeta, v, \varepsilon) \lesssim \tau(\zeta, v, \lambda\varepsilon) \lesssim \lambda \tau(\zeta, v, \varepsilon),$$

where  $m$  is the type of  $\Omega$ .

**Lemma 3.2** (Lemma 3.4 of [CD]). *For  $z$  close to  $\partial\Omega$ ,  $\varepsilon$  small and  $\zeta \in P_\varepsilon(z)$  or  $z \in P_\varepsilon(\zeta)$ , we have, for all  $1 \leq i \leq n$ :*

(1)  $\tau_i(z, \varepsilon) = \tau(z, v_i(z, \varepsilon), \varepsilon) \simeq \tau(\zeta, v_i(z, \varepsilon), \varepsilon)$  where  $(v_i(z, \varepsilon))_i$  is the  $(z, \varepsilon)$ -extremal basis;

(2)  $\tau_i(\zeta, \varepsilon) \simeq \tau_i(z, \varepsilon)$ ;

(3) *In the coordinate system  $(z_i)$  associated to the  $(z, \varepsilon)$ -extremal basis,  $\left| \frac{\partial \rho}{\partial z_i}(\zeta) \right| \lesssim \frac{\varepsilon}{\tau_i}$  where  $\tau_i$  is either  $\tau_i(z, \varepsilon)$  or  $\tau_i(\zeta, \varepsilon)$ .*

**Remark 3.1.** Clearly, for  $\delta_\Omega(z) \leq \delta_1$  and all non zero vector  $v$ , we can extend smoothly the functions  $\tau(z, v, \varepsilon)$  to all  $\varepsilon$  and we can also define vectors  $e_i(z, \varepsilon)$  and polydisks  $P(z, \varepsilon)$ , so that the above properties remain true with constants depending on  $A$  for  $\varepsilon$  and  $\lambda\varepsilon \in ]0, A]$ ,  $\delta_\Omega(z)$  and  $\delta_\Omega(\zeta) \leq \delta_1$ .

Of course the new  $(e_i(z, \varepsilon))_i$  are not extremal basis in the original sense but we will call them again extremal basis.

## 4. SOME PROPERTIES OF CARLESON CURRENTS

In the previous section, we defined the terminology of Carleson current of degree  $(1, 1)$  or 1. We extend it to general currents  $T$  of degree 2 with the same definition:

Let  $T = \sum_{i < j} T_{i,j}^0 dz^i \wedge dz^j + \sum T_{i,j}^1 dz^i \wedge d\bar{z}^j + \sum_{i < j} T_{i,j}^2 d\bar{z}^i \wedge d\bar{z}^j$  then

$$\|T\|_{W^1(\Omega)} = \sup_{u_1, u_2} \left\| \frac{\delta_\Omega |T(u_1, u_2)|}{k(\cdot, u_1)k(\cdot, u_2)} \right\|_{W^1(\Omega)} + (\delta_\Omega |T|)(\Omega) < +\infty,$$

where supremum is taken over all smooth vector fields  $u_1 = (u_1^i)_i$  and  $u_2 = (u_2^i)_i$  never vanishing in  $\Omega$ , and  $|T(u_1, u_2)|$  is the absolute value of the measure

$$T(u_1, u_2) = \sum_{i < j} T_{i,j}^0 u_1^i u_2^j + \sum T_{i,j}^1 u_1^i \bar{u}_2^j + \sum_{i < j} T_{i,j}^2 \bar{u}_1^i \bar{u}_2^j.$$

Moreover, let  $V$  be an open set in  $\Omega$  and  $T$  a current of degree 1 or 2 and order zero in  $V$ . We say that  $T$  is a Carleson current in  $\Omega$  if the current  $\chi_V T$ , where  $\chi_V$  is the characteristic function of  $V$ , is a Carleson current in  $\Omega$  and we denote  $\|T\|_{W^1(\Omega)} := \|\chi_V T\|_{W^1(\Omega)}$ .

Note that, if  $V$  is relatively compact in  $\Omega$ , a current  $T$  in  $V$  is a Carleson current (in  $\Omega$ ) if the coefficients of  $T$  are bounded measures.

In the two next sections we need to regularize Carleson currents to be able to write explicit formulas solving the  $d$  or the  $\bar{d}$  equation. This is done classically using convolutions (see M. Andersson and H. Carlsson, [AC90, page 472] in the case of strictly pseudo-convex domains), and, because of the definition of the  $W^1(\Omega)$  norm for currents, we give below some details (for currents of degree  $(1, 1)$  to simplify notations) when  $V$  is contained in a small neighborhood of a point of  $\partial\Omega$  ( $V \subset \{\delta_\Omega(z) < \beta\delta_1\}$ ).

For  $\varepsilon > 0$  sufficiently small, let  $\varphi_\varepsilon = \frac{1}{\varepsilon^{2n}} \varphi\left(\frac{\cdot}{\varepsilon}\right)$  where  $\varphi$  is a  $\mathcal{C}^\infty$ -smooth non negative function supported in the ball  $\{|z| < 1/2\}$  of  $\mathbb{C}^n$  such that  $\int \varphi = 1$ . Let  $T = \sum T_{I,J} dz^I \wedge d\bar{z}^J$  be a Carleson current of order zero in an open set  $V$  of  $\Omega$ . Let

$$V_\varepsilon = \{z \in V \text{ such that } \delta_{\partial V}(z) > \varepsilon\}.$$

Then for  $z \in V_\varepsilon$  define

$$T_\varepsilon = \sum_{I,J} T_{I,J} * \varphi_\varepsilon dz^I \wedge d\bar{z}^J$$

so that  $T_\varepsilon$  is a smooth form in  $V_\varepsilon$ .

**Proposition 4.1.** *With the above notations, if  $T$  is a closed Carleson current of degree 2 or 1 in  $V$ , then the forms  $T_\varepsilon$  are closed and  $\|T_\varepsilon\|_{W^1(\Omega)} \lesssim \|T\|_{W^1(\Omega)}$ .*

*Proof.* To simplify the notations, we make the proof for  $T$  of degree  $(1, 1)$ , that is  $T = \sum_{i,j} T_{i,j} dz^i \wedge d\bar{z}^j$  and  $T_\varepsilon = \sum_{i,j} T_{i,j}^\varepsilon dz^i \wedge d\bar{z}^j$  with  $T_{i,j}^\varepsilon(z) = \int_{\{|z-\zeta| < 1/2\}} \varphi_\varepsilon(\zeta - z) dT_{i,j}(\zeta)$ .

Let  $Z \in \partial\Omega$ . If  $t < c\varepsilon$  ( $c$  small enough depending only on  $\Omega$ ) then  $B(Z, t) \cap V_\varepsilon = \emptyset$ . Let us assume  $c\varepsilon < t \leq \varepsilon_0$ . We have to estimate

$$\begin{aligned} I &= \int \chi(z) \frac{\delta_\Omega(z)}{k(z, u)k(z, v)} \left| \sum_{i,j} T_{i,j}^\varepsilon(z) u_i(z) \bar{v}_j(z) \right| d\lambda(z) \\ &= \sup_{|f| \leq 1} \int f(z) \chi(z) \frac{\delta_\Omega(z)}{k(z, u(z))k(z, v(z))} \left( \sum_{i,j} T_{i,j}^\varepsilon(z) u_i(z) \bar{v}_j(z) \right) d\lambda(z) \end{aligned}$$

where  $\chi$  is the characteristic function of  $B(Z, t) \cap V_\varepsilon$  and  $u$  and  $v$  are smooth vector fields never vanishing in  $\Omega$ . Using the definition of  $T_\varepsilon$  we get

$$I = \sup_{|f| \leq 1} \int \sum_{i,j} \left( \int f(z) \chi(z) \frac{\delta_\Omega(z) u_i(z) \bar{v}_j(z)}{k(z, u(z))k(z, v(z))} \varphi_\varepsilon(\zeta - z) d\lambda(z) \right) dT_{i,j}(\zeta).$$

