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ON THE LARGEST PRIME FACTORS OF CONSECUTIVE INTEGERS

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ABSTRACT. Denote by $P^+(n)$ the largest prime factor of an integer n . One of Erdős–Turán’s conjectures asserts that the asymptotic density of integers n satisfying $P^+(n) < P^+(n + 1)$ is $1/2$. In this paper, we prove that this density is larger than 0.2017 , which improves the previous result “ 0.1356 ” of the second author.

1. INTRODUCTION

For each integer $n \geq 1$, let $P^+(n)$ denote the largest prime factor of n with the convention that $P^+(1) = 1$. An integer n is called y –friable (or y –smooth) if $P^+(n) \leq y$. In 1930, Dickman [6] obtained the well-known result: the following asymptotic formula

$$|\{n \leq x : P^+(n) \leq y\}| \sim x\rho(u) \quad (u \geq 1) \tag{1.1}$$

holds for $x \rightarrow \infty$ with $u = \log x / \log y$ fixed, where $\rho(u)$ is the Dickman–de Bruijn function. Later, the distribution of $P^+(\cdot)$ has been studied by many mathematicians, for example the work of Dartyge [2], La Bretèche and Tenenbaum [4, 5], Granville [12], Hildebrand [14, 15], Hildebrand and Tenenbaum [16, 17], Moree [18]. In recent years, friable integers play an important role in number theory, for example, an extraordinary breakthrough of Zhang [30] towards solving the Twin Prime Conjecture, remarkable progress of Vaughan and Wooley [23, 24, 25] in Waring’s problem.

In this paper, we aim to study the simultaneous distribution of the function $P^+(\cdot)$ at consecutive integers. In the 1930s, Erdős and Turán formulated the following conjecture in the correspondence.

Conjecture 1 (Erdős–Turán). *For $x \rightarrow \infty$, we have*

$$|\{n \leq x : P^+(n) < P^+(n + 1)\}| \sim \frac{1}{2}x. \tag{1.2}$$

Conjecture 1 is a constant concern for Erdős even though he thought it might be intractable by any technique at our disposal (see [8] or [21]). Later in 1978, Erdős and Pomerance conjectured that the largest prime factors of n and $n + 1$ are “independent events”.

Conjecture 2 (Erdős–Pomerance). *For any $a, b \in [0, 1]$, denote by $B(x; a, b)$ the number of $n \leq x$ with $P^+(n) \leq x^a$ and $P^+(n) \leq x^b$,*

$$B(x; a, b) := |\{n \leq x : P^+(n) \leq x^a, P^+(n + 1) \leq x^b\}|.$$

Then the asymptotic density

$$b(a, b) := \lim_{x \rightarrow \infty} B(x; a, b)$$

exists and equals $\rho(\frac{1}{a})\rho(\frac{1}{b})$.

In 2011, De Koninck and Doyon formulated a more general conjecture in order to study the distance between friable integers.

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Conjecture 3 (De Koninck-Doyon). *Fix an arbitrary integer $k \geq 2$ and let n be a large number. Let a_1, a_2, \dots, a_k be any permutation of the numbers $0, 1, \dots, k-1$. Then,*

$$\text{Prob}[P^+(n+a_1) < P^+(n+a_2) < \dots < P^+(n+a_k)] = \frac{1}{k!}.$$

There has been some progress towards these problems.

- For the Conjecture 1 (Erdős-Turán), in 1978 Erdős and Pomerance [9] first proved that there exists a positive asymptotic density of integers n with $P^+(n) < P^+(n+1)$. More precisely, they proved that

$$|\{n \leq x : P^+(n) < P^+(n+1)\}| > 0.0099x \quad (x \rightarrow \infty). \quad (1.3)$$

Later in 2005, the asymptotic density 0.0099 was improved to 0.05544 by La Bretèche, Pomerance and Tenenbaum [3], and to 0.5866 by Fouvry's arguments in "Further remarks" of the same paper [3]. Recently, the second author improved the asymptotic density to 0.1063 [26], and then further to 0.1356 [28].

On the other hand, in 2001 Rivat [20] proved a $P_y^+(n)$ -version of the Erdős-Turán conjecture for some small y . Here we just write a more concise version. Define $P_y^+(n) = \max\{p \mid n : p \leq y\}$ the greatest prime factor p of n which satisfies $p \leq y$. Then for $3 \leq y \leq \exp(\frac{\log x}{100 \log \log x})$, we have

$$|\{n \leq x : P_y^+(n) < P_y^+(n+1)\}| \sim \frac{1}{2}x \quad (x \rightarrow \infty).$$

Very recently Teräväinen [22] proved a logarithmic version of the Conjecture 1.

$$\delta(\{n \in \mathbb{N} : P^+(n) < P^+(n+1)\}) = \frac{1}{2}$$

where the logarithmic density δ of the set $A \subset \mathbb{N}$ is defined by

$$\delta(A) = \lim_{x \rightarrow \infty} \frac{1}{\log x} \sum_{n \leq x, n \in A} \frac{1}{n}$$

whenever it exists. Besides, he also proved a logarithmic version of Conjecture 2.

