



**HAL**  
open science

# Influence of Design Parameters on the Singularities and Workspace of a 3-RPS Parallel Robot

Ranjan Jha, Damien Chablat, Luc Baron

► **To cite this version:**

Ranjan Jha, Damien Chablat, Luc Baron. Influence of Design Parameters on the Singularities and Workspace of a 3-RPS Parallel Robot. Transactions of the Canadian Society for Mechanical Engineering, 2018, 42 (1), pp.30-37. 10.1139/tcsme-2017-0011 . hal-01797575

**HAL Id: hal-01797575**

**<https://hal.science/hal-01797575v1>**

Submitted on 22 May 2018

**HAL** is a multi-disciplinary open access archive for the deposit and dissemination of scientific research documents, whether they are published or not. The documents may come from teaching and research institutions in France or abroad, or from public or private research centers.

L'archive ouverte pluridisciplinaire **HAL**, est destinée au dépôt et à la diffusion de documents scientifiques de niveau recherche, publiés ou non, émanant des établissements d'enseignement et de recherche français ou étrangers, des laboratoires publics ou privés.

# Influence of Design Parameters on the Singularities and Workspace of a 3-RPS Parallel Robot

Ranjan Jha<sup>1</sup>, Damien Chablat <sup>\*2</sup>, and Luc Baron<sup>1</sup>

<sup>1</sup>Department of Mechanical Engineering,, École Polytechnique de Montréal, H3C A37 Québec, Canada

<sup>2</sup>Laboratoire des Sciences du Numérique de Nantes,, UMR CNRS 6004, Nantes, France

22 mai 2018

## Résumé

This paper presents the variations in the workspace, singularities and joint space with respect to the design parameter  $k$ , which is the ratio between the dimensions of the mobile platform to the dimensions of the base of the 3-RPS parallel manipulator. Also, the influence on the parasitic motions due to the design parameters is studied, which plays an important role in the selection of the manipulator for the desired task. The cylindrical algebraic decomposition method and Gröbner based computations are used to model the workspace and joint space with the parallel singularities in 2R1T and 3T projection spaces, where the orientation of the mobile platform is represented by using quaternions. These computations are useful to select the optimum value for the design parameter  $k$  such that the parasitic motions can be limited to specific values. Depending on the design parameter  $k$ , three different designs of the 3-RPS parallel robot are analyzed.

**Keywords :** Parallel Robot, Singularities, 3-RPS, Workspace, Kinematics , Parasitic motions

## Résumé

Cet article présente les variations de l'espace de travail, les singularités et de l'espace articulaire en fonction d'un paramètre de conception  $k$  qui est le ratio entre les dimensions de la plate-forme mobile et les dimensions de la base du manipulateur parallèle 3-RPS. On étudie également l'influence de ce paramètre de conception sur les mouvements parasites, qui joue un rôle important dans la sélection du robot pour une tâche souhaitée. La méthode de décomposition algébrique cylindrique et les bases de Gröbner sont utilisés pour modéliser l'espace de travail et l'espace

---

\*Corresponding Author

articulaire ainsi que les singularités parallèles dans les espaces de projection 2R1T et 3T, où l'orientation de la plate-forme mobile est représentée en utilisant des quaternions. Ces calculs sont utiles pour sélectionner la valeur optimale pour le paramètre de conception  $k$  de telle sorte que les mouvements parasites puissent être limités à des valeurs spécifiques. En fonction du paramètre  $k$ , trois configurations différentes du robot parallèle 3-RPS sont analysées.

**Mots clés :** Robot parallèle, Singularité, 3-RPS, Espace de travail, Cinématique, Mouvements parasites

# 1 Introduction

The study of workspace, joint space and singularities together assists the engineers and researchers in the efficient task planning and the selection of the particular configuration of the manipulator for a desired task. The accurate computation of the workspace and joint space for 3-RPS parallel robotic manipulator is a highly addressed research work across the world. It is well a known feature that this robot admits two operation modes. As there exists parasitic motions for 3-RPS parallel robot, a complete workspace analysis of this mechanism is possible, only when the analysis is done in both 2R1T (two rotational [2R] and one translational [1T]) and 3T (three translational) projection spaces for both the operation modes.