Note now that the function

$$z \mapsto f(z) \chi(z) \frac{\delta_\Omega(z) u_i(z) \bar{v}_j(z)}{k(z, u(z)) k(z, v(z))} \varphi_\varepsilon(\zeta - z)$$

is supported in  $B(Z, t) \cap \{\delta_\Omega > \varepsilon/2\}$ . Moreover, for  $z \in V_\varepsilon$  and  $|\zeta - z| < \varepsilon/2$ ,  $\delta_\Omega(z) \simeq \delta_\Omega(\zeta)$ , and,  $\zeta \in P(z, K_1 \delta_\Omega(z))$ . Then by (1) of Lemma 3.2 and (4) of Lemma 3.1,  $k(z, u(z)) \simeq k(\zeta, u(z))$  and  $k(z, v(z)) \simeq k(\zeta, v(z))$ , so

$$I \lesssim \sup_{|f| \leq 1} \int \sum_{i,j} \left( \int f(z) \chi(z) \frac{\delta_\Omega(\zeta) u_i(z) \bar{v}_j(z)}{k(\zeta, u(z)) k(\zeta, v(z))} \varphi_\varepsilon(\zeta - z) d\lambda(z) \right) dT_{i,j}(\zeta).$$

Making the change of variables  $\zeta - z = \xi$  and applying Fubini theorem, we get ( $\chi(\zeta - \xi) \neq 0$  implies  $\zeta \in P(Z, Kt)$ )

$$\begin{aligned} I &\lesssim \int \varphi_\varepsilon(\xi) \left[ \sup_{|f| \leq 1} \int f(\zeta - \xi) \chi(\zeta - \xi) \frac{\delta_\Omega(\zeta)}{k(\zeta, u(\zeta - \xi)) k(\zeta, v(\zeta - \xi))} \right. \\ &\quad \left. \sum_{i,j} u_i(\zeta - \xi) \bar{v}_j(\zeta - \xi) dT_{i,j}(\zeta) \right] d\lambda(\xi) \\ &= \int \varphi_\varepsilon(\xi) \left[ \int_{B(Z, Kt)} \chi(\zeta - \xi) \frac{\delta_\Omega(\zeta)}{k(\zeta, u(\zeta - \xi)) k(\zeta, v(\zeta - \xi))} \right. \\ &\quad \left. d \left| \sum_{i,j} \tau_\xi(u_i) \tau_\xi(\bar{v}_j) T_{i,j} \right|(\zeta) \right] d\lambda(\xi). \end{aligned}$$

where  $\tau_\xi(u_i)(\zeta) = u_i(\zeta - \xi)$ ,  $\tau_\xi(v_i)(\zeta) = v_i(\zeta - \xi)$ .

Finally, (3.6) gives, denoting  $u_\xi(\zeta) = u(\zeta - \xi)$  and  $v_\xi(\zeta) = v(\zeta - \xi)$

$$I \lesssim \sup_{|\xi| < \varepsilon/2} \int_{B(Z, Kt) \cap \{\delta_\Omega > \varepsilon/2\}} \frac{\delta_\Omega(\zeta)}{k(\zeta, u_\xi(\zeta)) k(\zeta, v_\xi(\zeta))} d|T(u_\xi, v_\xi)|$$

which concludes the proof, as the smooth vector fields  $u_\xi$  and  $v_\xi$  can be viewed as smooth vector fields in  $\Omega$  never vanishing, and, for  $\beta$  small enough,  $V$  is contained in the union of the tents.  $\square$

If  $T$  is globally defined in  $\Omega$ , then the forms  $T_\varepsilon$  are defined in  $\{z \in \Omega \text{ such that } \delta_\Omega(z) > \varepsilon\}$  so there exists a constant  $C$  (depending only on  $\rho$ ) such that they are defined in  $\Omega^\varepsilon = \{\rho < -C\varepsilon\}$ ,  $\varepsilon$  small enough, but *they are not* Carleson currents in  $\Omega^\varepsilon$  in general. Then to be able to use this regularization procedure in the last section we have to introduce a notion of  $s$ -Carleson current.

Let  $s > 0$  small. We say that a measure  $\mu$  in  $\Omega$  is a  $s$ -Carleson measure if

$$\|\mu\|_{W_s^1(\Omega)} := \sup_{z \in \partial\Omega, s < \varepsilon < \varepsilon_0} \frac{|\mu|(P_\varepsilon(z) \cap \Omega)}{\sigma(P_\varepsilon(z) \cap \partial\Omega)} + |\mu|(\Omega) < +\infty,$$

and we say that a 1-current  $\omega$  of order zero is a  $s$ -Carleson current in  $\Omega$  if

$$\|\omega\|_{W_s^1(\Omega)} := \sup_u \left\| \frac{|\omega(u)|}{k(\cdot, u)} \right\|_{W_s^1(\Omega)} + |\omega|(\Omega) < +\infty.$$

Then:

**Proposition 4.2.** *There exists a constant  $C$  depending only on  $\rho$  such that, if  $T$  is a closed Carleson current in  $\Omega$  then, for  $\varepsilon$  small, the closed forms  $T_\varepsilon$  are  $\varepsilon$ -Carleson currents in  $\Omega^\varepsilon = \{\rho < -C\varepsilon\}$  and  $\|T_\varepsilon\|_{W_\varepsilon^1(\Omega^\varepsilon)} \lesssim \|T\|_{W^1(\Omega)}$ .*

*Proof.* By Proposition 4.1 it suffices to show that  $\|T_\varepsilon\|_{W_\varepsilon^1(\Omega^\varepsilon)} \lesssim \|T_\varepsilon\|_{W^1(\Omega)}$ . Let  $z \in \partial\Omega^\varepsilon$  and let  $Z$  be the projection of  $z$  on  $\partial\Omega$ . For  $t > \varepsilon$ ,  $P(z, t) \subset P(Z, Kt)$  and the proposition follows (3.8) and Lemma 3.2.  $\square$

Finally, to solve the  $d$  and  $\bar{d}$  equations with good estimates, we need to compare the notion of Carleson current for smooth currents of degree 2 or 1  $T$  with a convenient punctual norm  $\|T(\zeta)\|_k d\lambda\|_{W^1(\Omega)}$ : we define

$$\|T(\zeta)\|_k = \sup_{v_i \in \mathbb{C}^n, \|v_i\|=1} \frac{|T(v_1, v_2)(\zeta)|}{k(\zeta, v_1)k(\zeta, v_2)},$$

for forms of degree 2 and

$$\|T(\zeta)\|_k = \sup_{\|v\|=1} \frac{|T(v)(\zeta)|}{k(\zeta, v)},$$

for forms of degree (0, 1).

If there were smooth vector fields  $(\tilde{e}_i)_{1 \leq i \leq n}$  such that, at each point  $z$ ,  $(\tilde{e}_i(z))_i$  is a  $(z, \delta_\Omega(z))$ -extremal basis, this comparison would be immediate and, as noted by several authors, many points of the theory of convex (and lineally convex) domains of finite type would be simplified. Unfortunately this is not the case, and, in the case of convex domains of finite type and smooth currents, W. Alexandre overcomes this difficulty using a base of the Bergman metric (and estimates of this metric proved by J. Mc Neal, see [Ale17, Proposition 2.12]). The same result could be proved in our context of lineally convex domains using the results of [CD14]. However, we do not use this method because it is quite easy to show, in general, that the  $W^1(\Omega)$ -norm of a current is controlled by vector fields “almost extremal”:

**Proposition 4.3.** *Let  $\psi$  be a current of order zero of degree 2 or 1 in an open set  $U \subset \{\delta_\Omega(z) < \beta\delta_1\}$  of  $\Omega$ .*

- (1) *There exist  $n$  smooth vector fields  $u_i$  never vanishing in  $\Omega$  such that, if  $\psi$  is of degree 2,*

$$\|\psi\|_{W^1(\Omega)} \simeq \sum_{i,j} \left\| \frac{\delta_\Omega |\psi(u_i, u_j)|}{k(\cdot, u_i)k(\cdot, u_j)} \right\|_{W^1(\Omega)},$$

*the vector fields  $u_i$  coinciding, outside a set of  $|\psi|$ -measure arbitrary small, with extremal basis in the sense of geometrically separated domains, and*

$$\|\psi\|_{W^1(\Omega)} \simeq \sum_i \left\| \frac{|\psi(u_i)|}{k(\cdot, u_i)} \right\|_{W^1(\Omega)}$$

*if it is of degree 1, the constants in the equivalence being independent of  $\psi$ .*

- (2) *Moreover, if  $\psi$  is smooth. Then:*

- (a)  $\|\psi\|_{W^1(\Omega)} \simeq \|\delta_\Omega \|\chi_U \psi\|_k d\lambda\|_{W^1(\Omega)}$  if  $\psi$  is of degree 2, and  $\|\psi\|_{W^1(\Omega)} \simeq \|\|\chi_U \psi\|_k d\lambda\|_{W^1(\Omega)}$  if not, the constants in the equivalence being independent of  $\psi$ ;
- (b) for  $s > 0$ , if  $\psi$  is of degree 1,  $\|\psi\|_{W_s^1(\Omega)} \simeq \|\|\chi_U \psi\|_k d\lambda\|_{W_s^1(\Omega)}$  the constants in the equivalence being independent of  $\psi$ .