- For the Conjecture 3 with $k = 3$, in 1978 Erdős and Pomerance [9] observed that the patterns

$$P^+(n-1) > P^+(n) < P^+(n+1) \quad (1.4)$$

and

$$P^+(n-1) < P^+(n) > P^+(n+1) \quad (1.5)$$

both occur infinitely often and conjectured that these two patterns occur for a positive density of n . In addition, they proved that there are infinitely many integers n such that

$$P^+(n-1) < P^+(n) < P^+(n+1)$$

by considering the integers n of the form $n = p^{2^{k_0}}$ with k_0 well defined. Finally, for the fourth pattern, in 2001 Balog [1] proved the following lower bound

$$|\{n \leq x : P^+(n-1) > P^+(n) > P^+(n+1)\}| \gg x^{1/2} \quad (x \rightarrow \infty).$$

Recently, the second author [28] made a progress towards proving that the patterns (1.4) and (1.5) both occur for a positive density of n :

$$|\{n \leq x : P^+(n-1) > P^+(n) < P^+(n+1)\}| > 1.063 \times 10^{-7}x \quad (*)$$

and

$$|\{n \leq x : P^+(n-1) < P^+(n) > P^+(n+1)\}| > 8.84 \times 10^{-4}x \quad (**).$$

In addition, we can get a nontrivial upper bound for the above four patterns respectively by considering the distance between friable integers. Just after the second author's submission [28], Teräväinen [22] obtained a similar results to (*) et (**) with an unspecified positive density by using another method.

- For the Conjecture 3 with $k \geq 4$, little is known. In the same paper [28], the second author proved the existence of a positive density of integers n with the largest prime factors $P^+(n+j)$, $j \leq k-1$ in the following two certain patterns. More precisely, for any fixed integer $k \geq 3$ and $j_0 \in \{0, \dots, k-1\}$, there exist two positive constants $C_3(k)$ and $C_4(k)$ such that

$$|\{n \leq x : P^+(n+j_0) = \min_{0 \leq j \leq k-1} P^+(n+j)\}| \geq C_3(k)x + o(x)$$

and

$$|\{n \leq x : P^+(n+j_0) = \max_{0 \leq j \leq k-1} P^+(n+j)\}| \geq C_4(k)x + o(x).$$

Besides consecutive integers, the second author recently studied the largest prime factors of consecutive integers with one of which without small prime factor. We refer the reader to [27] for more details.

Now we consider the pattern $P^+(n) < P^+(n+1)$, i.e. the conjecture of Erdős-Turán. In [28], the second author starts from the inclusion-exclusion principle

$$\begin{aligned} \sum_{\substack{n < x \\ P^+(n) < P^+(n+1)}} 1 &= \sum_{\substack{n < x \\ P^+(n+1) > x^{1-c}}} 1 - \sum_{\substack{n < x \\ P^+(n) > P^+(n+1) > x^{1-c}}} 1 + \sum_{\substack{n < x \\ P^+(n) < P^+(n+1) \leq x^{1-c}}} 1 \\ &=: \mathcal{S}_A - \mathcal{S}_B + \mathcal{S}_C, \end{aligned} \quad (1.6)$$

where $0 < c \leq 1/2$ is a parameter. Then the sum \mathcal{S}_A is estimated by a result of Hildebrand [15] and \mathcal{S}_B is estimated by the Rosser-Iwaniec sieve and a theorem of Bombieri-Vinogradov type. For the sum \mathcal{S}_C , the second author gives a nontrivial lower bound by introducing a well adapted system of weights $\omega(n; y, z)$ defined by

$$\omega(n; y, z) := \sum_{\substack{z < p \leq y \\ p|n}} 1 \leq \frac{\log x}{\log z} \quad (1.7)$$

for $n \leq x$, $z < y$.