For the parallel robots with several inverse and direct kinematic solutions, the aspects are defined as the maximal singularity-free sets in the workspace or the cross-product of the joint space by the workspace. An assembly mode is associated with a solution for the Direct Kinematic Problem (DKP) and a working mode for the Inverse Kinematic Problem (IKP). The notion of aspect, previously defined in [3] for serial robots and in [24] for parallel robot with one operation mode, and in [6, 7] extended for a parallel robot with several operation modes. There are different techniques based on geometric [13, 20], discretization [5, 2, 9], and algebraic methods [25, 22, 7, 8, 16] which can be used to compute the workspace of parallel robot. Practically, a change of assembly mode may occur during the execution of a trajectory between two configurations in the workspace for the 3-RPS parallel robot with similar base and mobile platforms which is shown in [7, 15].

This paper presents the results which are obtained by applying algebraic methods for the workspace, singularities and joint space analysis of the reconfigurable 3-RPS parallel robot with changeable size ratio of moving and base platform. The cylindrical algebraic decomposition (CAD) algorithm is used to study the workspace and joint space, and a Gröbner based elimination method is used to compute the parallel singularities. The outline of this paper is as follows. Section 2 describes the architecture and kinematic equations of the manipulator. Section 3 discusses the influence of design parameter on parallel singularities. Section 4 and 5 presents the workspace and joint space analysis of the manipulator. Section 6 finally concludes the paper.

## 2 Mechanism Under Study

The robot under study is the 3-RPS parallel robot with three degrees of freedom and has been studied recently by many researchers [14, 1, 23, 17]. It is the assembly of two equilateral triangles (the base and the moving platform) by three identical RPS legs where R is a revolute passive joint, P an actuated prismatic joint and S a passive spherical joint. Thus, the revolute joint is connected to the fixed base and the spherical joint to the mobile platform.

Considering the 3-RPS parallel manipulator, as shown in Fig. 1, the fixed base

consists of an equilateral triangle with vertices  $A_1$ ,  $A_2$  and  $A_3$ , and circumradius  $g$ . The moving platform is another equilateral triangle with vertices  $B_1$ ,  $B_2$  and  $B_3$  and varying circumradius  $h$ , and circumcenter  $P$ . The two design parameters  $g$  and  $h$  are positive numbers. Connecting each of the vertices' pairs  $A_i, B_i$  ( $i = 1, 2, 3$ ) by a limb, a revolute joint lies at  $A_i$  and a spherical joint lies at  $B_i$ .  $\rho_i$  denotes the length of each limb and the motion control is done through an actuated prismatic joint. Thus, there are five parameters, namely  $g$ ,  $h$ ,  $\rho_1$ ,  $\rho_2$  and  $\rho_3$ . The  $g$  and  $h$  parameters ( $h/g = k$ ) determine the design of the manipulator (shown in fig. 2) whereas the joint parameters  $\rho_1$ ,  $\rho_2$  and  $\rho_3$  determine the motion of the robot. Spatial rotations in three dimensions can be parametrized using Euler angles [12], unit quaternions [27] or dual quaternions [14]. The quaternion representation is used for modeling the orientation as quaternions do not suffer from singularities as Euler angles do. Moreover, to transform the trigonometric equations to algebraic equations, we may either introduce the singularity of the transformation  $t = \tan(\alpha/2)$  or replace the angle  $\alpha$  by two parameters  $\cos_\alpha$  and  $\sin_\alpha$  with  $\cos_\alpha^2 + \sin_\alpha^2 = 1$ . In addition, it is easier to represent workspace sections with the quaternions than the dual quaternions because the position is simply defined the Cartesian coordinates.

A quaternion  $\mathbf{q}$  is defined by

$$\mathbf{q} = q_1 + q_2\mathbf{i} + q_3\mathbf{j} + q_4\mathbf{k} \quad (1)$$

The quaternion rotation matrix is then

$$\mathbf{Q} = \begin{bmatrix} 2q_1^2 + 2q_2^2 - 1 & -2q_1q_4 + 2q_2q_3 & 2q_1q_3 + 2q_2q_4 \\ 2q_1q_4 + 2q_2q_3 & 2q_1^2 + 2q_3^2 - 1 & -2q_1q_2 + 2q_3q_4 \\ -2q_1q_3 + 2q_2q_4 & 2q_1q_2 + 2q_3q_4 & 2q_1^2 + 2q_4^2 - 1 \end{bmatrix} \quad (2)$$