*Remark.* (1) In [Ale17, Proposition 2.12] W. Alexandre proved (2) of the proposition for convex domains of finite type, using *universal* vector fields (i.e. depending only on  $\Omega$  but not on  $\psi$ ) related to the Bergman metric.

- (2) The extremal basis in the sense of geometrically separated domains ([CD14]) are not stricto-sensus extremal basis in J. Mc. Neal and M. Conrad sense, but they give the same homogeneous space.



- (3) The equivalences of (2) of the proposition can be proved directly without using any extremal basis, using simply the continuity of the functions  $z \mapsto \frac{\delta_\Omega(z)\psi(u,v)(z)}{k(z,u(z))k(z,v(z))}$  which gives an equivalent of Lemma 1 below on small euclidean balls. The final construction of the vectors fields  $(u_i)_i$  is analog (and easier).
- (4) Even if the proof of Proposition 5.1 needs only the second part of the proposition, we thought that it is interesting to present the assertion in the general case of non smooth currents.

*Proof of Proposition 4.3.* We only do the proof for currents  $\psi$  of degree 2. The inequality  $\gtrsim$  is trivial, so we prove the converse one.

**Lemma 1.** *Let  $w \in U$  and  $(e_i)_i$  be a  $\delta_\Omega(w)$ -extremal basis at  $w$ . Let  $u$  and  $v$  be two smooth non vanishing vectors fields.*

- (1) *Assume the coefficients of  $\psi$  are measures. Then, for every measurable set  $D \subset P(w, \delta_\Omega(w)) \cap U$ , we have*

$$(4.1) \quad \int_D \frac{\delta_\Omega(\xi) d|\psi(u,v)|(\xi)}{k(\xi, u(\xi))k(\xi, v(\xi))} \lesssim \sum_{i,j} \int_D \frac{\delta_\Omega(\xi) d|\psi(e_i, e_j)|(\xi)}{k(\xi, e_i)k(\xi, e_j)}.$$

- (2) *Moreover, if  $\psi$  smooth. Then, for  $\xi \in P(w, \delta_\Omega(w)) \cap U$  we have*

$$(4.2) \quad \frac{|\psi(u,v)(\xi)|}{k(\xi, u(\xi))k(\xi, v(\xi))} \lesssim \sum_{i,j} \frac{|\psi(e_i, e_j)(\xi)|}{k(\xi, e_i)k(\xi, e_j)}.$$

*Proof.* Decomposing  $u$  and  $v$  on the basis  $(e_i)_i$ , we get

$$\frac{|\psi(u,v)(\xi)|}{k(\xi, u(\xi))k(\xi, v(\xi))} \lesssim \frac{\sum |\psi(e_i, e_j)(\xi)| |u_i(\xi)| |v_j(\xi)|}{k(\xi, u(\xi))k(\xi, v(\xi))},$$

if  $\psi$  is smooth, and, if not

$$\int_D \frac{\delta_\Omega(\xi) d|\psi(u,v)|(\xi)}{k(\xi, u(\xi))k(\xi, v(\xi))} \lesssim \sum_{i,j} \int_D \frac{\delta_\Omega(\xi) |u_i(\xi)| |v_j(\xi)| d|\psi(e_i, e_j)|(\xi)}{k(\xi, u(\xi))k(\xi, v(\xi))}.$$

Now, by (4) of Lemma 3.1 and (2) of Lemma 3.2,

$$\begin{aligned} k(\xi, u(\xi)) &\simeq \frac{\delta_\Omega(\xi)}{\tau(\xi, u(\xi), \delta_\Omega(w))} \\ &\simeq \frac{\delta_\Omega(\xi)}{\tau(w, u(\xi), \delta_\Omega(w))} \end{aligned}$$

and, by (2) of Lemma 3.1,

$$\frac{1}{\tau(w, u(\xi), \delta_\Omega(w))} \simeq \max \frac{|u_i(\xi)|}{\tau_i(w, \delta_\Omega(w))} \text{ and } \frac{1}{k(\xi, u(\xi))} \lesssim \min \frac{\tau_i(w, \delta_\Omega(w))}{|u_i(\xi)|} \delta_\Omega(\xi)^{-1}.$$

Then

$$\begin{aligned} \frac{|u_i(\xi)| |v_j(\xi)|}{k(\xi, u(\xi))k(\xi, v(\xi))} &\lesssim \tau_i(w, \delta_\Omega(w)) \tau_j(w, \delta_\Omega(w)) \delta_\Omega(\xi)^{-2} \\ &\lesssim \frac{1}{k(\xi, e_i)k(\xi, e_j)} \end{aligned}$$

because  $\tau_i(w, \delta_\Omega(w)) = \tau(w, e_i, \delta_\Omega(w)) \simeq \tau(\xi, e_i, \delta_\Omega(\xi))$  and the lemma is proved.  $\square$

**Lemma 2.** *Under the conditions of the previous lemma, the inequalities (4.2) and (4.1) are still true replacing the basis  $(e_i)_i$  by the basis  $(e'_i)_i$  where  $e'_i = e_i + O(\delta_\Omega^2(w))$ .*

*Proof.* By (2) of Lemma 3.1,  $k(\xi, e'_i) \simeq k(\xi, e_i)$ , and

$$\langle \psi; e'_i, e'_j \rangle = \langle \psi; e_i, e_j \rangle + \sum_{s,t} O(\delta_\Omega^2(w)) \langle \psi; e_s, e_t \rangle$$

which proves the result for  $\beta$  small enough.  $\square$

We now finish the proof of the Proposition 4.3, proving both parts at the same time. Let  $P_i = P\left(Z_i, \frac{\delta_\Omega(Z_i)}{K}\right)$ ,  $i \in \mathbb{N}$ , be a minimal covering of  $U \cap \Omega$ . For each  $i$  fixed and  $N_i$  to be precised later, let

$$A_i^j = \left\{ z \in U \cap \Omega \text{ such that } d_e(z, P_i) < \frac{\delta_\Omega(Z_i)}{K} \frac{j}{N_i} \right\}, j = 1, \dots, N_i,$$

and  $A_i^0 = P_i$ . We assume that  $K$  is chosen so that, for all  $j$ ,  $A_i^j \subset P(Z_i, \delta_\Omega(Z_i))$ . Let  $B_i^j = A_i^j \setminus A_i^{j-1}$ .

Let  $I_k = \{i \in \mathbb{N} \text{ such that } \delta_\Omega(Z_i) \in [2^{-k}, 2^{-k+1}]\}$  and  $M_k = \#I_k$  the cardinal of  $I_k$ . For each  $i \in I_k$  let us choose  $N_i = N(k)$  sufficiently large so that there exists  $s(i) \geq 1$  such that

$$|\psi|(B_i^{s(i)}) \leq \frac{1}{2^{k+4}} \frac{1}{M_k} \|\psi\|_{W^1(\Omega)}.$$

Let  $C_i = A_i^{s(i)-1}$ . Note that  $(C_i)_i$  is an open covering of  $U \cap \Omega$ . Let  $\Delta = \bigcup_i B_i^{s(i)}$  and let  $D_j$  the connected components of  $(U \cap \Omega) \setminus \Delta$ .

Let  $J(j) = \{t \text{ such that } D_j \subset C_t\}$ . Let  $w_j$  one of the points  $Z_t$  such that  $t \in J(j)$ . Let  $(\psi_j)_j$  be a family of smooth functions such that  $0 \leq \psi_j \leq 1$ ,  $\psi_j \equiv 1$  on  $D_j$ ,  $\text{Supp}(\psi_j) \subset D_j \cup \{A_t^{s(t)} \text{ such that } t \in J(j)\}$ , and  $\sum_j \psi_j \equiv 1$  on  $U \cap \Omega$ .