In this paper, we focus on the sum \mathcal{S}_C . In fact, in [28], the second author only sieves out the $x^{0.414}$ -friable integers $n \leq x$ such that $n+1$ has a prime factor in the interval $(x^{0.414}, x^{1/2}/(\log x)^B]$ with $B > 0$, due to a formidable obstacle: the level of distribution $Q = x^{1/2}/(\log x)^B$ for friable integers (see Lemma 2.2). Here by using a switch skill, we can sieve out some friable integers n such that $n+1$ has a prime factor larger than $x^{1/2}$. So that we can improve the previous lower bound $0.0118x$ of \mathcal{S}_C in [28], and then prove that the density of the pattern $P^+(n) < P^+(n+1)$ is larger than $1/5$. More precisely, we have the following result.

Theorem 1. For $x \rightarrow \infty$, we have

$$|\{n < x : P^+(n) < P^+(n+1)\}| > 0.2017x. \quad (1.8)$$

The lower bound is also true for the pattern $P^+(n) > P^+(n+1)$.

Throughout this paper, we denote by ε an arbitrarily small positive constant, and p, p' primes.

2. LEMMAS

Denote by

$$S(x, y) := \{n \leq x : P^+(n) \leq y\} \quad (2.1)$$

the set of y -friable integers not exceeding x and

$$\Psi(x, y) := |S(x, y)| \quad (2.2)$$

the cardinality of $S(x; y)$. Then we have the following lemma.

Lemma 2.1 (Hildebrand, [15]). For any $\varepsilon > 0$, we have

$$\Psi(x, y) = x\rho(u) \left\{ 1 + O_\varepsilon \left(\frac{\log(u+1)}{\log y} \right) \right\}$$

uniformly, for each $\varepsilon > 0$, in the domain

$$x \geq x_0(\varepsilon), \quad \exp \{(\log \log x)^{5/3+\varepsilon}\} \leq y \leq x,$$

where $u = \log x / \log y$ and $\rho(u)$ is the Dickman–de Bruijn function which is defined by differential equation

$$\begin{cases} \rho(u) = 1, & 0 \leq u \leq 1, \\ u\rho(u)' = -\rho(u-1), & u > 1. \end{cases}$$

The second lemma concerns the distribution of friable integers in arithmetic progressions.

Lemma 2.2. For any given positive constant $A > 0$, there exists a constant $B = B(A) > 0$ such that the estimate

$$\sum_{q \leq x^{1/2}/(\log x)^B} \max_{t \leq x} \max_{\substack{(a,q)=1 \\ n \equiv a \pmod{q}}} \left| \sum_{\substack{n \in S(t,y) \\ n \equiv a \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{n \in S(t,y) \\ (n,q)=1}} \right| \ll \frac{x}{(\log x)^A} \quad (2.3)$$

holds uniformly for $2 \leq y \leq x$, where the constant implied by the symbol “ \ll ” depends only on A .

In [29], Wolke proved a theorem of Bombieri-Vinogradov type for sifted numbers without small prime factors, and in the same paper he also announced the similar result for friable integers. Fouvry and Tenenbaum [10] proved a slightly weaker form of the formula (2.3). In addition, for Lemma 2.2 we can also see the work of Fouvry-Tenenbaum [11], Harper [13] and Drapreau [7].

For $q \in \mathbb{N}$ and $(a, q) = 1$, define

$$\pi(x; \ell, a, q) = \sum_{\substack{\ell p \leq x \\ \ell p \equiv a \pmod{q}}} 1 \quad \text{and} \quad \pi'(x; \ell, a, q) = \sum_{\substack{p \leq x \\ \ell p \equiv a \pmod{q}}} 1.$$

Lemma 2.3. *For any given positive constant $A > 0$, there exists a constant $B = B(A) > 0$ such that the estimate*

$$\sum_{q \leq x^{1/2}/(\log x)^B} \max_{y \leq x} \max_{(a, q)=1} \left| \sum_{\substack{L_1 < \ell \leq L_2 \\ (\ell, q)=1}} f(\ell) \left(\pi(y; \ell, a, q) - \frac{\text{li}(y/\ell)}{\varphi(q)} \right) \right| \ll \frac{x}{(\log x)^A} \quad (2.4)$$

and

$$\sum_{q \leq x^{1/2}/(\log x)^B} \max_{y \leq x} \max_{(a, q)=1} \left| \sum_{\substack{L_1 < \ell \leq L_2 \\ (\ell, q)=1}} f(\ell) \left(\pi'(y/L_2; \ell, a, q) - \frac{\text{li}(y/L_2)}{\varphi(q)} \right) \right| \ll \frac{x}{(\log x)^A} \quad (2.5)$$

hold for $(\log y)^{2B} < L_1 \leq L_2 < x^{1-\varepsilon}$, where $|f(\ell)| \leq 1$ and the constant implied by the symbol “ \ll ” depends only on ε and A .