with  $q_1^2 + q_2^2 + q_3^2 + q_4^2 = 1$ . The transformation from the moving frame to the fixed frame can be described by a position vector  $OP$  and a  $3 \times 3$  rotation matrix  $\mathbf{R}$ . Let  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  be the three unit vectors defined along the axes of the moving frame, then the rotation matrix can be expressed in terms of the coordinates of  $\mathbf{u}$ ,  $\mathbf{v}$  and  $\mathbf{w}$  as :

$$\mathbf{R} = \begin{bmatrix} u_x & v_x & w_x \\ u_y & v_y & w_y \\ u_z & v_z & w_z \end{bmatrix} \quad (3)$$

The vertices of the base triangle and mobile platform triangle are

$$\mathbf{A}_1 = \begin{bmatrix} g \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{A}_2 = \begin{bmatrix} -g/2 \\ g\sqrt{3}/2 \\ 0 \end{bmatrix} \quad \mathbf{A}_3 = \begin{bmatrix} -g/2 \\ -g\sqrt{3}/2 \\ 0 \end{bmatrix} \quad (4)$$

$$\mathbf{b}_1 = \begin{bmatrix} h \\ 0 \\ 0 \end{bmatrix} \quad \mathbf{b}_2 = \begin{bmatrix} -h/2 \\ h\sqrt{3}/2 \\ 0 \end{bmatrix} \quad \mathbf{b}_3 = \begin{bmatrix} -h/2 \\ -h\sqrt{3}/2 \\ 0 \end{bmatrix} \quad (5)$$

The coordinates of  $\mathbf{b}_i$  with respect to fixed frame reference are obtained by  $\mathbf{B}_i = \mathbf{P} + \mathbf{R}\mathbf{b}_i$  for  $i = 1, 2, 3$ . Also the coordinates of the center of the mobile platform in the fixed reference is  $\mathbf{P} = [x \ y \ z]^T$ . The distance constraints yields :

$$\|\mathbf{A}_i - \mathbf{B}_i\| = \rho_i \quad \text{for } i = 1, 2, 3 \quad (6)$$

As  $A_i$  are revolute joints, the motion of the  $B_i$  are constrained in planes. This leads to the three constraint equations :

$$u_y h + y = 0 \quad (7)$$

$$y - u_y h/2 + \sqrt{3}v_y h/2 + \sqrt{3}x - \sqrt{3}u_x h/2 + 3v_x h/2 = 0 \quad (8)$$

$$y - u_y h/2 - \sqrt{3}v_y h/2 - \sqrt{3}x + \sqrt{3}u_x h/2 + 3v_x h/2 = 0 \quad (9)$$

Solving with respect to  $x$  and  $y$  we get :

$$y = -hu_y \quad (10)$$

$$x = h \left( \sqrt{3}u_x - \sqrt{3}v_y - 3u_y + 3v_x \right) \sqrt{3}/6 \quad (11)$$

In Eqs. (6, 8, 9), we substitute  $x, y$  using Eqs. (10) and (11), and  $\mathbf{u}, \mathbf{v}, \mathbf{w}$  by quaternion expressions using Eq. (2). Then Eqs. (8) and (9) become  $q_1 q_4 = 0$ . Thus, we have either  $q_1 = 0$  or  $q_4 = 0$ . This property is associated with the notion of operation mode. Let  $OM_1$  be the operation mode for  $q_1 = 0$  and  $OM_2$  for  $q_4 = 0$ . More information can be found in [7] for the definition of the Operation Modes. Figure 3 represents the two assembly modes plot for  $\rho_1 = \rho_2 = \rho_3 = 2$ , which corresponds to the operation modes  $OM^1$  and  $OM^2$ .

To obtain the algebraic equations, we replace  $\sqrt{3}$  by the variable  $s_3$  and add the equation  $s_3^2 - 3 = 0$  and the constraint  $s_3 > 0$ .