For each  $j$  let  $(e_l^j)_l$  be a  $\delta_\Omega(w_j)$ -extremal basis at  $w_j$ . Note that we can chose  $e_l^j$  so that the component of  $e_l^j$  on the first vector of the canonical basis  $(f_k)_k$  of  $\mathbb{C}^n$  is non negative. If we denote  $v_l^j = e_l^j + \delta_\Omega^2(w_j) f_1$  and  $u_l = \sum_j \psi_j v_l^j$ , the vector fields  $u_l$ ,  $1 \leq l \leq n$ , are smooth, don't vanish on  $U$ , and the proposition follows the lemmas.  $\square$

## 5. PROOF OF THEOREM 2.1

The main point in the proof is the following local version of the theorem:

**Proposition 5.1.** *For each point  $p \in \overline{\Omega}$  there exist two neighborhoods  $W$  and  $V$  of  $p$  in  $\overline{\Omega}$ ,  $W \Subset V$  (in  $\overline{\Omega}$ ) such that:*

- (1) *If  $\vartheta$  is a closed current of order 0 and degree 2 supported in  $V \cap \Omega$  such that  $\vartheta$  is a Carleson current in  $\Omega$ , there exists  $w$  a solution of the equation  $dw = \vartheta$  in  $W$  such that  $w$  is a Carleson current in  $\Omega$ .*
- (2) *If  $\omega$  is a closed current of order 0 and degree 1 supported in  $V \cap \Omega$  such that  $\omega$  is a Carleson current in  $\Omega$ , there exists  $f$  a solution of the equation  $df = \omega$  in  $W$  such that  $\delta_\Omega^{1/m-1} f$  is a Carleson measure in  $\Omega$ .*

For convenience of the reader, let us briefly indicate how Theorem 2.1 is a simple consequence of the Proposition (this follows [Sko76], [Var80], and [AC90]).

We consider the following three sheaves  $\mathcal{F}_0$ ,  $\mathcal{F}_1$  and  $\mathcal{F}_2$ :

Let  $U$  be an open set in  $\overline{\Omega}$ .  $\vartheta \in \Gamma(U, \mathcal{F}_2)$  if  $\vartheta$  is a closed 2-current supported in  $U \cap \Omega$  and  $\chi \vartheta$  is a 2-Carleson current supported in  $U \cap \Omega$  for all  $\chi \in \mathcal{C}_0^\infty(U)$ ;  $w \in \Gamma(U, \mathcal{F}_1)$  if  $w$  is a 1-current supported in  $U \cap \Omega$ ,  $dw \in \Gamma(U, \mathcal{F}_2)$  and  $\chi w$  is a 1-Carleson current supported in  $U \cap \Omega$  for all  $\chi \in \mathcal{C}_0^\infty(U)$ ;  $f \in \Gamma(U, \mathcal{F}_0)$  if  $f$  is a measure in  $U \cap \Omega$ ,  $df \in \Gamma(U, \mathcal{F}_1)$  and  $\chi \delta_\Omega^{1/m-1} f$  is a Carleson measure supported in  $U \cap \Omega$  for all  $\chi \in \mathcal{C}_0^\infty(U)$ . Then  $\mathcal{F}_0$  and  $\mathcal{F}_1$  are fine sheaves and, by Proposition 5.1, the sequence

$$0 \rightarrow \mathbb{C} \xrightarrow{i} \mathcal{F}_0 \xrightarrow{d} \mathcal{F}_1 \xrightarrow{d} \mathcal{F}_2 \rightarrow 0$$

is exact, and, by a standard cohomology argument,

$$\Gamma(\overline{\Omega}, \mathcal{F}_2) / d\Gamma(\overline{\Omega}, \mathcal{F}_1) \simeq H^2(\overline{\Omega}, \mathbb{C}) \simeq H^2(\Omega, \mathbb{C})$$

and Theorem 2.1 is proved.

For  $p \in \Omega$ , choosing  $V$  and  $W$  to be euclidean balls relatively compact in  $\Omega$ , the proposition only means that if  $\vartheta$  (resp  $\omega$ ) is a current whose coefficients are bounded measures in  $V$  there exists  $w$  (resp.  $f$ ) solution of  $dw = \vartheta$  (resp.  $df = \omega$ ) in  $W$  whose coefficients are bounded measures in  $W$ . As this is standard, we don't give any details here.

We now prove Proposition 5.1 for a fixed point  $p \in \partial\Omega$ .

There exists a strictly positive real number  $\delta_1$  such that, for  $z \in \Omega$  satisfying  $\delta_\Omega(z) \leq \delta_1$  the polydisk  $P(z, \varepsilon)$  are well defined. Then we choose the neighborhoods  $V$  and  $W$  of  $p$  as follows: let  $r_1, r_2$  and  $\eta_1$  three positive real numbers,  $\delta_1 > r_1 > 4r_2 > 8\eta_1$ , such that (denoting by  $B_e$  an euclidean ball):

- $B_e(p, r_1) \cap \Omega \subset \{z \in \Omega \text{ such that } \delta_\Omega(z) < \delta_1\}$ ;
- there exists a point  $A(p) \in B_e(p, r_1) \cap \{\xi \text{ such that } 2r_2 < \delta_\Omega(\xi) < r_1/2\} \cap \Omega$ ;
- for  $\zeta \in B_e(p, r_2)$ ,  $P(\zeta, \eta_1) \cap \Omega \subset B_e(p, 2r_2)$ ;
- for  $\zeta \in B_e(p, 2r_2)$  and all  $\xi \in B_e(p, 2r_2)$ , if  $v(\xi) = \nabla \rho(\xi)$  is the normal at  $\xi$ ,  

$$\left| \left\langle v(\xi), \overrightarrow{A(p)\zeta} \right\rangle \right| \geq \frac{1}{2} \left\| \overrightarrow{A(p)\zeta} \right\| \|v(\xi)\|.$$

Then we define  $V = B_e(p, r_1) \cap \Omega$  and  $W = B_e(p, r_2) \cap \{\zeta \in \Omega \text{ such that } \delta_\Omega(\zeta) < \eta_1\}$ .

First, we regularize the currents using Proposition 4.1 so that the regularized currents are smooth and closed in  $V_\varepsilon = \{\zeta \in V \text{ such that } \delta_{\partial V}(\zeta) > \varepsilon\}$ .

Thus, to finish the proof of Proposition 5.1 we assume the currents  $\vartheta$  and  $\omega$  supported and smooth in  $\overline{V_\varepsilon}$  and we will solve the equation  $dw = \vartheta$  and  $df = \omega$  in  $W_\varepsilon = B_e(p, r_2 - \varepsilon) \cap \{\zeta \in \Omega \text{ such that } \delta_\Omega(\zeta) < \eta_1\}$  (so that  $\cup_\varepsilon W_\varepsilon = W$ ), using Proposition 4.3, and the Proposition 5.1 will follow a standard weak limiting procedure.

*Proof of Proposition 5.1 for smooth currents in  $V_\varepsilon$ .* By translation we assume  $A(p) = 0$ .

Let  $(P_j)_j$  be a minimal covering of  $V_\varepsilon$  by polydisks centered on  $Z_j$ ,  $P_j = P(Z_j, \delta_\Omega(Z_j))$ . Let  $(\Phi_j)_j$  be a smoth partition of 1 associated to the  $P_j$  i.e.  $\Phi_j \geq 0$ ,  $\sum \Phi_j = 1$ ,  $\Phi_j$  identically zero outside  $2P_j$ , chosen so that  $\left| \frac{\partial \Phi_j}{\partial v} \right| \lesssim \tau(\cdot, v, \delta_\Omega(Z_j))^{-1}$ .

Let  $(\psi_k)_{k \geq 0}$  be a family of functions in  $\mathcal{C}^\infty(\mathbb{R})$  with support in  $]2^{-k-1}, 2^{-k+1}]$  and such that  $\psi_k \geq 0$ ,  $\sum \psi_k = 1$  on  $]0, 1]$  and  $\psi'_k(t) \lesssim 2^k$ .

Finally let  $\varphi \in \mathcal{C}^\infty(\mathbb{R})$ ,  $0 \leq \varphi \leq 1$  such that  $\varphi(x) = 1$  if  $x < 1/2$  and  $\varphi(x) = 0$  if  $x > 1$ .

Let us denote  $\mathbb{D}$  the unit disk of  $\mathbb{C}$ . For  $\Lambda = (\lambda_1, \dots, \lambda_n) \in \mathbb{D}^n$  and for  $c$  sufficiently small, to be precised, we consider the following function

$$(5.1) \quad h_\Lambda(t, z) = tz + ct \sum_{k,j} \psi_k(1-t) \Phi_j(tz) \sum_{i=1}^n A_{i,j,k}(z)$$

with

$$A_{i,j,k}(z) = \varphi\left(\frac{2^{-k}}{-\rho(z)}\right) \frac{2^{-k}}{\delta_\Omega(Z_j)} \lambda_i \tau_i(Z_j, \delta_\Omega(Z_j)) e_i(Z_j, \delta_\Omega(Z_j)) + \\ \left(1 - \varphi\left(\frac{2^{-k}}{-\rho(z)}\right)\right) \lambda_i \tau_i(Z_j, 2^{-k}) e_i(Z_j, 2^{-k})$$

where  $(e_i(Z_j, \delta_\Omega(Z_j)))_i$  is the  $(Z_j, \delta_\Omega(Z_j))$ -extremal basis used for the polydisk  $P_j$  and  $(e_i(Z_j, 2^{-k}))_i$  a  $(Z_j, 2^{-k})$ -extremal basis.