Proof. For (2.4), it is the Theorem 2 in [19]. To prove (2.5), we can follow step by step Pan-Ding-Wang’s arguments of Theorem 2 in [19], so we omit the proof. \square

The fourth lemma is the estimation of \mathcal{S}_A and \mathcal{S}_B in (1.6).

Lemma 2.4. *Taking $c = 0.2056$ in (1.6), one has*

$$\mathcal{S}_A - \mathcal{S}_B \geq 0.1238x.$$

Proof. See section 10 in [28]. \square

3. ESTIMATION OF \mathcal{S}_C

In this section, we mainly focus on the estimation of \mathcal{S}_C which is defined by (1.6) with $c = 0.2056$. The improvement in Theorem 1 comes from a switch skill to sieve out some shifted friable integers with a large prime factor and a modification of well adapted system of weights $\omega(n; y, z)$ with more delicate calculation of \mathcal{S}_C .

Recall the definitions of $S(x, y)$ and $\Psi(x, y)$ in (2.1) and (2.2) respectively. Define

$$P(y, z) = \prod_{z < p < y} p, \quad S^+(x; y, z) = \{n \leq x : z < P^+(n) \leq y\}.$$

Then we have

$$\begin{aligned}
\mathcal{S}_C &= \sum_{\substack{n < x \\ P^+(n) < P^+(n+1) \leq x^{1-c}}} 1 \\
&\geq \sum_{\substack{n \in S(x, x^{\delta_1}) \\ (n+1, P(x^{\frac{1}{2}}/(\log x)^B, x^{\delta_1})) > 1}} 1 + \sum_{\substack{n \in S^+(x; x^{\delta_2}, x^{\delta_1}) \\ (n+1, P(x^{\frac{1}{2}}/(\log x)^B, x^{\delta_2})) > 1}} 1 \\
&\quad + \sum_{\substack{n \in S(x, x^{1-\eta_1}) \\ (n+1, P(x^{1-c}, x^{1-\eta_1})) > 1}} 1 + \sum_{\substack{n \in S^+(x; x^{1-\eta_2}, x^{1-\eta_1}) \\ (n+1, P(x^{1-c}, x^{1-\eta_2})) > 1}} 1 \\
&=: \mathcal{S}_{C_1} + \mathcal{S}_{C_2} + \mathcal{S}_{C_3} + \mathcal{S}_{C_4},
\end{aligned} \tag{3.1}$$

where $\eta_1, \eta_2, \delta_1, \delta_2$ are four parameters satisfying

$$c < \eta_2 < \eta_1 < \delta_1 < \delta_2 < \frac{1}{2}. \tag{3.2}$$

For the inequality in (3.1), it is obvious that the integers n counted in \mathcal{S}_{C_1} and \mathcal{S}_{C_2} (or in \mathcal{S}_{C_3} and \mathcal{S}_{C_4}) are disjoint. Besides, noting that

$$x^{\delta_1} \cdot x^{1-\eta_1} > x, \quad x^{\delta_2} \cdot x^{1-\eta_2} > x$$

by the choice of $\eta_1, \eta_2, \delta_1, \delta_2$ in (3.2), so the integers n counted in \mathcal{S}_{C_1} and \mathcal{S}_{C_3} (or in \mathcal{S}_{C_2} and \mathcal{S}_{C_4}) are also disjoint. Then we deduce that the integers n counted in $\mathcal{S}_{C_1}, \mathcal{S}_{C_2}, \mathcal{S}_{C_3}$ and \mathcal{S}_{C_4} are disjoint each other.

Next we will estimate the four sums respectively.

1. Estimation of \mathcal{S}_{C_1} and \mathcal{S}_{C_2}

First we will follow the approach of [28] to estimate \mathcal{S}_{C_1} and \mathcal{S}_{C_2} . By the definition of $\omega(n; y, z)$ in (1.7), we have

$$\omega(n+1; x^{1/2}/(\log x)^B, x^{\delta_1}) \begin{cases} \leq \frac{1}{\delta_1}, & (n+1, P(x^{1/2}/(\log x)^B, x^{\delta_1})) > 1, \\ = 0, & \text{otherwise.} \end{cases} \tag{3.3}$$