### 3 Influence of Design Parameter on Parallel Singularities

Differentiating the constraints equations with respect to time leads to the velocity model :

$$\mathbf{A}\mathbf{t} + \mathbf{B}\dot{\mathbf{q}} = 0 \quad (12)$$

where  $\mathbf{A}$  and  $\mathbf{B}$  are the parallel and serial Jacobian matrices, respectively,  $\mathbf{t}$  is the velocity of  $P$  and  $\dot{\mathbf{q}}$  the joint velocities. The parallel singularities occur whenever  $\det(\mathbf{A}) = 0$ .

As there exist parasitic motions for 3RPS parallel robot, it is possible to analyze the singularities in two different projection spaces,  $[p_z \ q_2 \ q_3]$  and  $[p_x \ p_y \ p_z]$ . The singularities of the robot in the projection space  $[z \ q_2 \ q_3]$  are presented in [7, 6]. Figures 4 and 5 represent the projection of parallel singularities in projection space  $[p_x \ p_y \ p_z]$  for  $OM^1$  and  $OM^2$ , respectively. The variations in the parallel singularities surface for different design parameter  $k$  are shown in Figures 4 and 5. Also the top view of these singularity surfaces is presented to analyze the change in the singularities surface boundaries with the change in the value of  $k$ . As the value of  $k$  increases the connectivity in the workspace region decreases, can be seen in Figures 4(e-h) and 5(e-h).

## 4 Workspace Analysis of the 3-RPS Parallel Robot

The 3-RPS parallel robot is three degrees of freedom parallel robot with two rotational [2R] and one translational [1T] motions, but still there exist two other translational motions in the  $x$  and  $y$  direction. These motions are termed as parasitic motions as they depend on other existing rotational motions. A complete workspace analysis of this mechanism is possible, only when the analysis is done in both 2R1T  $[p_z q_2 q_3]$  and 3T  $[p_x p_y p_z]$  projection spaces for  $OM_1$  and  $OM_2$ .

The workspace (resp. Joint space) analysis classifies the number of solutions of the parametric system associated with the Inverse (resp. Direct) Kinematic Problem (IKP). This method was introduced for parallel robots in [8]. The three main steps involved in the analysis are (i) Computation of a subset of the joint space (resp. workspace) where the number of solutions changes : the *Discriminant Variety* [8], (ii) Description of the complementary of the discriminant variety in connected cells : the *Generic Cylindrical Algebraic Decomposition*. and (iii) Connecting the cells belonging to the same connected component in the counterpart of the discriminant variety : *interval comparisons*. Discriminant varieties can be computed using basic and well known tools from computer algebra such as Gröbner bases [11] and a full package computing such objects in a general framework is available in Maple software through the RootFinding[Parametric] package. The CAD implemented in the SIROPA library has been used to compute the aspects into a set of cells where algebraic equations define its boundaries and a sample point in each one [8] for the 2PRR-RPR parallel robot. For example, the CAD can provide a formal decomposition of the joint space in cells where the polynomials ( $A$ ) and  $\det(\mathbf{B})$  have a constant sign and the number of solutions for the DKP is constant [10].

The workspace of the robot is a cylinder in the projection space  $[z q_2 q_3]$  if there are no joint limits on the actuated joints, a is presented in [6]. The workspace analysis can be done by dividing it into a set of aspects. In other words, an aspect is the largest connected region without any singularity of the corresponding operation mode. As there are several solutions for the DKP in the same aspect, non-singular assembly mode trajectories are possible, which is shown in [14, 7] for 3RPS parallel robot. The variation in the workspace boundaries due to the design parameter  $k$  for both the operation modes is shown in Fig. 6. Blue and red regions corresponds to the four solutions for the IKP for  $\det(\mathbf{A}) > 0$  and  $\det(\mathbf{A}) < 0$ , respectively. These analysis can be useful for the researchers or engineers to select the optimum value for the design parameter such that the parasitic motions can be limited to specific values. From Fig. 6, it can be depicted that as the value of  $k$  increases, singularity becomes more complex and there exists a larger area (aspects) without singularities for larger values of  $k$ .

## 5 Joint space Analysis of the 3-RPS Parallel Robot

The joint space analysis allows the characterization of the regions where the number of real solutions for the direct kinematic model is constant. Using CAD, we can do this study on sections of the joint space. The calculation for the full joint space is

possible, but the number of cells obtained is too large for the display capabilities of Maple. Without taking into account the notion of operation mode, Figure 7 depicts the regions with 4, 8, 12 or 16 solutions for the DKP. The maximum number for the DKP for each operation mode is 8.