This function satisfies the following properties:

- Lemma 5.1.** (1)  $h_\Lambda(0, z) = 0$ ,  $h_\Lambda(1, z) = z$  and  $d_z h_\Lambda(1, z) = dz$ ;  
 (2) If  $\Phi_j(tz) \neq 0$ , then  $\delta_\Omega(tz) \simeq \delta_\Omega(Z_j)$  and  $tz \in P_j(Z_j, K\delta_\Omega(Z_j))$ ;  
 (3) If  $\Phi_j(tz) \neq 0$ , if  $c_0$  ((3.2)) and  $c$  ((5.1)) are small enough,

- (a) if  $\varphi\left(\frac{2^{-k}}{-\rho(z)}\right) \neq 0$ , then  $\delta_\Omega(h_\Lambda(t, z)) \simeq \delta_\Omega(tz) \simeq \delta_\Omega(z) \simeq \delta_\Omega(Z_j)$  and  $h_\Lambda(t, z) \in P(Z_j, K\delta_\Omega(Z_j))$ ,  
 (b) If  $\varphi\left(\frac{2^{-k}}{-\rho(z)}\right) = 0$ , then  $\delta_\Omega(h_\Lambda(t, z)) \simeq 2^{-k}$  and  $h_\Lambda(t, z) \in P(Z_j, K2^{-k})$ ;  
 (4) Denoting

$$Q(t, z) = \begin{cases} \mathcal{B}(tz, 1-t) & \text{if } 1-t \geq -\rho(z) \\ \frac{1-t}{-\rho(z)} \mathcal{B}(tz, -\rho(z)) & \text{if } 1-t \leq -\rho(z) \end{cases}$$

and  $Q_1(t, z) = \{w = h_\Lambda(t, z) \text{ for } \Lambda \in \mathbb{D}^n\}$ .

- (a) Then  $c$  in (5.1) being small enough, for  $z \in W$  and  $\forall t \in [0, 1]$ ,  $Q_1(t, z) \subset Q(t, z)$ ,  $Q_1(t, z) \subset B_\varepsilon(p, r_1) \cap \Omega$ ,  
 (b)  $\forall t_0 > 0$ , there exists  $c_1 = c_1(\Omega, c, t_0) > 0$  so that  $c_1 Q(t, z) \subset Q_1(t, z)$ ,  $\forall t \geq t_0$ .

The homotopy operator  $H$  for  $d$  is then defined on smooth forms  $\vartheta$  taking the average over  $\mathbb{D}^n$  of the integrals of the  $dt$ -component of  $h_\Lambda^* \vartheta$ : for example, if  $\vartheta$  is a smooth form of degree 2,  $H(\vartheta) = H_\vartheta$  is the form of degree 1

$$H_\vartheta(z) = \oint H_\Lambda(\vartheta)(z) d\Lambda,$$

where  $H_\Lambda(\vartheta)(z)$  is defined on every vector  $v$  by

$$H_\Lambda(\vartheta)(z)(v) = \int_0^1 h_\Lambda^* \vartheta dt = \int_0^1 \vartheta(h_\Lambda(t, z))(Y_t, Z_{t,v}) dt,$$

where  $Y_t = \frac{\partial}{\partial t} h_\Lambda(t, z)$  and  $Z_{t,v} = \frac{\partial}{\partial v} h_\Lambda(t, z)$ . Then  $H(\vartheta)$  is smooth and  $dH + Hd = Id$ .

To get the required Carleson estimate for  $H(\vartheta)$ ,  $\vartheta$  and  $\omega$  must be zero around the origin  $A(p)$ , so we follow a classical procedure (see [AC90]). Let  $R > 0$  such that  $B(0, 2R) \subset V_\varepsilon \setminus W_\varepsilon$  ( $\varepsilon > 0$  small). Let  $\psi$  be a smooth cut of function equal to 0 in  $B(0, R)$  and 1 in  $V_\varepsilon \setminus B(0, 2R)$ . Then, with  $T = \vartheta$  or  $\omega$ , we have

$$dH(\psi T) = \psi T - H(d\psi \wedge T),$$

and  $H(d\psi \wedge T)$  is closed in  $W_\varepsilon$  and, if  $d\tau = H(d\psi \wedge T)$  then  $d(H(\psi T) + \tau) = \vartheta$  in  $W_\varepsilon$ .

Then the conclusion follows the next proposition because, using a standard Poincaré homotopy (reducing eventually  $W_\varepsilon$ ),  $\tau$  can be chosen satisfying

$$\|\chi_{W_\varepsilon} \delta_\Omega^{1/m-1} |\tau|\|_{W^1(\Omega)} \lesssim \|\tau\|_{L^\infty(W_\varepsilon)} \leq \|H(d\psi \wedge T)\|_{L^\infty(W_\varepsilon)}.$$

**Proposition 5.2.** *There exists a constant  $C$  such that:*

- (1) with the above notation,  $\|H(d\psi \wedge T)\|_{L^\infty(W_\varepsilon)} \leq C \|T\|_{W^1(\Omega)}$ ;
- (2) if  $\vartheta$  is a smooth current of degree 2 identically zero in  $B(0, R)$ , then  $\|\chi_{W_\varepsilon} \|H(\vartheta)\|_k d\lambda\|_{W^1(\Omega)} \leq C \|\delta_\Omega \|\vartheta\|_k d\lambda\|_{W^1(\Omega)}$ ;
- (3) if  $\omega$  is a smooth current of degree 1 identically zero in  $B(0, R)$ , then  $\|\chi_{W_\varepsilon} \delta_\Omega^{1/m-1} H(\omega) d\lambda\|_{W^1(\Omega)} \leq C \|\omega\|_k d\lambda\|_{W^1(\Omega)}$ ;

*Proof.* We begin to prove (2). Note that there exists  $t_0 > 0$  such that,  $\vartheta(h_\Lambda(t, z)) \neq 0$  implies  $t > t_0$  so that

$$H_\Lambda(\vartheta)(z)(v) = \int_{t_0}^1 \vartheta(h_\Lambda(t, z))(Y_t, Z_{t,v}) dt.$$

**Lemma 1.** *Let  $T$  be the operator defined on non negative functions  $f$  on  $V_\varepsilon$ , by*

$$T(f)(z) = \delta_\Omega(z)^{1/m-1} \int_{t_0}^1 (1-t)^{1/m-1} \left( \oint_{Q(t,z)} f(w) \delta_\Omega^{1-2/m}(w) d\lambda(w) \right) dt, z \in W_\varepsilon.$$

Then, for  $z \in W_\varepsilon$ ,

$$\|H_\vartheta(z)\|_k \lesssim T(\delta_\Omega(\cdot) \|\vartheta(\cdot)\|_k)(z).$$

*Proof of the lemma.* By the definition of the “norm”  $\|\cdot\|_k$  we have

$$|H_\vartheta(z)(v)| \leq \oint \left[ \int_{t_0}^1 \|\vartheta(h_\Lambda(t, z))\|_k k(h_\Lambda(t, z), Y_t) k(h_\Lambda(t, z), Z_{t,v}) dt \right] d\Lambda.$$

By the definition of  $h_\Lambda(t, z)$

$$Y_t = z + c \sum_{k,j} \psi_k(1-t) \Phi_j(tz) \sum_{i=1}^n A_{i,j,k}(z) + \\ ct \left[ \sum_{k,j} \langle d\Phi_j(tz); z \rangle \psi_k(1-t) - \Phi_j(tz) \psi'_k(1-t) \right] \sum_i A_{i,j,k}(z).$$

Assume  $\Phi_j(tz) \neq 0$  and  $\psi_k(1-t) \neq 0$ .