Hence we detect the condition $(n+1, P(x^{1/2}/(\log x)^B, x^{\delta_1})) > 1$ in \mathcal{S}_{C_1} by (3.3) and we obtain a lower bound of \mathcal{S}_{C_1} :

$$\begin{aligned}
\mathcal{S}_{C_1} &\geq \delta_1 \sum_{n \in S(x, x^{\delta_1})} \omega(n+1; x^{1/2}/(\log x)^B, x^{\delta_1}) \\
&= \delta_1 \sum_{n \in S(x, x^{\delta_1})} \sum_{\substack{p|(n+1) \\ x^{\delta_1} < p \leq x^{1/2}/(\log x)^B}} 1 \\
&= \delta_1 \sum_{\delta_1 < p \leq x^{1/2}/(\log x)^B} \sum_{\substack{n \in S(x, x^{\delta_1}) \\ n \equiv -1 \pmod{p}}} 1 \\
&= \delta_1 (\mathcal{S}_{C_1}^{(1)} + \mathcal{S}_{C_1}^{(2)}),
\end{aligned} \tag{3.4}$$

where

$$\begin{aligned}\mathcal{S}_{C_1}^{(1)} &= \sum_{x^{\delta_1} < p \leq x^{1/2}/(\log x)^B} \frac{1}{\varphi(p)} \sum_{\substack{n \in S(x, x^{\delta_1}) \\ (n, p)=1}} 1, \\ \mathcal{S}_{C_1}^{(2)} &= \sum_{\delta_1 < p \leq x^{1/2}/(\log x)^B} \left(\sum_{\substack{n \in S(x, x^{\delta_1}) \\ n \equiv -1 \pmod{p}}} 1 - \frac{1}{\varphi(p)} \sum_{\substack{n \in S(x, x^{\delta_1}) \\ (n, p)=1}} 1 \right).\end{aligned}$$

For $\mathcal{S}_{C_1}^{(2)}$, by Lemma 2.2, we have

$$\begin{aligned}\mathcal{S}_{C_1}^{(2)} &\ll \sum_{p \leq x^{1/2}/(\log x)^B} \left| \sum_{\substack{n \in S(x, x^{\delta_1}) \\ n \equiv -1 \pmod{p}}} 1 - \frac{1}{\varphi(p)} \sum_{\substack{n \in S(x, x^{\delta_1}) \\ (n, p)=1}} 1 \right| \\ &\ll x(\log x)^{-A}.\end{aligned}\tag{3.5}$$

for any $A > 0$, which is admissible.

We use Lemma 2.1 to estimate $\mathcal{S}_{C_1}^{(1)}$, and have

$$\begin{aligned}\mathcal{S}_{C_1}^{(1)} &= \sum_{x^{\delta_1} < p \leq x^{1/2}/(\log x)^B} \frac{1}{p-1} \Psi(x, x^{\delta_1}) \\ &= x\rho\left(\frac{1}{\delta_1}\right) \log \frac{1}{2\delta_1} + o(x).\end{aligned}\tag{3.6}$$

Therefore we have from (3.5) and (3.6) that

$$\mathcal{S}_{C_1} \geq x\rho\left(\frac{1}{\delta_1}\right) \delta_1 \log \frac{1}{2\delta_1} + o(x).\tag{3.7}$$

Since

$$\begin{aligned}\mathcal{S}_{C_2} &= \sum_{\substack{n \in S(x, x^{\delta_2}) \\ (n+1, P(x^{\frac{1}{2}}/(\log x)^B, x^{\delta_2})) > 1}} 1 - \sum_{\substack{n \in S(x, x^{\delta_1}) \\ (n+1, P(x^{\frac{1}{2}}/(\log x)^B, x^{\delta_2})) > 1}} 1 \\ &\geq \delta_2 \left(\sum_{n \in S(x, \delta_2)} - \sum_{n \in S(x, \delta_1)} \right) \omega(n+1; x^{1/2}/(\log x)^B, x^{\delta_2}),\end{aligned}$$

A similar manipulation allows us to deduce that

$$\mathcal{S}_{C_2} \geq x \left(\rho\left(\frac{1}{\delta_2}\right) - \rho\left(\frac{1}{\delta_1}\right) \right) \delta_2 \log \frac{1}{2\delta_2} + o(x).\tag{3.8}$$

2. Estimation of \mathcal{S}_{C_3} and \mathcal{S}_{C_4}

We now turn to consider the sums \mathcal{S}_{C_3} and \mathcal{S}_{C_4} . Noting that $x^{1-\eta_1} > x^{1/2}$, $x^{1-\eta_2} > x^{1/2}$, so we can not use Lemme 2.2, i.e. the theorem of Bombieri-Vinogradov type for friable integers directly as we have done for \mathcal{S}_{C_1} and \mathcal{S}_{C_2} .