Cuspidal configurations are associated with second-order degeneracies that appear for triply coalesced configurations. These configurations play an important role in the path planning because they are directly linked to the non-singular assembly mode changing trajectories [19, 26, 18, 4]. A state of the art for the computation of the cusp points is given in [21]. The variation in the joint space boundaries due to the design parameter  $k$  is shown in Figure 8. The slice of joint space is computed for the three different values of  $\rho_1$  and  $k$ . For  $\rho_1 = 2$ , as the value of  $k$  increases the cusp points corresponding to the region with 16 numbers of solution for DKP disappears (in Fig. 8(d), 8(e) and 8(f)). The increase in the area of the joint space with an increase in the value of  $k$ , shows the more feasible actuation sets compared with the configuration with lesser value of  $k$ .

## 6 Conclusions

This work reports the variations in the workspace and the joint space with respect to the design parameter  $k$  of the 3-RPS parallel manipulator. The cylindrical algebraic decomposition method and Gröbner based computations are used to model the workspace and joint space with the parallel singularities in 3T and 2R1T projection spaces, where the orientation of the mobile platform is represented using quaternions. Depending on the design parameter  $k$ , three different designs of the 3-RPS parallel manipulator are analyzed. A comparative study on the workspace and joint space for different design parameter  $k$ , plays an important role in the selection of the manipulator for the specific task or for the trajectory planning. It is shown that as the value of  $k$  increases, singularity becomes more complex and there exists a larger area (aspects) without singularities for larger values of  $k$ .

## Références

- [1] Sabbavarapu Ramana Babu, Vegesina Ramachandra Raju, and Koona Ramji. Design for optimal performance of 3-rps parallel manipulator using evolutionary algorithms. *Transactions of the Canadian Society for Mechanical Engineering*, 37(2):135, 2013.
- [2] Ilian A Bonev and Jeha Ryu. A new approach to orientation workspace analysis of 6-dof parallel manipulators. *Mechanism and machine theory*, 36(1):15–28, 2001.
- [3] Paul Borrel and Alain Liégeois. A study of multiple manipulator inverse kinematic solutions with applications to trajectory planning and workspace determination. In *Robotics and Automation. Proceedings. 1986 IEEE International Conference on*, volume 3, pages 1180–1185. IEEE, 1986.



- [4] Stéphane Caro, Philippe Wenger, and Damien Chablat. Non-singular assembly mode changing trajectories of a 6-dof parallel robot. In *Proceedings of the ASME 2012 International Design Engineering Technical Conferences & Computers and Information in Engineering Conference IDETC/CIE*, pages 12–15, 2012.
- [5] Gianni Castelli, Erika Ottaviano, and Marco Ceccarelli. A fairly general algorithm to evaluate workspace characteristics of serial and parallel manipulators#. *Mechanics based design of structures and machines*, 36(1) :14–33, 2008.
- [6] D Chablat, R Jha, F Rouillier, and G Moroz. Workspace and joint space analysis of the 3-rps parallel robot. In *ASME 2014 International Design Engineering Technical Conferences and Computers and Information in Engineering Conference*, pages V05AT08A056–V05AT08A056. American Society of Mechanical Engineers, 2014.
- [7] Damien Chablat, Ranjan Jha, Fabrice Rouillier, and Guillaume Moroz. Non-singular assembly mode changing trajectories in the workspace for the 3-rps parallel robot. In *Advances in Robot Kinematics*, pages 149–159. Springer, 2014.
- [8] Damien Chablat, Guillaume Moroz, and Philippe Wenger. Uniqueness domains and non singular assembly mode changing trajectories. In *Robotics and Automation (ICRA), 2011 IEEE International Conference on*, pages 3946–3951. IEEE, 2011.
- [9] Damien Chablat, Ph Wenger, Félix Majou, and J-P Merlet. An interval analysis based study for the design and the comparison of three-degrees-of-freedom parallel kinematic machines. *The International Journal of Robotics Research*, 23(6) :615–624, 2004.
- [10] George E Collins. Quantifier elimination for real closed fields by cylindrical algebraic decomposition. In *Automata Theory and Formal Languages 2nd GI Conference Kaiserslautern, May 20–23, 1975*, pages 134–183. Springer, 1975.
- [11] David A Cox, John Little, and Donal O’shea. *Using algebraic geometry*, volume 185. Springer Science & Business Media, 2006.
- [12] J Gallardo, H Orozco, JM Rico, CR Aguilar, and L Perez. Acceleration analysis of 3-rps parallel manipulators by means of screw theory. In *Parallel Manipulators, New Developments*. InTech, 2008.
- [13] Clement Gosselin. Determination of the workspace of 6-dof parallel manipulators. *ASME Journal of Mechanical Design*, 112(3) :331–336, 1990.
- [14] M Husty, J Schadlbauer, S Caro, and P Wenger. Non-singular assembly mode change of 3-rps manipulators. In *Proceedings of CK2013, International Workshop on Computational Kinematics, Barcelone*, 2013.
- [15] R Jha. *Contributions to the Performance Analysis of Parallel Robots*. PhD thesis, Ph. D. Thesis, École Centrale de Nantes, 2016.
- [16] Ranjan Jha, Damien Chablat, Fabrice Rouillier, and Guillaume Moroz. Workspace and singularity analysis of a delta like family robot. In *Robotics and Mechatronics*, pages 121–130. Springer, 2016.