Let us first estimate  $k(h_\Lambda(t, z), Y_t)$ . We have  $\langle d\Phi_j(tz); z \rangle \lesssim \delta_\Omega(Z_j)^{-1} \simeq \delta_\Omega(tz)^{-1} \lesssim \frac{1}{1-t}$ ,  $|\psi'_k(1-t)| \lesssim \frac{1}{1-t}$  and, by (3) (a) of Lemma 5.1,

$$\tau(h_\Lambda(t, z), e_i(Z_j, 2^{-k}), \delta_\Omega(h_\Lambda(t, z))) \gtrsim \tau(Z_j, e_i(Z_j, 2^{-k}), \delta_\Omega(h_\Lambda(t, z))).$$

As  $s > 1$  implies  $\tau(p, v, s\delta) \gtrsim s^{1/m} \tau(p, v, \delta)$  we get, if  $2^{-k} \leq -\rho(z)$ ,

$$\tau_i(Z_j, 2^{-k}) k(h_\Lambda(t, z), e_i(Z_j, 2^{-k})) \lesssim \delta_\Omega(h_\Lambda(t, z)) \left( \frac{\delta_\Omega(h_\Lambda(t, z))}{2^{-k}} \right)^{-1/m}.$$

Similarly, if  $2^{-k} \geq -\rho(z)/2$ ,

$$\tau_i(Z_j, \delta_\Omega(Z_j)) k(h_\Lambda(t, z), e_i(Z_j, \delta_\Omega(Z_j))) \lesssim \delta_\Omega(Z_j) \simeq \delta_\Omega(h_\Lambda(t, z)),$$

and (because  $\delta_\Omega(h_\Lambda(t, z)) \simeq 2^{-k}$ )

$$\frac{2^{-k}}{\delta_\Omega(Z_j)} \tau_i(Z_j, \delta_\Omega(Z_j)) k(h_\Lambda(t, z), e_i(Z_j, \delta_\Omega(Z_j))) \lesssim \delta_\Omega(h_\Lambda(t, z)) \left( \frac{\delta_\Omega(h_\Lambda(t, z))}{2^{-k}} \right)^{-1/m}.$$

These estimates give

$$k(h_\Lambda(t, z), Y_t) \lesssim \left( \frac{\delta_\Omega(h_\Lambda(t, z))}{2^{-k}} \right)^{1-1/m} \simeq \left( \frac{\delta_\Omega(h_\Lambda(t, z))}{1-t} \right)^{1-1/m}.$$

The estimate  $k(h_\Lambda(t, z), Z_{t,v})$  is easy: recording that ((1) of Lemma 3.1)

$$\left| \frac{\partial \Phi_j(\cdot)}{\partial v} \right| \lesssim \tau(\cdot, v, \delta_\Omega(Z_j))^{-1}, \quad \left| \frac{\partial}{\partial v}(-\rho)(z) \right| \lesssim \frac{\delta_\Omega(z)}{\tau(z, v, \delta_\Omega(z))}$$

and that  $\varphi'(\frac{2^{-k}}{-\rho(z)}) \neq 0$  implies  $2^{-k} \simeq -\rho(z)$ , one easily gets

$$k(h_\Lambda(t, z), Z_{t,v}) \lesssim \frac{\delta_\Omega(z)^{1/m} \delta_\Omega(h_\Lambda(t, z))^{1-1/m}}{\tau(z, v, \delta_\Omega(z))}.$$

Then we obtain

$$\|H(\vartheta)(z)\|_k \lesssim \delta_\Omega(z)^{1/m-1} \oint d\Lambda \int_0^1 \|\vartheta(h_\Lambda(t, z))\|_k \delta_\Omega(h_\Lambda(t, z))^{2-2/m} (1-t)^{1/m-1},$$

and the lemma is obtained making the change of variables  $\Lambda \mapsto h_\Lambda(t, z)$ , the jacobian being proportional to the volume of  $Q_1(t, z)$  which is equivalent to  $Q(t, z)$  because  $t \geq t_0$  implies  $c_1 Q(t, z) \subset Q_1(t, z) \subset Q(t, z)$ .  $\square$

**Lemma 2.** *The operator  $T$  of Lemma 1 satisfies the following estimate*

$$\|\chi_{W_\varepsilon} T(f) d\lambda\|_{W^1(\Omega)} \lesssim \|\chi_{V_\varepsilon} f d\lambda\|_{W^1(\Omega)}.$$

*Proof of the lemma.* Let  $B(\xi, \varepsilon)$  be a pseudo-ball on  $\partial\Omega$  and  $\widehat{B}(\xi, \varepsilon)$  the tent over  $B(\xi, \varepsilon)$ . For  $z \in \widehat{B}(\xi, \varepsilon)$  we decompose  $T(f)$  into two pieces (to simplify the notation we write  $\|fd\lambda\|_{W^1}$  instead of  $\|\chi_{V_\varepsilon}fd\lambda\|_{W^1(\Omega)}$ ):

$$T_1(f)(z) = \delta_\Omega(z)^{1/m-1} \int_{t_0}^{1-\varepsilon} (1-t)^{1/m-1} \left( \oint_{Q(t,z)} f(w) \delta_\Omega^{1-2/m}(w) d\lambda(w) \right) dt,$$

and

$$T_2(f)(z) = \delta_\Omega(z)^{1/m-1} \int_{1-\varepsilon}^1 (1-t)^{1/m-1} \left( \oint_{Q(t,z)} f(w) \delta_\Omega^{1-2/m}(w) d\lambda(w) \right) dt.$$

Consider first  $T_1(f)$ . As  $t < 1 - \varepsilon$  and  $\delta_\Omega(z) \lesssim \varepsilon$ , we have  $\delta_\Omega(w) \simeq \delta_\Omega(tz) \simeq 1 - t + \delta_\Omega(z) \simeq 1 - t$ ,  $d(w, \xi) \lesssim 1 - t$  and  $Q(t, z) \subset \widehat{B}(\xi, K(1 - t))$ . Then (note that  $\widehat{B}(\xi, 1 - t) \subset KQ(t, z)$ , because  $-\rho(z) \simeq \delta_\Omega(z) \lesssim \varepsilon \lesssim 1 - t$ )

$$\int_{Q(t,z)} f(w) d\lambda(w) \lesssim \|fd\lambda\|_{W^1} \frac{\text{Vol}(\widehat{B}(\xi, 1 - t))}{1 - t} \lesssim \|fd\lambda\|_{W^1} \frac{\text{Vol}(Q(t, z))}{1 - t},$$

and, using  $\delta_\Omega(w) \simeq 1 - t$ , we get

$$T_1(f)(z) \lesssim \|fd\lambda\|_{W^1} \delta_\Omega(z)^{1/m-1} \int_{t_0}^{1-\varepsilon} (1-t)^{-1/m-1} dt \lesssim \|fd\lambda\|_{W^1} \delta_\Omega(z)^{1/m-1} \varepsilon^{-1/m}$$

and

$$\int_{\widehat{B}(\xi, \varepsilon)} T_1(f)(z) d\lambda(z) \lesssim \|fd\lambda\|_{W^1} \varepsilon^{-1/m} \int_{\widehat{B}(\xi, \varepsilon)} \delta_\Omega(z)^{1/m-1} d\lambda(z) \lesssim \|fd\lambda\|_{W^1} \sigma(B(\xi, \varepsilon)).$$

Consider now  $T_2(f)$ .  $\widehat{B}(\xi, \varepsilon)$  is equivalent to the set

$$\{r\eta \text{ such that } 1 - \varepsilon \leq r \leq 1 \text{ and } \eta \in B(\xi, \varepsilon)\},$$

and if  $z = r\eta$ ,  $\delta_\Omega(z) \simeq 1 - r$ , and for  $w \in Q(t, z)$ ,  $w \in \widehat{B}(\xi, K\varepsilon)$  and  $\delta_\Omega(w) \simeq \delta_\Omega(tz) \simeq 1 - tr$ . Then

$$I = \int_{\widehat{B}(\xi, \varepsilon)} T_2(f)(z) d\lambda(z) \lesssim \int_{\widehat{B}(\xi, K\varepsilon)} f(w) \delta_\Omega(w)^{-1/m} \left( \int_{\mathcal{D}_w} \frac{(1-r)^{1/m-1} (1-t)^{1/m-1} (1-tr)^{1-1/m}}{\text{Vol}(Q(t, r\eta))} dr dt d\sigma(\eta) \right) d\lambda(w)$$

where

$$\mathcal{D}_w = \{(t, r, \eta) \text{ such that } w \in Q(t, r\eta), (t, r) \in [1 - \varepsilon, 1]^2 \text{ and } \eta \in B(\xi, \varepsilon)\}.$$

Note that  $1 - tr \simeq \max\{(1 - t), (1 - r)\}$  and let us cut  $\mathcal{D}_w$  into two parts

$$\mathcal{D}_w^1 = \mathcal{D}_w \cap \{r \geq t\} \text{ and } \mathcal{D}_w^2 = \mathcal{D}_w \cap \{r < t\},$$

and define  $I_i$ ,  $i = 1, 2$ , replacing in the definition of  $I$   $\mathcal{D}_w$  by  $\mathcal{D}_w^i$ .