For \mathcal{S}_{C_3} , it counts the number of $x^{1-\eta_1}$ -friable integers n such that $n+1$ has at least one prime factor in the interval $(x^{1-\eta_1}, x^{1-c})$ with $c < \eta_1 < 1/2$. Then we have

$$\begin{aligned}
\mathcal{S}_{C_3} &= \sum_{\substack{n \in S(x, x^{1-\eta_1}) \\ (n+1, P(x^{1-c}, x^{1-\eta_1})) > 1}} 1 \\
&= \sum_{x^{1-\eta_1} < p < x^{1-c}} \sum_{\substack{dp \leq x \\ P^+(dp-1) \leq x^{1-\eta_1}}} 1 \\
&= \sum_{x^{1-\eta_1} < p < x^{1-c}} \sum_{dp \leq x} 1 - \sum_{x^{1-\eta_1} < p' < x} \sum_{x^{1-\eta_1} < p < x^{1-c}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1.
\end{aligned} \tag{3.9}$$

For the sum \mathcal{S}_{C_4} , it counts the number of $(x^{1-\eta_1}, x^{1-\eta_2}]$ -friable integers n , i.e. $x^{1-\eta_1} < P^+(n) \leq x^{1-\eta_2}$, such that $n+1$ has at least one prime factor in the interval $(x^{1-\eta_2}, x^{1-c})$ with $c < \eta_2 < \eta_1 < 1/2$. So we have

$$\begin{aligned}
\mathcal{S}_{C_4} &= \sum_{\substack{n \in S^+(x; x^{1-\eta_2}, x^{1-\eta_1}) \\ (n+1, P(x^{1-c}, x^{1-\eta_2})) > 1}} 1 \\
&= \sum_{x^{1-\eta_2} < p < x^{1-c}} \sum_{\substack{dp \leq x \\ x^{1-\eta_1} < P^+(dp-1) \leq x^{1-\eta_2}}} 1 \\
&= \sum_{x^{1-\eta_1} < p' \leq x^{1-\eta_2}} \sum_{x^{1-\eta_2} < p < x^{1-c}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1.
\end{aligned} \tag{3.10}$$

Combing (3.9) and (3.10), we have

$$\begin{aligned}
\mathcal{S}_{C_3} + \mathcal{S}_{C_4} &= \sum_{x^{1-\eta_1} < p < x^{1-c}} \sum_{dp \leq x} 1 - \sum_{x^{1-\eta_1} < p' < x} \sum_{x^{1-\eta_1} < p < x^{1-c}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 \\
&\quad + \sum_{x^{1-\eta_1} < p' < x^{1-\eta_2}} \sum_{x^{1-\eta_2} < p < x^{1-c}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 \\
&= \widetilde{\mathcal{F}}_{C_3} + \widetilde{\mathcal{F}}_{C_4},
\end{aligned} \tag{3.11}$$

where

$$\widetilde{\mathcal{F}}_{C_3} := \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{dp \leq x} 1 - \sum_{x^{1-\eta_1} < p' < x} \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1$$

and

$$\widetilde{\mathcal{F}}_{C_4} := \sum_{x^{1-\eta_2} < p < x^{1-c}} \sum_{dp \leq x} 1 - \sum_{x^{1-\eta_2} < p' < x} \sum_{x^{1-\eta_2} < p < x^{1-c}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1.$$

Invoking the identity

$$\log(dp - 1) = \sum_{m|(dp-1)} \Lambda(m)$$

and prime number theory, we find

$$\begin{aligned} \sum_{x^{1-\alpha} < p < x^{1-\beta}} \sum_{dp \leq x} 1 &= \frac{1}{\log x} \sum_{x^{1-\alpha} < p < x^{1-\beta}} \sum_{dp \leq x} \log(dp - 1) + o(x) \\ &= \frac{1}{\log x} \sum_{p' < x} \log p' \sum_{x^{1-\alpha} < p < x^{1-\beta}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 \\ &\quad + \frac{1}{\log x} \sum_{k \geq 2} \sum_{p'^k < x} \log p' \sum_{x^{1-\alpha} < p < x^{1-\beta}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'^k}}} 1 + o(x) \\ &\geq \frac{1}{\log x} \sum_{p' < x} \log p' \sum_{x^{1-\alpha} < p < x^{1-\beta}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 + o(x). \end{aligned} \tag{3.12}$$

for $0 < \beta < \alpha < 1$.