- [17] Binbin Lian, Tao Sun, Yimin Song, and Xiaoli Wang. Passive and active gravity compensation of horizontally-mounted 3-rps parallel kinematic machine. *Mechanism and Machine Theory*, 104 :190–201, 2016.
- [18] E Macho, O Altuzarra, Ch Pinto, and A Hernandez. Transitions between multiple solutions of the direct kinematic problem. *Advances in Robot Kinematics : Analysis and Design*, pages 301–310, 2008.
- [19] P Ross McAree and R William Daniel. An explanation of never-special assembly changing motions for 3–3 parallel manipulators. *The International Journal of Robotics Research*, 18(6) :556–574, 1999.
- [20] Jean-Pierre Merlet. Geometrical determination of the workspace of a constrained parallel manipulator. *Advances in Robot Kinematics, Ferrare, Italy, Sept*, pages 7–9, 1992.
- [21] Guillaume Moroz, Fabrice Rouiller, Damien Chablat, and Philippe Wenger. On the determination of cusp points of 3-rpr parallel manipulators. *Mechanism and Machine Theory*, 45(11) :1555–1567, 2010.
- [22] Erika Ottaviano, Manfred Husty, and Marco Ceccarelli. Identification of the workspace boundary of a general 3-r manipulator. *Journal of Mechanical Design*, 128(1) :236–242, 2006.
- [23] J Schadlbauer, DR Walter, and ML Husty. The 3-rps parallel manipulator from an algebraic viewpoint. *Mechanism and Machine Theory*, 75 :161–176, 2014.
- [24] Philippe Wenger and Damien Chablat. Definition sets for the direct kinematics of parallel manipulators. In *Advanced Robotics, 1997. ICAR'97. Proceedings., 8th International Conference on*, pages 859–864. IEEE, 1997.
- [25] Mazen Zein, Philippe Wenger, and Damien Chablat. An exhaustive study of the workspace topologies of all 3r orthogonal manipulators with geometric simplifications. *Mechanism and Machine Theory*, 41(8) :971–986, 2006.
- [26] Mazen Zein, Philippe Wenger, and Damien Chablat. Non-singular assembly-mode changing motions for 3-rpr parallel manipulators. *Mechanism and Machine Theory*, 43(4) :480–490, 2008.
- [27] Wan-Yong Zhou, Wu-Yi Chen, Xian-Guo Han, and Fang Zou. The kinematics algorithm of 3rps mechanism with compound sphere joint. *Journal of Harbin Institute of Technology*, 6 :013, 2013.