If  $(t, r, \eta) \in \mathcal{D}_w^1$ ,  $\delta_\Omega(tr\eta) \simeq 1 - t$  and  $\delta_\Omega(w) \simeq 1 - t$ . Let  $w_1$  be the intersection of  $\partial\Omega$  with the half real line passing through 0 and  $w$ . Then  $\eta \in B(w_1, K_1(1 - t))$ ,  $t \in [1 - K_2\delta_\Omega(w), 1 - c_2\delta_\Omega(w)]$  and  $r \geq t \geq 1 - K_2\delta_\Omega(w)$ . As (by (2) of Proposition 4.3 and (3.7))

$$\text{Vol}(Q(t, r\eta)) \simeq \text{Vol}(\mathcal{B}(w, 1 - t)) \simeq (1 - t)\sigma(B(w_1, 1 - t)) \simeq (1 - t)\sigma(B(w_1, K_1(1 - t))),$$

we get

$$\begin{aligned}
I_1 &\lesssim \int_{\widehat{B}(\xi, K\varepsilon)} f(w) \delta_\Omega(w)^{-1/m} \left( \int_{\mathcal{D}_w^1} \frac{(1-r)^{1/m-1}}{(1-t) \sigma(B(w_1, K_1(1-t)))} dr dt \sigma(\eta) \right) d\lambda(w) \\
&\lesssim \int_{\widehat{B}(\xi, K\varepsilon)} f(w) \delta_\Omega(w)^{-1/m} \int_{1-K_2\delta_\Omega(w)}^{1-c_2\delta_\Omega(w)} \left( \int_{B(w_1, K_1(1-t))} \frac{d\sigma(\eta)}{\sigma(B(w_1, K_1(1-t)))} \right. \\
&\quad \left. \int_{1-K_2\delta_\Omega(w)}^1 (1-r)^{1/m-1} dr \right) \frac{dt}{1-t} d\lambda(w) \\
&\lesssim \int_{\widehat{B}(\xi, K\varepsilon)} f(w) d\lambda(w) \lesssim \|fd\lambda\|_{W^1} \sigma(B(\xi, K\varepsilon)) \lesssim \|fd\lambda\|_{W^1} \sigma(B(\xi, \varepsilon)).
\end{aligned}$$

Finally, if  $(t, r, \eta) \in \mathcal{D}_w^2$ ,  $\delta_\Omega(w) \simeq 1-r$  and

- if  $\frac{1-t}{-\rho(r\eta)} \leq 1$ , then  $w \in \frac{1-t}{-\rho(r\eta)} \mathcal{B}(tr\eta, -\rho(r\eta)) \subset \frac{1-t}{1-r} \mathcal{B}(tr\eta, 1-r)$  so  $tr\eta \in K_1 \frac{1-t}{1-r} \mathcal{B}(w, 1-r)$ , and, moreover

$$\text{Vol} \left( \frac{1-t}{-\rho(r\eta)} \mathcal{B}(tr\eta, -\rho(r\eta)) \right) \simeq \text{Vol} \left( \frac{1-t}{1-r} \mathcal{B}(w, 1-r) \right);$$

- if  $\frac{1-t}{-\rho(r\eta)} \geq 1$ , then  $w \in \mathcal{B}(tr\eta, 1-t) \subset K' \frac{1-t}{1-r} \mathcal{B}(w, 1-r)$ , because in this case  $1-t \simeq 1-r$ , so  $tr\eta \in K_1 \frac{1-t}{1-r} \mathcal{B}(w, 1-r)$ , and

$$\text{Vol}(\mathcal{B}(tr\eta, 1-r)) \simeq \text{Vol}(\mathcal{B}(w, 1-r)).$$

Now, for  $t, r$  and  $w$  fixed,  $\sigma(\{\eta \text{ such that } (t, r, \eta) \in \mathcal{D}_w^2\}) \lesssim \frac{1}{1-t} \text{Vol} \left( \frac{1-t}{1-r} \mathcal{B}(w, 1-r) \right)$ , and,  $\frac{1-t}{1-r} \mathcal{B}(w, 1-r) \subset K_2 \mathcal{B}(w, 1-t)$  by (3.7) and (3.8), and, by the last property of  $W$ , for  $w$  and  $t$  fixed, the length of the set of  $r$  such that there exists  $\eta$  such that  $tr\eta \in \mathcal{D}_w^2$  is  $\lesssim 1-t$ .

Then

$$\begin{aligned}
I_2 &\lesssim \int_{\widehat{B}(\xi, K\varepsilon)} f(w) \delta_\Omega(w)^{-1/m} \\
&\quad \int_{1-K_3\delta_\Omega(w)}^1 \left( \int_{\{(r, \eta) \text{ s. t. } tr\eta \in \mathcal{D}_w^2\}} \frac{d\sigma(\eta) dr}{\text{Vol} \left( \frac{1-t}{1-r} \mathcal{B}(w, 1-t) \right)} \right) (1-t)^{1/m-1} dt d\lambda(w),
\end{aligned}$$

and we get  $I_2 \lesssim \int_{\widehat{B}(\xi, K\varepsilon)} f(w) d\lambda(w) \lesssim \|fd\lambda\|_{W^1} \sigma(B(\xi, \varepsilon))$  finishing the proof of the lemma.  $\square$

(2) of the proposition follows immediately the lemmas.

Assertion (3) of the proposition is proved in a similar and easier way. We will not give more details.

We finish giving briefly the proof of (1) of the proposition. There exists  $\delta > 0$  such that  $t < \delta$  or  $t > 1 - \delta$  implies  $d\psi \wedge \vartheta(h_\Lambda(z, t)) = 0$  so

$$H(d\psi \wedge \vartheta)(z)(v_1, v_2) = \oint_\Lambda \int_\delta^{1-\delta} d\psi \wedge \vartheta(h_\Lambda(z, t))(Y_t, Z_{t, v_1}, Z_{t, v_2}) dt.$$

Note that, for  $t \in [\delta, 1 - \delta]$  and  $|v_i| \leq 1$ ,  $|Z_{t,v_i}|$  and  $|Y_t|$  are bounded by  $C = C(\delta)$ ,  $i = 1, 2$ . Then after the change of variables  $w(\Lambda) = h_\Lambda(z, t)$  we get

$$\begin{aligned} |H(d\psi \wedge \vartheta)(z)(v_1, v_2)| &\lesssim \int_{\delta}^{1-\delta} dt \int_{Q(t,z)} \frac{|d\psi \wedge \vartheta|(w)}{\text{Vol}(Q(t,z))} d\lambda(w) \\ &\lesssim \int_{\delta}^{1-\delta} \int_{Q(t,z)} \delta_\Omega(w) |d\psi \wedge \vartheta|(w) d\lambda(w) dt \\ &\lesssim \int_{\widehat{B}} \delta_\Omega(w) |d\psi \wedge \vartheta|(w) d\lambda(w) \\ &\lesssim \|\delta_\Omega d\psi \wedge \vartheta\|_{W^1} \lesssim \|\delta_\Omega \vartheta\|_{W^1} \end{aligned}$$

the first inequality coming from the fact that  $\text{Vol}(Q(t,z))$  and  $\delta_\Omega(w)$  are bounded from below,  $\widehat{B}$  in the third inequality being a tent containing  $Q(t,z)$  and the last because  $\sigma(B)$  is bounded from below.  $\square$

The proof of Proposition 5.1 is now complete.  $\square$

## 6. PROOF OF THEOREM 2.2

Let us introduce the notion of  $BMO_s$  functions, similarly to  $s$ -Carleson measures and currents defined before Proposition 4.2:

$f$  is in  $BMO_s(\partial D)$  if

$$\|f\|_{BMO_s(\partial D)} := \sup_{z \in \partial D, s < t < \varepsilon_0} \int_{P_t(z) \cap \partial D} \left| f - \oint_{P_t(z) \cap \partial D} f \right| < +\infty.$$

It is easy to see that, if  $\omega$  is *smooth*, then, the proof of Proposition 4.3 shows that

$$\|\omega\|_{W_s^1(D)} \simeq \| \|\omega\|_k d\lambda \|_{W_s^1(D)}.$$

We start the proof of the theorem regularizing the current  $\omega$  using Proposition 4.2: then (simplifying the notations), for  $\varepsilon > 0$  small enough, the regularized current  $\omega_\varepsilon$  is smooth  $\bar{\partial}$ -closed in  $\Omega^\varepsilon = \{\rho < -C\varepsilon\}$  and satisfies  $\|\omega_\varepsilon\|_{W_s^1(\Omega^\varepsilon)} \lesssim \|\omega\|_{W^1(\Omega)}$ .