Now for $\widetilde{\mathcal{F}}_{C_3}$, we have

$$\begin{aligned} \widetilde{\mathcal{F}}_{C_3} &= \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{dp \leq x} 1 - \sum_{x^{1-\eta_1} < p' < x} \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 \\ &\geq \frac{1}{1-\eta_1} \frac{1}{\log x} \sum_{p' < x} \log p' \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 \\ &\quad + \left(1 - \frac{1}{1-\eta_1}\right) \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{dp \leq x} 1 \\ &\quad - \sum_{x^{1-\eta_1} < p' < x} \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 + o(x) \end{aligned} \tag{3.13}$$

where we have used the estimation (3.12) with $(\alpha, \beta) = (\eta_1, \eta_2)$. For the second term on the right-hand side of (3.13), we use Mertens' formula. And for the the first term on the right-hand side of (3.13), we split it into two sums according to whether $p' > x^{1-\eta_1}$ or $p' \leq x^{1-\eta_1}$.

So we obtain

$$\begin{aligned}
\widetilde{\mathcal{F}}_{C_3} &= \frac{1}{1-\eta_1} \frac{1}{\log x} \sum_{p' \leq x^{1-\eta_1}} \log p' \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 \\
&\quad + \frac{1}{\log x} \sum_{x^{1-\eta_1} < p' < x} \left(\frac{1}{1-\eta_1} \log p' - \log x \right) \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 \\
&\quad - \frac{\eta_1}{1-\eta_1} \log \frac{1-\eta_2}{1-\eta_1} x + o(x) \\
&= \frac{1}{\log x} \left(\frac{1}{1-\eta_1} S_A + S_B \right) - \frac{\eta_1}{1-\eta_1} \log \frac{1-\eta_2}{1-\eta_1} x + o(x),
\end{aligned} \tag{3.14}$$

where

$$\begin{aligned}
S_A &:= \sum_{p' \leq x^{1-\eta_1}} \log p' \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1, \\
S_B &:= \sum_{x^{1-\eta_1} < p' < x} \left(\frac{1}{1-\eta_1} \log p' - \log x \right) \sum_{x^{1-\eta_1} < p \leq x^{1-\eta_2}} \sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1.
\end{aligned}$$

For S_A , we have lower bound

$$\begin{aligned}
S_A &\geq \sum_{p' \leq x^{1-\eta_1}} \log p' \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, p')=1}} \sum_{\substack{x^{1-\eta_1} < p \leq x/d \\ dp-1 \equiv 0 \pmod{p'}}} 1 \\
&\geq S_{A_1} + S_{A_2} - S_{A_3},
\end{aligned} \tag{3.15}$$

where

$$\begin{aligned}
S_{A_1} &:= \sum_{p' < x^{1/2}/(\log x)^B} \frac{\log p'}{\varphi(p')} \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, p')=1}} \sum_{x^{1-\eta_1} < p \leq x/d} 1, \\
S_{A_2} &:= \sum_{p' < x^{1/2}/(\log x)^B} \log p' \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, p')=1}} \left(\sum_{\substack{dp \leq x \\ dp-1 \equiv 0 \pmod{p'}}} 1 - \frac{1}{\varphi(p')} \sum_{dp \leq x} 1 \right), \\
S_{A_3} &:= \sum_{p' < x^{1/2}/(\log x)^B} \log p' \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, p')=1}} \left(\sum_{\substack{p < x^{1-\eta_1} \\ dp-1 \equiv 0 \pmod{p'}}} 1 - \frac{1}{\varphi(p')} \sum_{p < x^{1-\eta_1}} 1 \right).
\end{aligned}$$

for some $B > 0$.

For S_{A_2} , taking $(L_1, L_2, \ell) = (x^{\eta_2}, x^{\eta_1}, d)$ and $f(\ell) \equiv 1$ in (2.4) of Lemma 2.3, one has upper bound

$$S_{A_2} \leq \sum_{q < \frac{x^{1/2}}{(\log x)^B}} \log q \left| \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, q) = 1}} \sum_{\substack{dp \leq x \\ dp \equiv 1 \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, q) = 1}} \sum_{dp \leq x} 1 \right| \quad (3.16)$$

$$\ll \frac{x}{(\log x)^A}$$

for any $A > 0$, which is admissible. Similarly, for S_{A_3} , taking $(L_1, L_2, \ell) = (x^{\eta_2}, x^{\eta_1}, d)$ and $f(\ell) \equiv 1$ in (2.5) of Lemma 2.3, one has upper bound

$$S_{A_3} \leq \sum_{q < \frac{x^{1/2}}{(\log x)^B}} \log q \left| \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, q) = 1}} \sum_{\substack{p < x^{1-\eta_1} \\ dp \equiv 1 \pmod{q}}} 1 - \frac{1}{\varphi(q)} \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, q) = 1}} \sum_{p < x^{1-\eta_1}} 1 \right| \quad (3.17)$$