# Figures

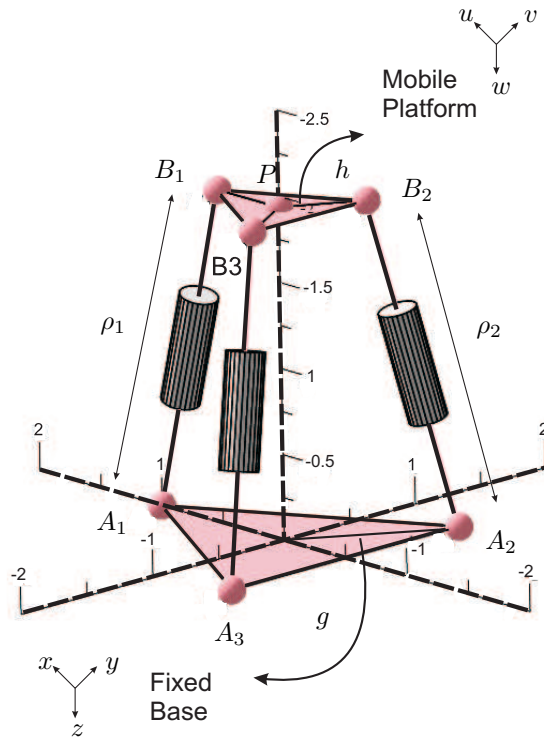


FIGURE 1 – 3-RPS parallel robot

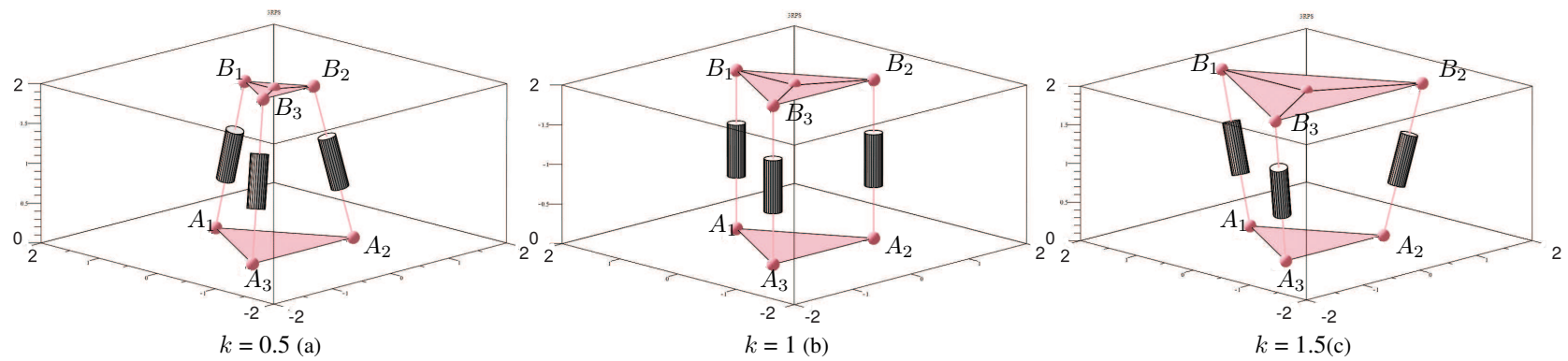


FIGURE 2 – Virtual model of 3-RPS parallel manipulator with different design parameter  $k$ ,  $k = 0.5$  (a),  $k = 1$  (b),  $k = 1.5$  (c)

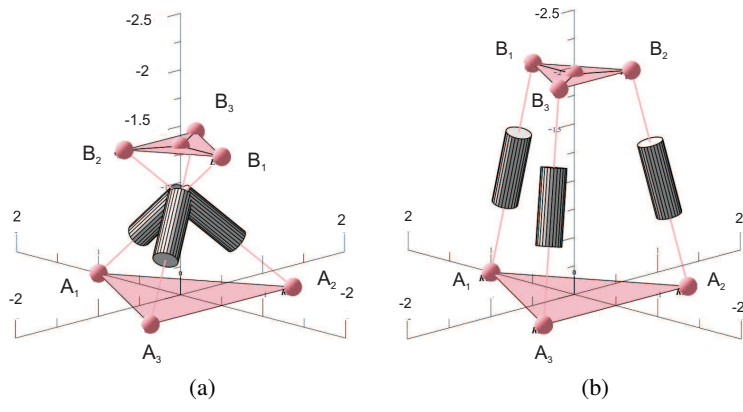


FIGURE 3 – Two assembly modes plot corresponds to operation modes  $OM_1$  (a) and  $OM_2$  (b) for  $\rho_1 = \rho_2 = \rho_3 = 2$