Now we solve the equation  $\bar{\partial}u_\varepsilon = \omega_\varepsilon$  using the method described in the proof of [CD, Theorem 2.4]:  $u_\varepsilon$  is given by the formula

$$u_\varepsilon(z) = \int_{\Omega^\varepsilon} K_\varepsilon^1(z, \zeta) \wedge \omega_\varepsilon(\zeta) - \bar{\partial}^* \mathcal{N}_\varepsilon \left( \int_{\Omega^\varepsilon} P_\varepsilon(z, \zeta) \wedge \omega_\varepsilon(\zeta) \right)$$

the kernels  $K_\varepsilon^1$  and  $P_\varepsilon$  are associated to the defining function  $\rho + C\varepsilon$  as described in [CDM14, CD]. We proceed as H Skoda in [Sko76, Section 8]. Clearly, the  $\mathcal{C}^\infty$ -smooth kernels  $P_\varepsilon$ , as well as their derivatives, converge uniformly to the corresponding kernel  $P$  of  $\Omega$  so that  $\int_{\Omega^\varepsilon} P_\varepsilon(z, \zeta) \wedge \omega_\varepsilon(\zeta)$  converges in every Sobolev norm to a  $\bar{\partial}$ -closed form  $g$  on  $\Omega$  such that, for all integer  $k$ ,  $\|g\|_{H^k}$  is bounded by the total mass of  $\omega$ . Then

$$\bar{\partial}^* \mathcal{N}_\varepsilon \left( \int_{\Omega^\varepsilon} P_\varepsilon(z, \zeta) \wedge \omega_\varepsilon(\zeta) \right)$$

converges in  $\mathcal{C}^1(\overline{\Omega})$  to a function  $h$ .

Let  $\Phi_\varepsilon : \partial\Omega \rightarrow \partial\Omega^\varepsilon$  be a family of  $\mathcal{C}^\infty$  diffeomorphisms such that  $\Phi_\varepsilon$  converges to the identity uniformly in  $\mathcal{C}^\infty$  norm on  $\partial\Omega$ .

As  $\bar{\partial}u_\varepsilon = \omega_\varepsilon$ , denoting  $v_\varepsilon = \int_{\Omega^\varepsilon} K_\varepsilon^1(z, \zeta) \wedge \omega_\varepsilon(\zeta)$ , if  $v_\varepsilon \circ \Phi_\varepsilon$  converges in  $L^1(\partial\Omega)$  to  $v = \int_{\Omega} K^1(z, \zeta) \wedge \omega(\zeta)$ , for  $z \in \partial\Omega$ , the function

$$u(z) = \int_{\Omega} K^1(z, \zeta) \wedge \omega(\zeta) - h$$



is a solution of the equation  $\bar{\partial}_b u = \omega$ . By the properties of the kernels  $K_\varepsilon^1$  and  $K^1$  ( $K_\varepsilon^1$  converges uniformly on  $\partial\Omega \times \bar{\Omega}_\eta$  ( $\eta > 0$  fixed) to  $K^1$ ) this convergence follows exactly the proof made by H Skoda in [Sko76, p. 272].

To conclude the proof of Theorem 2.2 we have to show that  $\|v\|_{BMO(\partial\Omega)} \lesssim \|\omega\|_{W^1(\Omega)}$ . The proof of [CD, Theorem 2.4] gives

$$\|u_\varepsilon\|_{BMO_\varepsilon} \leq C_1 \| \|\omega\|_k \|_{W_\varepsilon^1(\Omega^\varepsilon)} \lesssim \|\omega\|_{W^1(\Omega)}$$

with a constant  $C_1$  uniform in  $\varepsilon$  (small enough) because the estimates of [CD, lemmas 3.4, 3.5 and 3.6] are uniform in a neighborhood of  $\partial\Omega$ . Thus the end of the proof is the following lemma

**Lemma 6.1.** *With the previous notations  $\|u\|_{BMO(\partial\Omega)} \lesssim \sup_\varepsilon \|u_\varepsilon\|_{BMO_\varepsilon(\partial\Omega^\varepsilon)}$ .*

*Proof.* Let  $\xi \in \partial\Omega$  and let  $B(\xi, t)$  be a pseudo-ball on  $\partial\Omega$ . Then  $\sigma_\varepsilon(B(\Phi_\varepsilon(\xi), t))$  converges to  $\sigma(B(\xi, t))$  and

$$\oint_{B(\Phi_\varepsilon(\xi), t)} u_\varepsilon = \frac{1}{\sigma_\varepsilon(B(\Phi_\varepsilon(\xi), t))} \int_{B(\xi, t)} u_\varepsilon \circ \Phi_\varepsilon |J\Phi_\varepsilon| \xrightarrow{\varepsilon \rightarrow 0} \oint_{B(\xi, t)} u.$$

The lemma follows easily.  $\square$

## REFERENCES

- [AC90] M. Andersson and H. Carlsson, *On varopoulos's theorem about zero sets of  $\mathcal{H}^p$ -functions*, Bull. Sci. Math. **114** (1990), no. 4, 463–484.
- [Ale17] W. Alexandre, *Zero sets of  $\mathcal{H}^p$  functions in convex domains of finite type*, Math. Z. **287** (2017), no. 1–2, 85–115.
- [BCD98] J. Bruna, Ph. Charpentier, and Y. Dupain, *Zeros varieties for the Nevanlinna class in convex domains of finite type in  $\mathbb{C}^n$* , Ann. of Math. **147** (1998), 391–415.
- [BG99] J. Bruna and S. Grellier, *Zero sets of  $\mathcal{H}^p$  functions in convex domains of strict finite type in  $\mathbb{C}^n$* , Complex Var. **38** (1999), 243–261.
- [CD] P. Charpentier and Y. Dupain, *Weighted and boundary  $l^p$  estimates for solutions of the  $\bar{\partial}$ -equation on lineally convex domains of finite type and applications*, to appear in Math Z.
- [CD14] ———, *Extremal bases, geometrically separated domains and applications*, Algebra i Analiz **26** (2014), no. 1, 196–269.
- [CDM14] P. Charpentier, Y. Dupain, and M. Mounkaila, *Estimates for Solutions of the  $\bar{\partial}$ -Equation and Application to the Characterization of the Zero Varieties of the Functions of the Nevanlinna Class for Lineally Convex Domains of Finite Type*, J. Geom. Anal. **24** (2014), no. 4, 1860–1881.
- [CNS92] D. C. Chang, A. Nagel, and E. Stein, *Estimates for the  $\bar{\partial}$ -Neumann problem for pseudoconvex domains in  $\mathbb{C}^2$  of finite type*, Acta Math. **169** (1992), 153–228.
- [Con02] M. Conrad, *Anisotrope optimale Pseudometriken für lineal konvex Gebeite von endlichem Typ (mit Anwendungen)*, PhD thesis, Berg.Universität-GHS Wuppertal (2002).
- [Cum01] A. Cumenge, *Zero sets of functions in the Nevanlinna or the Nevanlinna-Djrbachian classes*, Pacific J. Math. **199** (2001), no. 1, 79–92.
- [DM01] K. Diederich and E. Mazzilli, *Zero varieties for the Nevanlinna class on all convex domains of finite type*, Nagoya Math. Journal **163** (2001), 215–227.
- [Hen75] G. M. Henkin, *Solutions with bounds for the equations of H. Lewy and Poincaré-Lelong. Construction of functions of Nevanlinna class with given zeros in a strongly pseudoconvex domain*, Dokl. Akad. Nauk SSSR **224** (1975), no. 4, 771–774.
- [JN61] F. John and L. Nirenberg, *On functions of bounded mean oscillation*, Comm. Pure Appl. Math. **14** (1961), 415–426.
- [Ngu01] N. Nguyen, *Un théorème de la couronne  $H^p$  et zéros des fonctions de  $H^p$  dans les convexes de type fini*, Prépublication n° 224, Laboratoire Emile Picard, Université de Toulouse III (2001).
- [Sko76] H. Skoda, *Valeurs au bord pour les solutions de l'opérateur  $\bar{\partial}$ , et caractérisation des zéros des fonctions de la classe de Nevanlinna*, Bull. Soc. Math. France **104** (1976), 225–299.
- [Var80] N. Th. Varopoulos, *Zeros of  $h^p$  functions in several complex variables*, Pacific J. Math. **88** (1980), no. 1, 189–246.

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