$$\ll \frac{x}{(\log x)^A}$$

for any $A > 0$, which is also admissible. Then for S_{A_1} , it follows from partial summation and prime number theorem that

$$S_{A_1} = \sum_{p' < x^{1/2}/(\log x)^B} \frac{\log p'}{\varphi(p')} \sum_{\substack{x^{\eta_2} < d < x^{\eta_1} \\ (d, p') = 1}} \sum_{x^{1-\eta_1} < p < x/d} 1, \quad (3.18)$$

$$= \left(\frac{1}{2} \log \frac{1-\eta_2}{1-\eta_1} + o(1) \right) x \log x.$$

For the sum S_B , we have trivially

$$S_B \geq 0. \quad (3.19)$$

So from (3.14)–(3.19) we obtain a lower bound of $\widetilde{\mathcal{F}}_{C_3}$:

$$\widetilde{\mathcal{F}}_{C_3} \geq x \frac{1-2\eta_1}{2(1-\eta_1)} \log \frac{1-\eta_2}{1-\eta_1} + o(x). \quad (3.20)$$

By the same method of estimation of (3.20), we can get a lower bound of $\widetilde{\mathcal{F}}_{C_4}$:

$$\widetilde{\mathcal{F}}_{C_4} \geq x \frac{1-2\eta_2}{2(1-\eta_2)} \log \frac{1-c}{1-\eta_2} + o(x). \quad (3.21)$$

Hence from (3.11), (3.20) and (3.21), we have

$$\mathcal{I}_{C_3} + \mathcal{I}_{C_4} \geq \widetilde{\mathcal{F}}_{C_3} + \widetilde{\mathcal{F}}_{C_4}$$

$$\geq \left(\frac{1-2\eta_1}{2(1-\eta_1)} \log \frac{1-\eta_2}{1-\eta_1} + \frac{1-2\eta_2}{2(1-\eta_2)} \log \frac{1-c}{1-\eta_2} + o(1) \right) x. \quad (3.22)$$

Finally, combing (3.7), (3.8) and (3.22), we get

$$\begin{aligned} \mathcal{S}_C \geq & \left(\frac{1-2\eta_1}{2(1-\eta_1)} \log \frac{1-\eta_2}{1-\eta_1} + \frac{1-2\eta_2}{2(1-\eta_2)} \log \frac{1-c}{1-\eta_2} \right. \\ & \left. + \delta_1 \rho\left(\frac{1}{\delta_1}\right) \log \frac{1}{2\delta_1} + \delta_2 \rho\left(\frac{1}{\delta_2}\right) \log \frac{1}{2\delta_2} - \delta_2 \rho\left(\frac{1}{\delta_1}\right) \log \frac{1}{2\delta_2} \right) x + o(x) \end{aligned} \quad (3.23)$$

with $\eta_2, \eta_1, \delta_1, \delta_2$ satisfying the condition (3.2).

4. PROOF OF THEOREM 1

We conclude from Lemma 2.4 and (3.23) that

$$\sum_{\substack{n < x \\ P^+(n) < P^+(n+1)}} 1 \geq C(\eta_2, \eta_1, \delta_1, \delta_2)x + o(x) \quad (4.1)$$

where

$$\begin{aligned} C(\eta_2, \eta_1, \delta_1, \delta_2) := & 0.1238 + \frac{1-2\eta_1}{2(1-\eta_1)} \log \frac{1-\eta_2}{1-\eta_1} + \frac{1-2\eta_2}{2(1-\eta_2)} \log \frac{1-c}{1-\eta_2} \\ & + \delta_1 \rho\left(\frac{1}{\delta_1}\right) \log \frac{1}{2\delta_1} + \delta_2 \rho\left(\frac{1}{\delta_2}\right) \log \frac{1}{2\delta_2} - \delta_2 \rho\left(\frac{1}{\delta_1}\right) \log \frac{1}{2\delta_2} \end{aligned} \quad (4.2)$$

with $\eta_2, \eta_1, \delta_1, \delta_2$ satisfying

$$0.2056 < \eta_2 < \eta_1 < \delta_1 < \delta_2 < \frac{1}{2}. \quad (4.3)$$

With the help of *Mathematica 8.0*, we find

$$C := \max_{\eta_2, \eta_1, \delta_1, \delta_2 \text{ satisfy (4.3)}} C(\eta_2, \eta_1, \delta_1, \delta_2) \geq 0.2017 \quad (4.4)$$

by taking $\eta_2 = 0.3190$, $\eta_1 = 0.4098$, $\delta_1 = 0.4099$ and $\delta_2 = 0.4576$. Now Theorem 1 comes from (4.1), (4.2) and (4.4).

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